Commuting for meetings

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Abstract

Urban congestion causes travel times to exhibit considerable variability, which leads to coordination problems when people have to meet. We analyze a game for the timing of a meeting between two players who must each complete a trip of random duration to reach the meeting, which does not begin until both are present. Players prefer to depart later and also to arrive sooner, provided they do not have to wait for the other player. We find a unique Nash equilibrium, and a continuum of Pareto optima that are strictly better than the Nash equilibrium for both players. Pareto optima may be implemented as Nash equilibria by penalty or compensation schemes.

Keywords: congestion; random travel time variability; coordination game

JEL codes: C7, D1, R4

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1 Introduction

Urban traffic congestion is a significant burden on developed economies. As most people know from experience, the costs of congestion are not only related to the (average) delay. The difficulty or even impossibility of predicting travel time is also an inherent feature of urban congestion and should be taken into account by economic analysis.[1]

Many efforts to incorporate the cost of random delays into policy assessment have drawn on a model of preferences regarding the timing and duration of a trip. Such scheduling preferences were first introduced in Vickrey (1969, 1973) with Small (1982) providing the first empirical estimates.[2] This has led to a substantial literature on the value of random travel time variability. This literature considers a traveler about to undertake a trip where the travel time is random; he chooses departure time to maximize expected utility, where utility depends on the departure time and the arrival time. It is then possible to examine how indirect utility depends on the distribution of random travel time.[3]

When we take into consideration that a traveler might be on his way to a meeting of some kind, it becomes clear that there are interactions involved that seem quite important. There are many situations where this is relevant. We can think of meetings at work, appointments with friends and family, a date with a potential partner or generally any situation where people have to meet. These interactions

[1] Random delays are often related to traffic incidents such as accidents. According to one estimate (Schrank and Lomax, 2009, Appendix B, p. B-27), incident-related delays alone contribute 52 - 58 percent of total delay in urban areas in the United States. On bad days, delay can easily be as large as undelayed travel time itself (Fosgerau and Fukuda, 2012).

[2] Vickrey (1969, 1973) introduced two specific forms of scheduling preferences in the context of the bottleneck model, de Palma and Fosgerau (2011) discuss a general form.

[3] See, among others, Noland and Small (1995), Bates et al. (2001), Fosgerau and Karlstrom (2010), Fosgerau and Engelson (2011), and Engelson and Fosgerau (2011).

are overlooked by the literature just reviewed, which takes the perspective of a single individual.

We develop an economic model for a meeting between two people. The model describes two players each initially engaged in some activity from which they each derive utility at a declining rate. Each must choose a departure time from his activity, and after a random travel time with known distribution each arrives at the meeting. The players only derive utility at the meeting after both have arrived, and thus waiting for the other player is costly. Players choose departure time to maximize their payoff, the expected utility.[4] We consider Nash equilibrium where neither player has incentive to change departure time given the departure time of the other and compare this to the set of Pareto optima.

Our findings may be summarized as follows. We find that Nash equilibrium exists in our model and is unique. A player's payoff depends on the joint travel time distribution of both players. Specifically, payoffs are non-increasing in the variance of the difference of travel times, which means that not only the variance of the individual travel times but also their correlation matters. These conclusions are natural but do not arise in the extant literature discussed above. Moreover, there is a continuum of Pareto optima in the model, and these Pareto optima correspond one-to-one to the probability that the first player is late. Nash equilibrium is not Pareto optimal, and there exists a continuum of Pareto optima that yield strict increases in payoff for both players relative to Nash equilibrium. With penalties to each player for arriving later than the other, it is possible to implement any Pareto optimum as a Nash equilibrium. Some Pareto optima may also be implemented

[4]This is a kind of coordination game. The standard coordination game has discrete strategy set, whereas the strategy set here is continuous.

through a scheme that compensates players for waiting for the other.

These results have implications that seem not to have been discussed before. First, evaluation of measures to reduce travel time variability could seek to take into account the interaction with other people than the travelers who are directly affected, namely those who might be waiting for the travelers when they arrive late. Second, there might be occasions where alternative policy measures have different effects on the distribution of travel times within a city. In such cases, the present results suggest that measures that have greater effect on the variance of *differences* in travel times for different transport corridors should be given more emphasis, ceteris paribus. Finally, employers could conceivably implement penalty or compensation schemes for their employees that lead to a Pareto optimum as the Nash outcome, where the penalty or compensation depends on the difference in arrival times.

As the variability of the difference of travel times matters in our model, it is relevant to examine empirically the joint distribution of travel times for different travel relations. Since it is necessary to use data at the level of trips and not just roads, data requirements for a comprehensive empirical analysis are quite severe and we have not been able to find relevant studies in the literature. Instead, Appendix C provides a cursory examination using data from cameras installed on major arterial roadways in the Stockholm urban area. We have identified nine pairs of paths having cameras at both ends, each pair having different upstream locations and a single downstream location in the city center. The paths are rather short, but are the best we could find. Using data from the a.m. and p.m. peaks of all weekdays from September and October, during 2005 to 2007, we produced the table shown in Appendix C. The data reveal substantial travel time variability with

a standard deviation of travel time ranging up to 75% of the mean travel time. The correlation within pairs ranges widely from -0.4 to 0.4, indicating that it would be clearly inadequate to assume travel times to be independent in order to compute the variance of the travel time difference.

Our model does not comprise the concept of a designated meeting time. The basic mechanism driving our model is the most fundamental property of in-person meetings, namely that the meeting does not in fact occur until both participants are present. The present model is the simplest we can conceive that comprises this mechanism. Extending the model with a designated meeting time requires some other elements to be included as well. In particular, there must be some penalty (e.g. accounting for embarrassment) for being late relative to the meeting time, and there must also be some mechanism for agreeing on a meeting time. Hence, including a designated meeting time would be a significant complication of the present model.

Our model is related to Ostrovsky and Schwarz (2006), who consider a manager who schedules simultaneous production processes of random duration where it is costly if the processes do not finish at the same time. Assuming independent random activity durations and linear costs for arrival earlier or later than the last completed activity, they characterize the socially optimal target arrival times in terms of the probability of arriving last, and they show how a penalty for last arrival can be determined that internalizes the total cost and results in the most efficient target arrivals. In contrast, the present paper considers individually rational agents facing trips with dependent durations and nonlinear costs for early departure.

