Mental Accounting: A Closed-Form Alternative to the Black Scholes Model

Siddiqi, Hammad

University of Queensland

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Hammad Siddiqi[1]

University of Queensland

h.siddiqi@uq.edu.au

Abstract

Experimental evidence and opinions of market professionals suggest that people rely on mental accounting while valuing a call option. I show that mental accounting generates a closed-form alternative to the Black Scholes formula that does not require a complete market. The new formula is arbitrage free. The new formula differs from the Black Scholes formula only due to the appearance of a parameter in the formula that captures the risk premium on the underlying. The new formula, called the analogy option pricing formula, provides a new explanation for the implied volatility skew puzzle. I also show that the key aspects of the analogy formula are consistent with empirical evidence.

Keywords: *Mental Accounting, Analogy Making, Incomplete Markets, Implied Volatility, Implied Volatility Skew, Option Prices, Risk Premium, Black Scholes Model*

JEL Classifications: G13; G12

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Mental Accounting: A Closed Form Alternative to the Black Scholes Model

Market professionals typically consider a call option to be a surrogate for the underlying because of the similarity in their respective payoffs.[2] Does such mental framing affect option valuation? Rockenbach (2004) finds that mental accounting hypothesis is the best predictor of behavior in laboratory experiments. The mental accounting hypothesis implies that participants demand the same expected return from a call option as available on the underlying stock. Experiments in Siddiqi (2012) and Siddiqi (2011) replicate and extend these findings to trinomial and other settings. The experiments suggest that when people cannot replicate a call option by using the underlying and a risk-free bond, they co-categorize it with the underlying and demand the same expected return as available on the underlying as the two assets have related payoffs. It appears that participants in laboratory markets consider a call option to be a surrogate for the underlying without receiving any coaching to this effect due to the similarity in their payoffs. Arguably, investors in financial markets are even more likely to consider a call a surrogate for the underlying as they receive such advice from professional traders.

In general, a large literature in economics and finance acknowledges the importance of mental framing for investment decisions. A well known application of mental accounting to asset pricing is behavioral portfolio theory of Shefrin and Statman (2000). Despite strong experimental evidence of the relevance of mental accounting for option valuation and clear precedents of the application of mental accounting to asset pricing (such as Barberis and Huang (2001), and Shefrin and Statman (2000)), the notion of mental accounting has not been directly incorporated in option pricing models. (Some recent exceptions in the related area of prospect theory are Nordon and Pianca (2012), and Versluis, Lehnert, and Wolff (2010)). This is quite surprising given the fact the option pricing models are typically relative pricing models in which the price of the underlying is taken as given and options are valued relative to that. Hence, the underlying provides a natural mental frame for valuation. In this article, I attempt to bridge this gap in the literature. I derive option pricing formulas which incorporate the experimentally observed influence of mental accounting on option valuation. Specifically, I derive option pricing formulas based on the notion

[2] As illustrative examples, see the following:
http://ezinearticles.com/?Call-Options-As-an-Alternative-to-Buying-the-Underlying-Security&id=4274772,
http://www.investingblog.org/archives/194/deep-in-the-money-options/,
http://www.triplescreenmethod.com/TradersCorner/TC052705.asp,
http://daytrading.about.com/od/stocks/a/OptionsInvest.htm

that a call option is placed in the same mental account as the underlying; hence, the same expected return is demanded from a call option as available on the underlying.

A call option is co-categorized with its underlying due to the similarity between their respective payoffs. Such co-categorization is an aspect of mental accounting (see Hendersen and Peterson (1992)). Thaler (1999) defines mental accounting as the set of cognitive operations used by individuals to organize, evaluate, and keep track of financial activities. Mental accounting of a call with its underlying implies that individuals co-categorize a call with its underlying and set similar targets (returns) for the two assets due to similarity in their payoffs.

As in the Black Scholes model, I assume that the underlying follows a geometric Brownian motion. The formulas for call and put options (via put-call parity) thus obtained can be considered a generalization of the Black Scholes formulas. I refer to them as *mental accounting Black Scholes formula* or *analogy option pricing formulas.* One can assume that the underlying follows jump diffusion or stochastic volatility processes and derive *analogy based jump diffusion* and *analogy based stochastic volatility* formulas respectively. As the notion of mental accounting is independent of the distributional assumptions regarding the underlying, it can be added to a variety of models with different distributional specifications.

I show that the mental accounting option pricing formulas provide an explanation for the implied volatility skew puzzle. Specifically, if the market prices are determined in accordance with the analogy formula and the Black Scholes formula is used to back-out implied volatility, then the skew is observed.

The sudden emergence of the implied volatility skew in index options worldwide after the crash of 1987 is a major puzzle in option pricing. Broadly speaking, three types of extensions of the Black Scholes model have been proposed in an attempt to capture the observed skew. One approach is *stochastic volatility* approach in which the underlying's volatility is assumed to be a mean reverting diffusion process typically correlated with the stochastic process of the underlying itself. See Heston (1993), Stein and Stein (1991), and Hull and White (1987) among others. Stochastic volatility models generate a variety of skews and smiles. However, in order to generate an implied volatility skew consistent with what is observed for traded options, unrealistically high value of negative correlation between volatility and the index is required.

Another approach extends the original suggestion of Merton (1976) to generate a variety of smiles and skews by allowing the underlying to follow a jump diffusion process and by carefully selecting the jump (Poisson process) parameters. An incomplete list of papers include Bakshi, Cao, and Chen (1997), Bates (1996), and Das and Foresi (1996). However, jump diffusion models cannot generate the skew without assuming that jumps are distributed asymmetrically around the current stock price. This assumption adds to computational complexity. Empirically, in order to match the model generated skew to the skew observed from traded options (with asymmetric jumps), one needs to assume that the market is pricing-in larger and more frequent jumps then what has been historically observed. See Andersen and Andreasen (2002).

The third approach due to Derman and Kani (1994), Dupire (1994), and Rubinstein (1994) and extended by Brotherton-Ratcliffe (1998), Andreasen (1997), Lagnado and Osher (1997), Brown and Toft (1999), Jackwerth (1996), Chriss (1996) and others assumes that the volatility is a deterministic function of the stock price and time. However, fitting such models to quite steep short-term skews that are typically observed does not give convincing results.

Even though the approach taken in this article is fundamentally different from the approaches described above, it should be seen as complementary to all three approaches mentioned above. In principle, one can supplement any of the above approaches with the idea of mental accounting. In fact, supplementing these models with mental accounting makes the skews generated by these models more pronounced. This may result in implied volatility skews in the above models with more reasonable parameter values. Showing this is the subject of future research. Here, we only consider the case in which the underlying follows geometric Brownian motion, as in the Black Scholes model. The purpose is to demonstrate the practicality of the idea of mental accounting of a call option with its underlying (due to payoff similarity) by showing that it can be formally incorporated in an option pricing model.

