Liquidity Effects of Central Banks' Asset Purchase Programs

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Abstract

This paper constructs a model of the monetary economy with multiple nominal assets. Assets differ in terms of the liquidity services they provide, and money is the most liquid asset. The central bank can implement policies by adjusting the relative supply of money and other assets. I show that the central bank can control the overall liquidity and welfare of the economy by changing the relative supply of assets with different liquidity characteristics. A liquidity trap exists away from the Friedman rule that has a positive real interest rate; the central bank's asset purchase/sale programs may be ineffective in instances of low enough inflation rates. My model also enables me to study the welfare effects of a restriction on trading with government bonds.

JEL Classification Numbers: E0, E4, E5

Keywords: Open-Market Operation, Liquidity Effects, Liquidity Trap

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1 Introduction

The key tool for implementing monetary policy is the interest rate on overnight loans between banks. In normal times, this rate is sensitive to the quantity of excess reserves. A central bank can control the rate on overnight loans by regular open-market operations. During and after the financial crisis of 2007, many central banks implemented policies that involved central banks' participation in a variety of markets. Large-scale asset programs that change the size and the composition of central banks' balance sheets were a major part of these policies. Traditional frictionless models of the monetary economy are not able to capture the real effects of these policies. Tobin (1969) P. 29 noticed the inability of traditional monetary models and states "there is no reason to think that the impact [of monetary policy] will be captured in any single [variable]... whether it is a monetary stock or a market interest rate". Later Tobin and Buiter (1981) used the arguments in Tobin (1969) and state that central banks should actively participate in the private capital market to impact the economy in the long run. These policies would directly impact rates of return on capital and therefore affect capital formation. This was the first time that economists were thinking about unconventional monetary policies. In order to investigate these policies, we need models with frictions in the asset market. Later economists started to build models that can generate real effects of central banks' asset purchase (or sale) programs and are able to capture the real effects of unconventional monetary policies.

Central banks' asset purchase programs involve purchasing assets and paying with assets that are different in terms of the liquidity services they provide. The liquidity characteristics of the central banks' and households' balance sheets are affected by this practice. To investigate the liquidity effects of these policies, I construct a microfounded model of monetary economy where households can trade goods with different types of assets. I use the theoretical model to show that within a specific set of parameters, open-market operations may affect the decision of households in the economy and welfare. In these cases, the central bank can affect the amount of produced goods in the economy by trading illiquid assets with money. There is an optimum supply of bonds that maximizes welfare in the economy. In an economy with two types of government-issued assets with different liquidity characteristics, the central bank is able to use open-market operations to change the liquidity characteristics of agents' portfolios. The central bank's asset purchase/sale programs can improve welfare by increasing (decreasing) liquidity in periods of low (high) liquidity.

During the period $2008 - 2011$ many central banks implemented a series of unconventional monetary policies in response to the financial crisis. A major part of these policies was the large-scale asset purchase programs (known as quantitative easing). The Bank of

Japan implemented similar policies from $2000 - 2006$[1]. These programs are basically open-market operations that change the size or the composition of central banks' balance sheets. Similarly, the Federal Reserve implemented two sets of policies in response to the financial crisis: 1-Quantitative easing: expanding the asset side of the central bank's balance sheet by purchasing conventional assets[2] and issuing reserves on the liability side. 2-Credit easing: changing the composition of the Fed's balance sheet by selling conventional assets and buying unconventional assets[3]. While academics discuss several channels through which these policies can affect the real economy (e.g. Krishnamurthy and Vissing-Jorgensen (2011)), policy makers (e.g. Bernanke and Reinhart (2004)) mainly highlight two: 1-Signaling lower interest rate in the long-term. 2-Increasing demand for other assets in the economy and decreasing yield on these assets[4].

The literature on open-market operations and quantitative easing falls into two categories. First, there are earlier papers that show open-market operations are irrelevant for the real economy. In these models assets are perfectly substitutable in terms of liquidity services, and open-market operations do not change the liquidity characteristics of households' asset portfolios. In a model with liquid bonds, since bonds and money are perfectly substitutable, households have a similar liquidity preference toward holding bonds and money. Households cannot use bonds to affect the liquidity characteristics of their portfolios. Following Wallace (1981), a branch of literature uses a Modigliani-Miller argument to show that the size and the composition of central banks' balance sheets and thus open-market operations do not have any real effect on the economy. In order to capture real effects of asset purchase programs, we need models with frictions in the asset market in which assets are imperfect substitutes.

Second, papers show that open-market operations can affect the real economy. In a model in which interest bearing assets provide different liquidity services compared to money, open-market operations change the liquidity characteristics of households' portfolios and have real effects on the economy. Kocherlakota (2003) uses a similar argument and shows that in a centralized market, agents use illiquid bonds to smooth consumption.

[1] Bernanke et al. (2004), Marumo et al. (2003), Okina and Shiratsuka (2004), Baba et al. (2005), and Oda and Ueda (2005) study QE in Japan and find similar significant effects on asset prices.

[2] In the US this mainly takes the form of treasuries.

[3] Credit easing is also called an asset sterilizing program or Operation Twist. The first studied unconventional monetary policy is called "operation twist". In 1961, in response to the recession, the Kennedy administration and the Federal Reserve decided to flatten the yield curve on treasury debt by keeping the short-term rate constant and lowering the long-term rates. Under this policy the Federal Reserve kept its federal funds rate constant and purchased long-term Treasury debt and agency-backed private debt. On the other side, the Treasury reduced its issuance of long-term debt and increased its issuance of short-term debt. These policies affected the short-term and long-term rates, agency, and the corporate bond market (Modigliani and Sutch (1966), Modigliani and Sutch (1967) and Swanson et al. (2011) discuss these policies.)

[4] Agents rebalanced their portfolios towards other assets in the economy.

Curdia and Woodford (2011) add an intermediary sector to a canonical New Keynesian model[5]. The model in Curdia and Woodford (2011) is able to analyze three separate central banks' policies regarding quantity of reserves, interest paid on reserves, and the combination of central banks' balance sheets. This allows them to study a rich set of central bank policies. They find that quantitative easing in the strict sense is likely to be ineffective. Williamson (2012) adds an intermediary sector[6] to a New Monetarist monetary framework[7]. In a version of the model with public and private assets, Williamson (2012) shows that a policy similar to the first round of quantitative easing pursued by the Federal Reserve is, at best, ineffective. Kiyotaki and Moore (2012) study a model of monetary economy with differences in liquidity across assets. They show that open-market operations are effective when the central bank purchases the assets with partial resaleability and a substantial liquidity premium during negative liquidity shocks. The illiquid asset in their model are mainly capital and securities that are issued based on capital, and their analysis focuses on the role of open-market operations on privately provided liquidity.

I expand the existing literature on the effects of open-market operations by building a micro-founded model of a monetary economy. The basic model is a variation of Shi (2008), who uses a similar framework to study the legal restrictions on trade with nominal bonds. Agents can trade with different government-issued assets that provide different liquidity services. Contrary to Shi (2008), here the argument is not based on parameters in the utility function. Shi (2008) assumes that agents can use bonds to trade certain types of goods that yield a higher utility when consumed. Here, consumption of different goods yield the same amount of utility and my analysis hinges on the liquidity characteristics of assets. Assets are different in terms of the liquidity services they provide. Central bank's open-market purchase of assets increases liquidity in the economy by injecting money and purchasing interest-bearing assets.

How does this model investigate the liquidity effects of central bank's policies? First, I use a household structure, which helps to build a tractable model that avoids the evolving distribution of asset holding. In this structure, households do not face any intertemporal uncertainty. Therefore, there is no precautionary motive for saving. Households buy assets only for the liquidity services they provide. The yield on assets is a pure liquidity premium. Second, I model liquidity services provided by assets. Households do not gain utility by holding assets. This allows me to investigate the effects of different central policies on the liquidity premium on assets and the overall welfare in the economy.

[5] Similar to the framework in Woodford (2011)

[6] Williamson (2012) uses a model based on Diamond and Dybvig (1983).

[7] A model based on Lagos and Wright (2005) with heterogeneous agents similar to Rocheteau and Wright (2005).

In the literature on monetary economics and policy liquidity traps are mostly associated with the Friedman rule[8]. Williamson (2012) studies a liquidity trap in cases where the economy is away from the Friedman rule and when the real interest rate is zero. He discusses the liquidity channel of open-market operations in a model with public and private liquidity in which it is costly to operate a monetary system. In this paper, we can have the properties of the liquidity trap equilibrium when the real interest rate is positive. In this case, marginal open-market operations do not have real effects on the economy. In an extension of the model with three assets I show that a policy of credit easing can affect welfare.

In section 2, I develop a micro-founded model of the monetary economy. I then study the optimal choices and discuss different equilibria and welfare effects of different policies. In section 3, I study the model with two types of government issued assets. Section 4 offers concluding remarks and possible extensions.

2 Model environment

Time is discrete and has infinite horizons. There are H types of households ($H \geq 3$). Each household consumes a good that is produced by some other type of household, type h household consumes good h but produces good $h+1$. There is no double coincidence of wants, and goods are perishable. Each household consists of a large number of members (measure one). These members could be sellers (measure σ), buyers (measure $N-\sigma$), or leisure seekers (measure $1-N$). Buyers and sellers trade goods while leisure seekers are inactive. There is perfect consumption insurance between household members; members of a household share consumption and regard utility of the household as the common objective.