Basu and Weibull (2003) discuss the habit of punctuality in the context of

choosing departure time for meetings with random travel times. They find that two stable Nash equilibria can arise, where either both persons are punctual or both tardy, and they conclude that the same society may be caught in a punctual or in a tardy equilibrium. The strategy set in their paper consists of the two strategies, the punctual and the tardy, and it is that discreteness which leads to multiple equilibria. In our paper, the strategy set is continuous, and this leads to just one equilibrium.

Our model does not represent congestion but merely takes a consequence of congestion, travel time variability, as given. By the same token, we have no congestion externality in our model. Our model could conceivably be used to extend models of congestion.

The paper is organized as follows. We formulate our model in Section 2, and Nash equilibrium is analyzed in Section 3, with some results on the cost of travel time variability in Section 3.1. Section 4 analyzes Pareto optimum with results on the implementation of Pareto optimum through penalty or compensation schemes in Section 4.1 and a numerical example is given in Section 4.2. Section 5 concludes. Proofs are in the Appendix, which also provides a summary of the notation employed as well as of travel time data from Stockholm.

2 A meeting game

Consider two players, labeled 1 and 2, who are going to a joint meeting. They each have a utility function describing their preferences regarding the timing of their meeting. If the players depart at times d_1 and d_2 from their previous activities and experience travel times of T_1 and T_2, then their meeting can start at time

$\max\{d_1 + T_1, d_2 + T_2\}$. Their utility functions are defined as functions of their own departure time and the time at which the meeting starts by

$$U_i[d_1, d_2] = \int_0^{d_i} h_i[s]\, ds - w_i \max\{d_1 + T_1, d_2 + T_2\}. \tag{1}$$

When d_i, T_1 and T_2 are fixed, this is an instance of trip timing preferences introduced by Vickrey (1973). The players derive utility at the individual-specific time varying rate h_i until the time of departure; then they derive zero utility while traveling and while possibly waiting for the other player to arrive at the meeting; then they derive utility at the individual-specific constant rate $w_i > 0$ until the meeting ends. Note that in general, h_i is not required to be greater than zero; such an assumption would reflect that staying longer at the origin is always preferred to traveling, ceteris paribus. Some results later, however, will require positivity of h_i. The points in time when the previous activity begins and when the meeting ends are constant; they are set to zero to economize on notation, and at no loss of generality. This leads to the form in (1).

Throughout the paper, we impose the following condition on the utility rates h_i.

Condition 1 *The utility rates h_i are differentiable with $h_i' < 0$ and $w_i \in range[h_i], i = 1, 2$.*

It is possible to establish unique existence of Nash equilibrium with increasing marginal utility rates of time spent at the meeting. However, other conclusions that we obtain in the paper rely on the tractability that we get from the constant utility rate at the meeting.

Travel times $T_i \geqslant 0$ are random variables possessing means $E[T_i] = \mu_i$, they

are independent of the departure times but may be mutually dependent. Players choose a departure time to maximize expected utility, taking the departure time of the other player as given, with payoffs

$$u_i\left[d_1, d_2\right] = EU_i\left[d_1, d_2\right] = \int_0^{d_i} h_i\left[s\right] ds - w_i E \max\left\{d_1 + T_1, d_2 + T_2\right\}. \quad (2)$$

Denoting the difference of random travel times $\Delta = T_2 - T_1$, the payoff of player i can be written[5] as

$$u_i\left[d_1, d_2\right] = \int_0^{d_i} h_i\left[s\right] ds - w_i \frac{d_1 + d_2}{2} - w_i E\left[\frac{T_1 + T_2}{2}\right] - w_i E\left[\frac{|d_1 - d_2 - \Delta|}{2}\right]. \quad (3)$$

Hence, the mean average travel time and the distribution of the travel time difference Δ are the only relevant attributes of travel time for the payoffs. To guarantee existence and uniqueness of Nash equilibrium (as will be shown) we assume the following condition throughout the paper, which rules out mass points in the distribution of Δ.[6]

Condition 2 *Δ has compact support with continuous cumulative distribution function F.*

If there is zero probability that players arrive simultaneously, $d_1 + T_1 = d_2 + T_2$,

[5] Using that $\max\{x, y\} = \frac{x+y}{2} + \frac{|x-y|}{2}$.

[6] If Δ has a discrete distribution, then h_i can be found such that there is a continuum of Nash equilibria.

then the derivatives[7] of the payoffs are

$$\frac{\partial u_1}{\partial d_1} = h_1[d_1] - w_1 F[d_1 - d_2], \tag{4a}$$

$$\frac{\partial u_2}{\partial d_2} = h_2[d_2] - w_2(1 - F[d_1 - d_2]), \tag{4b}$$

and by Condition 1 they attain values both larger and smaller than zero. Otherwise, the derivatives do not exist at points (d_1, d_2) with non-zero probability of simultaneous arrival. Moreover, $\frac{\partial u_i}{\partial d_i}$ is decreasing in d_i. Therefore the best response of each player exists and is unique. The best response of player i always satisfies

$$0 \leqslant h_i[d_i] \leqslant w_i, \tag{5}$$

since otherwise the player can increase her payoff by unilaterally advancing or postponing her departure.

3 Nash equilibrium

We begin by analyzing Nash equilibrium. The derivatives (4) attain the value 0, and the response functions $d_i = r_i[d_j]$ satisfy the first-order conditions

$$\begin{cases} h_1[r_1[d_2]] = w_1 F[r_1[d_2] - d_2], \\ h_2[r_2[d_1]] = w_2(1 - F[d_1 - r_2[d_1]]). \end{cases} \tag{6}$$

We will establish necessary and sufficient conditions for existence and uniqueness of a Nash equilibrium. Equilibrium requires that $r_1[r_2[d_1]] = d_1$, i.e. that if player 1 plays d_1, and player 2 responds with $r_2[d_1]$, then player 1 will not want

[7] Starting from (2), use that the derivative of $\max\{d_1 + T_1, d_2 + T_2\}$ is 1 when $d_1 + T_1 > d_2 + T_2$, which happens with probability $F(d_1 - d_2)$. It is zero otherwise.

to change. We are thus looking for points d_1 where the response to the response $r_1[r_2[\cdot]]$ meets the 45 degree line. If function F is differentiable everywhere, then the uniqueness can be proved by differentiating the first-order conditions, solving with respect to $r_2'[d_1]$ and noting that the response to the response, $r_1[r_2[\cdot]]$, has derivative less than 1, hence at most one equilibrium can exist. The theorem here is slightly more general.