Some cognitive scientists argue that similarity spotting or analogy making forms the core of cognition and it is the fuel and fire of thinking (see Hofstadter and Sander (2013)). Hofstadter and Sander (2013) write, *"[…], at every moment of our lives, our concepts are selectively triggered by analogies that our brain makes without letup, in an effort to make sense of the new and unknown in terms of the old and known." (Hofstadter and Sander (2013), Prologue page1).*

They define analogy making as the act of placing objects in the same mental category due to a perceived similarity between them. In economics literature, grouping assets in mental categories is an aspect of mental accounting, a termed attributed to Thaler (1980). In fact, equivalence between the categorization theories of cognitive science (analogy making is one example) and mental accounting has been proposed and studied (see Henderson and Peterson (1992)).

The recognition of analogy making as an important decision principle is not new. Hume wrote in 1748, "*From causes which appear similar, we expect similar effects. This is the sum of all our experimental conclusions*". (Hume 1748, Section IV). Similar ideas have been expressed in economic literature by Keynes (1921), Selten (1978), and Cross (1983) among others. To our knowledge, two formal approaches have been proposed to incorporate analogy making into economics: 1) case based decision theory of Gilboa and Schmeidler (2001) in which preferences are determined by the cases in a decision maker's memory and their similarity with the decision problem being considered, and 2) coarse thinking/analogy making model of Mullainathan, Schwartzstein, and Shleifer (2008) in which expectations about an attribute are formed by co-categorizing a situation with analogous situations and transferring the information content of the attribute across co-categorized situations. The approach in this paper, if broadly interpreted, relates to the model of Mullainathan et al (2008). The attribute of concern here is return on a call option, which is influenced by the return on the underlying as investors co-co-categorize a call with the underlying stock.

This paper adds to the literature in several ways. 1) We put forward a closed-form alternative to the Black Scholes formula. Having a closed-form is advantageous for a number of reasons. Most importantly, it greatly simplifies computation enabling one to develop intuition about the impact of various parameters. The analogy approach results in a closed-form with a formula that differs from the Black Scholes formula due to the appearance of only one additional parameter, which is the risk premium on the underlying. Hence, the analogy formula is as simple as the Black Scholes formula. 2) The analogy formula provides a new explanation for the implied volatility skew puzzle. Specifically, if the market prices are determined by the analogy formula, and the Black Scholes formula is used to back-out implied volatility, the skew is observed. 3) In an interesting paper, Derman (2002) writes, "*If options prices are generated by a Black–Scholes equation whose rate is greater than the true riskless rate, and if these options prices are then used to produce implied volatilities via the Black–Scholes equation with a truly riskless rate, it is not hard to check that the resultant implied volatilities will produce a negative volatility skew*." (Derman (2002) page 295).

This paper provides a reason for the above mentioned effect.[3] The analogy formula is exactly identical to the Black Scholes formula apart from replacing the risk free rate with the return on the underlying stock (that is, the risk free is supplemented with the risk premium). Our approach is also broadly consistent with Shefrin (2008) who provides a systematic treatment of how behavioral assumptions impact the pricing kernel at the heart of modern asset pricing theory. 4) We provide a number of testable predictions of the model and summarize existing evidence. Existing evidence strongly supports the analogy approach. 5) Duan and Wei (2009) use daily option quotes on the S&P 100 index and its 30 largest component stocks, to show that, after controlling for the underlying asset's total volatility, a higher amount of systematic risk leads to a higher level of implied volatility and a steeper slope of the implied volatility curve. In the analogy option pricing model, higher risk premium on the underlying for a given level of total volatility generates this result. As risk premium is related to systematic risk, this prediction of the analogy model is quite intriguing. 6) Our approach is also an example of behavioralization of finance. Shefrin (2010) argues that finance is in the midst of a paradigm shift, from a neoclassical based framework to a psychologically based framework. Behavioralizing finance is the process of replacing neoclassical assumptions with behavioral counterparts while maintaining mathematical rigor. 7) One limitation of the Black Scholes model is that it requires a complete market. In contrast, the analogy formula does not require a complete market. In an incomplete market there is no unique no-arbitrage price; rather a wide interval of arbitrage-free prices is obtained as the martingale measure is not unique. Which price to pick then? Two approaches have been developed to search for solutions in an incomplete market. One is to pick a specific martingale measure according to some optimal criterion. See Follmer and Schweizer (1991), Miyahara (2001), Fritelli (2002), Bellini and Fritelli (2002), and Goll and Ruschendorf (2001) among others. The other approach is utility based option pricing. See Hodges and Neuberger (1989), Davis (1997), and Henderson (2002) for early treatment. Our approach relates to the former as it effectively specifies analogy making as a mechanism for picking a specific martingale measure.

Section 2 illustrates the application of mental accounting to option pricing through a discrete time numerical example. It then discusses a general trinomial model and shows the mental accounting picks out a particular Equivalent Martingale Measure (EMM); hence, prices generated via mental accounting are arbitrage-free. Section 3 discusses the continuous limit and derives the analogy based option pricing formulas under the assumption that the underlying follows geometric Brownian motion. Section 4 shows that if the market prices are determined in accordance with the

[3] I am grateful to Emanuel Derman for pointing this out.

analogy formulas and the Black Scholes formula is used to back-out implied volatility, then the skew is observed. Section 5 shows that the key properties of the analogy model are consistent with empirically observed features. Duan and Wei (2009) use daily option quotes on the S&P 100 index and its 30 largest component stocks, to show that, after controlling for the underlying asset's total volatility, a higher amount of systematic risk leads to a higher level of implied volatility and a steeper slope of the implied volatility curve. In the mental accounting model developed here, higher risk premium on the underlying for a given level of total volatility generates this result. As risk premium is related to systematic risk, this feature of the analogy model is quite intriguing. Section 6 concludes.

2. Analogy Making: An Incomplete Market Example

Consider a simple incomplete market in which there are two assets and three states. Each state is equally likely to occur. Asset "S" has a price of 100 today and the risk free asset "B" also has a price of 100 today. The state-wise payoffs are summarized in table 1.

Table 1

Asset Type	Price	Red	Blue	Green
S	100	200	90	50
B	100	110	110	110

Suppose a new asset, "A" is introduced with the following payoffs: Red state payoff is 140; Blue state payoff is 30; and Green state payoff is 0. This claim is equivalent to a call option on "S" with a strike price of 60. "A" cannot be replicated with S and B. Hence, there is no unique no-arbitrage price. However, an arbitrage free interval can be specified: $45.5 < arbitrage\ free\ price < 54.5$.

