On the pseudo-equilibrium manifold in semi-algebraic economies with real financial assets

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Abstract

The aim of this paper is to prove that if the consumption set of an economy with incomplete financial markets is semi-algebraic, then the corresponding pseudo-equilibrium manifold is also semi-algebraic. For this, we proceed by constructing an incomplete financial economy with real assets and semi-algebraic utility functions. Then, we show that the spot-equilibrium set and the pseudo-equilibrium set are smooth semi-algebraic manifolds. We extent this results by showing that the pseudo-equilibrium natural projection is a semi-algebraic diffeomorphism in each regular point of the semi-algebraic pseudo-equilibrium manifold. It is directly related with the local determinacy of pseudo-equilibrium.

Key words: semi-algebraic, finance, spot-equilibrium, pseudo-equilibrium

JEL classification: D50, D51, D52

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1 Introduction

Real algebraic geometry has important and interesting tools for economic analysis. Kubler and Schmedders (2010) use Gröbner bases to compute equilibria in economies with complete and incomplete markets. It simplifies the problem of calculating multiple equilibria to compute and count the solutions of algebraic equations. It applies to many economic models with indeterminacy in the competitive equilibrium.

Blume and Zame (1992, 1994) have made fundamental contributions to the tame topology of economic equilibrium. They have shown that if the consumption set is definable, then the utility and demand functions are also definable. It implies the walrasian equilibrium set is definable. That allows them to prove local determinacy of equilibrium as a triviality theorem in general equilibrium theory and game theory.

On the other hand, Zhou (1997) proves that the complement of the spot-equilibrium manifold is semi-algebraic without assuming semi-algebraicness. In the present paper we are interested in studying the pseudo-equilibrium manifold in incomplete financial markets by assuming a semi-algebraic consumption set. It allows us to analyse the algebraic properties of this manifold and the pseudo-equilibrium natural projection.

2 Semi-algebraic sets and functions

We present some general definitions of real algebraic geometry and real analysis for better understanding our results. We refer the reader to Bochnak, Coste and Roy (BCR)(1998) for a complete treatment about semi-algebraic sets and functions.

Definition 1 *(BCR, 1998, P. 24) A semi-algebraic subset of* $\mathbb{R}^n$ *is a subset of the form*

$$\bigcup_{i=1}^{s}\bigcap_{j=1}^{r_i}\{\boldsymbol{x}\in\mathbb{R}^n : \boldsymbol{f}_{i,j} *_{ij} 0\}$$

where $\boldsymbol{f}_{ij}\in\mathbb{R}[X_1,...,X_n]$ *and* $*_{ij}$ *is either* $(<)$ *or* $(=)$ *for* $i=1,...,s$ *and* $j=1,...,r_i$.

This sets are defined by polynomials and/or inequalities in $\mathbb{R}$. The real algebraic sets are semi-algebraic. Boolean operations of semi-algebraic sets are semi-algebraic.

Definition 2 *(BCR, 1998, P. 28) The function* $f:\mathbb{R}^m\supseteq A\to B\subseteq\mathbb{R}^n$ *is semi-algebraic if its graph* $[\Gamma_f=\{\boldsymbol{a}=(x,y): y=f(x)\}\subseteq\mathbb{R}^{m+n}]$ *is semi-algebraic.*

The continuous semi-algebraic function f is a *semi-algebraic homeomorphism* if it is a bijection with f^{-1} continuous. A semi-algebraic function is C^n if there exist its n continuous partial derivatives. If f, f^{-1} are C^∞, then f is a *semi-algebraic diffeomorphism*. A function is *proper* if the pre-image of compacta is compact, and it is *semi-algebraically proper* if it is proper.

Definition 3 *(BCR, 1998, P. 30) A decomposition of a semi-algebraic set is a partition in simpler semi-algebraic subsets, semi-algebraically homomorphic to a open hypercubes.*

The real semi-algebraic manifolds are defined naturally from differential topology as (BCR, 1998, P. 57). Let f be a C^∞-proper function and A, B semi-algebraic manifolds. A point $a\in A$ is *regular* if the Jacobian matrix of f is non-singular at a; a point $b\in B$ is a *regular value* if $f^{-1}(b)$ has only regular points.