Theorem 1 *There is at most one Nash equilibrium.*

The existence of Nash equilibrium is not guaranteed because the response curves may not cross. Indeed, it follows by letting $d_i = r_i[d_j]$ and adding the conditions in (6) that Nash equilibrium satisfies the equation $h_1[d_1]/w_1 + h_2[d_2]/w_2 = 1$, the possibility of which requires the following condition to hold.

Condition 3 $\lim_{x\to+\infty}\left[\frac{h_1[x]}{w_1}+\frac{h_2[x]}{w_2}\right] < 1.$

This condition is also sufficient for the existence of Nash equilibrium.

Theorem 2 *Condition 3 is necessary and sufficient for the existence of a Nash equilibrium.*

Intuitively, failure of Condition 3 implies that the players are not able to find a solution that is optimal for both, because optimality of departure time for each of them requires the ratio of utility rates $h_i[d_i]/w_i$ to equal the probability of arriving later than the other player, and yet these ratios never sum to 1; there would always be one of the players who would have incentive to depart later. Condition 3 will be maintained throughout the paper.

3.1 The cost of travel time variability

In this section we consider the connection between the players' payoff and the distribution of random travel times. As noted before, what matters about the travel time distribution is the average expected travel time, and the distribution of the difference of travel times. We parameterize the difference of travel times as $\Delta = T_2 - T_1 = \mu + \sigma Y$.

Theorem 3 *The players' payoffs in the Nash equilibrium $u_i\left[d_1^*, d_2^*\right]$ are non-increasing functions of the scale σ of the difference of travel times. In particular, they are decreasing if both functions h_i are positive.*

Thus, given fixed mean travel times, increasing the variability of difference of travel times reduces the payoffs of both players.

As mentioned in the Introduction, the extant literature on valuing travel time variability takes the perspective of a single player. A result like Theorem 3 could not be formulated within this literature. This paper describes a situation where a change in travel time variability for one player affects also the other player. As shown in Theorem 3, the effect occurs fully through the variance of the travel time difference. The change in the variance of the travel time difference that follows a change in travel time variance for one player depends crucially on the correlation between the travel times. Depending on this correlation, an increase in travel time variability for one player may either increase or decrease the variance of the travel time difference. Reparameterize the travel time distributions as $T_i = \mu_i + \sigma_i X_i$, where now X_i are standardized with mean zero and variance one, $\rho_{1,2}$ is the correlation coefficient for X_1, X_2 and note that $\frac{\partial \sigma}{\partial \sigma_i} = 2\sigma_i - 2\sigma_j \rho_{1,2}$. Then the following corollary is immediate.

Corollary 1 *Assume Condition 3 and that functions h_i are positive. Then the change in payoff for player 1 from a marginal change in travel time variability for player i is*

$$\frac{du_1\left[d_1^*, d_2^*\right]}{d\sigma_i} = \frac{du_1\left[d_1^*, d_2^*\right]}{d\sigma} 2\left(\sigma_i - \sigma_j \rho_{1,2}\right),$$

and it is negative if and only if $\rho_{1,2} < \sigma_i / \sigma_j$.

If travel times are negatively correlated, then an increase in travel time variability for either player is always costly for both players. Further, since $\rho_{1,2}$ is never greater than 1, then if player i already has the highest travel time variability then increasing his travel time variability will always make both players worse off. However, if the correlation is sufficiently large and player i has the smallest travel time variability, then an increase in his travel time variability, ceteris paribus, is beneficial for both players. This effect arises because the increase will make their random delays and their resulting departure decisions better synchronized. Appendix C provides some empirical evidence on the correlation of travel times in Stockholm. In none of the cases we examine there is an increase in travel time variability beneficial.

4 Pareto optimum

We now turn to consider Pareto optima, defined as pairs of departure times where it is not possible to increase payoff for one player without reducing it for the other.

The payoff functions (2) for the players have gradients

$$\nabla u_1 [d_1, d_2] = \begin{pmatrix} h_1 [d_1] - w_1 F [d_1 - d_2] \\ -w_1 (1 - F [d_1 - d_2]) \end{pmatrix}, \tag{7a}$$

$$\nabla u_2 [d_1, d_2] = \begin{pmatrix} -w_2 F [d_1 - d_2] \\ h_2 [d_2] - w_2 (1 - F [d_1 - d_2]) \end{pmatrix}. \tag{7b}$$

According to Ehrgott (2005, Thms 3.21 and 3.27), the first-order conditions for Pareto optimum can be expressed via the vector equation

$$\lambda_1 \nabla u_1 [d_1, d_2] + \lambda_2 \nabla u_2 [d_1, d_2] = 0. \tag{8}$$

Existence of positive (λ_1, λ_2) satisfying (8) is a sufficient condition for (d_1, d_2) to be a Pareto optimum while existence of non-negative (λ_1, λ_2) satisfying (8), with at least one of them positive, is a necessary condition for Pareto optimality. Using the gradient expressions (7) above, the sufficient condition can be expressed[8] as

$$\left(\frac{h_1}{w_1} - F\right)\left(\frac{h_2}{w_2} - (1 - F)\right) = (1 - F) F \tag{9a}$$

$$\left\{\frac{h_1}{w_1} = F = 0\right\} \vee \left\{\left(\frac{h_1}{w_1} - F\right) F > 0\right\} \tag{9b}$$

$$\left\{\frac{h_2}{w_2} = (1 - F) = 0\right\} \vee \left\{\left(\frac{h_2}{w_2} - (1 - F)\right)(1 - F) > 0\right\}, \tag{9c}$$

while the necessary condition is

$$\left(\frac{h_1}{w_1} - F\right)\left(\frac{h_2}{w_2} - (1 - F)\right) = (1 - F) F \tag{10a}$$

$$\left(\frac{h_1}{w_1} - F\right) F \geq 0 \tag{10b}$$

$$\left(\frac{h_2}{w_2} - (1 - F)\right)(1 - F) \geq 0. \tag{10c}$$

[8] Omitting function arguments for visual clarity. The symbol $\vee$ denotes logical "or".

It turns out there is a continuum of Pareto optima in this game. In fact, for each possible realization of the travel time difference there is a Pareto optimum that has the same difference between the departure times of the players. Hence, all possible divisions of the risk of having to wait for the other player are attained as Pareto optima.