There are two markets in this economy, a centralized market for assets and a decentralized market for goods. Money and bonds are supplied by the central bank. The central bank implements policies by printing money at rate γ and changing the relative composition of the stock of bonds and money in the economy. In the centralized market for assets, the government bonds are sold for money. In the decentralized market for goods, search frictions exist. Buyers and sellers of different households are randomly matched in pairs. The number of matches for each household is αN, where α is a parameter of the environment and N is the aggregate number of traders in the market. According to this matching technology, the matching rate for buyers is $\frac{\alpha N}{N-\sigma}$ and the matching rate for the sellers is $\frac{\alpha N}{\sigma}$ [9]. The matching process of a three household economy is shown in figure 1. Because of the assumed structure of the environment, a successful match is between a buyer of household "h" and a seller of

[8] Money grows at the rate that agents discount future consumption.

[9] I assume α is low enough that matching rates are less than 1.

household "$h+1$."

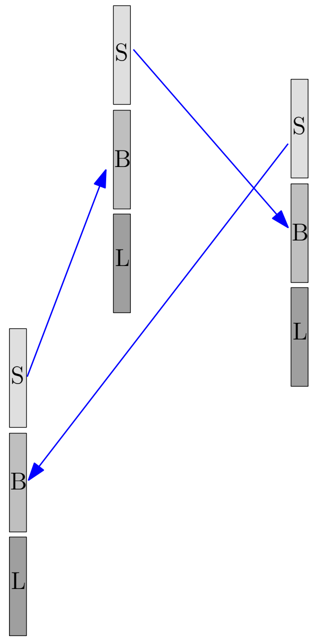

Figure 1: Matching process in a 3 household economy

In the centralized market for assets, households trade government bonds for money. In the decentralized market for goods, household members trade goods for money or government bonds. Trade history is private information, agents are anonymous, and the population is large. Therefore, there is no credit. After household members are matched, a matching shock determines the type of assets they can use for trade. With probability $1-l$, they can only use money (this trade is indexed by subscript "m") to purchase goods, and with probability l they can use both money and bonds (this trade is indexed by subscript "b") to purchase goods[10].

2.1 Household decisions

The representative household solves the following maximization problem:

$$v(m,b) = \max_{c_i,q_i,x_i,n,m_{+1},b_{+1},} \{u(c_m) - \alpha N(1-l)\psi(Q_m)$$
$$+u(c_b) - \alpha N l \psi(Q_b) + h(1-n) + \beta v(m_{+1}, b_{+1})\}, \quad i \in \{m,b\} \tag{1}$$

[10]Different types of matches can be interpreted as "monitored" and "non-monitored" matches as in Williamson (2012).

subject to the following constrains:

$$x_m \leq \frac{m}{n-\sigma}, \tag{2}$$

$$x_b \leq \frac{m+b}{n-\sigma}, \tag{3}$$

where lower case letters are choices of the household under consideration and capital letters are per capita variables that individual households cannot affect.

Households choose consumption (c_i), terms of trade (q_i, x_i), number of traders (n), and asset portfolio (m_{+1}, b_{+1}) for the next period to maximize the above value function. The utility from trade is the sum of the net utility in each type of trade. In each trade, household shares the utility from consumption of the purchased goods and the cost of production of the sold goods. In a money trade a representative household consumes c_m and produces Q_m. The total number of money trades for the representative household is $\alpha N(1-l)$. Similarly, in a money and bond trade, a representative household consumes c_b and prodeces Q_b. Since buyers have all the bargaining power, the amount sold is shown by capital letters Q_i. $u()$ is continuous and twice differentiable, and $u'() > 0$, $u''() < 0$. I assume $\psi'() > 0$, $\psi''() > 0$; $h'() > 0$ and $h''() < 0$. Each household divides its members into three groups: sellers/producers (measure σ), buyers (measure $n-\sigma$), and leisure seekers (measure $1-n$). Households choose n, and σ is fixed[11]. In each type of trade, buyers are constrained by the portfolio of assets that they have. In a money trade, buyers are constrained by the amount of money they have (2). In a money and bond trade, buyers are constrained by the total portfolio of assets they carry(3). Goods are divisible and perishable. Consumption in each type of trade is the matching rate times the total amount of goods bought by the buyers in that trade

$$c_b = \underbrace{\frac{\alpha N}{(N-\sigma)}}_{\text{Matching rate}} (n-\sigma) l q_b$$

$$c_m = \frac{\alpha N(n-\sigma)(1-l)}{(N-\sigma)} q_m.$$

Let us define $\omega_i, \quad i \in \{m, b\}$ as the marginal value of assets

$$\omega_m = \frac{\beta}{\gamma}\frac{\partial v(m,b)}{\partial m_{+1}}$$

[11]This assumption is for simplicity. I can allow households to choose σ and the main results hold.

$$\omega_b = \frac{\beta}{\gamma}\frac{\partial v(m,b)}{\partial b}.$$

Ω_m is the per capita value of money in the economy. In each trade, sellers sell goods for a portfolio of assets, which has a marginal value of Ω_m. Seller's surplus is $x_i\Omega_m - \psi(q_i), \quad i \in \{m, b\}$. Since buyers have all the bargaining power, the offer sets sellers' surplus to 0. Thus, the participation constraint is $x_i\Omega_m - \psi(q_i) = 0$, and can be written as:

$$x_i = \psi(q_i)/\Omega_m \quad i \in \{m, b\}. \tag{4}$$

The value of household's asset portfolio in terms of money follows equation 5:

$$\begin{aligned}(m_{+1} + s_{+1}b_{+1} + T_{+1})\gamma = \\ m + b + \alpha N l X^b + \alpha N(1-l)X^m - \frac{\alpha N(n-\sigma)}{N-\sigma} l x^b - \frac{\alpha N(n-\sigma)}{N-\sigma}(1-l)x^m,\end{aligned} \tag{5}$$

where s_{+1} is the price of bond in the asset market. Money balance plus the amount spent on the assets in the next period and the tax (or transfers) is equal to the portfolio of assets in the current period plus the assets that the sellers bring back minus the assets that buyers have spent on their purchases of goods.

2.1.1 Timing

The timing of the events is shown in figure 2.

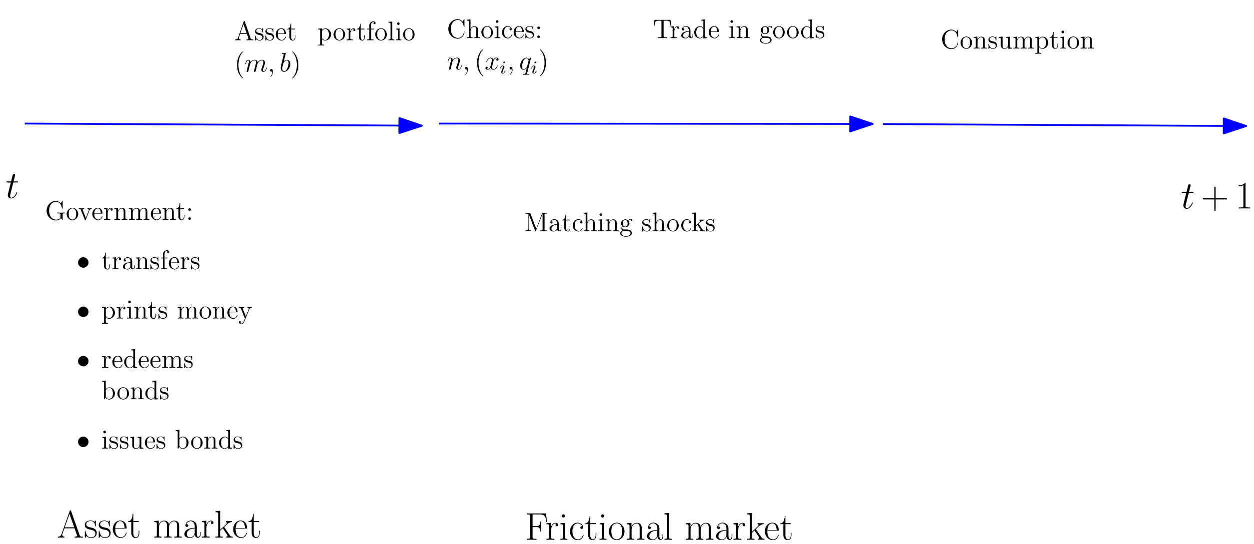

Figure 2: Timing

At the beginning of each period, the asset market opens. Households redeem nominal bonds from the previous period for one unit of money, trade assets for money, receive transfer

T, and adjust their portfolios to (m, b). The asset market is closed until the beginning of the next period. Households choose the amount of total traders n and give buyers instructions on how to trade in different types of trade (Goods: q_i, assets: $x_i, i \in \{m, b\}$). Buyers and sellers search in the goods market and match according to the linear matching function. Matched sellers produce and trade and then bring goods and assets back to the household and members of the household share consumption.

2.2 Optimal choices

The first order condition for q_i is

$$u'(c_i) = (\omega^m + \lambda^i)\frac{\psi'(q_i)}{\Omega} \quad i \in \{b, m\}, \tag{6}$$

where λ^b and λ^m are the Lagrange multipliers on trades with bond and money, respectively. I can solve for bond prices by taking the first order conditions with respect to b:

$$b_{+1} : s_{+1} = \frac{\omega_b}{\omega_m}. \tag{7}$$

Households' choices of the measure of traders (n) solves the following:

$$h'(1-n) = \frac{\alpha N}{N-\sigma}\left[lu'(c_b)(q_b - \frac{\psi(q_b)}{\psi'(q_b)}) + (1-l)u'(c_m)(q_m - \frac{\psi(q_m)}{\psi'(q_m)})\right]. \tag{8}$$

The envelope conditions for m_{+1}, b_{+1} are:

$$m_{+1} : \frac{\gamma}{\beta}\omega^m_{-1} = \omega_m + \frac{\alpha N l}{N-\sigma}\lambda^b + \frac{\alpha N(1-l)}{N-\sigma}\lambda^m \tag{9}$$

$$b_{+1} : \frac{\gamma}{\beta}\omega^b_{-1} = \omega_m + \frac{\alpha N l}{N-\sigma}\lambda^b. \tag{10}$$

At the end of each period, each unit of bond is redeemed for a unit of money, therefore the value of an asset is the value of money in the following period plus the liquidity services that the asset provides, accounting for discounting and inflation. Money provides liquidity services in all types of trades, and bonds are used in certain types of trade. Expressions 9 and 10 show that money has a liquidity premium over bonds.

