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Abstract

The acyclicity of individual improvements in a generalized congestion game (where the sums of local utilities are replaced with arbitrary aggregation rules) can be established with a Rosenthal-style construction if aggregation rules of all players are "quasi-separable." Every universal separable ordering on a finite set can be represented as a combination of addition and lexicography.

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1 Introduction

Rosenthal's (1973) construction proved extremely fruitful in the study of networks games, group formation, etc.; it was also a source of inspiration for researchers, even in areas rather distant in a technical sense (Milchtaich, 1996; Holzman and Law-Yone, 1997; Konishi, Le Breton, and Weber, 1997; Sandholm, 2010; McLennan, Monteiro, and Tourky, 2011; Harks, Klimm, and Möhring, 2011). Monderer and Shapely (1996) built their theory of potential games around it.

Kukushkin (2007) showed the crucial role of addition by introducing the notion of a generalized congestion game, where the players may aggregate local utilities in an arbitrary way rather than just summing them up. Then the sum was proven to be necessary (assuming continuity and strict monotonicity) to ensure the existence of a Nash equilibrium regardless of all other characteristics of the game.

This paper follows the same line of inquiry, but with even more purely technical flavor. First, we show that Rosenthal's construction can be reproduced, with only trivial modification, if the preferences of all players are "quasi-separable," i.e., consistent with a *universal separable ordering* (Proposition 1). This applies, e.g., to the minimum ("weakest-link") aggregation, which is not separable, but is consistent with the leximin ordering.

Second, we show that every universal separable ordering on a finite set can be represented as a combination of addition and lexicography (Theorem 2). This result can be viewed as a discrete analogue of the famous Debreu–Gorman Theorem (Debreu, 1960; Gorman, 1968; see also Wakker,

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1989), even though it cannot claim anything approaching the importance of the latter. We abandon the assumption that the ordering is continuous on a connected domain; instead, we require the ordering to be applicable to *every* Cartesian power of a given finite set (like the sum, or leximin/leximax).

The message of this paper can be summarized in two opposite ways. On one hand, we found that Rosenthal's construction can work for a broader class of preferences than originally envisaged. On the other hand, this generalization can be viewed as inessential, and hence Theorem 2 interpreted as an impossibility result.

The following section contains basic definitions; Sections 3 and 4, the main results. Some questions of secondary importance are discussed in Sections 5 and 6. Section 7 summarizes the message of the paper.

2 Basics

A *strategic game* Γ is defined by a finite set N of players, and a set X_i of *strategies* and a *utility function* u_i on the set $X_N := \prod_{i \in N} X_i$ of *strategy profiles* for each $i \in N$. We always assume each X_i to be finite.

Given $i \in N$, we denote $X_{-i} := \prod_{j \neq i} X_j$. With every strategic game, we associate the *individual improvement* relation on X_N ($i \in N$, y_N, $x_N \in X_N$):

$$y_N \triangleright_i x_N \rightleftharpoons [y_{-i} = x_{-i} \; \& \; u_i(y_N) > u_i(x_N)];$$
$$y_N \triangleright x_N \rightleftharpoons \exists i \in N \, [y_N \triangleright_i x_N].$$

A *Nash equilibrium* is a *maximizer* of $\triangleright$, i.e., a strategy profile $x_N \in X_N$ such that $y_N \triangleright x_N$ holds for no $y_N \in X_N$.

In the terminology of Monderer and Shapley (1996), a function $P : X_N \to \mathbb{R}$ is an *exact potential* of Γ if $u_i(y_N) - u_i(x_N) = P(y_N) - P(x_N)$ whenever $i \in N$, $y_N, x_N \in X_N$, and $y_{-i} = x_{-i}$. A function $P : X_N \to \mathbb{R}$ is a *generalized ordinal potential* of the game if $P(y_N) > P(x_N)$ whenever $y_N, x_N \in X_N$ and $y_N \triangleright x_N$. Clearly, every exact potential is also a generalized ordinal potential.

Being interested in games with *ordinal* preferences here and following Kukushkin (1999), we define a *potential* of Γ as an irreflexive and transitive relation $\succ\!\!\succ$ on X_N satisfying

$$\forall x_N, y_N \in X_N \, [y_N \triangleright x_N \Rightarrow y_N \succ\!\!\succ x_N]. \tag{1}$$

Since X_N is finite, the existence of a potential in our sense is equivalent to the existence of a generalized ordinal potential (Monderer and Shapley 1996, Lemma 2.5); and it obviously implies the existence of a Nash equilibrium.