3 The semi-algebraic financial economy

Suppose two-periods $t = \{0, 1\}$ with 0 today and 1 tomorrow. There is uncertainty represented by $s = \{0, 1, ..., S\}$ states of nature. There are $l = \{1, ..., L\}$ commodities and $i = \{1, ..., I\}$ individuals. There are $L(S+1)$ available commodities. Individuals have some endowments in each state. Let $\omega_i^s(l)$ be the i's endowment of l in s and $\boldsymbol{\omega}_i^s = (\omega_i^s(l))_{l=1}^L$ the i's vector of endowments in s.

Let $\boldsymbol{\omega}_i = (\boldsymbol{\omega}_i^s)_{s=0}^S \in \mathbb{R}_{++}^{L(S+1)}$ the i's total endowment and $\boldsymbol{\omega} = (\boldsymbol{\omega}_i)_{i=1}^I \in \mathbb{R}_{++}^{IL(S+1)}$ the total endowment. Following Blume and Zame (1992) let $\boldsymbol{X} \subseteq \mathbb{R}_{++}^{IL(S+1)}$ be the semi-algebraic consumption set. Let $x_i^s(l)$ be the i's consumption of l in s and $\boldsymbol{x}_i^s = (x_i^s(l))_{l=1}^L$ the i's vector of consumption in s. Let $\boldsymbol{x}_i = (\boldsymbol{x}_i^s)_{s=0}^S \in \boldsymbol{X}$ the i's total consumption and $\boldsymbol{x} = (\boldsymbol{x}_i)_{i=1}^I \in \boldsymbol{X}$ the total consumption.

Assumption 1 *The individual utility function* $u_i : \mathbb{R}_{++}^{L(S+1)} \to \mathbb{R}$ *is smooth, semi-algebraic, strictly increasing and strictly quasi-concave.*

Let $p^s(l)$ be l's price in s; $\boldsymbol{p}^s = (p^s(l))_{l=1}^L$ the prices in s, and $\boldsymbol{p} = (\boldsymbol{p}^s)_{s=0}^S \in \mathbb{R}_{++}^{L(S+1)}$ the vector of prices. There are $j \in \{1, ..., J\}$ real assets. A *real asset* pays $\boldsymbol{y}_j^s = (y_j^s(l))_{l=1}^L \in \mathbb{R}^L$ where $y_j^s(l) \in \mathbb{R}$ are l's units given by j in s. Let $\boldsymbol{y}_j = (\boldsymbol{y}_j^s)_{s=0}^S \in \mathbb{R}^{SL}$ and $\boldsymbol{y} = (\boldsymbol{y}_j)_{j=1}^J \in \mathbb{R}^{SLJ}$ be total pay-off's. Let $\boldsymbol{r}_j^s : \mathbb{R}_{++}^{L(S+1)} \to \mathbb{R}$ be the return of real asset j with $\boldsymbol{r}_j^s(\boldsymbol{p}) = \boldsymbol{p}^s \boldsymbol{y}_j^s$ and the return matrix $\mathbb{V}(\boldsymbol{p}, \boldsymbol{y}_j^s)$.

Let $Image\mathbb{V}(\boldsymbol{p}, \boldsymbol{y}_j^s) \subseteq \mathbb{R}^S$ be a linear subspace and denote $\rho(\mathbb{V}(\boldsymbol{p}, \boldsymbol{y}_j^s))$ as the rank of $\mathbb{V}(\boldsymbol{p}, \boldsymbol{y}_j^s)$ with $\rho(\mathbb{V}(\boldsymbol{p}, \boldsymbol{y}_j^s)) = Dimension[Image\mathbb{V}(\boldsymbol{p}, \boldsymbol{y}_j^s)]$. A *semi-algebraic financial economy* is $(\boldsymbol{\omega}, \boldsymbol{y}) \in \mathbb{R}_{++}^{IL(S+1)} \times \mathbb{R}^{SLJ}$ by fixing semi-algebraic utilities. Let $\boldsymbol{x}_i^{\boldsymbol{1}} = (\boldsymbol{x}_i^s)_{s=1}^S \in \mathbb{R}_{++}^{SL}$, $\boldsymbol{\omega}_i^{\boldsymbol{1}} = (\boldsymbol{\omega}_i^s)_{s=1}^S \in \mathbb{R}_{++}^{SL}$ and $\boldsymbol{p}^{\boldsymbol{1}} = (\boldsymbol{p}^s)_{s=1}^S \in \mathbb{R}_{++}^{SL}$ be the consumption, endowment and price vectors without the unique state in $t = 0$.