"A" and "S" have related payoffs as "A" is a call option on "S". Mental accounting of "A" with "S" means that one demands the same expected return from "A" as available from "S". The expected return from "S" is: $1.13333 = \frac{\frac{1}{3}(200+90+50)}{100}$. If "A" offers the same expected return as "S", its price must be: $\frac{\frac{1}{3}(140+30+0)}{1.13333} = 50.$

Hence, analogy making picks out the following price from the arbitrage- free interval: 50. It can be shown that this price corresponds to the following martingale measure: (0.363633, 0.136364, 0.5). Hence, in this example, analogy making picks out a specific martingale measure from a set of possibilities. Consequently, it can be considered a selection mechanism.

The problem of pricing "A" in an incomplete market can be stated as: Given the actual probability measure $\boldsymbol{P}$, there is a set $\boldsymbol{Q}$ of *equivalent martingale measures* such that the price of "A" is in an arbitrage-free interval:

$\left(inf_{Q\in\boldsymbol{Q}}E_Q[max(S_T-K,0)], sup_{Q\in\boldsymbol{Q}}E_Q[max(S_T-K,0)]\right)$. Where E_Q gives the expected discounted value under Q.

Up	2					Stock	160
Down	0.5					Call	120
Middle	1					Covered Call	40
Up Prob.	0.495						
Down Prob.	0.495			Stock	80	Stock	80
Middle Prob.	0.01			Call	47.93587	Call	40
E(Return)	1.2475			Covered Call	32.06413	Covered Call	40
K	40						
		Stock	40	Stock	40	Stock	40
		Call	19.14787	Call	15.87174	Call	0
		Covered Call	20.85213	Covered Call	24.12826	Covered Call	40
				Stock	20	Stock	20
				Call	0	Call	0
				Covered Call	20	Covered Call	20
						Stock	10
						Call	0
						Covered Call	10

Figure 1

2.1 Mental Accounting and Delta-Hedging

Consider a two period trinomial situation as shown in figure 1. The parameters are: Up factor=2, Down factor=0.5, Middle factor=1, Risk free interest rate per trinomial period=0, Strike price=40, Probabilities of up, down, and middle movements are 0.495, 0.495, and 0.01 respectively. It follows that the expected return on the stock per period is 1.2475.

Figure 1 shows the analogy price of the call option in each node. It also shows the corresponding payoffs from the covered call strategy in each node. The covered call strategy involves buying one unit of the underlying and writing one call option. It is easy to verify that the expected one period return from the covered call strategy throughout the tree is equal to the expected return from the underlying: 1.2475. That is, under analogy making, the expected return from $S - C$ is equal to the expected return from the underlying.

Consider a portfolio in which one buys 0.5 unit of the underlying and writes one call option. The value of such a portfolio at the beginning when the stock price is 40 is 0.85215. In the next period, if the stock price goes up to 80, the value becomes -7.93587. If the stock price goes down to 20, the value is 10. If the stock price remains unchanged at 40, the next period value of the portfolio is 4.128257. Once again, it is easy to verify that the expected return from the portfolio $0.5S - C$ is equal to the expected return from the stock: 1.2475. In fact, the expected return from $Sx - C$ is equal to the expected return from the stock for all values of x (as long as one is careful to exclude division by 0). We will use this fact to derive the option pricing formula in the continuous limit.

It is also interesting to consider a scenario in which one is only aware of the up and down states while being oblivious of the middle state. This corresponds to the binomial model of Cox, Ross, and Rubinstein (1979). For such a person, the delta-hedged portfolio is equal to $0.798931S - C$. The payoff from this portfolio if either the up or the down state is realized is 15.97851. Hence, he would misperceive the portfolio to be risk-free and would incorrectly spot arbitrage opportunities.

3. Mental Accounting and Option Valuation: The Discrete Trinomial Case

The basic set-up of the model described here is a generalization of Cox, Ross, and Rubinstein (1979) binomial model (CRR model). Assume that trade occurs only on discrete dates indexed by 0, 1, 2, 3, 4, 5,………T. Initially, there are only two assets. One is a riskless bond that pays $\dot{r}$ every period meaning that if B dollars are invested at time i, the payoff at time i+1 is $\dot{r}B$. The second asset is a risky stock. If the stock price at a given time i is $S_j(i)$, then in the next time period, it can change to either $S_{j+1}(i+1)$ or $S_{j-1}(i+1)$. The stock price can also remain unchanged at $S_j(i+1)$. The variable j is an index for state and the variable i is an index for time. In this set-up, the state-space for a two-period model is shown in figure 2. The transition probabilities in this state-space are represented by Q.

In each time period, the stock price can undergo either a state change of ± 1 unit or remain the same. Sometimes, the price changes correspond to a state change of one unit. That is, if at time i, the state is $S_j(i)$, then at time i+1, it changes either to $S_{j+1}(i+1)$ or $S_{j-1}(i+1)$. Such changes, termed *unit changes*, correspond to the binomial changes assumed in CRR model. On other occasions, the state remains unchanged. That is, the stock price stays the same. These are referred to as *no changes*. So the structure of the state-space is that of *no changes* super-imposed on the binomial model of CRR. For simplicity, we assume that there are no dividends.

Assume that a new asset, $C_j(i)$, which is a call option on the stock, is introduced, with maturity at τ. Without loss of generality, assume that j=0. Consider the following portfolio:

$$V(i) = S_0(i)x - C_0(i) \tag{1}$$

Where $x = \frac{C_{+1}(i+1) - C_{-1}(i+1)}{S_{+1}(i+1) - S_{-1}(i+1)}$

Time 0	Time 1	Time 2
		$S_{+2}(2)$
	$S_{+1}(1)$	$S_{+1}(2)$
$S_0(0)$	$S_0(1)$	$S_0(2)$
	$S_{-1}(1)$	$S_{-1}(2)$
		$S_{-2}(2)$

The state space over two periods.

Figure 2

The portfolio in (1) is called the delta-hedged portfolio because such a portfolio gives the same value if either of the adjacent states is realized in the next period. That is, conditional on *unit changes* in the state, the portfolio is risk-free. In what follows, for ease of reading, we suppress the subscripts and/or time index, wherever doing so is unambiguous.

If only *unit changes* happen, then, in the next period:

$$V_{+1} = V_l(i+1) = S_{+1}(i+1)x - C_{+1}(i+1) \tag{2}$$

Or

$$V_{-1} = V_l(i+1) = S_{-1}(i+1)x - C_{-1}(i+1) \tag{3}$$

Define the single period capital gain return on the underlying stock as follows:

$\Delta_k = \frac{S_{+k}(i+1)}{S(i)}$ where k= -1 ,0, 1

Substituting the value of x in either (2) or (3) leads to:

$$V_{\pm}(i+1) = \frac{C_{+1}(i+1)\Delta_{-1} - C_{-1}(i+1)\Delta_{+1}}{\Delta_{+1} - \Delta_{-1}} \tag{4}$$

If only *unit changes* are allowed, then the portfolio in (1) takes the value shown in (4) in the next period. That is, the delta-hedged portfolio is *locally* risk free; however, it is not *globally* risk free as it takes a different value if the stock price does not change.