A *congestion game* (Rosenthal, 1973) may have an arbitrary finite set N of players, while strategies and preferences are defined by the following construction. There are a finite set A of *facilities* and a *local utility function* $\varphi_\alpha \colon \mathbb{N} \to \mathbb{R}$ for each $\alpha \in \mathrm{A}$; each $x_i \in X_i$ ($i \in N$) is a subset of A. Given $x_N \in X_N$, we denote $N(\alpha, x_N) := \{i \in N \mid \alpha \in x_i\}$ and $n(\alpha, x_N) := \#N(\alpha, x_N)$ for each $\alpha \in \mathrm{A}$. Now the utility function of each player i is

$$u_i(x_N) := \sum_{\alpha \in x_i} \varphi_\alpha(n(\alpha, x_N)). \tag{2}$$

In the most popular interpretation, A is the set of edges of a (directed) graph and each X_i consists of paths with a given origin and a given target. Under this interpretation, it is natural to assume each φ_α to be decreasing (congestion proper). However, such assumptions are not needed for the most basic results about congestion games; they were not made in Rosenthal (1973), and are not made here.

For every congestion game, the function

$$P(x_N) := \sum_{\alpha \in \mathrm{A}} \sum_{k=1}^{n(\alpha, x_N)} \varphi_\alpha(k) \tag{3}$$

is an exact potential (Rosenthal, 1973); therefore, the strict ordering on X_N represented by the function, $y_N \ggcurly x_N \rightleftharpoons P(y_N) > P(x_N)$, is a potential in the sense of (1).

The notion of a *generalized congestion game* was introduced in Kukushkin (2007). Roughly speaking, it is a game with the same structure of strategy sets as a congestion game proper, but with the sum in (2) replaced with an arbitrary function. Somewhat simplifying that notion, we assume that each player is characterized by a *universal aggregation rule*, i.e., an infinite sequence of symmetric functions $U_i^{(m)} \colon V^m \to \mathbb{R}$ ($m \in \mathbb{N}$), where $V \subseteq \mathbb{R}$, and that her utility function is

$$u_i(x_N) := U_i^{(\#x_i)}\big(\langle \varphi_\alpha(n(\alpha, x_N)) \rangle_{\alpha \in x_i}\big).$$

Naturally, this formula only makes sense if all values of the functions φ_α belong to V. Since $U_i^{(\#x_i)}$ is symmetric, there is no need to specify an order on x_i. In the following, we employ the term *an unordered cortege* (of the length $\#x_i$), i.e., a collection of real numbers with possible repetitions. It is natural to assume every $U_i^{(m)}$ to be increasing in its arguments, but such an explicit assumption is not needed here.

3 Quasiseparable aggregation

A *universal separable ordering* $\succeq$ on $V \subseteq \mathbb{R}$ is an infinite sequence of orderings, i.e., reflexive, transitive, and total binary relations, $\succeq^m$ on V^m ($m \in \mathbb{N}$; we denote $\succ^m$ and $\sim^m$, respectively, its asymmetric and symmetric components) such that

1. $\succeq^1$ is the standard order $\geq$ on V induced from $\mathbb{R}$;
2. for every permutation σ of $\{1, \ldots, m\}$,

   $$\langle v_1, \ldots, v_m \rangle \sim^m \langle v_{\sigma(1)}, \ldots, v_{\sigma(m)} \rangle$$

   (symmetry); by this condition, every relation $\succeq^m$ can be perceived as defined on the set of unordered corteges of the length m;
3. for every $m' > m \geq 1$, every $\langle v_1, \ldots, v_{m'} \rangle \in V^{m'}$, and every $\langle v'_1, \ldots, v'_m \rangle \in V^m$,

   $$\langle v_1, \ldots, v_m, v_{m+1}, \ldots, v_{m'} \rangle \succeq^{m'} \langle v'_1, \ldots, v'_m, v_{m+1}, \ldots, v_{m'} \rangle \iff \langle v_1, \ldots, v_m \rangle \succeq^m \langle v'_1, \ldots, v'_m \rangle$$

   (separability).

A universal aggregation rule U is *consistent* with a universal separable ordering $\succeq$ if there is an infinite sequence $\{\bar{v}_m \in V\}_{m=2,3,\ldots}$ such that for every $m' \geq m$, every $\langle v_1, \ldots, v_m \rangle \in V^m$, and every $\langle v'_1, \ldots, v'_{m'} \rangle \in V^{m'}$,

$$U^{(m')}(v'_1, \ldots, v'_{m'}) > U^{(m)}(v_1, \ldots, v_m) \Rightarrow \langle v'_1, \ldots, v'_{m'} \rangle \succ^{m'} \langle v_1, \ldots, v_m, \bar{v}_{m+1}, \ldots, \bar{v}_{m'} \rangle \tag{4a}$$

and

$$U^{(m)}(v_1, \ldots, v_m) > U^{(m')}(v'_1, \ldots, v'_{m'}) \Rightarrow \langle v_1, \ldots, v_m, \bar{v}_{m+1}, \ldots, \bar{v}_{m'} \rangle \succ^{m'} \langle v'_1, \ldots, v'_{m'} \rangle. \tag{4b}$$

It seems reasonable to call such aggregation rules *quasiseparable*.