2.3 Definition of the equilibrium

Definition 1 *An equilibrium is households' choices* $(c^{i\in\{m,b\}}, q^{i\in\{m,b\}}, x^{i\in\{m,b\}}, n, m_{-1}, b_{+1})$*, the value function* $(v(m,b))$*, shadow value of assets* (ω^m, ω^b)*, asset price* (s)*, and other households' choices, such that*

1. *Given bond price* (s)*, and choices of others, household choices are optimal (1).*
2. *The choices and shadow prices are the same across households, i.e.,* $q^i = Q^i, x^i = X^i, n = N, \omega^i = \Omega^i$*.*
3. *Bonds market clear* $(b = B)$*.*
4. *Positive and finite values of assets* $(0 < \omega < \infty)$*.*
5. *Stationarity: quantities and prices are constant over time.*

2.4 Welfare analysis

The envelope conditions show that the only point at which all of the constraints are non-binding is where $\gamma = \beta$. Let us call the Lagrange multiplier on the constraint of a money trade λ^m and similarly the Lagrange multiplier on a constraint of a bond and money trade λ^b.

Lemma 1 *At Friedman rule* $(\gamma = \beta)$*,* $\lambda^m = \lambda^b = 0$*. For* $\gamma > \beta$*,* $\exists i \in \{m, b\}$ *such that* $\lambda^i > 0$*.*

In order to study open-market operations, let us define the ratio of stock of bonds to stock of money as:

$$z = \frac{B}{M}.$$

The central bank implements policies by changing the inflation rate (γ) and relative supply of assets (z). Changes in z are the effects of open-market operations. Open-market purchase (sale) of bonds decreases (increases) z.

I define the welfare function as the utility function of a representative household.

$$w = u(c_b) - \alpha N l \psi(Q_b) + u(c_m) - \alpha N(1-l)\psi(Q_m) + h(1-n). \tag{11}$$

By using the above measure of welfare, I can study the welfare effects of policies. We have four types of equilibria based on the set of binding liquidity constraints. The only point at which all of the liquidity constraints are non-binding is at the Friedman rule. The Friedman rule is shown to be optimal in a wide variety of models. As the next proposition shows, the Friedman rule is optimal in this framework.

Proposition 1 *The Friedman rule is optimal.*

The proof of the above proposition is intuitive. Since buyers' bargaining power is 1, households send too many buyers compared to the planner's choice. Increasing γ punishes unmatched buyers and the representative households. On the other hand, inflation decreases the amount of goods in each trade. The former effect is known as the extensive margin of trade, and the latter is known as the intensive margin of trade. Both intensive margin (q) and extensive margin (n) decrease with inflation (γ). The planner chooses the lowest possible level for γ to maximize welfare. Therefore, the Friedman rule is optimal.

Based on the set of liquidity constraints that are binding, we can have four types of equilibria. As shown before, an equilibrium where both of the liquidity constraints are non-binding can only happen at the Friedman rule and this equilibrium is efficient. In Appendix A, I have characterized different types of equilibria and proved the following proposition:

Proposition 2 *Open-market operations can only have welfare effects when both of the liquidity constraints are binding. The properties of the equilibria are shown in table 2.*

Table 1: Different types of equilibria for a 2-asset economy

Case	λ^m	λ^b	s	$\frac{\partial W}{\partial z}$
I	$+$	0	$\frac{\beta}{\gamma}$	0
II	0	$+$	1	0
III	$+$	$+$	$\frac{\beta}{\gamma} < s < 1$	$+, -$
IV	0	0	1	0

According to proposition 2, marginal open market operations may have real effects when we have a type III equilibrium. the central bank's asset purchase/sale programs may be ineffective in other cases. The literature calles these situations "liquidity traps". In the literature on monetary economics and policy, liquidity traps are mostly associated with the Friedman rule (type IV equilibrium where $\gamma = \beta$). Williamson (2012) studies liquidity traps in cases where the economy is away from the Friedman rule and when the real interest rate is zero. In this paper, we can have properties of the liquidity trap equilibrium even when the real interest rate is positive[12]. In an equilibrium where only the money constraint is

[12]The nominal interest rate is $\frac{1}{s}$. The real interest rate is the difference between the nominal interest rate an the inflation rate

binding (Case I), open-market operations have no real effects on the economy and the real interest rate is positive ($s = \frac{\beta}{\gamma}$). In this case, marginal open-market operations do not have real effects on the economy.

2.5 Numerical example

Using the following functional forms and parameters, I simulate the model. The calculations of the different types of equilibria are in the Appendix B and the results of the simulation are reported in figures 3 and 4:

$$u(c) = \log(c); \quad \psi(q) = \frac{q^2}{2}; \quad h(n) = 2a(n)^{1/2},$$

where a is a parameter of the model. Table 2 shows the properties of the equilibrium for different amounts of the liquidity parameter (l).

Table 2: Properties of the equilibrium for a 2-asset economy

Case	λ^m	λ^b	Criteria
I	+	0	$l > \bar{l} = \frac{(\frac{\gamma}{\beta}-1)(N-\sigma)+\alpha N}{(2+z)\alpha N}$
II	0	+	$l < \underline{l} = \frac{\alpha N-(\frac{\gamma}{\beta}-1)(N-\sigma)(1+z)}{\alpha N(2+z)}$
III	+	+	$\underline{l} \leq l \leq \bar{l}$
IV	0	0	N/A

Figure 3 shows the range of parameters for different types of equilibria. For high enough values of l, the liquidity constraint for money binds and the constraint on trade with bond is slack. For low enough l, the constraint on bond binds and for $\underline{l} < l < \bar{l}$ both of the constraints are binding.

As I have proven in Proposition 2, increasing z will only affect welfare when we are in type III equilibrium with both liquidity constraints binding. Figure 4 shows the welfare properties of the equilibrium for values of l that cause both liquidity constraints binding (type III equilibrium). As shown in figure 4, for each inflation rate there exists an optimal level of bond supply (z) that maximizes welfare.

2.5.1 The case for legal restrictions on trading with bonds

As shown in figure 4, in a type II equilibrium, increasing z from zero increases the overall welfare. Similar to the argument in Shi (2008), an increase in z can be interpreted as imposing legal restrictions on trade with bonds. An economy with zero supply of bond is a

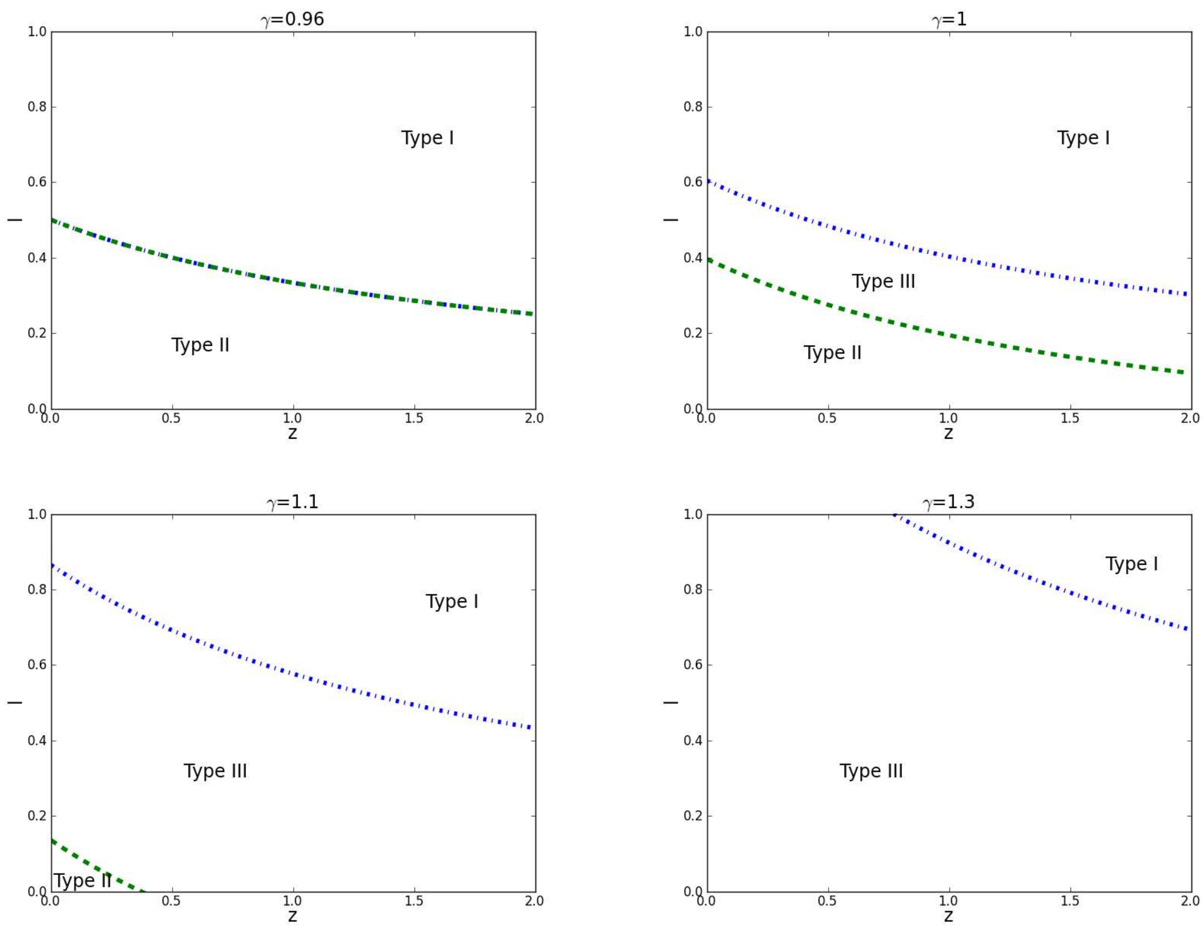

Figure 3: Range of parameters for different types of equilibrium for a 2-asset economy