If only *unit changes* are allowed, then the delta-hedged portfolio is risk-free. Consequently, in accordance with the principle of no-arbitrage, it should earn the risk-free rate of return.

$$V_{\pm}(i+1) = \dot{r}V(i) \tag{5}$$

Substituting (1) and (5) in (4) and simplifying leads to:

$$\dot{r}\left[\frac{C_{+1}(i+1) - C_{-1}(i+1)}{\Delta_{+1} - \Delta_{-1}}\right] - \left[\frac{C_{+1}(i+1)\Delta_{-1} - C_{-1}(i+1)\Delta_{+1}}{\Delta_{+1} - \Delta_{-1}}\right] = \dot{r}C(i) \tag{6}$$

Starting from time $i = \tau$, recursive application of (6) leads to the current price of the call option.

Re-arranging (6):

$$\left(\frac{\dot{r} - \Delta_{-1}}{\Delta_{+1} - \Delta_{-1}}\right) C_{+1}(i+1) + \left(\frac{\Delta_{+1} - \dot{r}}{\Delta_{+1} - \Delta_{-1}}\right) C_{-1}(i+1) = \dot{r}C(i) \tag{7}$$

In (7), the terms in brackets in front of $C_{+1}(i+1)$ and $C_{-1}(i+1)$ are the risk neutral probabilities.

Consider the value of the delta-hedged portfolio in the case of *no state change*. The value of the delta-hedged portfolio conditional on *no state change* is:

$$V(i+1) \mid no\ state\ change = S_0(i+1)x - C_0(i+1) \tag{8}$$

The delta-hedged portfolio is no longer risk free. In the case of *unit state changes*, its value is risk free and is given by (4), and in the case of *no state change,* its value is given by (8). Assume that the true probability (under Q) of there being *no state change* is γ. The expected value of the delta-hedged portfolio can now be written as:

$$E[V(i+1)] = \gamma[S_0(i+1)x - C_0(i+1)] + (1-\gamma)V_{\pm}(i+1) \tag{9}$$

As the delta-hedged portfolio can no longer be considered identical to the risk-free asset, the principle of no-arbitrage cannot be applied to determine a unique price for the call option. A call option is similar to the underlying stock. In accordance with experimental evidence (Siddiqi (2012, Siddiqi (2011), and Rockenback (2004)), we assume that the same expected return is demanded from a call option as is available on the underlying. It follows that the delta-hedged portfolio should also offer the same expected return as the underlying stock. Proposition 1 shows the recursive pricing equation that the call option must satisfy under analogy making.

Proposition 1 ***If analogy making determines the price of the call option, then the following recursive pricing equation must be satisfied:***

$$\boldsymbol{(1-\gamma)\left\{C_{+1}(i+1)\left[\frac{\frac{(\dot{r}+\delta-\gamma)}{1-\gamma}-\Delta_{-1}}{\Delta_{+1}-\Delta_{-1}}\right]+C_{-1}(i+1)\left[\frac{\Delta_{+1}-\frac{(\dot{r}+\delta-\gamma)}{1-\gamma}}{\Delta_{+1}-\Delta_{-1}}\right]\right\}+\gamma C(i+1) = (\dot{r}+\delta)C(i)} \qquad \boldsymbol{(10)}$$

Where $\boldsymbol{\delta}$ is the risk premium on the underlying stock.

Proof.

Analogy making implies that $(\dot{r}+\delta)V(i)=E[V(i+1)]$. Substituting (4) and (8) in (9) and collecting terms together leads to (10).

■

(7) can be obtained from (10) by making γ and δ equal to zero. If the delta-hedged portfolio in (1) is considered identical to the riskless asset, which corresponds to unawareness of a part of the state space, then according to the principle of no-arbitrage, it should offer the risk free return. In that case the pricing equation for the call option is given in (7). However, with awareness of full state space, and under mental accounting of a call with its underlying, the correct pricing equation is given in (10).

Mental accounting results in an arbitrage-free price for the call option. To see this, one just needs to realize that the existence of the risk neutral measure or the equivalent martingale measure is both necessary and sufficient for prices to be arbitrage-free. See Harrison and Kreps (1979). One can simply multiply payoffs with the corresponding risk neutral probabilities to get the price of an asset times the risk free rate. Proposition 2 shows the equivalent martingale measure associated with the analogy model developed here.

Proposition 2 ***The equivalent martingale measure or the risk neutral pricing measure associated with the analogy model is given by:***

Risk neutral probability of a +1 change in state: $\boldsymbol{(1-\gamma_n)(q_n)}$

Risk neutral probability of a -1 change in state: $\boldsymbol{(1-\gamma_n)(1-q_n)}$

Risk neutral probability of no change in state: $\boldsymbol{\gamma_n}$

Where $\boldsymbol{q_n = \frac{(\dot{r}-\Delta_{-1})(C(i+1)-C_{-1}(i+1))-(1-\Delta_{-1})(\dot{r}C(i+1)-C_{-1}(i+1))}{(\dot{r}-1)(C_{+1}(i+1)-C_{-1}(i+1))-(\Delta_{+1}-\Delta_{-1})(\dot{r}C(i+1)-C(i+1))}}$

And $\boldsymbol{\gamma_n = \frac{(\dot{r}-\Delta_{-1})-q_n(\Delta_{+1}-\Delta_{-1})}{(1-\Delta_{-1})-q_n(\Delta_{+1}-\Delta_{-1})}}$

Proof.

By the definition of equivalent martingale measure, the following equations must hold:

$$(1-\gamma_n)\{C_{+1}(i+1)q_n + C_{-1}(i+1)(1-q_n)\} + \gamma_n C(i+1) = \dot{r}C(i)$$

$$(1-\gamma_n)\{S_{+1}(i+1)q_n + S_{-1}(i+1)(1-q_n)\} + \gamma_n S(i+1) = \dot{r}S(i)$$

Substituting the values of q_n and γ_n in the above equations and simplifying shows that the left hand sides of the above equations are equal to $\dot{r}C(i)$ and $\dot{r}S(i)$ respectively.