Theorem 4 *Assume h_1 and h_2 are positive. Assume F is continuous and increasing on $\operatorname{supp}\Delta$. Then there is a continuum of Pareto optima, each uniquely corresponding to different $d_1 - d_2 \in \operatorname{supp}\Delta$, and no other Pareto optima. The set of Pareto optima coincides with the set of solutions to (9a) such that $d_1 - d_2 \in \operatorname{supp}\Delta$.*

It is natural to ask whether the Nash equilibrium can be a Pareto optimum. When utility rates h_i are positive, i.e. when time at the origin is preferred to traveling, then this turns out never to be the case. Some intervention is required to reach a Pareto optimum.

Theorem 5 *With positive h_1 and h_2, Nash equilibrium is not Pareto optimal.*

The reason behind Theorem 5 is that both players in Nash equilibrium could gain if they both departed a little earlier. This insight is generalized in the next theorem, which states that Pareto optimum requires at least one of the players to depart so early that the utility rate at the origin is at least as high as at the meeting. Otherwise both players would gain if both departed a little earlier.

Theorem 6 *With $h_1[d_1] < w_1$ and $h_2[d_2] < w_2$, then (d_1, d_2) can not be a Pareto optimum.*

The next theorem affirms that there always exists Pareto optima that are improvements relative to Nash equilibrium for both players. It is hard to give an intuition for this theorem, it relies on properties of concave optimization over compact sets.

Theorem 7 *Let the travel time difference admit a density. Then there is a continuum of Pareto optima that are improvements relative to Nash equilibrium for both players.*

Section 4.2 below presents an example where all Pareto optima are improvements relative to Nash equilibrium for both players. That section also shows an example, where there exists a Pareto optimum that is worse than Nash equilibrium for one of the players.

4.1 Implementing Pareto optimum

There are two simple ways to implement a Pareto optimum as a Nash equilibrium through a pricing mechanism: a penalty or a compensation. We take into account the effect of payment on behavior but disregard that payment or compensation affects welfare and assume this effect is neutralized in some way. We consider first a penalty and discuss a compensation scheme later. Denote the random arrival time of player i at the meeting as $A_i = T_i + d_i$. If each player i pays a positive penalty for being late of $\beta_i w_i \max\{A_i - A_j, 0\}$, then the first-order conditions for Nash equilibrium become $h_1[d_1] - w_1 F[d_1 - d_2] = \beta_1 F[d_1 - d_2]$ and $h_2[d_2] - w_2(1 - F[d_1 - d_2]) = \beta_2(1 - F[d_1 - d_2])$, which satisfy the necessary conditions for Pareto optimum in (10) if and only if $\beta_1\beta_2 = 1$. We show that Nash equilibrium exists uniquely in the game with these penalties, and that this Nash

equilibrium is Pareto optimum in the original game without penalties. Increasing β_1 benefits the first player and hurts the other.

Theorem 8 *Consider the game with penalties* $\beta_i w_i \max\{A_i - A_j, 0\}, \beta_i > 0, i, j = 1, 2$. *If* $\beta_1\beta_2 = 1$, *then the Nash equilibrium of this game exists uniquely and it is a Pareto optimum for the original game without penalties. In this equilibrium, the second player's payoff* u_2 *of the original game is non-decreasing in* β_1 *while the payoff* u_1, *the departure time difference* $d_1 - d_2$ *and the probability* $F[d_1 - d_2]$ *for player 1 of arriving later than player 2 are non-increasing in* β_1. *If* $\beta_1\beta_2 \neq 1$ *then Nash equilibrium cannot be a Pareto optimum in the original game without penalties.*

If alternatively each player 1 receives a positive compensation $\alpha_1 w_1 \max\{A_2 - A_1.0\}$ when player 2 is late, and player 2 similarly receives a positive compensation $\alpha_2 w_2 \max\{A_1 - A_2, 0\}$ when player 1 is late, then the first-order conditions for Nash equilibrium become

$$h_1[d_1] - w_1(\alpha_1 + (1-\alpha_1)F[d_1 - d_2]) = 0, \tag{11a}$$

$$h_2[d_2] - w_2(\alpha_2 + (1-\alpha_2)(1 - F[d_1 - d_2])) = 0, \tag{11b}$$

which satisfy the necessary conditions for Pareto optimum in (10) if and only if $\alpha_1\alpha_2 = 1$. So we may as well define $\alpha_1 = \alpha > 0$ and $\alpha_2 = \alpha^{-1}$. From (11) we see that the condition

$$h_1[d_1]/w_1 = \alpha h_2[d_2]/w_2, \tag{12}$$

must hold at Nash equilibrium, which implies that Nash equilibrium can only exist if

$$\frac{\inf h_1}{w_1}\frac{w_2}{\sup h_2} < \alpha < \frac{\sup h_1}{w_1}\frac{w_2}{\inf h_2}. \tag{13}$$

Thus the conditions for existence of Nash equilibrium in the game with compensations are more restrictive than in the game with penalties. However, if $\alpha = 1$, then it is easy to check that there is a unique Nash equilibrium $\left(h_1^{-1}[w_1], h_2^{-1}[w_2]\right)$.

The following theorem partly parallels Theorem 8 with the difference that we can only be sure the Nash equilibrium exists for α near 1.

Theorem 9 *Suppose that the density of the travel time difference is continuous. Consider the game with compensations defined as above. There is an interval around* 1 *such that Nash equilibrium of this game exists uniquely for any* α *in this interval.*

The following theorem strengthens the last result concerning existence of Nash equilibrium in the game with compensations at the price of not guaranteeing uniqueness of the equilibrium.

Theorem 10 *Suppose*

$$\max\left\{\frac{\inf h_1}{w_1}, \frac{w_2}{\sup h_2}\right\} < \alpha < \min\left\{\frac{\sup h_1}{w_1}, \frac{w_2}{\inf h_2}\right\}.$$

Then there exists a Nash equilibrium in the game with compensations.

In particular, if $range\,(h_1) = range\,(h_2) = \mathbb{R}_+$, then Nash equilibrium exists for any $\alpha > 0$. Moreover, note that (13) is satisfied for α in a neighborhood of 1, such that the conclusion of Theorem 9 regarding existence is incorporated in Theorem 10.

Even when Nash equilibrium exists in the game with compensations, it does not necessarily constitute Pareto optimum in the original game. It follows from Theorem 4 that $d_1 - d_2 \in \operatorname{supp} \Delta$ in Pareto optimum. This relationship may not be valid for all α.