Proposition 1. *Let $\succeq$ be a universal separable ordering on $V \subseteq \mathbb{R}$, and Γ be a generalized congestion game where $\varphi_\alpha(\mathbb{N}) \subseteq V$ for each $\alpha \in \mathrm{A}$ and each player's aggregation rule U_i is consistent with $\succeq$. Then Γ admits a potential in the sense of (1).*

Proof. Let $\bar{v}^i_k$ be constants associated with the aggregation rule used by player i; we denote $M_i := \max_{x_i \in X_i} \# x_i$ and $M := \sum_{i \in N} M_i$. With every $x_N \in X_N$, we associate an unordered cortege:

$$\varkappa(x_N) := \Big\langle \langle \varphi_\alpha(k) \rangle_{\alpha \in \mathrm{A}, k=1,\ldots,n(\alpha,x_N)}, \langle \bar{v}^i_k \rangle_{i \in N, k=\#x_i+1,\ldots,M_i} \Big\rangle$$

(assuming the convention that facilities $\alpha \in \mathrm{A}$ with $n(\alpha, x_N) = 0$ are not represented in $\varkappa(x_N)$ at all). It is easy to check that $\sum_i \# x_i = \sum_\alpha n(\alpha, x_N)$; therefore, the length of $\varkappa(x_N)$ is M for every $x_N \in X_N$. If we show that $y_N \triangleright x_N$ implies $\varkappa(y_N) \succ^M \varkappa(x_N)$, (1) will be proven with $\succ^M$ as $\ggcurly$.

Let $y_N \triangleright_i x_N$, i.e., $u_i(y_N) > u_i(x_N)$ and $y_{-i} = x_{-i}$. A is partitioned into four disjoint subsets: $\mathrm{A}^0 := x_i \cap y_i$, $\mathrm{A}^+ := y_i \setminus x_i$, $\mathrm{A}^- := x_i \setminus y_i$, $\mathrm{A}^* := \mathrm{A} \setminus (x_i \cup y_i)$; thus, $x_i = \mathrm{A}^0 \cup \mathrm{A}^-$ and $y_i = \mathrm{A}^0 \cup \mathrm{A}^+$. Denoting $\bar{m} := \max\{\# x_i, \# y_i\}$, we define

$$\begin{aligned} \varkappa_{-i} := \Big\langle & \langle \varphi_\alpha(k) \rangle_{\alpha \in \mathrm{A}^0, k=1,\ldots,n(\alpha,x_N)-1=n(\alpha,y_N)-1}, \langle \varphi_\alpha(k) \rangle_{\alpha \in \mathrm{A}^+, k=1,\ldots,n(\alpha,x_N)=n(\alpha,y_N)-1}, \\ & \langle \varphi_\alpha(k) \rangle_{\alpha \in \mathrm{A}^-, k=1,\ldots,n(\alpha,y_N)=n(\alpha,x_N)-1}, \langle \varphi_\alpha(k) \rangle_{\alpha \in \mathrm{A}^*, k=1,\ldots,n(\alpha,x_N)=n(\alpha,y_N)}, \\ & \langle \bar{v}^j_k \rangle_{j \in N, j \neq i, k=\#x_j+1,\ldots,M_j}, \langle \bar{v}^i_k \rangle_{k=\bar{m}+1,\ldots,M_i} \Big\rangle \end{aligned}$$

(under a similar convention).

Let $\# y_i \geq \# x_i$. We define

$$\varkappa_i(x_N) := \Big\langle \langle \varphi_\alpha(n(\alpha, x_N)) \rangle_{\alpha \in \mathrm{A}^0 \cup \mathrm{A}^-}, \langle \bar{v}^i_k \rangle_{k=\#x_i+1,\ldots,\#y_i} \Big\rangle$$

and

$$\varkappa_i(y_N) := \Big\langle \varphi_\alpha(n(\alpha, y_N)) \Big\rangle_{\alpha \in \mathrm{A}^0 \cup \mathrm{A}^+} \ \big[= \Big\langle \varphi_\alpha(n(\alpha, y_N)) \Big\rangle_{\alpha \in y_i} \big].$$

Since $u_i(y_N) > u_i(x_N)$, we have $\varkappa_i(y_N) \succ^{\bar{m}} \varkappa_i(x_N)$ by condition (4a).