■

4. Mental Accounting and Option Valuation: The Continuous Limit

The previous section considers a discrete trinomial situation and derives the pricing relations for a European call option under mental accounting. It also shows that the prices generated by the mental accounting approach are arbitrage-free as an equivalent martingale measure can be found. In this section, we derive the option pricing formulas in the continuous limit. It is well known that the discrete trinomial state space discussed earlier converges to geometric Brownian motion in the continuous limit. However, as the trinomial state space implies that an option cannot be replicated by using a combination of the underlying and the risk-free asset, the market is incomplete (three sources of uncertainty and two assets). We keep all the other assumptions of the Black Scholes model apart from market completeness. For clarity, we list all the assumptions common to the Black Scholes model and the analogy model developed here:

1) The underlying follows constant coefficient geometric Brownian motion

2) The risk free rate of borrowing and lending is r

3) There are no dividends.

4) Assets are infinitely divisible

5) There are no transaction costs

6) There are no taxes

7) All options are European style

As we are considering the continuous limit of the discrete trinomial process described in section 3, we are assuming that the options are not perfectly replicable by some combination of the underlying and the risk-free asset. That is, we do not assume market completeness. This is our main point of departure from the Black Scholes model as Cox, Ross, and Rubinstein (1979) show that the Black Scholes model is the continuous limit of the two-state (binomial) model. If an option cannot be replicated by some combination of the underlying and the risk-free asset then it follows that there is no combination of the underlying and the option that generates the risk-free asset. That is, the delta-hedged portfolio is no longer risk-free in our case. In fact, with mental accounting of a call with its underlying, the expected return on the portfolio is equal to the expected return from the underlying stock. In contrast, under the Black Scholes model, the delta-hedged portfolio is risk-free.

It is easy to see (see the example in section 2) that the portfolio $Sx - C$ grows at the expected rate of $r + \delta$ for all values of x under analogy making/mental accounting. δ is the risk premium on the underlying stock. It follows:

$$E[dV] = E[dS]x - E[dC] \quad \forall x$$

$$=> (r + \delta)Vdt = E[dS]x - E[dC] \tag{11}$$

Where $V = Sx - C$.

Proposition 3 shows the appropriate partial differential equation which must be satisfied under analogy making.

Proposition 3 ***If analogy makers set the price of a European call option, the analogy option pricing partial differential Equation (PDE) is***

$$(\boldsymbol{r} + \boldsymbol{\delta})\boldsymbol{C} = \frac{\partial \boldsymbol{C}}{\partial \boldsymbol{t}} + \frac{\partial \boldsymbol{C}}{\partial \boldsymbol{S}}(\boldsymbol{r} + \boldsymbol{\delta})\boldsymbol{S} + \frac{\partial^2 \boldsymbol{C}}{\partial \boldsymbol{S}^2}\frac{\boldsymbol{\sigma}^2 \boldsymbol{S}^2}{\mathbf{2}}$$

Proof.

See Appendix A ■

The analogy option pricing PDE can be solved by transforming it into the heat equation. Proposition 4 shows the resulting call option pricing formula for European options.

Proposition 4 ***The formula for the price of a European call is obtained by solving the analogy based PDE. The formula is*** $\boldsymbol{C} = \boldsymbol{SN}(\boldsymbol{d_1}) - \boldsymbol{Ke}^{-(\boldsymbol{r}+\boldsymbol{\delta})(\boldsymbol{T}-\boldsymbol{t})}\boldsymbol{N}(\boldsymbol{d_2})$ ***where*** $\boldsymbol{d_1} = \frac{\boldsymbol{ln}(\boldsymbol{S}/\boldsymbol{K})+(\boldsymbol{r}+\boldsymbol{\delta}+\frac{\boldsymbol{\sigma}^2}{2})(\boldsymbol{T}-\boldsymbol{t})}{\boldsymbol{\sigma}\sqrt{\boldsymbol{T}-\boldsymbol{t}}}$ ***and*** $\boldsymbol{d_2} = \frac{\boldsymbol{ln}\left(\frac{\boldsymbol{S}}{\boldsymbol{K}}\right)+\left(\boldsymbol{r}+\boldsymbol{\delta}-\frac{\boldsymbol{\sigma}^2}{2}\right)(\boldsymbol{T}-\boldsymbol{t})}{\boldsymbol{\sigma}\sqrt{\boldsymbol{T}-\boldsymbol{t}}}$

Proof.

See Appendix B.

■

Corollary 4.1 ***The formula for the analogy based price of a European put option is***

$$\boldsymbol{Ke^{-r(T-t)}\left\{1 - e^{-\delta(T-t)}N(d_2)\right\} - SN(-d_1)}$$

Proof. Follows from put-call parity.

■

Note that put-call parity does not require a complete market. That is, corresponding European call and put options, even if not perfectly replicable with the underlying and the risk-free asset, should satisfy put-call parity.

The analogy option pricing formula is different from the Black-Scholes formula due to the appearance of risk premium on the underlying in the analogy formula. It suggests that the risk premium on the underlying stock does matter for option pricing. The analogy formula is derived by keeping all the assumptions behind the Black-Scholes formula except one: the assumption that a replicating portfolio exists which perfectly replicates a call option is dropped.

5. The Implied Volatility Skew

All the variables in the Black Scholes formula are directly observable except for the standard deviation of the underlying's returns. So, by plugging in the values of observables, the value of standard deviation can be inferred from market prices. This is called implied volatility. If the Black Scholes formula is correct, then the implied volatility values from options that are equivalent except for the strike prices should be equal. However, in practice, for equity index options, a skew is observed in which in-the-money call options' (out-of-the money puts) implied volatilities are higher than the implied volatilities from at-the-money and out-of-the-money call options (in-the-money puts).

The analogy approach developed here provides an explanation for the skew. If the analogy formula is correct, and the Black Scholes model is used to infer implied volatility then skew arises as table 2 shows.