Example 1 *Let* $h_i(d_i) = \exp(-d_i)$ *and* $w_i = 1$, $i = 1, 2$. *Then* $\inf h_i = 0, \sup h_i = \infty$ *and Nash equilibrium in the game with compensations exists for any* $\alpha > 0$ *by Theorem 10. Equation (12) implies* $d_1 - d_2 = -\ln \alpha$ *at Nash equilibrium. Thus the game with compensation does not implement Pareto optimum if* $-\ln \alpha \notin \operatorname{supp} F$. *In particular, the fair compensation rule with* $\alpha = 1$ *does not implement Pareto optimum if* $0 \notin \operatorname{supp} F$.

As for β in the case with penalties, we find that the difference between equilibrium departure times depends monotonically on α, at least in the range of α where the Nash equilibrium is unique. The intuition is straightforward: a larger α decreases the cost for player 1 of arriving early and increases it for player 2.

Theorem 11 *Let* $d^*[\alpha] = (d_1^*[\alpha], d_2^*[\alpha])$ *be the unique Nash equilibrium in the game with compensations for* α *near* 1. *Then* $d_1^*[\alpha] - d_2^*[\alpha]$ *is decreasing.*

We complete the discussion by asking whether penalties and compensations may be defined in a single scheme that implements Pareto optimum as a Nash equilibrium. Suppose then that penalties are introduced with weights β_1 and β_2 and also that compensations are introduced with weights α_1 and α_2. Suppose also that we are looking for Pareto optima with $0 < F(d_1 - d_2) < 1$. Then the first necessary condition for Pareto optimum (10) implies that either $\alpha_1 = \alpha_2^{-1} > 0$ with $\beta_1 = \beta_2 = 0$, or conversely that $\beta_1 = \beta_2^{-1} > 0$ with $\alpha_1 = \alpha_2 = 0$, meaning that penalties and compensations cannot be combined to implement Pareto optimum.

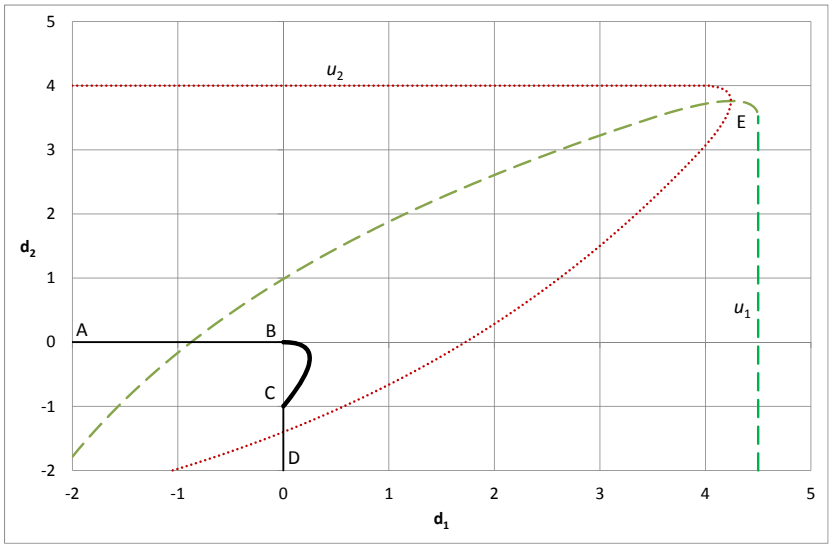

Figure 1: Numerical illustration as discussed in the text

4.2 Numerical illustration

This section provides a numerical example to illustrate the concepts involved. The example is shown in Figure 1. Let $h_i[x] = 4/(x+4)$ for $x \geq -2$, and $h_i[x] = -x$ for $x < -2$, $i = 1, 2$. Let $w_1 = w_2 = 1$. Further assume that the travel time difference is uniformly distributed on $[0, 1]$ and note that this assumption introduces asymmetry between players . To find the unique Nash equilibrium, solve the system of equations

$$\begin{aligned} \frac{4}{d_1+4} - d_1 + d_2 &= 0 \\ \frac{4}{d_2+4} - 1 + d_1 - d_2 &= 0, \end{aligned}$$

which reduces to a third degree equation. The solution is found numerically to be (4.25, 3.76), shown as point E in the figure.

The set of Pareto optima (using $S = d_1 - d_2$) is

$$\left\{ (d_1, d_2) \mid \left(\frac{4}{d_1 + 4} - S \right) \left(\frac{4}{d_2 + 4} - 1 + S \right) = S(1 - S), 0 \leq S \leq 1 \right\},$$

which reduces to $\left\{ \left(d_2 + \sqrt{-d_2}, d_2 \right) \mid -1 \leq d_2 \leq 0 \right\}$ and is indicated in the figure by the thick solid curve from B to C. Note that the payoffs u_1 and u_2 attain their global maxima at the endpoints $(0, 0)$ and $(0, -1)$ of the Pareto set, corresponding to B and C.

In the game with penalties, the Nash equilibrium runs along the Pareto set approaching point C $(0, -1)$ as $\alpha \to 0$ and the point B $(0, 0)$ as $\alpha \to \infty$. According to Theorem 8, all Nash equilibria of the game with penalties are Pareto optima of the original game. Moreover, all but two Pareto allocations can be implemented in this way.

In the game with compensations, the Nash equilibrium runs up along the vertical axis from D to C as α runs from 0 to $3/4$, then follows the Pareto set from C to B for $3/4 \leq \alpha \leq 1$, and finally continues left along the horizontal axis from B to A for $\alpha > 1$, as demonstrated by the thin solid line in the figure 1. Hence in this example, all Pareto optima can be implemented as Nash equilibria in the game with compensations, but not all these Nash equilibria are Pareto optima.