If $\# y_i \leq \# x_i$, we define

$$\varkappa_i(x_N) := \Big\langle \varphi_\alpha(n(\alpha, x_N)) \Big\rangle_{\alpha \in \mathrm{A}^0 \cup \mathrm{A}^-} \ \big[= \Big\langle \varphi_\alpha(n(\alpha, x_N)) \Big\rangle_{\alpha \in x_i} \big]$$

and

$$\varkappa_i(y_N) := \Big\langle \langle \varphi_\alpha(n(\alpha, y_N)) \rangle_{\alpha \in \mathrm{A}^0 \cup \mathrm{A}^+}, \langle \bar{v}^i_k \rangle_{k=\#y_i+1,\dots,\#x_i} \Big\rangle.$$

In either case, $\varkappa(x_N) = \langle \varkappa_{-i}, \varkappa_i(x_N) \rangle$ and $\varkappa(y_N) = \langle \varkappa_{-i}, \varkappa_i(y_N) \rangle$. Conditions $u_i(y_N) > u_i(x_N)$ and (4b) imply $\varkappa_i(y_N) \succ^{\bar{m}} \varkappa_i(x_N)$. Now we have $\varkappa(y_N) \succ^M \varkappa(x_N)$ by separability. □

The simplest and most important example of a universal separable ordering is given by the additive aggregation rule:

$$v' \succeq^m v \iff \sum_{k=1}^{m} \nu(v'_k) \geq \sum_{k=1}^{m} \nu(v_k), \tag{5}$$

where $\nu : \mathbb{R} \to \mathbb{R}$ is strictly increasing. Thus, Rosenthal's (1973) congestion games are covered by Proposition 1 with this ordering, $\nu(v) = v$, and $\bar{v}_m = 0$ for all m. Moreover, the construction in the proof generates exactly the potential (3) in this case.

At a first glance, different $\nu(\cdot)$ and $\bar{v}_m \neq 0$ provide a more general result, but this is, in a sense, an illusion; it may be worthwhile to consider the situation in more detail. Let each player i in a generalized congestion game Γ use a universal aggregation rule U_i consistent with the additive ordering (5); the conditions (4) imply that player i's utility function is (up to a monotonic transformation)

$$u_i(x_N) = \sum_{\alpha \in x_i} \nu(\varphi_\alpha(n(\alpha, x))) \; + \sum_{k=\#x_i+1}^{M_i} \bar{v}^i_k.$$

Obviously, we can represent Γ as a congestion game, redefining $\varphi^*_\alpha(k) = \nu(\varphi_\alpha(k))$, adding to A new facilities (i, m), $i \in N$, $1 \leq m \leq M_i$, defining $\varphi^*_{(i,m)}(1) = \bar{v}^i_m$, and replacing each $x_i \in X_i$ with $x_i \cup \{(i, \#x_i + 1), \dots, (i, M_i)\}$.

A number of similar "pseudo-generalizations" of Rosenthal's construction were discussed in Kukushkin (2007, Section 4).

Another example of a universal separable ordering is the leximin: when comparing two lists of local utility values, we start with the worst in either list; in the case of equality, we move to the second worst, etc. The minimum ("weakest-link") aggregation,

$$U^{(m)}(v_1, \dots, v_m) := \min_{k=1,\dots,m} \nu(v_k),$$

is consistent with this ordering (strictly speaking, we must modify our definition in this case, allowing $\bar{v}^i_k = +\infty$) although not separable itself. A similar connection exists between the maximum ("best-shot") aggregation and the leximax ordering. Thus, Rosenthal's construction of a potential, as modified in the proof of Proposition 1, can work in both these cases.

Finally, we may go ordinal the whole way, and assume that the preferences of the players may be described by orderings without numeric representations. Then the leximin or leximax orderings themselves will be eligible.

4 Universal separable orderings on a finite set

Theorem 2. *For every finite set $V \subseteq \mathbb{R}$ and every universal separable ordering $\succeq$ on V, there are a natural number $n < \#V$ and a mapping $\mu\colon V \to \mathbb{R}^n$ such that*

$$\langle v'_1, \ldots, v'_m \rangle \succeq^m \langle v_1, \ldots, v_m \rangle \iff \sum_{k=1}^{m} \mu(v'_k) \geq_{\text{Lex}} \sum_{k=1}^{m} \mu(v_k) \tag{6}$$

for every $m \in \mathbb{N}$ and $v'_1, \ldots, v'_m, v_1, \ldots, v_m \in V$, where the sums in the right-hand side are understood coordinate-wise, and $\geq_{\text{Lex}}$ denotes the lexicographic order on $\mathbb{R}^n$: first the first coordinate matters, then the second, etc.

Proof. Whenever $v, v' \in V$, it will be convenient to denote the ordered pair $[v, v']$ and call it an *interval*. An interval $[v, v']$ is *positive* if $v' \geq v$. A formal sum $\sum_{k=1}^{m} [v_k, v'_k]$ is called *positive* if

$$\langle v'_1, \ldots, v'_m \rangle \succeq^m \langle v_1, \ldots, v_m \rangle. \tag{7}$$

The empty sum is also assumed positive. Since some intervals in such a sum may be identical, we also have a notion of a positive linear combination $\sum_{k=1}^{m} \theta_k [v_k, v'_k]$ with nonnegative integer θ_k. Assuming $-[v_k, v'_k] = [v'_k, v_k]$, we extend the notion to negative θ_k as well.