Definition 4 *A vector* $(\boldsymbol{x}, \boldsymbol{p}) \in \boldsymbol{X} \times \mathbb{R}_{++}^{L(S+1)}$ *is an* **spot-equilibrium** *for the economy* $(\boldsymbol{\omega}, \boldsymbol{y}) \in \mathbb{R}_{++}^{IL(S+1)} \times \mathbb{R}_{++}^{SLJ}$ *if it satisfies the following three conditions:*

1. *The vector* $\boldsymbol{x}_1 \in \boldsymbol{X}$ *solves the problem* $max_{\boldsymbol{x}_1 \in \boldsymbol{X}} u_1(\boldsymbol{x}_1) s.t. \boldsymbol{p}(\boldsymbol{x}_i - \boldsymbol{\omega}_i) = 0$

2. *For* $i = 2, ..., I$, $\boldsymbol{x}_i \in \boldsymbol{X}$ *solves the following problem:*

$$max_{\boldsymbol{x}_i \boldsymbol{X}} u_i(\boldsymbol{x}_i) s.t. \boldsymbol{p}(\boldsymbol{x}_i - \boldsymbol{\omega}_i) = 0; \boldsymbol{p}^{\boldsymbol{1}} \boxtimes (\boldsymbol{x}_i^{\boldsymbol{1}} - \boldsymbol{\omega}_i^{\boldsymbol{1}}) \in Image\mathbb{V}(\boldsymbol{p}, \boldsymbol{y}_j^s)$$

3. *The vector* $\boldsymbol{x} \in \boldsymbol{X}$ *satisfies* $\sum_{i=1}^I (\boldsymbol{x}_i - \boldsymbol{\omega}_i) = 0$

From the assumption 1 and definition 4 items 1-2, there exist smooth demand functions $D_1(\boldsymbol{p}, \boldsymbol{p}\boldsymbol{\omega}_1)$ and $D_i(\boldsymbol{p}, Image\mathbb{V}(\boldsymbol{p}, \boldsymbol{y}_j^s), \boldsymbol{\omega}_i)$. These functions have well known mathematical properties. Additionally, we could construct the aggregate excess demand function $Z(\boldsymbol{p}, Image\mathbb{V}(\boldsymbol{p}, \boldsymbol{y}_j^s), \boldsymbol{\omega}_i)$ as a continuous composition of individual demand functions, also mathematically well behaved:

$$Z(\boldsymbol{p}, Image\mathbb{V}(\boldsymbol{p}, \boldsymbol{y}_j^s), \boldsymbol{\omega}_i) = D_1(\boldsymbol{p}, \boldsymbol{p}\boldsymbol{\omega}_1) - \boldsymbol{\omega}_1 + \sum_{i=2}^{I} (D_i(\boldsymbol{p}, Image\mathbb{V}(\boldsymbol{p}, \boldsymbol{y}_j^s), \boldsymbol{\omega}_i) - \boldsymbol{\omega}_i)$$

Definition 5 *The* **spot-equilibrium manifold** *for* $(\boldsymbol{\omega}, \boldsymbol{y}) \in \mathbb{R}_{++}^{IL(S+1)} \times \mathbb{R}^{SLJ}$ *is:*

$$\Omega_{\boldsymbol{\omega}} = \{\boldsymbol{p} \in \mathbb{R}_{++}^{L(S+1)} : Z_{\boldsymbol{\omega}}(\boldsymbol{p}) = 0\}$$

It is known the spot-equilibrium in this semi-algebraic financial economy exists generically. For this, we relax our definition. Instead of $Image\mathbb{V}(\boldsymbol{p}, \boldsymbol{y}_j^s)$ we will use the linear space $\mathbb{R}^S \supseteq \boldsymbol{F} \in \mathbb{G}_{L,S}$ with a $\mathbb{G}_{L,S}$ grassmanian manifold. We are letting independent the dimension of assets to changes in spot prices.