Finally, we will find the level sets for the payoffs of the Nash equilibrium of the original game. The payoffs are $u_i[d_1, d_2] = 4 \ln [d_i + 4] - \Phi[d_1, d_2] + C$ where

$$\Phi[d_1, d_2] = \begin{cases} d_2, & d_1 - d_2 \leq 0 \\ (d_1 - d_2)^2 / 2 + d_2, & 0 < d_1 - d_2 \leq 1 \\ d_1 - 1/2, & 1 < d_1 - d_2 \end{cases}$$

and the constant C depends on the mean travel times. The value of C is irrelevant

and we just use $C = 0$. The Nash equilibrium payoffs are then $u_1(4.25, 3.76) = 4.56$ and $u_2(4.25, 3.76) = 4.32$. Within the band $0 < d_1 - d_2 \leq 1$, the equations for the level sets of u_1 and u_2 are $d_2 = d_1 - 1 + \sqrt{8\ln[d_1 + 4] - 2d_1 - 8.12}$ and $d_1 = d_2 + \sqrt{8\ln[d_1 + 4] - 2d_2 - 8.64}$, respectively. Above this band, the corresponding equations are $d_2 = 4\ln[d_1 + 4] - 4.56$ and $d_2 = 4.00$, while below the band they are $d_1 = 4.50$ and $d_1 = 4\ln[d_2 + 4] - 3.82$. The level sets are shown in Figure 1 as dashed for u_1 and dotted for u_2. For both level sets the whole Pareto set lies on the same side of the level set as the corresponding global optimum. Thus all Pareto optima are improvements relative to Nash equilibrium for both players in this example (cf. Theorem 7).

If the travel time difference is assumed to be uniformly distributed on $[9, 10]$ (instead of $[0, 1]$) then the Nash equilibrium is $(10.758, 1.487)$ with payoffs $u_1 = 44.20$ and $u_2 = 40.24$. Furthermore, one Pareto optimum (which is a global optimum for player 1) is located at $(0, -10)$ where the payoffs are $u_1 = 50$ and $u_2 = -0.77$. Thus in this case there is a Pareto optimum which is an improvement for player 1 but not for player 2, compared to the Nash equilibrium.

5 Conclusion

This paper has formulated and analyzed a game theoretic model to describe the essential features of a physical meeting between two persons facing random travel time variability. Even though the issue is clearly important, it has so far been ignored by the literature, which has exclusively taken the perspective of individual travelers. As discussed in the Introduction, the present analysis opens the door for new considerations on policies to address urban congestion.

Future work could extend the analysis in several directions. One extension would be to include the concept of a designated meeting time; this would require some specification of a process to determine the meeting time as well as incorporation of penalties (accounting, e.g., for embarrassment) for being late relative to the meeting time. Another extension to the model would allow for meetings between more than two players; we have not been able to carry out such an extension but others might be more successful. Moreover, the model could perhaps be extended to take into account that waiting at the meeting place might be more productive than traveling even if less productive than meeting.

There is also much empirical work that could be done. As large-scale datasets of GPS traces of moving cars become available, we are approaching the point where the pattern of travel time variability across cities becomes observable and amenable to modeling. Then network models could be developed to predict changes in the pattern of travel time variability resulting from various policies. Travel surveys could be extended to include information about meetings of various kinds, and this information could be used in the empirical application of the meeting model.

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A Notation

Notation	Explanation
A_i	Random arrival time for individual i
d_i	Departure time for individual i
d_i^*	Nash equilibrium departure time for individual i
F, f	CDF and PDF for Δ
i, j	Individuals from the set $(1, 2)$
h_i	Marginal utility of individual i spending time at the origin
r_i	Response function for individual i
S	Difference in departure times, $d_1 - d_2$
T_i	Random travel time for individual i
U_i	Utility for individual i
u_i	Expected utility for individual i
w_i	Marginal utility for individual i spending time at the meeting
X_i	Standardized travel time for i
Y	Standardized difference in travel times
α_i	Individual i's earliness compensation
β_i	Individual i's lateness penalty
Δ	Random variable for the difference in travel times, $T_2 - T_1$
μ	Mean of the difference in travel times, Δ
λ_i	Parameter in the first-order condition for Pareto optimum
μ_i	Mean travel time for individual i
ρ_{ij}	Correlation coefficient of travel times for individuals i and j
σ	Standard deviation of the difference in travel times, Δ
σ_i	Standard deviation of travel time for individual i
Φ	A part of the function for payoffs under Nash equilibrium

B Proofs

Proof of Theorem 1. We claim that $r_1[d_2] \leqslant r_1[d_2+\eta] < r_1[d_2]+\eta$ for any d_2 and $\eta > 0$. Indeed, denote $d_1 = r_1[d_2]$, $d_1' = r_1[d_2+\eta]$ and assume $d_1' \geqslant d_1+\eta$. Use (6), the monotonicity of F and the strict monotonicity of h_1 to obtain that $h_1[d_1'] = w_1 F[d_1'-d_2-\eta] \geqslant w_1 F[d_1-d_2] = h_1[d_1] > h_1[d_1']$, which is a contradiction. Now assume $d_1' < d_1$. Then $h_1[d_1'] = w_1 F[d_1'-d_2-\eta] \leqslant w_1 F[d_1-d_2] = h_1[d_1] < h_1[d_1']$, which is again a contradiction. Thus the claim is established.

It follows from the claim that r_1 is a weak contraction mapping, i.e. $|r_1[d_2'] - r_1[d_2]| < |d_2'-d_2|$ for any d_2, d_2'. In a similar way one can show that r_2 is a weak contraction as well. Therefore the composition $r_1[r_2[\cdot]]$ is a weak contraction too and cannot have more than one fixed point. ■

Proof of Theorem 2. The necessary part has already been proved. To show the sufficiency, denote $d_2 = r_2[d_1]$. We claim that $r_1[r_2[d_1]] < d_1$ for d_1 sufficiently large.

First, if $h_1[d_1^0] = 0$ for some d_1^0, then (4a) implies that $r_1[d_2] \leqslant d_1^0$ for any d_2. Therefore for any $d_1 > d_1^0$ we have $r_1[r_2[d_1]] < d_1^0 < d_1$.

If $h_2[d_2^0] = 0$ for some d_2^0, then choose $d_1 > r_1[d_2^0]$ and $d_1 > d_2^0 + b$ where $F[b] = 1$. Due to (4a), $d_2 \leqslant d_2^0$ and therefore $h_2[d_2] = w_2(1-F[d_1-d_2]) \leqslant w_2(1-F[d_1-d_2^0]) = 0$. Hence $d_2 = d_2^0$ and again $r_1[r_2[d_1]] < d_1^0 < d_1$.