We denote $\mathbb{Q}$ the field of rational numbers and $\mathfrak{Q}$ the vector space (over $\mathbb{Q}$) of all formal linear combinations $\sum_{k=1}^{m} r_k [v_k, v'_k]$ of positive intervals in V with rational coefficients. Then we define $\mathfrak{Q}_+$ as the set of $I \in \mathfrak{Q}$ that are positive in the sense of (7) or become positive after multiplication by an $n \in \mathbb{N}$.

Lemma 4.1. *$\mathfrak{Q}_+$ is a half-space in $\mathfrak{Q}$, i.e.,*

$$\forall I, I' \in \mathfrak{Q}_+ \left[(I + I') \in \mathfrak{Q}_+\right]; \tag{8a}$$

$$\forall I \in \mathfrak{Q}_+ \, \forall r \in \mathbb{Q} \left[r \geq 0 \Rightarrow rI \in \mathfrak{Q}_+\right]; \tag{8b}$$

$$\forall I \in \mathfrak{Q} \left[I \in \mathfrak{Q}_+ \;\; or \;\; (-I) \in \mathfrak{Q}_+\right]. \tag{8c}$$

Proof. We start with (8a). If nI and $n'I'$ are positive integer combinations, then so are $n'nI$ and $n'nI'$ too. Let $n'nI = \sum_{k=1}^{m} [v_k, v'_k]$ and $n'nI' = \sum_{k=m+1}^{m'} [v_k, v'_k]$. Applying (7) and the separability of $\succeq^m$, we obtain

$$\langle v'_1, \ldots, v'_m, v'_{m+1}, \ldots, v'_{m'} \rangle \succeq^{m'} \langle v_1, \ldots, v_m, v'_{m+1}, \ldots, v'_{m'} \rangle \succeq^{m'} \langle v_1, \ldots, v_m, v_{m+1}, \ldots, v_{m'} \rangle,$$

i.e., $n'n(I + I')$ is a positive integer combination. The proof of (8b) is even simpler. (8c) immediately follows from the completeness of $\succeq^m$. □

Now we define

$$I' \geqslant I \rightleftharpoons (I' - I) \in \mathfrak{Q}_+$$

for all $I', I \in \mathfrak{Q}$. Clearly, $\geqslant$ is an ordering; we define $\gg$ and $\simeq$ as its asymmetric and symmetric components, respectively. By Lemma 4.1, $\geqslant$ is consistent with addition in a natural sense.

Since every $\succeq^m$ is symmetric, we have

$$[v, v'] + [v', v''] \simeq [v, v''] \tag{9}$$

whenever $v, v', v'' \in V$ and $v \leq v' \leq v''$.

Let $I', I \in \mathfrak{Q}$ and $I \gg 0$; we say that I' *is not Archimedean dominated by* I, $I' \ggg I$, if there is an integer k such that $kI' \gg I$. For $I \ll 0$, we define $I' \ggg I \rightleftharpoons \exists k\,[kI' \gg -I]$. Adding $I \ggg 0$ by definition for all $I \in \mathfrak{Q}$, we obtain an ordering; its asymmetric and symmetric components are denoted $\ggg$ and $\approx$, respectively. When $I' \approx I$, we say that I' and I *have the same Archimedean rank*. Thus, $\mathfrak{Q}$ is partitioned into equivalence classes of $\approx$. By definition, I and $-I$ have the same Archimedean rank for every $I \in \mathfrak{Q}$. If $I' \ggg I$, then $I' \gg I \gg -I'$ if $I' \gg 0$, while $-I' \gg I \gg I'$ otherwise.

Whenever $I_0 \gg 0$ and $I_0 \ggg I \geqslant 0$, we define

$$I/I_0 := \sup\{r \in \mathbb{Q} \mid I \geqslant rI_0\} \in \mathbb{R}$$

(an attempt to apply the definition to $I \ggg I_0$ would produce $I/I_0 = +\infty$). When $I \ll 0$, we define $I/I_0 := -[(-I)/I_0] = \inf\{r \in \mathbb{Q} \mid rI_0 \geqslant I\}$.

Lemma 4.2. *Let $I, I', I_0 \in \mathfrak{Q}$, $I_0 \gg 0$, $I_0 \ggg I'$, $I_0 \ggg I$, and $r \in \mathbb{Q}$. Then*