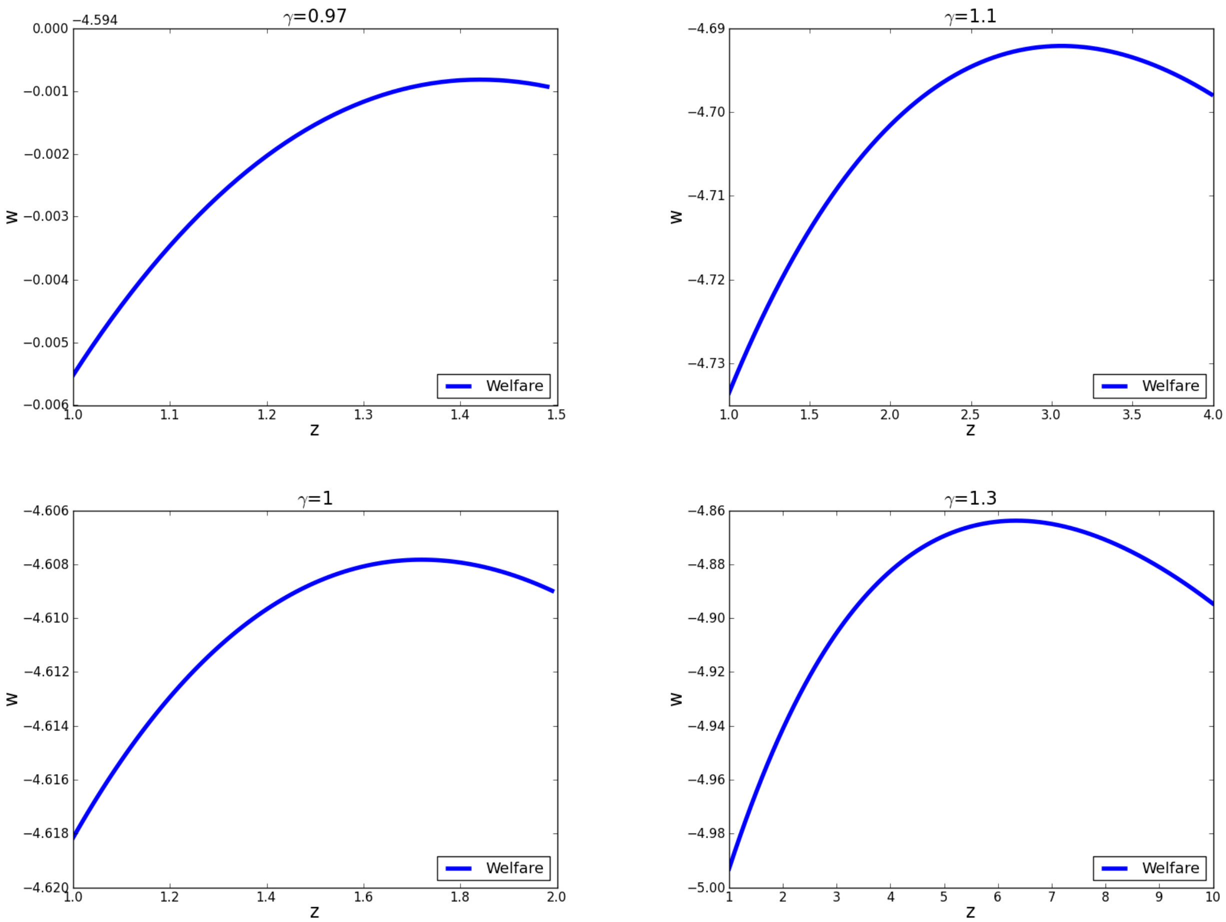

Figure 4: Welfare effects of open-market operations

pure monetary economy. An increase in z from zero represents imposing legal restrictions on trades with bonds. As figure 4 shows, this can improve welfare for a range of parameters.

Contrary to the argument in Shi (2008), the argument here is not based on parameters in the utility function. Shi (2008) assumes that agents can use bond to trade certain types of goods that yield a higher utility when consumed. Here, consumption of different goods yields the same amount of utility.

3 Model with 3 assets

In this section, I add a third asset to the model. This extension of the model allows me to study the effects of a change in the composition of the central bank's balance sheet on the real economy. All three assets provide liquidity services, and, similar to the previous section, money is the most liquid asset in the economy. I call the least liquid asset in the economy "long-term bond" and the other asset "short-term bond."[13] The matching shock works as follows:

- Shock n: With probability l, agents can trade with money and short term bond and long term bond.
- Shock s: With probability k, agents can trade with money and short term bond.
- Shock l: With probability $1-l-k$, agents can only trade with money.

In each trade buyers make take-it-or-leave-it offers on the amount of goods $q_{i\in\{n,s,l\}}$ and the portfolio of assets to be traded for goods $x_{i\in\{n,s,l\}}$. Note that the portfolio of assets could be a combination of money, short-term bond, and long-term bond depending on the type of trade/shock.

Households solve the following maximization problem:

$$\begin{aligned} v(m,b_l,b_s) = \max_{c_{i\in\{n,s,l\}},q_{i\in\{n,s,l\}},x_{i\in\{n,s,l\}},n,m_{-1},b^s_{-1},b^l_{+1}} & \{u(c_l)-\alpha N(1-l-k)\psi(Q_l) \\ & +u(c_s)-\alpha Nk\psi(Q_s)+u(c_n)-\alpha Nl\psi(Q_n) \\ & +h(1-n)+\beta v(m_{-1},b^l_{+1},b^s_{+1})\}. \end{aligned} \tag{12}$$

[13]Here, it is assumed that a short-term bond is more liquid than a long-term bond. A short-term bond can be used in more transactions to purchase goods.

subject to the following constraints:

$$x_n \leq \frac{m + b_l + b_s}{n - \sigma} \tag{13}$$

$$x_s \leq \frac{m + b_s}{n - \sigma} \tag{14}$$

$$x_l \leq \frac{m}{n - \sigma}. \tag{15}$$

According to constraints 13, 14 and 15, in each type of trade (money and short-term bond trade, money and long-term bond trade, and money trade), buyers are constrained by the portfolio of assets that they have. Consumption in each type of trade is characterized by the following:

$$c_n = \frac{\alpha N(n - \sigma)l}{(N - \sigma)} q_n$$

$$c_s = \frac{\alpha N(n - \sigma)k}{(N - \sigma)} q_s$$

$$c_l = \frac{\alpha N(n - \sigma)(1 - k - l)}{(N - \sigma)} q_l.$$

where $\frac{\alpha N}{(N - \sigma)}$ is the matching rate and $(n-\sigma)l$, $(n-\sigma)k$ and $(n-\sigma)(1-k-l)$ are the number of buyers in each type of trade.

Let's define $\omega_i \quad i \in \{m, b_s, b_l\}$ as the marginal value of assets

$$\omega_m = \frac{\beta}{\gamma} \frac{\partial}{\partial m_{+1}} v(m, b_l, b_s)$$

$$\omega_{b_s} = \frac{\beta}{\gamma} \frac{\partial}{\partial b^s_{+1}} v(m, b_l, b_s)$$

$$\omega_{b_l} = \frac{\beta}{\gamma} \frac{\partial}{\partial b^l_{+1}} v(m, b_l, b_l).$$

Since buyers have all the bargaining power, the offer sets sellers' surplus to 0. Thus, the participation constraint is:

$$x_i = \psi(q_i)/\Omega_m \quad i \in \{l, s, n\}. \tag{16}$$

The value of household's asset portfolio in terms of money follows equation 17.

$$
\begin{aligned}
&(m_{-1} + s^l_{+1} b^l_{+1} + s^s_{+1} b^s_{+1} + T_{-1})\gamma = \\
&\quad m + b_l + b_s + \alpha N l X^n + \alpha N k X^s + \alpha N(1-k-l)X^l \\
&\quad -\frac{\alpha N(n-\sigma)}{N-\sigma} l x^n - \frac{\alpha N(n-\sigma)}{N-\sigma} k x^s - \frac{\alpha N(n-\sigma)}{N-\sigma}(1-l-k)x^l. \qquad (17)
\end{aligned}
$$

where s^l_{+1} and s^s_{+1} are the prices of long-term bonds and short-term bonds respectively. $\alpha N l X^n + \alpha N k X^s + \alpha N(1-k-l)X^l$ is the amount of assets that households' sellers spend and $\frac{\alpha N(n-\sigma)}{N-\sigma} l x^n - \frac{\alpha N(n-\sigma)}{N-\sigma} k x^s - \frac{\alpha N(n-\sigma)}{N-\sigma}(1-l-k)x^l$ is the amount of asets that households' buyers buy.