Definition 6 *A vector* $(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{F}) \in \boldsymbol{X} \times \mathbb{R}_{++}^{L(S+1)} \times \mathbb{G}_{L,S}$ *is a* **pseudo-equilibrium** *for* $(\boldsymbol{\omega}, \boldsymbol{y}) \in \mathbb{R}_{++}^{IL(S+1)} \times \mathbb{R}_{++}^{SLJ}$ *if it satisfies the following three conditions:*

1. *The vector* $\boldsymbol{x}_1 \in \boldsymbol{X}$ *solves the problem* $max_{\boldsymbol{x}_1 \in \boldsymbol{X}} u_1(\boldsymbol{x}_1) s.t. \boldsymbol{p}(\boldsymbol{x}_i - \boldsymbol{\omega}_i) = 0$

2. *Given* $(\boldsymbol{p}, \boldsymbol{F}, \boldsymbol{\omega}) \in \mathbb{R}_{++}^{L(S+1)} \times \mathbb{G}_{L,S} \times \mathbb{R}_{++}^{IL(S+1)}$, *for* $i = 2, ..., I$ *the vector* $\boldsymbol{x}_i \in \boldsymbol{X}$ *solves the following problem:*

$$max_{\boldsymbol{x}_i \in \boldsymbol{X}} u_i(\boldsymbol{x}_i) s.t. \boldsymbol{p}(\boldsymbol{x}_i - \boldsymbol{\omega}_i) = 0; \boldsymbol{p}^{\boldsymbol{1}} \boxtimes (\boldsymbol{x}_i^{\boldsymbol{1}} - \boldsymbol{\omega}_i^{\boldsymbol{1}}) \in \boldsymbol{F}$$

3. *The vector* $\boldsymbol{x} \in \boldsymbol{X}$ *satisfies* $\sum_{i=1}^{I} (\boldsymbol{x}_i - \boldsymbol{\omega}_i) = 0$

4. $Image\mathbb{V}(\boldsymbol{p}, \boldsymbol{y}_j^s) \subseteq \boldsymbol{F}$

Definition 7 *The* **pseudo-equilibrium manifold** *for* $(\boldsymbol{\omega}, \boldsymbol{y}) \in \mathbb{R}_{++}^{IL(S+1)} \times \mathbb{R}^{SLJ}$ *is:*

$$\Phi_{\boldsymbol{\omega}} = \{(\boldsymbol{p}, \boldsymbol{F}) \in \mathbb{R}_{++}^{L(S+1)} \times \mathbb{G}_{L,S} : Image\mathbb{V}(\boldsymbol{p}, \boldsymbol{y}_j^s) \subseteq \boldsymbol{F}\}$$

The manifold Φ has some interesting mathematical properties. Chichilnisky and Heal (1996) proves it is an structure topologically equivalent to $\mathbb{G}_{L,S}$. They also show it is not contractible. Moreover, when $Image\mathbb{V}(\boldsymbol{p}, \boldsymbol{y}_j^s) = \boldsymbol{F}$ then $\Phi_{\boldsymbol{\omega}}$ is equivalent to $\Omega_{\boldsymbol{\omega}}$.

4 The results

Theorem 1 *If* $\boldsymbol{X} \subseteq \mathbb{R}_{++}^{IL(S+1)}$ *is semi-algebraic, then* $\Omega_{\boldsymbol{\omega}}$ *is a semi-algebraic spot-equilibrium manifold for* $(\boldsymbol{\omega}, \boldsymbol{y}) \in \mathbb{R}_{++}^{IL(S+1)} \times \mathbb{R}^{SLJ}$.

Proof : Fix $(\boldsymbol{\omega}^*, \boldsymbol{y}^*) \in \mathbb{R}_{++}^{IL(S+1)} \times \mathbb{R}_{++}^{SLJ}$. Let $D_1(\boldsymbol{p}, \boldsymbol{p}\boldsymbol{\omega}_1^*)$ be the $i = 1$'s demand function and $D_i(\boldsymbol{p}, Image\mathbb{V}(\boldsymbol{p}, \boldsymbol{y}_j^{*s}), \boldsymbol{\omega}_i^*)$ the $i \neq 1$'s individual demand function. From the assumption, theorem 1 of Blume and Zame (1992) implies $D_1(\boldsymbol{p}, \boldsymbol{p}\boldsymbol{\omega}_1^*)$ and $D_i(\boldsymbol{p}, Image\mathbb{V}(\boldsymbol{p}, \boldsymbol{y}_j^{*s}), \boldsymbol{\omega}_i^*)$ are semi-algebraic demand functions. The aggregate excess demand function $Z(\boldsymbol{p}, Image\mathbb{V}(\boldsymbol{p}, \boldsymbol{y}_j^{*s}), \boldsymbol{\omega}^*)$ is as follows,