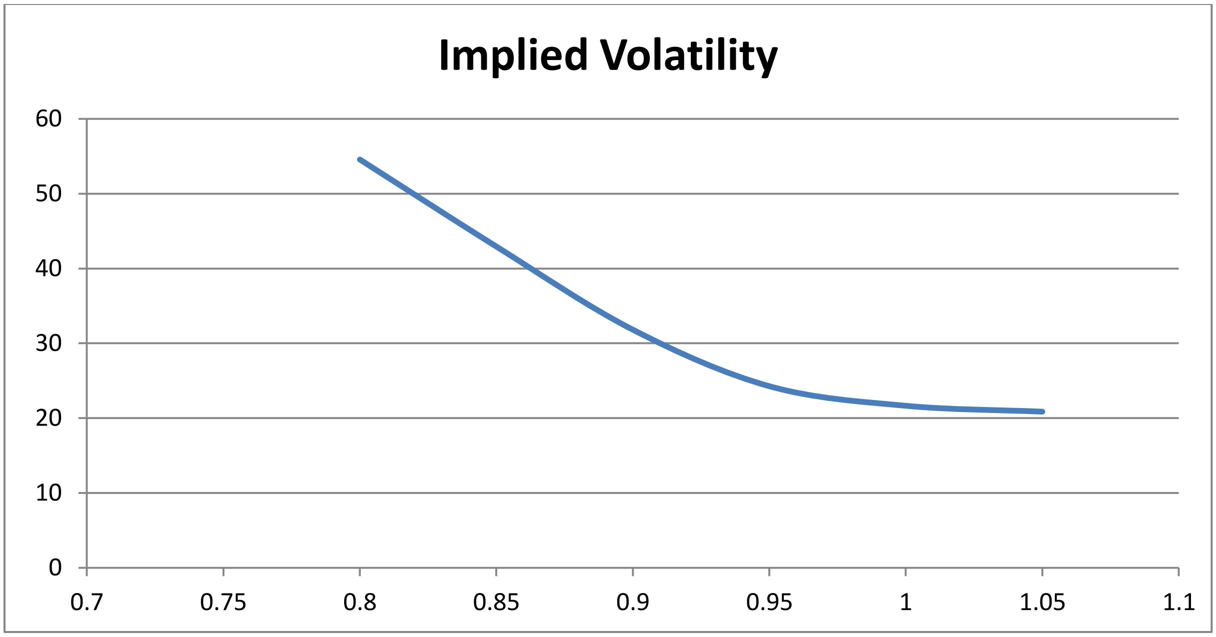
Table 2

Implied Volatility Skew

Underlying's Price=100, Volatility=20%, Risk Premium on the Underlying=5%, Time to Expiry=0.06 year

K	**Black Scholes Price**	**Analogy Price**	**Difference**	**Implied Volatility**
105	0.5072	0.5672	0.06	20.87
100	2.160753	2.326171	0.165417	21.6570
95	5.644475	5.901344	0.25687	24.2740
90	10.30903	10.58699	0.277961	31.8250
85	15.26798	15.53439	0.266419	42.9400
80	20.25166	20.50253	0.250866	54.5700

As table 2 shows, implied volatility skew is seen if the analogy formula is correct, and the Black Scholes formula is used to infer implied volatility. Notice that in the example considered, difference between the Black Scholes price and the analogy price is quite small even when implied volatility gets more than double the value of actual volatility.



K/S

Figure 3

Figure 3 is the graphical illustration of table 2. It is striking to observe from table 2 and figure 3 that the implied volatility skew is quite steep even when the price difference between the Black Scholes price and the analogy price is small. In the next section, we outline a number of key empirical predictions that follow from the analogy making model.

6. Key Aspects of the Analogy Model and Empirical Evidence

Feature#1 ***After controlling for the underlying asset's total volatility, a higher amount of risk premium on the underlying leads to a higher level of implied volatility and a steeper slope of the implied volatility curve.***

Risk premium on the underlying plays a key role in analogy option pricing formula. Figure 4 illustrates this. In the figure, implied volatility skews for two different values of risk premia are plotted. Other parameters are the same as in table 2.

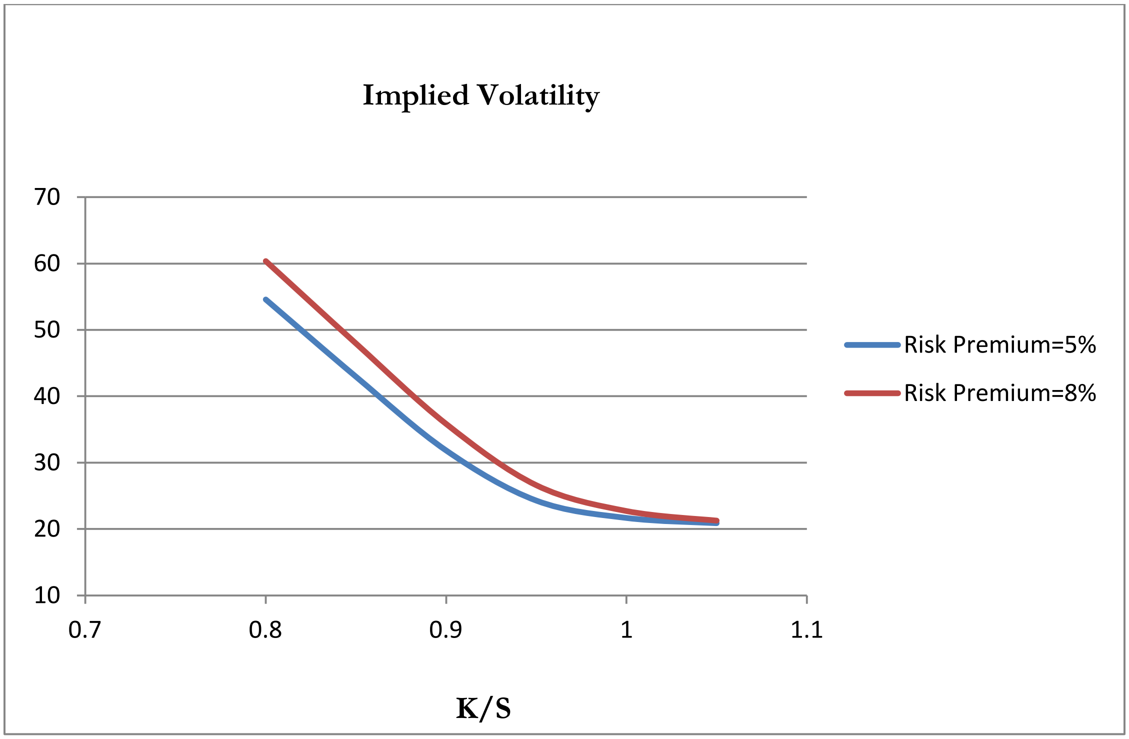


Figure 4

Duan and Wei (2009) use daily option quotes on the S&P 100 index and its 30 largest component stocks, to show that, after controlling for the underlying asset's total risk, a higher amount of systematic risk leads to a higher level of implied volatility and a steeper slope of the implied volatility curve. As risk premium is related to systematic risk, the prediction of the analogy model is quite intriguing.

Feature #2 ***Implied volatility should typically be higher than realized/historical volatility***

It follows directly from the analogy formula that as long as the risk premium on the underlying is positive, implied volatility should be higher than actual volatility. Existing evidence is strongly in favor of this prediction. Rennison and Pederson (2012) calculate implied volatilities from at-the-money options in 14 different options markets over a period ranging from 1994 to 2012. They show that implied volatilities are typically higher than realized volatilities.