3.1 Optimal choices

The first order condition for q_i is:

$$
u'(c_i) = (\omega^m + \lambda^i)\frac{\psi'(q_i)}{\Omega} \quad i \in l, s, n. \qquad (18)
$$

I can solve for bond prices by taking the first order conditions with respect to b_l^+, b_s^+.

$$
b^l_{+1} : s^l_{+1} = \frac{\omega_{b_l}}{\omega_m}. \qquad (19)
$$

$$
b^s_{+1} : s^s_{+1} = \frac{\omega_{b_s}}{\omega_m}. \qquad (20)
$$

The envelope conditions for $m_{+1}, b^s_{+1}, b^l_{+1}$ are:

$$
m_{+1} : \frac{\gamma}{\beta}\omega_m^{-1} = \omega_m + \frac{\alpha N l}{N-\sigma}\lambda^n + \frac{\alpha N k}{N-\sigma}\lambda^s + \frac{\alpha N(1-l-k)}{N-\sigma}\lambda^l \qquad (21)
$$

$$
b^s_{+1} : \frac{\gamma}{\beta}\omega_{b_s}^{-1} = \omega_m + \frac{\alpha N l}{N-\sigma}\lambda^n + \frac{\alpha N k}{N-\sigma}\lambda^s \qquad (22)
$$

$$
b^l_{+1} : \frac{\gamma}{\beta}\omega_{b_l}^{-1} = \omega_m + \frac{\alpha N l}{N-\sigma}\lambda^n. \qquad (23)
$$

At the end of each period, each asset is redeemed for a unit of money, therefore the value of an asset is the value of money in the next period plus the transaction services of each asset accounting for discounting and inflation. Money provides transaction service in all types of trades, but bonds are used as medium of exchange in certain types of trade. Similar to the 2-asset economy, the Friedman rule ($\gamma = \beta$) is optimal. Here, the central bank has 3 policy variables: money growth rate (γ), long-term bond supply ($z_l = \frac{B_l}{M}$), and short-term bond supply ($z_s = \frac{B_s}{M}$).

In order to study the different equilibria and welfare effects of policy, I will focus on the log-utility and quadratic cost functions:

$$u(c) = \log(c)$$

$$\psi(q) = q^2/2.$$

Lemma 2 *With $u(c) = \log(c)$ and $\psi(q) = q^2/2$, $N = n$ is the same for different cases of equilibrium. An equilibrium exists if $h'(1-N) = \frac{3}{2(N-\sigma)}$ has a real solution for N.*

With log-utility and quadratic cost functions, the first order condition for n becomes $h'(1-N) = \frac{3}{2(N-\sigma)}$ in all of the cases. These functional forms shut down variations in the extensive margin of trade.

In the next proposition, I characterize these different types of equilibrium.

Proposition 3 *Eight types of equilibrium exist, all of which have different sets of binding liquidity constraints, as defined in table 3.*

Table 3: Different types of equilibria for the 3-asset economy

	I	II	III	IV	V	VI	VII	IIX
λ^n	+	0	+	+	0	0	+	0
λ^s	0	+	+	0	0	+	+	0
λ^l	+	+	0	0	+	0	+	0

The properties of these equilibria are summarized in table 4.

As the proposition shows, in equilibriums with at least two binding liquidity constraints, there exists a set of parameters that indicate that open-market operations affect welfare. In these cases, replacing less liquid bonds in household portfolios with liquid money would increase the intensive margin of trade and welfare.

Table 4 also shows that a policy of changing the relative supply of bonds while keeping the size of the central bank's balance sheet growing with the rate of inflation can affect the overall welfare when we have a type II or VII equilibrium. Credit easing can be implemented by changing the relative supply of bonds while the following relationship holds

$$s_l dz_l + s_s dz_s = 0$$

Table 4: Properties of the equilibrium for the 3-asset economy

Case	Prices	$\frac{\partial q_i}{\partial z_i}$	$\frac{\partial W}{\partial z_i}$
I	$\beta/\gamma < s_l = s_s < 1$	$\frac{\partial q_s}{\partial z_i} = 0$ $\frac{\partial q_n}{\partial z_i} > 0$ $\frac{\partial q_l}{\partial z_i} < 0$	$\frac{\partial W}{\partial z_i} >=< 0$
II	$s_l = \beta/\gamma < s_s < 1$	$\frac{\partial q_n}{\partial z_l} = 0$ $\frac{\partial q_l}{\partial z_s} < 0$ $\frac{\partial q_s}{\partial z_s} >=< 0$	$\frac{\partial W}{\partial z_l} = 0$ $\frac{\partial W}{\partial z_s} >=< 0$
III	$\beta/\gamma < s_l < s_s = 1$	$\frac{\partial q_l}{\partial z_i} = 0$ $\frac{\partial q_s}{\partial z_s} > 0$ $\frac{\partial q_s}{\partial z_l} < 0$ $\frac{\partial q_n}{\partial z_i} >=< 0$	$\frac{\partial W}{\partial z_i} >=< 0$
IV	$s_l = s_s = 1$	0	0
V	$s_l = s_s = \beta/\gamma < 1$	0	0
VI	$s_l = \beta/\gamma < s_s = 1$	0	0
VII	$\beta/\gamma < s_l < s_s < 1$	$\frac{\partial q_j}{\partial z_i} >=< 0$	$\frac{\partial W}{\partial z_i} >=< 0$
IIX	$s_l = s_s = 1$	0	0

The above relationship allows the central bank to keep the size of its balance sheet growing at rate γ[14].

Figures 5, 6, and 7 show different types of equilibria for different bond supply (z_s, z_l) and inflation rates (γ).

4 Conclusion

In this paper, I construct a model of the monetary economy in which different assets provide liquidity services. Adding illiquid nominal bonds to a microfounded model of monetary economy allows me to study the welfare effects of central banks asset purchase programs. I show that the central bank can change the overall liquidity and welfare in the economy by changing the relative supply of assets with different liquidity characteristics. My model also enables me to study the welfare effects of a restriction on trade with government bonds. I show that in a non-empty set of parameters restricting trade with government bonds can affect welfare. A liquidity trap can exist away from the Friedman rule and with a positive real interest rate. One possible extension of the model is to add privately issued assets to the model and investigate the liquidity effects of a richer set of central banks' asset purchase (or sale) programs.

[14]An example of a policy of credit easing in a type II equilibrium is

$$\frac{\beta}{\gamma} dz_l + s_l dz_s = 0$$

This policy has real effects on the economy and the central bank keeps the size of its balance sheet growing at a constant rate.

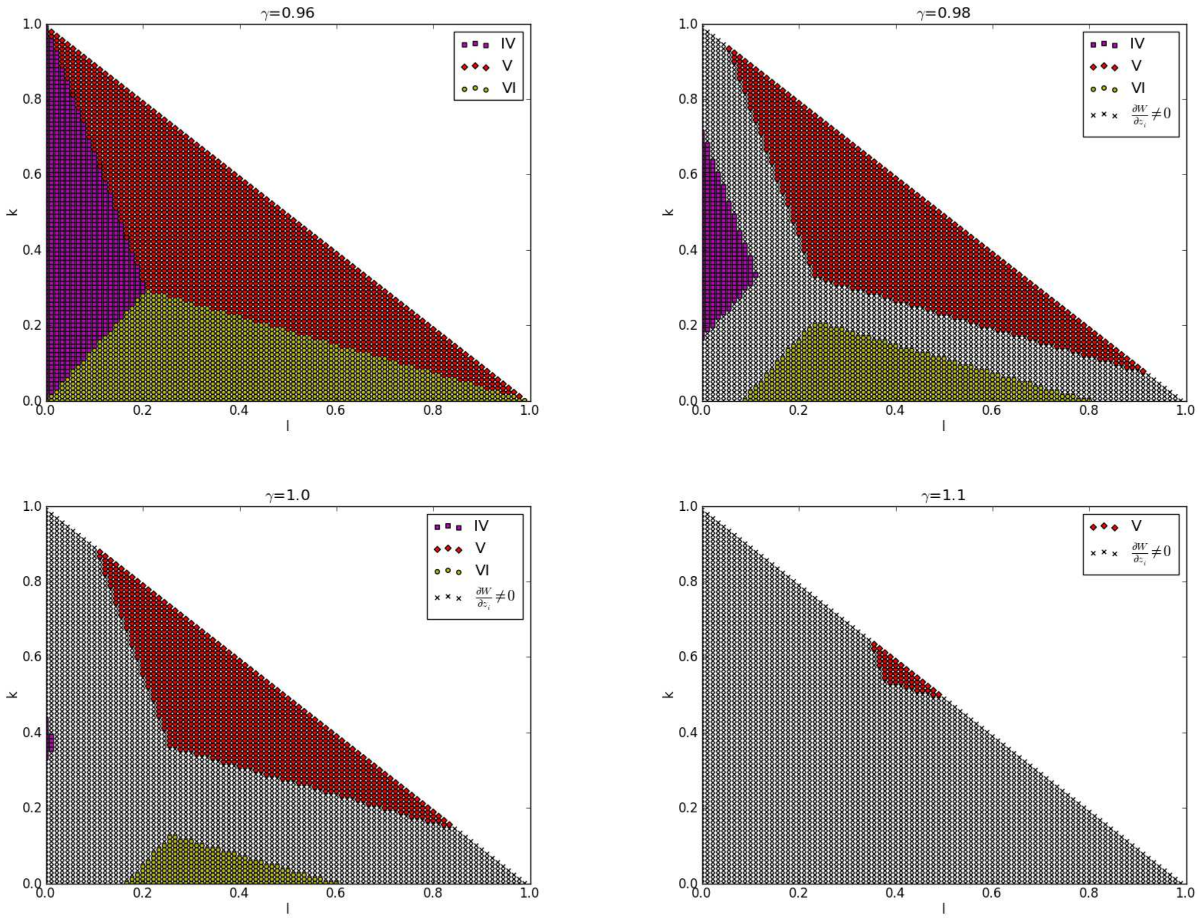

Figure 5: Range of parameters for different types of equilibrium for the 3-asset economy ($z_s = z_l = 0.7$)

Appendix

A The 2-asset economy

I characterize 3 cases of the equilibria based on the set of liquidity constraints that are binding. In all of these cases at least one of the constraints are binding. The case where none of them are binding only happens at the Friedman rule, and it is shown to be efficient.