$$Z(\boldsymbol{p}, Image\mathbb{V}(\boldsymbol{p}, \boldsymbol{y}_j^{*s}), \boldsymbol{\omega}^*) = D_1(\boldsymbol{p}, \boldsymbol{p}\boldsymbol{\omega}_1^*) - \boldsymbol{\omega}_1^* + \sum_{i=2}^{I} (D_i(\boldsymbol{p}, Image\mathbb{V}(\boldsymbol{p}, \boldsymbol{y}_j^{*s}), \boldsymbol{\omega}^{*i}) - \boldsymbol{\omega}^{*i})$$

The function $Z(\boldsymbol{p}, Image\mathbb{V}(\boldsymbol{p}, \boldsymbol{y}_j^{*s}), \boldsymbol{\omega}^*)$ is semi-algebraic because it is a continuous composition of semi-algebraic functions. It defines $\Omega_{\boldsymbol{\omega}^*}$ as the zeros of a semi-algebraic function, which is a real algebraic manifold. Therefore, the spot-equilibrium manifold $\Omega_{\boldsymbol{\omega}^*}$ for the financial economy $(\boldsymbol{\omega}^*, \boldsymbol{y}^*)$ is a semi-algebraic manifold. □

Let us use some properties of grassmanian manifolds. Let Σ be the set of permutations from s. Let $\boldsymbol{P}_{\sigma^*\in\Sigma}$ be the $S\times S$-matrix of the permutation $\sigma^*\in\Sigma$. Take a matrix $\mathbb{M}_{(S-J)\times J}$ for $[\mathbb{I}\mid\mathbb{M}]\boldsymbol{P}_{\sigma^*}\in\boldsymbol{F}\subseteq\mathbb{G}_{L,S}$ being $\mathbb{I}_{(S-J)\times(S-J)}$ the identity matrix. Write $Image\mathbb{V}(\boldsymbol{p},\boldsymbol{y}_j^s)\subseteq\boldsymbol{F}$ as $[\mathbb{I}\mid\mathbb{M}]\boldsymbol{P}_{\sigma^*}\mathbb{V}(\boldsymbol{p},\boldsymbol{y}_j^s)=0$. Let $\lambda_1,\lambda_{2i},\lambda_{3i}\in\mathbb{R}_+$ be the Lagrange multipliers of a concave optimization problem from definition 6.

Theorem 2 *If $\boldsymbol{X}\subseteq\mathbb{R}_{++}^{L(S+1)}$ is semi-algebraic, then $\Phi_{\boldsymbol{\omega}}$ is a semi-algebraic pseudo-equilibrium manifold for $(\boldsymbol{\omega},\boldsymbol{y})\in\mathbb{R}_{++}^{IL(S+1)}\times\mathbb{R}^{SLJ}$.*

Proof : Let $\boldsymbol{\varphi}=(\varphi_1,...,\varphi_I)\in\mathbb{R}^I$ be a polynomial vector. Kubler and Schmedders (2011) remarks it is possible to write $\partial u_i(\boldsymbol{x}_i)/\partial\boldsymbol{x}_i$ as $\varphi_i(\boldsymbol{x}_i,\partial u_i(\boldsymbol{x}_i)/\partial\boldsymbol{x}_i)=0$ by assuming a free-square $\boldsymbol{\varphi}$. It allows us to solve the optimization problem of definition 6 in polynomial equations. Consider the FOC's with semi-algebraic marginal utilies:

$$\begin{aligned}\varphi_1(\boldsymbol{x}_1;\lambda_1\boldsymbol{p})&=0\\ i=2,...,I,\varphi_i(\boldsymbol{x}_i^0;\lambda_{2i}\boldsymbol{p}^0)&=0\\ k=2,...,I,\varphi_k\left(\boldsymbol{x}_i^{\boldsymbol{1}};\lambda_{2i}\boldsymbol{p}^{\boldsymbol{1}}-\lambda_{3i}[\mathbb{I}\mid\mathbb{M}]\boldsymbol{P}_{\sigma^*}\begin{bmatrix}\boldsymbol{p}^1&...&0\\0&\ddots&0\\0&...&\boldsymbol{p}^S\end{bmatrix}\right)&=0\end{aligned}$$