$$(I' + I)/I_0 = (I'/I_0) + (I/I_0);$$

$$(rI)/I_0 = r(I/I_0);$$

$$I_0 \ggg I \iff I/I_0 = 0.$$

Proof. The proof consists of rather tedious checks. Let $I' \gg 0$ and $I \gg 0$; then for every $r \in \mathbb{Q}$ such that $r < (I'/I_0) + (I/I_0)$, we can find $r_1, r_2 \in \mathbb{Q}$ such that $r_1 + r_2 = r$, $r_1 < I'/I_0$, and $r_2 < I/I_0$. By definition, $I' \gg r_1 I_0$ and $I \gg r_2 I_0$, hence $(I' + I) \gg rI_0$; since r was arbitrary, $(I' + I)/I_0 \geq (I'/I_0) + (I/I_0)$. Conversely, for every $r \in \mathbb{Q}$ such that $r > (I'/I_0) + (I/I_0)$, we can find $r_1, r_2 \in \mathbb{Q}$ such that $r_1 + r_2 = r$, $r_1 > I'/I_0$, and $r_2 > I/I_0$. By definition, $I' \ll r_1 I_0$ and $I \ll r_2 I_0$, hence $(I' + I) \ll rI_0$; since r was arbitrary, $(I' + I)/I_0 \leq (I'/I_0) + (I/I_0)$.

Turning to negative intervals, it is enough to consider $I' \gg 0$, $I \gg 0$, and $I' - I \gg 0$; then for every $r \in \mathbb{Q}$ such that $r < (I'/I_0) - (I/I_0)$, we can find $r_1, r_2 \in \mathbb{Q}$ such that $r_1 - r_2 = r$, $r_1 < I'/I_0$, and $r_2 > I/I_0$. By definition, $I' \gg r_1 I_0$ and $I \ll r_2 I_0$, hence $(I' - I) \gg rI_0$; since r was arbitrary, $(I' - I)/I_0 \geq (I'/I_0) - (I/I_0)$. The converse inequality is obtained in a similar way.

Checking the second equality, we may assume $I \gg 0$ and $r > 0$; then $rI \geqslant rr'I_0 \iff I \geqslant r'I_0$.

As to the last equivalence, it is again sufficient to consider $I \gg 0$. If $nI \geqslant I_0$, then $I/I_0 \geq 1/n > 0$. Conversely, if $I/I_0 > 0$, then $I \geqslant rI_0$ for every $r \in \mathbb{Q}$ such that $0 < r < I/I_0$, hence $(1/r)I \geqslant I_0$, hence $I \ggg I_0$. □

Lemma 4.3. *For every finite-dimensional vector subspace $L \subseteq \mathfrak{Q}$, there is a natural number $n \leq \dim L$ and a mapping $\lambda\colon L \to \mathbb{R}^n$ such that λ is linear over $\mathbb{Q}$ and*

$$\forall I', I \in L\,[I' \geqslant I \iff \lambda(I') \geq_{\text{Lex}} \lambda(I)]. \tag{11}$$

Proof. Let $I_1, \dots, I_h$ be a basis of L; without restricting generality, $I_1 \ggg I_k \gg 0$ for all k. For every $I \in L$, we set $q(I) := I/I_1$. By Lemma 4.2, $q\colon L \to \mathbb{R}$ is linear over $\mathbb{Q}$. If $\dim L = 1$, then $L = \{rI_1\}_{r\in\mathbb{Q}}$ and we are home, by (8b), with $\lambda := q$.

Otherwise, we argue by induction in $\dim L$. Since $q(I_1) = 1$, the kernel of q, $K = \{I \in L \mid q(I) = 0\}$, is a proper vector subspace of L. By the induction hypothesis, there is a linear operator $\lambda'\colon K \to \mathbb{R}^m$ with $m < \dim L$ representing $\geqslant$ on K in the sense of (11). Now we fix a projection $p\colon L \to K$, i.e., a linear operator such that $p(I) = I$ whenever $I \in K$, and define $\lambda\colon L \to \mathbb{R}^{m+1}$ by $\lambda(I) := \langle q(I), \lambda'(p(I))\rangle$ for every $I \in L$. Checking that λ represents $\geqslant$ on L is straightforward: if $q(I') > q(I)$, then obviously $I' \gg I$; if $q(I') = q(I)$, then $(I' - I) \in K$, hence $\lambda(I') \geq_{\text{Lex}} \lambda(I) \iff \lambda'(I') \geq_{\text{Lex}} \lambda'(I) \iff I' \geqslant I$. □

Let $V = \{v^0, v^1, \dots, v^{\bar{m}}\}$ with $v^k < v^{k+1}$ for every relevant k. We call each $[v^k, v^{k+1}]$ an *elementary interval* and denote $\mathcal{E}$ the set of elementary intervals. For each $v^k \in V$, we set $\varkappa(v^k) := \sum_{h=0}^{k-1}[v^h, v^{h+1}] \in \mathfrak{Q}$, so $\varkappa(v^0) = 0$. Applying Lemma 4.3 with $L = \mathbb{Q}(\mathcal{E})$, we obtain appropriate $n \leq \#\mathcal{E} = \#V - 1$ and λ.