Case I: $\lambda^m > 0$ and $\lambda^b = 0$

The first order conditions are

$$u'(c_b) = \psi'(q_b)$$

$$u'(c_m) = \psi'(q_m) + \frac{\psi'(q_m)}{\Omega^m}\lambda^m$$

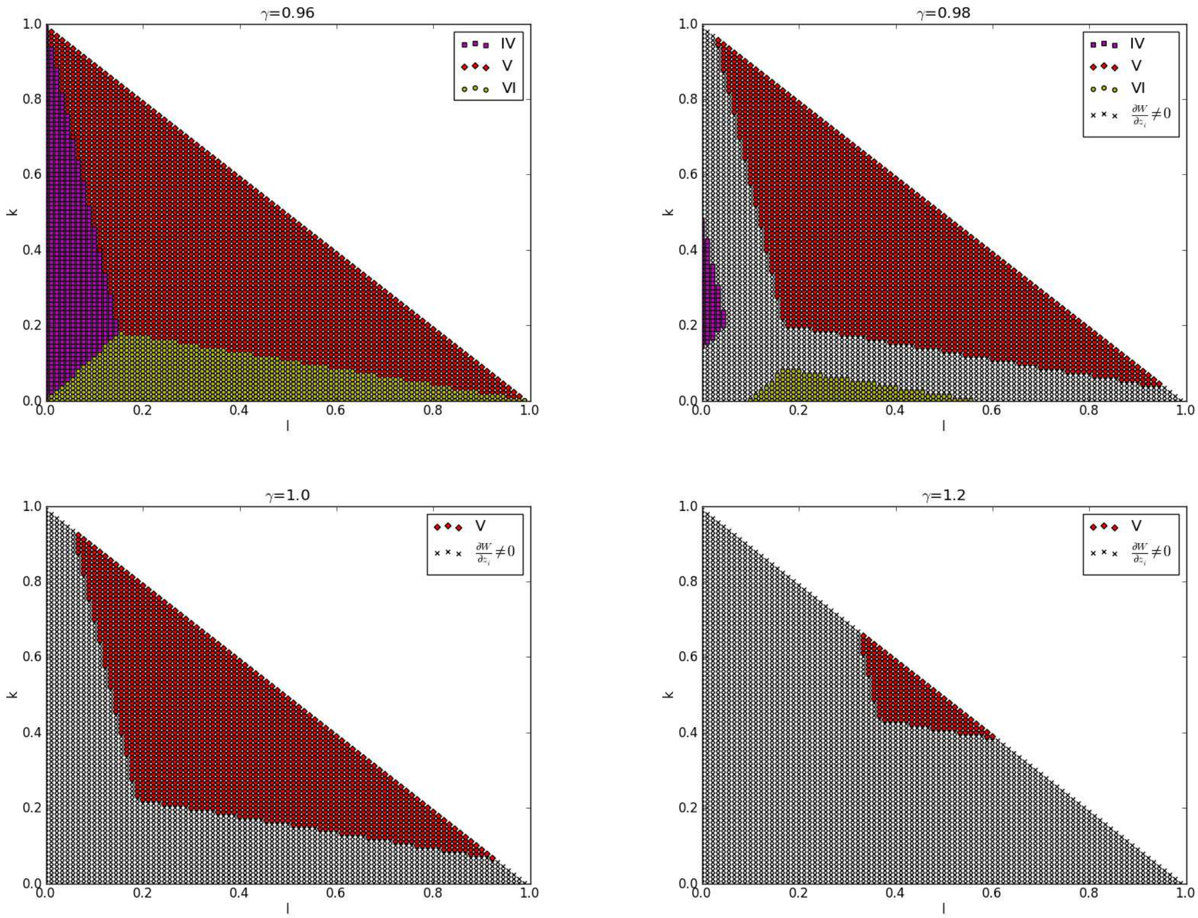

Figure 6: Range of parameters for different types of equilibrium for the 3-asset economy $(z_s = 2.7, z_l = 0.7)$

I can rewrite the above equation as

$$\lambda^m = (\frac{u'(c_m)}{\psi(q_b)} - 1)\Omega^m$$

Envelope condition gives

$$\frac{\gamma}{\beta}\omega^m_{-1} = \omega^m + \frac{\alpha N(1-l)}{N-\sigma}\left[(\frac{u'(c_m)}{\psi'(q_m)} - 1)\Omega^m\right]$$

By applying stationarity, I can write the envelope as

$$\frac{\gamma}{\beta} - 1 = \frac{\alpha N(1-l)}{N-\sigma}[\frac{u'(c_m)}{\psi'(q_m)} - 1]$$

The price for nominal bond is

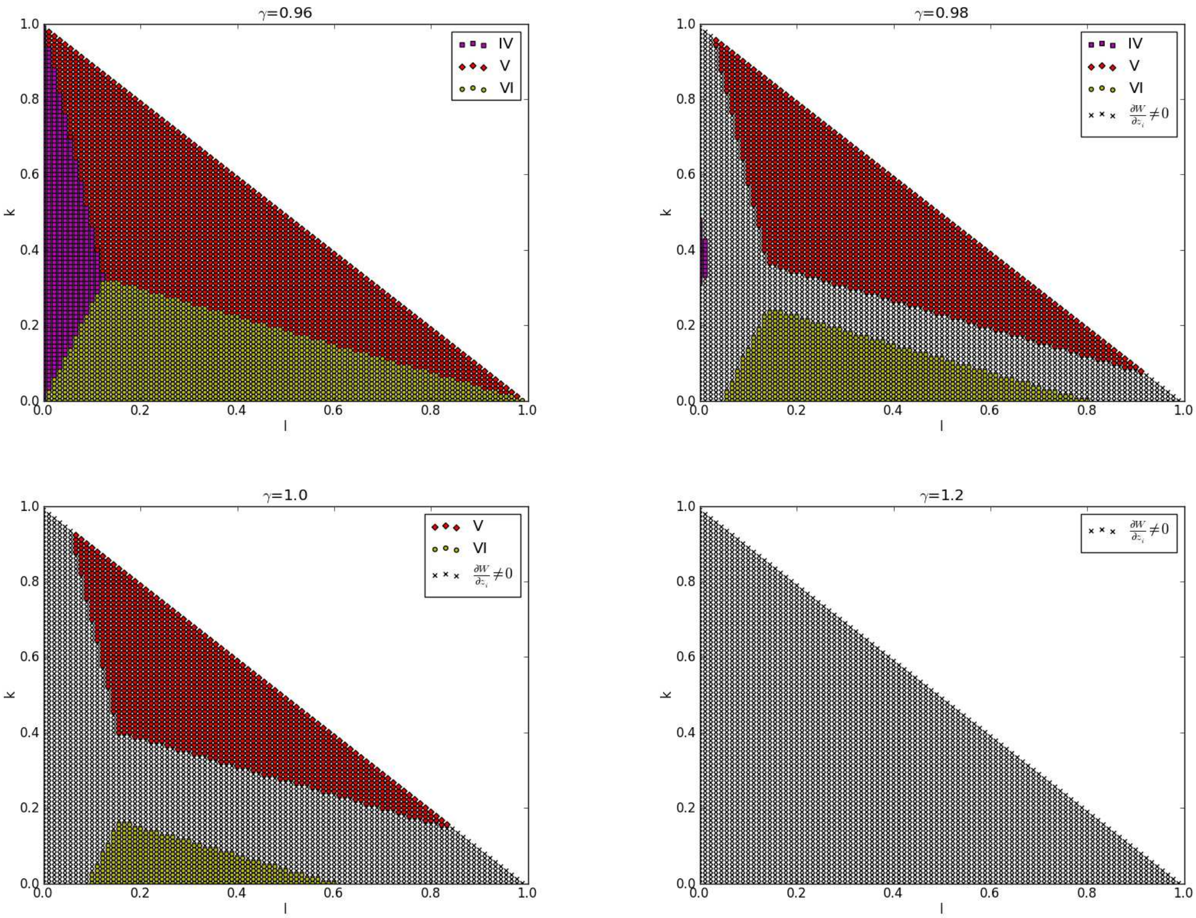

Figure 7: Range of parameters for different types of equilibrium for the 3-asset economy $(z_s = 0.7, z_l = 2.7)$

$$s = \frac{\omega^b}{\omega^m} = \frac{\beta}{\gamma}$$

Case II: $\lambda^m = 0$ and $\lambda^b > 0$

The first order conditions are

$$u'(c_m) = \psi'(q_m)$$

$$\lambda^b = (\frac{u'(c_b)}{\psi(q_b)} - 1)\Omega^b$$

Envelope condition gives

$$\frac{\gamma}{\beta}\omega^m_{-1} = \omega^m + \frac{\alpha N l}{N - \sigma}\left[(\frac{u'(c_b)}{\psi'(q_b)} - 1)\Omega^b\right]$$

The price for the nominal bond is 1. By applying stationarity, I can write the envelope as

$$\frac{\gamma}{\beta} - 1 = \frac{\alpha N l}{N - \sigma}[\frac{u'(c_b)}{\psi'(q_b)} - 1]$$

It is straightforward to see that changing z would not affect households' decision and welfare when at least one of the liquidity constraints is not binding.