It remains to prove the claim when h_1 and h_2 are always positive. Denote $q_i = \lim_{x\to+\infty} h_i[x]/w_i$ and choose positive $\varepsilon < \frac{1-q_1-q_2}{2}$. There exists M such that $h_i[x]/w_i < q_i+\varepsilon$ for all $x > M$, $i = 1,2$. Choose $d_1 > M+|a|+|b|$ where $F[a] = 0$. Then from the first-order condition for the best response of player 2

we obtain $F[d_1 - d_2] = 1 - h_2[d_2]/w_2 < 1$ whence $d_2 > d_1 - b \geqslant d_1 - |b| \geqslant M + |a| \geqslant M$. By the choice of M this implies

$$1 - F[d_1 - d_2] = h_2[d_2]/w_2 < q_2 + \varepsilon$$

Denote $d_1' = r_1[d_2]$. From the first-order condition for the best response of player 1 we obtain $F[d_1' - d_2] = h_1[d_1']/w_1 > 0$ which implies $d_1' > d_2 + a \geqslant M + |a| + a \geqslant M$, whence $F[d_1' - d_2] = h_1[d_1']/w_1 < q_1 + \varepsilon$.

Adding the two inequalities yields

$$F[d_1' - d_2] + 1 - F[d_1 - d_2] < q_1 + q_2 + 2\varepsilon < 1$$

or

$$F[d_1' - d_2] < F[d_1 - d_2],$$

which is only possible if $d_1' < d_1$ i.e. $r_1[r_2[d_1]] - d_1 < 0$. Thus we have established the claim made above.

By (6) we have $r_1[x] \geqslant h_1^{-1}[w_1]$ for any x. Therefore for any $d_1 < h_1^{-1}[w_1]$ we have $r_1[r_2[d_1]] - d_1 > 0$. Thus $r_1[r_2[d_1]] - d_1$ is positive for $d_1 < h_1^{-1}[w_1]$ and negative for d_1 large enough. Then continuity of $r_1[r_2[d_1]] - d_1$ (implied by the weak contraction property shown in the proof of Theorem 1) yields existence of a departure time d_1 such that $r_1[r_2[d_1]] - d_1 = 0$, i.e. $(d_1, r_2[d_1])$ is a Nash equilibrium. ■

Proof of Theorem 3. We need only prove the theorem for player 1. Using the envelope theorem we find that

$$\frac{du_1[d_1^*, d_2^*]}{d\sigma} = \frac{\partial u_1}{\partial d_2}\frac{dd_2^*}{d\sigma} + \frac{\partial u_1}{\partial \sigma}.$$

From (2) and (3) and denoting $q = \frac{d_1^* - d_2^* - \mu}{\sigma}$, the derivatives of the payoff function are obtained as

$$\frac{\partial u_1}{\partial d_2} = -w_1 P\left[\Delta > d_1^* - d_2^*\right] = -w_1 \int_q^{+\infty} \varphi\left[x\right] dx$$

and

$$\begin{aligned}
\frac{\partial u_1}{\partial \sigma} &= -\frac{w_1}{2} E\left[\frac{\partial}{\partial \sigma}\left|\mu + \sigma X - d_1^* + d_2^*\right|\right] \\
&= -\frac{w_1}{2}\int_q^{+\infty} x\varphi\left[x\right] dx + \frac{w_1}{2}\int_{-\infty}^{q} x\varphi\left[x\right] dx \\
&= -w_1 \int_q^{+\infty} x\varphi\left[x\right] dx.
\end{aligned}$$

To compute the derivative $\frac{dd_2^*}{d\sigma}$, differentiate the first-order conditions

$$\begin{cases} h_1\left[d_1^*\right]/w_1 + h_2\left[d_2^*\right]/w_2 = 1 \\ h_1\left[d_1^*\right] = w_1 \Phi\left[q\right] \end{cases} \tag{14}$$

to obtain

$$\begin{cases} \frac{h_1'\left[d_1^*\right]}{w_1}\frac{dd_1^*}{d\sigma} + \frac{h_2'\left[d_2^*\right]}{w_2}\frac{dd_2^*}{d\sigma} = 0 \\ \frac{h_1'\left[d_1^*\right]}{w_1}\frac{dd_1^*}{d\sigma} = \varphi\left[q\right]\frac{\left(\frac{dd_1^*}{d\sigma} - \frac{dd_2^*}{d\sigma}\right) - q}{\sigma} \end{cases}$$

and solve to obtain

$$\frac{dd_2^*}{d\sigma} = -\frac{\varphi\left[q\right]\frac{h_1'\left[d_1^*\right]}{w_1}}{\varphi\left[q\right]\left(\frac{h_1'\left[d_1^*\right]}{w_1} + \frac{h_2'\left[d_2^*\right]}{w_2}\right) - \frac{h_1'\left[d_1^*\right]}{w_1}\frac{h_2'\left[d_2^*\right]}{w_2}\sigma} q = -\lambda q,$$

where $0 \leqslant \lambda < 1$ since $h_i'\left[d_i^*\right] < 0$. Therefore

$$
\begin{aligned}
\frac{du_1\left[d_1^*, d_2^*\right]}{d\sigma} &= w_1 \int_q^{+\infty} \varphi\left[x\right] dx \cdot \lambda q - w_1 \int_q^{+\infty} x\varphi\left[x\right] dx \\
&= w_1 \int_q^{+\infty} \left(\lambda q - x\right) \varphi\left[x\right] dx \\
&\leqslant w_1 \int_q^{+\infty} \left(\lambda - 1\right) x\varphi\left[x\right] dx \leqslant 0,
\end{aligned}
$$

where the last inequality is due to $EX = 0$. It follows from the non-positivity of the derivative that the payoff of the first player in the equilibrium is non-increasing in σ.

To prove the second assertion of the theorem note that the last inequality can be satisfied as an equality only when $\Phi\left[q\right] = 0$ or $\Phi\left[q\right] = 1$. Due to the first-order conditions (14), this is only possible if $h_1\left[d_1^*\right] = 0$ or $h_2\left[d_2^*\right] = 0$. Thus when both functions h_i are positive we have $\frac{du_1\left[d_1^*, d_2^*\right]}{d\sigma} < 0$, which proves that the payoff of the first player in the equilibrium is decreasing as a function of σ. ■

Proof of Theorem 4. First we will show that for each $0 < \phi < 1$, there is a Pareto optimum (d_1^*, d_2^*) such that $F\left[d_1^* - d_2^*\right] = \phi$. There is a unique δ such that $F\left[\delta\right] = \phi$. Consider a function defined by

$$
\begin{aligned}
M\left[d_1\right] &= \left(\frac{h_1\left[d_1\right]}{w_1} - \phi\right)\left(\frac{h_2\left[d_1 - \delta\right]}{w_2} - 1 + \phi\right) - \phi\left(1 - \phi\right) \\
&= \frac{h_1\left[d_1\right]}{w_1}\frac{h_2\left[d_1 - \delta\right]}{w_2} - \left(\phi\frac{h_2\left[d_1 - \delta\right]}{w_2} + \left(1 - \phi\right)\frac{h_1\left[d_1\right]}{w_1}\right).
\end{aligned}
$$

Let $a = \min\left\{h_1^{-1}\left[w_1\right], h_2^{-1}\left[w_2\right] + \delta\right\}$ and $b = \max\left\{h_1^{-1}\left[w_1\right], h_2^{-1}\left[w_2\right] + \delta\right\}$.