Now let $m \in \mathbb{N}$ and $v'_1, \dots, v'_m, v_1, \dots, v_m \in V$ be given. By the definition (7), $\langle v'_1, \dots, v'_m\rangle \succeq^m \langle v_1, \dots, v_m\rangle$ if and only if $\sum_{k=1}^m [v_k, v'_k] \geqslant 0$; by (9), $[v_k, v'_k] \simeq \big(\varkappa(v'_k) - \varkappa(v_k)\big)$. Therefore,

$$\langle v'_1, \dots, v'_m\rangle \succeq^m \langle v_1, \dots, v_m\rangle \iff \sum_{k=1}^m \varkappa(v'_k) \geqslant \sum_{k=1}^m \varkappa(v_k).$$

By Lemma 4.3,

$$\sum_{k=1}^m \varkappa(v'_k) \geqslant \sum_{k=1}^m \varkappa(v_k) \iff \sum_{k=1}^m \lambda(\varkappa(v'_k)) \geq_{\text{Lex}} \sum_{k=1}^m \lambda(\varkappa(v_k)).$$

Defining $\mu\colon V \to \mathbb{R}^n$ as $\mu = \lambda \circ \varkappa$, we obtain (6). Theorem 2 is proven. □

Proposition 3. *Given $V \subseteq \mathbb{R}$, a natural number n, and a mapping $\mu\colon V \to \mathbb{R}^n$, the sequence of relations $\succeq^m$ defined by (6) constitutes a universal separable ordering on V if and only if μ is strictly increasing.*

A straightforward proof is omitted.

Theorem 2 and Proposition 3 together provide a characterization of universal separable orderings on a finite set.

As an example, let us consider how representation (6) can be obtained for a universal separable ordering where, at a first glance, addition has no place, viz. the leximin ordering. We set $n := \#\mathcal{E} = \#V - 1$, i.e., $V = \{v^0, v^1, \dots, v^n\}$ with $v^k < v^{k+1}$ for all k. For each $v \in V$ and $k = 1, \dots, n$, we define $\mu_k(v) := 1$ if $v \geq v^k$ and $\mu_k(v) := 0$ otherwise. Given a list $v_1, \dots, v_m$, we immediately see that $\sum_{k=1}^m \mu_1(v_k) = m - \#\{k \in \{1, \dots, m\} \mid v_k = v^0\}$, $\sum_{k=1}^m \mu_2(v_k) = m - \#\{k \in \{1, \dots, m\} \mid v_k \leq v^1\}$, etc. Therefore, the lexicographic comparison of these sums produces the leximin ordering indeed.

5 Separability on infinite domains

The finiteness of the set V was not needed in Proposition 3; however, the proof of Theorem 2 collapses without the assumption. Moreover, it seems implausible that finite lexicography could be sufficient to represent every separable ordering, e.g., the leximin ordering, even on $V = \mathbb{N}$.

To keep the hope for a characterization result alive, we need a notion of lexicography with infinitely many indices. Two, at least, independent versions of such a notion are available.

Let $\mathcal{B}$ be a well ordered set. Then a lexicographic order $\geq_{\text{Lex}}$ on $\mathbb{R}^{\mathcal{B}}$ is defined in essentially the same way as on $\mathbb{R}^n$. Comparing two vectors from $\mathbb{R}^{\mathcal{B}}$, we find the least coordinate $\beta \in \mathcal{B}$ where they differ ($\mathcal{B}$ is well ordered!) and decide accordingly.

Proposition 4. *Given a set $V \subseteq \mathbb{R}$, a well ordered set $\mathcal{B}$, and a mapping $\mu\colon V \to \mathbb{R}^{\mathcal{B}}$, the sequence of relations $\succeq^m$ defined by (6) constitutes a universal separable ordering on V if and only if μ is strictly increasing.*

A straightforward proof is omitted.

Unfortunately, there is no ground to expect the class of universal separable orderings described in Proposition 4 to be exhaustive. An alternative way to define lexicography without finiteness looks more promising although there is no clear-cut result as yet.

Let there be a set W and a list of functions $\mu_\beta\colon W \to \mathbb{R}$ indexed by a parameter $\beta \in \mathcal{B}$; we call the list *pseudo-finite* if for every $w \in W$, $\mu_\beta(w) = 0$ except for a finite number of $\beta \in \mathcal{B}$. Given a pseudo-finite list with $\mathcal{B}$ linearly ordered, we define the lexicographic order on W in a natural way:

$$w' >_{\text{Lex}} w \rightleftharpoons \exists \beta \in \mathcal{B} \left[\mu_\beta(w') > \mu_\beta(w) \;\&\; \forall \beta' < \beta \left[\mu_{\beta'}(w') \geq \mu_{\beta'}(w)\right]\right];$$

$$w' \geq_{\text{Lex}} w \rightleftharpoons \left[w' >_{\text{Lex}} w \;\text{ or }\; \forall \beta \in \mathcal{B}\left[\mu_\beta(w') = \mu_\beta(w)\right]\right].$$

It is easy to see that $\geq_{\text{Lex}}$ is an ordering.