Case III: $\lambda^m > 0$ and $\lambda^b > 0$

The first order conditions are

$$\lambda^m = (\frac{u'(c_m)}{\psi'(q_m)} - 1)\Omega^m$$

$$\lambda^b = (\frac{u'(c_b)}{\psi'(q_b)} - 1)\Omega^m$$

Envelope conditions give

$$\frac{\gamma}{\beta}\omega^m_{-1} = \omega^m + \frac{\alpha N l}{N - \sigma}\left[\left(\frac{u'(c_b)}{\psi'(q_b)} - 1\right)\Omega^m\right] + \frac{\alpha N(1-l)}{N - \sigma}\left[\left(\frac{u'(c_m)}{\psi'(q_m)} - 1\right)\Omega^m\right]$$

$$\frac{\gamma}{\beta}\omega^m_{-1} = \omega^m + \frac{\alpha N l}{N - \sigma}\left[\left(\frac{u'(c_b)}{\psi'(q_b)} - 1\right)\Omega^m\right]$$

By applying stationarity, I can write the envelope conditions as

$$\frac{\gamma}{\beta}s - 1 = \frac{\alpha N l}{N - \sigma}\left[\frac{u'(c_b)}{\psi'(q_b)} - 1\right]$$

$$\frac{\gamma}{\beta} - 1 = \frac{\alpha N l}{N - \sigma}\left[\frac{u'(c_b)}{\psi'(q_b)} - 1\right] + \frac{\alpha N(1-l)}{N - \sigma}\left[\frac{u'(c_m)}{\psi'(q_m)} - 1\right]$$

It is straightforward to see that changing z affects the decisions of households and has real effects on the economy.

B Numerical example for the 2-asset economy

I solve the model for the following functional forms

$$u(c) = \log(c)$$

$$\psi(q) = \frac{q^2}{2}$$

$$h(n) = 2an^{1/2}$$

Now I solve the model for 3 different cases of liquidity constrains

Case I:

$$q_b = \frac{1}{(\alpha N l)^{1/2}}$$

$$q_m = \frac{1}{\Big((\frac{\gamma}{\beta} - 1)(N - \sigma) + \alpha N(1 - l)\Big)^{1/2}}$$

$$\frac{a}{(1 - N)^{1/2}} = N - \sigma$$

By using the constraints it is straightforward to show that this equilibrium happens for high enough l

$$l > \bar{l} = \frac{(\frac{\gamma}{\beta} - 1)(N - \sigma) + \alpha N}{(2 + z)\alpha N}$$

Case II:

$$q_m = \frac{1}{(\alpha N(1 - l))^{1/2}}$$

$$q_b = \frac{1}{\Big((\frac{\gamma}{\beta} - 1)(N - \sigma) + \alpha N l\Big)^{1/2}}$$

$$\frac{a}{(1 - N)^{1/2}} = N - \sigma$$

By using the constraints it is straightforward to show that this equilibrium happens for low enough l

$$l < \underline{l} = \frac{\alpha N - (\frac{\gamma}{\beta} - 1)(N - \sigma)(1 + z)}{\alpha N(2 + z)}$$

Case III:

$$q_m = \frac{1}{\Big(\frac{\gamma}{\beta}(1 - s)(N - \sigma) + \alpha N(1 - l)\Big)^{1/2}}$$

$$q_b = \frac{1}{\Big((\frac{\gamma}{\beta}s - 1)(N - \sigma) + \alpha N l\Big)^{1/2}}$$

$$s = \frac{\frac{\gamma}{\beta}(N - \sigma) + \alpha N(1 - l) + (1 + z)(N - \sigma - \alpha N l)}{\frac{\gamma}{\beta}(2 + z)(N - \sigma)}$$

$$\frac{a}{(1-N)^{1/2}} = N - \sigma$$

C The 3-asset economy

Define

$$\zeta(q_i) = \psi'(q_i)q_i - \psi(q_i) \quad i \in \{l, s, n\}$$

In what follows I solve the problem in different cases of equilibrium.

Cases:

I: $\lambda^s = 0 < \lambda^n, \lambda^l$

From the envelope conditions it follows

$$s_s = s_l < 1$$

The first order conditions are

$$u'(c_s) = \psi'(q_s)$$

$$\frac{\gamma}{\beta}s_s - 1 = \frac{\alpha N l}{N-\sigma}[\frac{u'(c_n)}{\psi'(q_n)} - 1]$$

$$\frac{\gamma}{\beta}(1-s_s) = \frac{\alpha N(1-l-k)}{N-\sigma}[\frac{u'(c_l)}{\psi'(q_l)} - 1]$$

$$\begin{aligned} h'(1-N) = (\frac{\gamma}{\beta}s_s - 1 + \frac{\alpha N l}{N-\sigma})\zeta(q_n) + \frac{\alpha N k}{N-\sigma}\zeta(q_s) + \\ (\frac{\gamma}{\beta}(1-s_s) + \frac{\alpha N(1-k-l)}{N-\sigma})\zeta(q_l) \end{aligned}$$

Solution for $u(c) = \log(c)$ and $\psi(q) = q^2/2$

$$1/q_s^2 = \alpha N k$$

$$1/q_n^2 = (\frac{\gamma}{\beta}s_s - 1)(N-\sigma) + \alpha N l$$

$$1/q_l^2 = \frac{\gamma}{\beta}(1-s_s)(N-\sigma) + \alpha N(1-l-k)$$

$$h'(1-N) = \frac{3}{2(N-\sigma)}$$

The solution to the above equations is

$$s_s = \frac{1 + \frac{\beta\alpha N(1-l-k)}{\gamma(N-\sigma)} + \beta/\gamma(1+z_s+z_l)(1-\frac{\alpha Nl}{N-\sigma})}{2+z_l+z_s}$$

$$1/q_n^2 = \frac{\alpha N(1-l-k) + \gamma/\beta(N-\sigma) + \alpha Nl - (1+z_l+z_s)(N-\sigma)}{2+z_l+z_s}$$

$$1/q_l^2 = \frac{1+z_l+z_s}{2+z_l+z_s}(\alpha N(1-k) + (N-\sigma)(\gamma/\beta - 1))$$

Using some algebra I can solve for the criteria for this equilibrium

$$\frac{1+z_l+z_s}{(1+z_s)(2+z_l+z_s)+(1+z_l+z_s)}(1+\frac{N-\sigma}{\alpha N}(\gamma/\beta-1)) \leq k$$

$$\frac{(1+z_l+z_s)(1+\frac{N-\sigma}{\alpha N}(\gamma/\beta-(1+z_s+z_l)))}{(1+z_s)(2+z_l+z_s)+(1+z_l+z_s)} \leq k$$

The left hand side of the second equation is greater than the first.

II: $\lambda^n = 0 < \lambda^s, \lambda^l$

$$s_l = \beta/\gamma < s_s < 1$$

$$u'(c_n) = \psi'(q_n)$$

$$\frac{\gamma}{\beta}s_s - 1 = \frac{\alpha Nk}{N-\sigma}[\frac{u'(c_s)}{\psi'(q_s)} - 1]$$

$$\frac{\gamma}{\beta}(1-s_s) = \frac{\alpha N(1-l-k)}{N-\sigma}[\frac{u'(c_l)}{\psi'(q_l)} - 1]$$

$$h'(1-N) = \frac{\alpha Nl}{N-\sigma}\zeta(q_n) + (\frac{\gamma}{\beta}s_s - 1 + \frac{\alpha Nk}{N-\sigma})\zeta(q_s) + (\frac{\gamma}{\beta}(1-s_s) + \frac{\alpha N(1-k-l)}{N-\sigma})\zeta(q_l)$$

Solution for $u(c) = \log(c)$ and $\psi(q) = q^2/2$

$$1/q_s^2 = (\frac{\gamma}{\beta}s_s - 1)(N-\sigma) + \alpha Nk$$

$$1/q_n^2 = \alpha Nl$$

$$1/q_l^2 = \frac{\gamma}{\beta}(1-s_s)(N-\sigma) + \alpha N(1-l-k)$$

$$h'(1-N) = \frac{3}{2(N-\sigma)}$$

The solution to the above equations is

$$s_s = \frac{1 + \frac{\alpha N(1-l-k)}{N-\sigma}\beta/\gamma - (\alpha Nk - \beta/\gamma)(1+z_s)}{2+z_s}$$

$$1/q_s^2 = \frac{\alpha N(1-l) + (1+z_s)(\alpha NK(1-\gamma/\beta(N-\sigma)) - (N-\sigma)) - (N-\sigma)(\gamma/\beta - 1)}{2+z_s}$$

$$1/q_l^2 = \frac{1+z_s}{2+z_s}(\gamma/\beta(N-\sigma)(1-\alpha Nk) + \alpha N(1-l-k) - (N-\sigma))$$

Using some algebra I can solve for the criteria for this equilibrium

$$k \leq \frac{((1+z_s+z_l)(2+z_s) + 1 + z_s)l - (1+z_s)(1 - \frac{N-\sigma}{\alpha N}(\gamma/\beta - 1)) + \frac{(1+z_s)^2(N-\sigma)}{\alpha N}}{(1+z_s)^2(1-\gamma/\beta(N-\sigma))}$$

$$\frac{(1 + \frac{N-\sigma}{\alpha N}(\gamma/\beta - 1))(1+z_s) - ((2+z_s)(1+z_l+z_s) + (1+z_s)l)}{(1+\gamma/\beta(N-\sigma))(1+z_s)} \leq k$$