Feature #3 ***Implied volatility curve should flatten out with expiry***

Figure 5 plots implied volatility curves for two different expiries. All other parameters are the same as in table 2. It is clear from the figure that as expiry increases, the implied volatility curve flattens out.

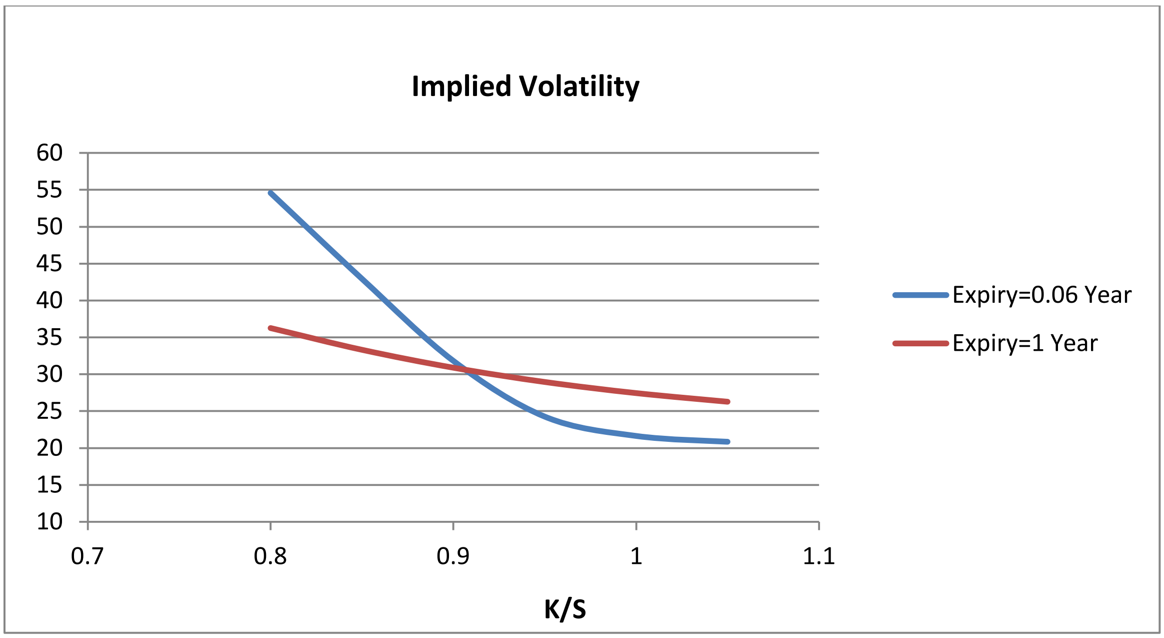


Figure 5

Empirically, implied volatility curve typically flattens out with expiry (see Greiner (2013) as one example). Hence, this match between a key prediction of the analogy model and empirical evidence is quite intriguing.

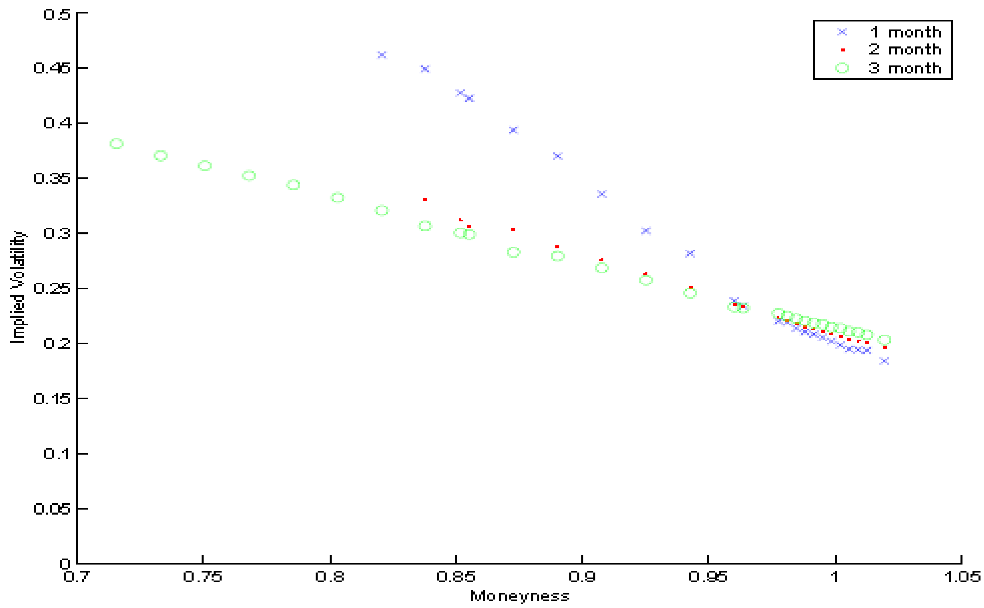


Figure 6 *Implied volatility as a function of moneyness on January 12, 2000, for options with at least two days and at most three months to expiry.*

As an illustration of the fact that implied volatility curve flattens with expiry, figure 6 is a reproduction of a chart from Fouque, Papanicolaou, Sircar, and Solna (2004) (figure 2 from their paper). It plots implied volatilities from options with at least two days and at most three months to expiry. The flattening is clearly seen.

7. Conclusions

Even though this paper only considers the case of a call and its underlying asset, it is interesting to note that the idea of analogy making is potentially extendable to a general class of assets. In this regard, the following two approaches may be taken. Firstly, any equity claim can be considered a call option on the underlying firm's assets with the face value of debt as the striking price. This line of inquiry may open up new ways of exploring the relationship between the economic decisions by a firm and their impact on share prices. It is not hard to see that decisions that would matter in one way without similarity based co-categorizations may impact the share prices differently with similarity based co-categorizations. Secondly, similarity based reasoning, when extended to a general class of assets, typically, either leads to an underestimation or overestimation of risk. Exploring the consequences of such misperceptions for investor behavior is another interesting line on inquiry.

There is also an interesting link between research in the growing area of unawareness (agents are unaware of the full state space) and the principle of analogy making. Analogy making is an inductive principle and the intuitive appeal of inductive reasoning when faced with unawareness is undeniable. Exploration of this connection is the subject of future research.

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Appendix A

To deduce the analogy based PDE consider:

$V = Sx - C$

$\Rightarrow E[dV] = E[dS]x - E[dC]$

$=> (r+\delta)Vdt = E[dS]x - E[dC] \quad \forall x$

Where $E[dS] = uSdt$ and by Ito's Lemma $E[dC] = \left(uS\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2}\right)dt$

Choosing $x = \frac{\partial C}{\partial S}$:

$\Rightarrow (r+\delta)Vdt = (uSdt)\frac{\partial C}{\partial S} - \left(uS\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 C}{\partial S^2}\right)dt$

$$(r+\delta)Vdt = -\left(\frac{\partial C}{\partial t} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 C}{\partial S^2}\right)dt$$

$$\Rightarrow (r+\delta)\left(S\frac{\partial C}{\partial S} - C\right) = -\left(\frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2}\right)$$

$$\Rightarrow (r+\delta)C = (r+\delta)S\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2} \qquad (A1)$$

The above is the analogy based PDE.