These functions are clearly semi-algebraic. The other first order conditions are market clearing conditions. Let $OC([\mathbb{I}\mid\mathbb{M}]\boldsymbol{P}_{\sigma^*}\mathbb{V}(\boldsymbol{p},\boldsymbol{y}_j^s))$ be the vector of ordered columns of $\mathbb{V}(\boldsymbol{p},\boldsymbol{y}_j^s)$. Each coordinate of $\mathbb{V}(\boldsymbol{p},\boldsymbol{y}_j^s)$ is a real linear polynomial, so it is semi-algebraic. In consequence, $OC([\mathbb{I}\mid\mathbb{M}]\boldsymbol{P}_{\sigma^*}\mathbb{V}(\boldsymbol{p},\boldsymbol{y}_j^s))=0$ is a semi-algebraic vectorial function. The other FOC's are:

$$\begin{aligned}\boldsymbol{p}(\boldsymbol{x}_1-\boldsymbol{\omega}_1)&=0\\ i=2,...,I,\boldsymbol{p}(\boldsymbol{x}_i-\boldsymbol{\omega}_i)&=0\\ [\mathbb{I}\mid\mathbb{M}]\boldsymbol{P}_{\sigma^*}\begin{bmatrix}\boldsymbol{p}^1&...&0\\0&\ddots&0\\0&...&\boldsymbol{p}^S\end{bmatrix}(\boldsymbol{x}_i^{\boldsymbol{1}}-\boldsymbol{\omega}_i^{\boldsymbol{1}})&=0\\ \sum_{i=1}^{I}\left([(x_i^s(l))_{l=1}^{L-1}]_{s=1}^S-[(\omega_i^s(l))_{l=1}^{L-1}]_{s=1}^S\right)&=0\\ OC([\mathbb{I}\mid\mathbb{M}]\boldsymbol{P}_{\sigma^*}\mathbb{V}(\boldsymbol{p},\boldsymbol{y}_j^s))&=0\end{aligned}$$

Let $\phi_{\sigma^*}(\boldsymbol{x},\boldsymbol{p},\mathbb{M},\lambda_1,\lambda_2,\lambda_3,\boldsymbol{\omega},\boldsymbol{y})$ represent the left-side of the semi-algebraic first order conditions for σ^*. Fix an economy $(\boldsymbol{\omega}^*,\boldsymbol{y}^*)$. A vector $(\boldsymbol{x}^*,\boldsymbol{p}^*,\mathbb{M}^*,\lambda_1^*,\lambda_2^*,\lambda_3^*)$ is a pseudo-equilibrium for $(\boldsymbol{\omega}^*,\boldsymbol{y}^*)$ if $\phi_{\sigma^*}(\boldsymbol{x}^*,\boldsymbol{p}^*,\mathbb{M}^*,\lambda_1^*,\lambda_2^*,\lambda_3^*,\boldsymbol{\omega}^*,\boldsymbol{y}^*)=0$. Notice that the function ϕ_σ is semi-algebraic. It implies $\Phi_{\boldsymbol{\omega}^*}$ is a real algebraic manifold for the economy $(\boldsymbol{\omega}^*,\boldsymbol{y}^*)$ so it is a semi-algebraic manifold. $\square$

Following Debreu (1970), the map $\pi:\Phi_{\boldsymbol{\omega}}\to\mathbb{R}_{++}^{L(S+1)}$ for $(\boldsymbol{\omega},\boldsymbol{y})\in\mathbb{R}_{++}^{IL(S+1)}\times\mathbb{R}^{SLJ}$ is the vectorial function of the natural projection:

$$\pi(\boldsymbol{p},\boldsymbol{p}\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,...,\boldsymbol{\omega}_I)=[D_1(\boldsymbol{p},\boldsymbol{p}\boldsymbol{\omega}_1)+\sum_{i=2}^{I}(D_i(\boldsymbol{p},\boldsymbol{F},\boldsymbol{\omega}_i)-\boldsymbol{\omega}_i),...,\boldsymbol{\omega}_I]$$

We call it *pseudo-equilibrium natural projection*. From this, $(\boldsymbol{p},\boldsymbol{F})$ is a pseudo-equilibrium for $(\boldsymbol{\omega},\boldsymbol{y})$ if $\pi(\boldsymbol{p},\boldsymbol{p}\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,...,\boldsymbol{\omega}_I)=(\boldsymbol{\omega}_1,...,\boldsymbol{\omega}_n)$ and $Image\mathbb{V}(\boldsymbol{p},\boldsymbol{y}_j^s)\subseteq\boldsymbol{F}$.