Proposition 5. *Let there be a set $V \subseteq \mathbb{R}$, a chain $\mathcal{B}$, and a pseudo-finite list of functions $\mu_\beta\colon V \to \mathbb{R}$ ($\beta \in \mathcal{B}$) such that $v' >_{\text{Lex}} v$ whenever $v', v \in V$ and $v' > v$. For each $m \in \mathbb{N}$, we denote $\succeq^m$ the ordering $\geq_{\text{Lex}}$ defined by the list of functions*

$$\mu_\beta^m(v_1, \ldots, v_m) := \sum_{k=1}^{m} \mu_\beta(v_k). \tag{12}$$

Then the sequence of $\succeq^m$ is a universal separable ordering on V.

A straightforward proof is omitted.

The leximin ordering can be represented by pseudo-finite lexicography as in Proposition 5 without any restrictions on the domain. Given a set $V \subseteq \mathbb{R}$, we set $\mathcal{B} := V$ and define, for every $v \in V$, $\mu_v(v) = -1$ and $\mu_w(v) = 0$ for all $w \neq v$. It is easily checked that the list is pseudo-finite. It is equally easily to see that the functions μ_v^m defined by (12) count how many times every particular $v \in V$ enters the list $v_1, \ldots, v_m$; hence lexicographic comparison leads to the leximin ordering in essentially the same way as at the end of Section 4.

The leximax ordering, and hence the maximum aggregation rule, can be treated dually.

6 On symmetry

In our definition of a universal aggregation rule, we demanded every $U^{(m)}$ to be symmetric. In principle, we could proceed without this restriction, assuming that the strategies in a generalized congestion game are corteges rather than subsets. Without symmetry, however, we would not be able to reproduce Rosenthal's construction of a potential; moreover, even the existence of a Nash equilibrium could not be guaranteed.

Proposition 6. *Let $U^{(m)}: V^m \to \mathbb{R}$, where $V \subseteq \mathbb{R}$, have the property that every generalized congestion game where $\#X_i = m$ for each strategy of each player, $\varphi_\alpha(\mathbb{N}) \subseteq V$ for each $\alpha \in \mathrm{A}$, and each player aggregates local utilities with $U^{(m)}$ possesses a Nash equilibrium. Then $U^{(m)}$ must be symmetric.*

Proof. Otherwise, swapping two arguments over would change the value of $U^{(m)}$. Without restricting generality, we may assume $u^+ = U^{(m)}(v_1, v_2, v_3, \dots, v_m) > U^{(m)}(v_2, v_1, v_3, \dots, v_m) = u^-$ for some $v_1, v_2, v_3, \dots, v_m \in V$.

Now let us consider a generalized congestion game where: $N := \{1, 2\}$; the facilities $\mathrm{A} := \{a, b, c, d\} \cup \{e_k\}_{k=3,\dots,m}$; $X_1 := \big\{\langle a, b, e_3, \dots, e_m\rangle,\ \langle c, d, e_3, \dots, e_m\rangle\big\}$; $X_2 := \big\{\langle d, a, e_3, \dots, e_m\rangle,\ \langle b, c, e_3, \dots, e_m\rangle\big\}$; $\varphi_t(1) := v_1$ and $\varphi_t(2) := v_2$ for each $t \in \{a, b, c, d\}$, while $\varphi_{e_k}(2) := v_k$ for all $k = 3, \dots, m$; each player $i \in N$ aggregates local utilities with $U^{(m)}$.

The 2×2 matrix of the game looks as follows:

	da	bc
ab	(u^-, u^+)	(u^+, u^-)
cd	(u^+, u^-)	(u^-, u^+).

There is no Nash equilibrium in the game. □

Remark. Neither continuity, nor monotonicity of $U^{(m)}$ were needed in the proof. Thus, this result generalizes Lemmas B.1 and B.2 from Kukushkin (2007).

7 Conclusion

To summarize, Proposition 1 shows that Rosenthal's (1973) construction hinges on the separability of additive aggregation. Very technically speaking, we thus found that the same construction can work for a broader class of preferences than originally envisaged.

On the other hand, this generalization is, to a large extent, illusory. In a particular quasi-separable generalized congestion game (with a finite number of players and a finite number of facilities) only a finite number of $\succeq^m$ and a finite set V of possible values of local utilities can be relevant. By Theorem 2, every $\succeq^m$ admits a representation (6) on V; and the lexicographic ordering on $\mathbb{R}^n$ obviously admits a scalar additive representation (5) on every finite subset. Thus, the main findings of this paper might as well be described as an impossibility result: for Rosenthal's construction to be applicable to a generalized congestion game, the preferences must admit an additive representation.

On the other other hand, if more players or facilities are added to a game, the representation (5) may become invalid and have to be modified. Therefore, one can argue that a combination of addition and lexicography as in Theorem 2 does not admit a single representation (5) suitable for all occasions, and hence such combinations do, indeed, define a broader class of preferences for which Rosenthal's approach can work.

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