Appendix B

The analogy based PDE derived in Appendix A can be solved by converting to heat equation and exploiting its solution.

Start by making the following transformation:

$$\tau = \frac{\sigma^2}{2}(T-t)$$

$$x = ln\frac{S}{K} => S = Ke^x$$

$$C(S,t) = K \cdot\ c(x,\tau) = K \cdot c\left(ln\left(\frac{S}{K}\right), \frac{\sigma^2}{2}(T-t)\right)$$

It follows,

$$\frac{\partial C}{\partial t} = K \cdot \frac{\partial c}{\partial \tau} \cdot \frac{\partial \tau}{\partial t} = K \cdot \frac{\partial c}{\partial \tau} \cdot \left(-\frac{\sigma^2}{2}\right)$$

$$\frac{\partial C}{\partial S} = K \cdot \frac{\partial c}{\partial x} \cdot \frac{\partial x}{\partial S} = K \cdot \frac{\partial c}{\partial x} \cdot \frac{1}{S}$$

$$\frac{\partial^2 C}{\partial S^2} = K \cdot \frac{1}{S^2} \cdot \frac{\partial^2 C}{\partial x^2} - \ K \cdot \frac{1}{S^2}\frac{\partial C}{\partial x}$$

Plugging the above transformations into (A1) and writing $\tilde{r} = \frac{2(r+\delta)}{\sigma^2}$, we get:

$$\frac{\partial c}{\partial \tau} = \frac{\partial^2 c}{\partial x^2} + (\tilde{r} - 1)\frac{\partial c}{\partial x} - \tilde{r}c \qquad (B1)$$

With the boundary condition/initial condition:

$$C(S,T) = max\{S-K, 0\}\ becomes\ c(x,0) = max\{e^x - 1,0\}$$

To eliminate the last two terms in (B1), an additional transformation is made:

$$c(x,\tau) = e^{\alpha x + \beta \tau}u(x,\tau)$$

It follows,

$$\frac{\partial c}{\partial x} = \alpha e^{\alpha x + \beta \tau}u + e^{\alpha x + \beta \tau}\frac{\partial u}{\partial x}$$

$$\frac{\partial^2 c}{\partial x^2} = \alpha^2 e^{\alpha x + \beta \tau}u + 2\alpha e^{\alpha x + \beta \tau}\frac{\partial u}{\partial x} + e^{\alpha x + \beta \tau}\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial c}{\partial \tau} = \beta e^{\alpha x + \beta \tau}u + e^{\alpha x + \beta \tau}\frac{\partial u}{\partial \tau}$$

Substituting the above transformations in (B1), we get:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + (\alpha^2 + \alpha(\tilde{r} - 1) - \tilde{r} - \beta)u + \big(2\alpha + (\tilde{r} - 1)\big)\frac{\partial u}{\partial x} \qquad (B2)$$

Choose $\alpha = -\frac{(\tilde{r}-1)}{2}$ and $\beta = -\frac{(\tilde{r}+1)^2}{4}$. (B2) simplifies to the Heat equation:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \qquad (B3)$$

With the initial condition:

$$u(x_0, 0) = max\{(e^{(1-\alpha)x_0} - e^{-\alpha x_0}), 0\} = max\left\{\left(e^{\left(\frac{\tilde{r}+1}{2}\right)x_0} - e^{\left(\frac{\tilde{r}-1}{2}\right)x_0}\right), 0\right\}$$

The solution to the Heat equation in our case is:

$$u(x, \tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} e^{-\frac{(x-x_0)^2}{4\tau}} u(x_0, 0)dx_0$$

Change variables: $= \frac{x_0 - x}{\sqrt{2\tau}}$, which means: $dz = \frac{dx_0}{\sqrt{2\tau}}$. Also, from the boundary condition, we know that $u > 0\ iff\ x_0 > 0$. Hence, we can restrict the integration range to $z > \frac{-x}{\sqrt{2\tau}}$

$$u(x, \tau) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\pi}}}^{\infty} e^{-\frac{z^2}{2}} \cdot e^{\left(\frac{\tilde{r}+1}{2}\right)(x+z\sqrt{2\tau})}dz - \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{-\frac{z^2}{2}} \cdot e^{\left(\frac{\tilde{r}-1}{2}\right)(x+z\sqrt{2\tau})}dz$$

$$=: H_1 - H_2$$

Complete the squares for the exponent in H_1:

$$\frac{\tilde{r}+1}{2}\big(x + z\sqrt{2\tau}\big) - \frac{z^2}{2} = -\frac{1}{2}\left(z - \frac{\sqrt{2\tau}(\tilde{r}+1)}{2}\right)^2 + \frac{\tilde{r}+1}{2}x + \tau\frac{(\tilde{r}+1)^2}{4}$$

$$=: -\frac{1}{2}y^2 + c$$

We can see that $dy = dz$ and c does not depend on z. Hence, we can write:

$$H_1 = \frac{e^c}{\sqrt{2\pi}} \int_{-x/\sqrt{2\pi} - \sqrt{\tau/2}(\tilde{r}+1)}^{\infty} e^{-\frac{y^2}{2}} dy$$

A normally distributed random variable has the following cumulative distribution function:

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{y^2}{2}} dy$$

Hence, $H_1 = e^c N(d_1)$ where $d_1 = x/\sqrt{2\pi} + \sqrt{\tau/2}\,(\tilde{r} + 1)$

Similarly, $H_2 = e^f N(d_2)$ where $d_2 = x/\sqrt{2\pi} + \sqrt{\tau/2}\,(\tilde{r} - 1)$ and $f = \frac{\tilde{r}-1}{2}x + \tau\frac{(\tilde{r}-1)^2}{4}$

The analogy based European call pricing formula is obtained by recovering original variables:

$$Call = SN(d_1) - Ke^{-(r+\delta)(T-t)}N(d_2)$$

Where $\boldsymbol{d_1} = \frac{\boldsymbol{ln(S/K)+(r+\delta+\frac{\sigma^2}{2})(T-t)}}{\boldsymbol{\sigma\sqrt{T-t}}}$ $and\ \boldsymbol{d_2} = \frac{\boldsymbol{ln\left(\frac{S}{K}\right)+\left(r+\delta-\frac{\sigma^2}{2}\right)(T-t)}}{\boldsymbol{\sigma\sqrt{T-t}}}$