Theorem 3 *If $\Phi_{\boldsymbol{\omega}}$ is a semi-algebraic pseudo-equilibrium manifold for the financial economy $(\boldsymbol{\omega}, \boldsymbol{y}) \in \mathbb{R}_{++}^{IL(S+1)} \times \mathbb{R}^{SLJ}$, then $\pi : \Phi_{\boldsymbol{\omega}} \rightarrow \mathbb{R}_{++}^{L(S+1)}$ is a semi-algebraic diffeomorphism in each regular point of $\Phi_{\boldsymbol{\omega}}$.*

Proof : Fix a semi-algebraic economy $(\boldsymbol{\omega}^*, \boldsymbol{y}^*) \in \mathbb{R}_{++}^{IL(S+1)} \times \mathbb{R}^{SLJ}$. From our theorem 2, it is direct that $\pi : \Phi_{\boldsymbol{\omega}^*} \rightarrow \mathbb{R}_{++}^{L(S+1)}$ is a continuous semi-algebraic function. Moreover, from Villanacci, Carosi, Benevieri and Batinelli (2002, Proposition 23, P. 374) we know π is a proper function, so it is also a semi-algebraic proper function.

For any regular point $(\boldsymbol{p}^*, F^*) \in \Phi_{\boldsymbol{\omega}^*}$ the determinant of the Jacobian matrix of π is $\mid \nabla\pi \mid_{(\boldsymbol{p}^*, \boldsymbol{F}^*)} \neq 0$. Apply the implicit function theorem. Let $\boldsymbol{\omega}^* \in B \subseteq \mathbb{R}_{++}^{IL(S+1)}$ and $(\boldsymbol{p}^*, \boldsymbol{F}^*) \in A \subseteq \Phi_{\boldsymbol{\omega}^*}$ being A, B semi-algebraic open neighbourhoods. There exists a semi-algebraic homeomorphism $\pi \mid_A : A \rightarrow B$. From our theorems 1 and 2 we deduce $\pi \mid_A$ and $\pi^{-1} \mid_A$ are C^∞. It implies the claimed result. □

5 Remarks

Our theorems 1-2 are based on definable equilibrium set from Blume and Zame (1992, 1994). Our theorem 3 is robust by finite fundamental groups as Chichilnisky (1998).

References

[1] Blume, L. and Zame, W. (1992). *The algebraic geometry of competitive equilibrium.* Economic theory and international trade; essays in memoriam J. Trout Rader, ed. por W. Neuefeind y R. Reizman, Springer-Verlag, Berlin..

[2] Blume, L. and Zame, W. (1994). *The algebraic geometry of perfect and sequential equilibrium.* Econometrica 62, No. 4, pp. 783-794.

[3] Bochnak, J., Coste, M. and Roy, M. (1998). *Real Algebraic Geometry.* Springer Verlag-Berlin.

[4] Chichilnisky, G. and Heal, G. (1996). *The existence and structure of the pseudo-equilibrium manifold in incomplete asset markets.* Journal of mathematical economics vol. 26, 171-186.

[5] Chichilnisky, G. (1998). *Topology and invertible maps.* Advances in applied mathematics 21, 113-123.

[6] Debreu, G. (1970). *Economies with a finite set of equilibria.* Econometrica 38, 387-392.

[7] Villanacci, A., Carosi, L., Benevieri, P. and Battinelli, A. (2002). *Differential topology and general equilibrium with complete and incomplete markets.* Kluwer academic publishers.

[8] Kubler, F. and Schmedders, K. (2010). *Competitive equilibria in semi-algebraic economies.* Journal of economic theory 145, 301-330.

[9] Zhou, Y. (1997). *Genericity analysis on the pseudo-equilibrium manifold.* Journal of economic theory 73, 79-92.