III: $\lambda^l = 0 < \lambda^n, \lambda^s$

$$s_l < s_s = 1$$

$$u'(c_l) = \psi'(q_l)$$

$$\frac{\gamma}{\beta}s_l - 1 = \frac{\alpha Nl}{N-\sigma}[\frac{u'(c_n)}{\psi'(q_n)} - 1]$$

$$\frac{\gamma}{\beta}(1-s_l) = \frac{\alpha NK}{N-\sigma}[\frac{u'(c_s)}{\psi'(q_s)} - 1]$$

$$h'(1-N) = \frac{\alpha N(1-k-l)}{N-\sigma}\zeta(q_l) + (\frac{\gamma}{\beta}s_l - 1 + \frac{\alpha Nl}{N-\sigma})\zeta(q_n) + (\frac{\gamma}{\beta}(1-s_l) + \frac{\alpha Nk}{N-\sigma})\zeta(q_s)$$

Solution for $u(c) = \log(c)$ and $\psi(q) = q^2/2$

$$1/q_s^2 = \frac{\gamma}{\beta}(1-s_s)(N-\sigma) + \alpha Nk$$

$$1/q_n^2 = (\frac{\gamma}{\beta}s_s - 1)(N-\sigma) + \alpha Nl$$

$$1/q_l^2 = \alpha N(1-l-k)$$

$$h'(1-N) = \frac{3}{2(N-\sigma)}$$

And by some algebra

$$s_l = \frac{(1+z_s)(1+\frac{\alpha Nk}{N-\sigma}\beta/\gamma) - \beta/\gamma(\frac{\alpha Nk}{N-\sigma} - 1)(1+z_s+z_l)}{2+2z_s+z_l}$$

$$1/q_l^2 = \gamma/\beta(N-\sigma) + \alpha Nk - \frac{(1+z_s)(\gamma/\beta(N-\sigma)+\alpha Nk) - (\alpha Nl - (N-\sigma))(1+z_l+z_s)}{2+2z_s+z_l}$$

$$1/q_n^2 = \frac{\alpha N(1+z_s)(k+l) - (N-\sigma)(1+z_l+z_s)(\gamma/\beta - 1)}{2+2z_s+z_l}$$

Using some algebra I can solve for the criteria for this equilibrium

$$k + l \leq \frac{(2+2z_s+z_l) + \frac{N-\sigma}{\alpha N}(1+z_s+z_l)^2(\gamma/\beta - 1)}{(1+z_s)(1+z_s+z_l) + (2+2z_s+z_l)}$$

$$((1+z_s)(1+z_s+z_l) + 2 + 2z_s + z_l)k + (-(1+z_s)(1+z_s+z_l) + 2 + 2z_s + z_l)l \leq \\ 2 + 2z_s + z_l - (1+z_s+z_l)(1+z_s)(\frac{N-\sigma}{\alpha N}(\gamma/\beta + 1))$$

IV: $\lambda^s = \lambda^l = 0 < \lambda^n$

$$s_l = s_s = 1$$

$$u'(c_l) = \psi'(q_l)$$

$$u'(c_s) = \psi'(q_s)$$

$$\frac{\gamma}{\beta} - 1 = \frac{\alpha Nl}{N-\sigma}[\frac{u'(c_n)}{\psi'(q_n)} - 1]$$

$$h'(1-N) = \frac{\alpha N k}{N-\sigma}\zeta(q_s) + (\frac{\gamma}{\beta} - 1 + \frac{\alpha N l}{N-\sigma})\zeta(q_n) + \frac{\alpha N(1-k-l)}{N-\sigma}\zeta(q_l)$$

As the above equations show, marginal open-market operations (small changes in z_s and z_l) do not change the real decisions of the households and welfare. Solution for $u(c) = \log(c)$ and $\psi(q) = q^2/2$

$$1/q_s^2 = \alpha N k$$
$$1/q_n^2 = (\frac{\gamma}{\beta} - 1)(N-\sigma) + \alpha N l$$
$$1/q_l^2 = \alpha N(1-l-k)$$
$$h'(1-N) = \frac{3}{2(N-\sigma)}$$

Using some algebra I can solve for the criteria for this equilibrium

$$(\gamma/\beta - 1)(\frac{N-\sigma}{\alpha N})(1+z_s+z_l) \leq (1+z_s)k - (1+z_s+z_l)l$$

$$(2+z_s+z_l)l + k \leq 1 - (1+z_s+z_l)(\gamma/\beta - 1)\frac{N-\sigma}{\alpha N}$$

V: $\lambda^s = \lambda^n = 0 < \lambda^l$

$$s_l = s_s = \beta/\gamma < 1$$

$$u'(c_n) = \psi'(q_n)$$

$$u'(c_s) = \psi'(q_s)$$

$$\frac{\gamma}{\beta} - 1 = \frac{\alpha N(1-k-l)}{N-\sigma}[\frac{u'(c_n)}{\psi'(q_n)} - 1]$$

$$h'(1-N) = \frac{\alpha N l}{N-\sigma}\zeta(q_n) + (\frac{\gamma}{\beta} - 1 + \frac{\alpha N(1-k-l)}{N-\sigma})\zeta(q_l) + \frac{\alpha N k}{N-\sigma}\zeta(q_s)$$

As the above equations show, marginal open-market operations (small changes in z_s and z_l)

do not change the real decisions of the households and welfare.

Solution for $u(c) = \log(c)$ and $\psi(q) = q^2/2$

$$1/q_s^2 = \alpha N k$$

$$1/q_n^2 = \alpha N l$$

$$1/q_l^2 = (\frac{\gamma}{\beta} - 1)(N - \sigma) + \alpha N(1 - l - k)$$

$$h'(1 - N) = \frac{3}{2(N - \sigma)}$$

Using some algebra I can solve for the criteria for this equilibrium:

$$1 + (\gamma/\beta - 1)\frac{N - \sigma}{\alpha N} \leq (2 + z_s + z_l)l + k$$

$$1 + (\gamma/\beta - 1)(\frac{N - \sigma}{\alpha N}) \leq (2 + z_s)k + l$$

VI: $\lambda^n = \lambda^l = 0 < \lambda^s$

$$s_l = \beta/\gamma < s_s = 1$$

$$u'(c_n) = \psi'(q_n)$$

$$u'(c_l) = \psi'(q_l)$$

$$\frac{\gamma}{\beta} - 1 = \frac{\alpha N k}{N - \sigma}[\frac{u'(c_s)}{\psi'(q_s)} - 1]$$

$$h'(1 - N) = \frac{\alpha N l}{N - \sigma}\zeta(q_n) + (\frac{\gamma}{\beta} - 1 + \frac{\alpha N k}{N - \sigma})\zeta(q_s) + \frac{\alpha N(1 - k - l)}{N - \sigma}\zeta(q_l)$$

As the above equations show, marginal open-market operations (small changes in z_s and z_l) do not change the real decisions of the households and welfare.

Solution for $u(c) = \log(c)$ and $\psi(q) = q^2/2$:

$$1/q_s^2 = (\frac{\gamma}{\beta} - 1)(N - \sigma) + \alpha N k$$

$$1/q_n^2 = \alpha N l$$

$$1/q_l^2 = \alpha N(1-l-k)$$

$$h'(1-N) = \frac{3}{2(N-\sigma)}$$

Using some algebra I can solve for the criteria for this equilibrium:

$$(2+z_s)k + l \leq 1 - (\gamma/\beta - 1)\frac{N-\sigma}{\alpha N}(1+z_s)$$

$$(\gamma/\beta - 1)(\frac{N-\sigma}{\alpha N})(1+z_s) \leq (1+z_s+z_l)l - (1+z_s)k$$

VII: $0 < \lambda^s, \lambda^l, \lambda^n$

$$s_l < s_s < 1$$

$$\frac{\gamma}{\beta}s_l - 1 = \frac{\alpha N l}{N-\sigma}[\frac{u'(c_n)}{\psi'(q_n)} - 1]$$

$$\frac{\gamma}{\beta}(s_s - s_l) = \frac{\alpha N k}{N-\sigma}[\frac{u'(c_s)}{\psi'(q_s)} - 1]$$

$$\frac{\gamma}{\beta}(1-s_s) = \frac{\alpha N(1-l-k)}{N-\sigma}[\frac{u'(c_l)}{\psi'(q_l)} - 1]$$

$$h'(1-N) = (\frac{\gamma}{\beta}s_l - 1 + \frac{\alpha N l}{N-\sigma})\zeta(q_n) + (\frac{\gamma}{\beta}(s_s - s_l) + \frac{\alpha N k}{N-\sigma})\zeta(q_s) + (\frac{\gamma}{\beta}(1-s_s) + \frac{\alpha N(1-l-k)}{N-\sigma})\zeta(q_l)$$

Solution for $u(c) = \log(c)$ and $\psi(q) = q^2/2$:

$$1/q_s^2 = \frac{\gamma}{\beta}(s_s - s_l)(N-\sigma) + \alpha N k$$

$$1/q_n^2 = (\frac{\gamma}{\beta}s_l - 1)(N-\sigma) + \alpha N l$$

$$1/q_l^2 = \frac{\gamma}{\beta}(1-s_s)(N-\sigma) + \alpha N(1-l-k)$$

$$h'(1-N) = \frac{3}{2(N-\sigma)}$$

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