Then $h_1[a] \geq w_1$ and $h_2[a-\delta] \geq w_2$. Hence,

$$\begin{aligned} M[a] &= \frac{h_1[a]}{w_1}\frac{h_2[a-\delta]}{w_2} - \left(\phi\frac{h_2[a-\delta]}{w_2} + (1-\phi)\frac{h_1[a]}{w_1}\right) \\ &\geq \frac{h_1[a]}{w_1}\frac{h_2[a-\delta]}{w_2} - \max\left\{\frac{h_1[a]}{w_1}, \frac{h_2[a-\delta]}{w_2}\right\} \geq 0. \end{aligned}$$

Similarly, $M[b] \leq 0$. Since M is continuous it follows that there is d_1^* such that $M[d_1^*] = 0$, i.e. (d_1^*, d_2^*) satisfy the first condition in (9). Now, either $h_1[b] = w_1$ or $h_2[b-\delta] = w_2$. In the first case $\frac{h_1[d_1^*]}{w_1} - \phi \geq \frac{h_1[b]}{w_1} - \phi = 1 - \phi > 0$, which implies $\frac{h_2[d_2^*]}{w_2} - (1-\phi) > 0$, since $\left(\frac{h_1[d_1^*]}{w_1} - \phi\right)\left(\frac{h_2[d_1^*-\delta]}{w_2} - (1-\phi)\right) = \phi(1-\phi) > 0$, so the second and the third conditions in (9). The second case is considered similarly and leads to the same conclusion. Thus (d_1^*, d_2^*) is a Pareto optimum for any $0 < \phi < 1$. Moreover, for each pair (d_1^*, d_2^*) there is a unique $\phi = F[d_1^* - d_2^*]$. Hence there is a continuum of Pareto optima.

Next we show that $d_1 - d_2 \notin \operatorname{supp}\Delta$ cannot be a Pareto optimum. If, in particular, $F[d_1 - d_2 + \varepsilon] = 0, \varepsilon > 0$, then replacement of d_1 by $d_1 + \varepsilon$ increases the payoff of the first player without effect on the payoff of the second player, hence this is not a Pareto optimum. The case $F[d_1 - d_2 - \varepsilon] = 1, \varepsilon > 0$ is similar.

Finally, we show that $d_1^* - d_2^*$ on the boundary of $\operatorname{supp}\Delta$ corresponds to a unique Pareto optimum. Assume $F[d_1^* - d_2^*] = 0$ and $F[\delta] > 0$ for $\delta > d_1^* - d_2^*$. Then the necessary conditions (10) give the unique d_2^* such that $h_2[d_2^*] = w_2$. The payoff of player 2 cannot be increased since he is guaranteed to arrive later than player 1 and his utility rate at the origin is the same as at the meeting. The only change he is indifferent to is unilateral earlier departure of player 1. However this change decreases payoff for player 1. Hence (d_1^*, d_2^*) is a PO. Formally, assume

$d_2 \neq d_2^*$ and estimate

$$\begin{aligned} u_2[d_1, d_2] - u_2[d_1^*, d_2^*] &= (u_2[d_1, d_2] - u_2[d_1 + d_2^* - d_2, d_2^*]) \\ &\quad + (u_2[(d_1 + d_2^* - d_2, d_2^*)] - u_2[d_1^*, d_2^*]) \end{aligned} \tag{15}$$

Based on (2), the first parenthesis in (15) can be bounded as follows:

$$\begin{aligned} &\int_{d_2^*}^{d_2} h_2[s]\, ds - w_2 E \max\{d_1 + T_1, d_2 + T_2\} + w_2 E \max\{d_1 + d_2^* - d_2 + T_1, d_2^* + T_2\} \\ = &\int_{d_2^*}^{d_2} h_2[s]\, ds + w_2(d_2^* - d_2) = \int_{d_2^*}^{d_2} (h_2[s] - w_2)\, ds < 0, \end{aligned}$$

where the last inequality follows from the fact that $h_2[s] > w_2$ for $s < d_2^*$ and $h_2[s] < w_2$ for $s > d_2^*$. By the mean value theorem, there exists s strictly between d_1^* and $d_1 + d_2^* - d_2$ such that the second parenthesis in (15) can be written as

$$(d_1 + d_2^* - d_2 - d_1^*) \frac{\partial u_2}{\partial d_1}[s, d_2^*] = -(d_1 + d_2^* - d_2 - d_1^*) w_2 F[s - d_2^*].$$

This is negative if $(d_1 + d_2^* - d_2 - d_1^*) > 0$ and 0 otherwise, because $F[s - d_2^*] > 0$ for $s > d_1^*$ and $F[s - d_2^*] = 0$ for $s \leq d_1^*$. Thus the whole expression (15) is negative.

It remains to consider the case $d_2 = d_2^*$. If $d_1 > d_1^*$ then, similarly, we have

$$u_2[d_1, d_2^*] - u_2[d_1^*, d_2^*] = (d_1 - d_1^*) \frac{\partial u_2}{\partial d_1}[s - d_2^*] = -(d_1 - d_1^*) w_2 F[s - d_2^*] < 0$$

while $d_1 < d_1^*$ yields $u_2[d_1, d_2^*] - u_2[d_1^*, d_2^*] < 0$ since

$$\frac{\partial u_1}{\partial d_1}[s - d_2^*] = h_1[s] - w_1 F[s - d_2^*] = h_1[s] > 0$$

for $s < d_1^*$. ■

Proof of Theorem 5. Suppose (d_1, d_2) is both Nash equilibrium and Pareto

optimal. The first-order Nash equilibrium necessary condition is

$$h_1[d_1] - w_1 F[d_1 - d_2] = h_2[d_2] - w_2(1 - F[d_1 - d_2]) = 0.$$

Then it follows from (10) that either $F[d_1 - d_2] = 0$ or $F[d_1 - d_2] = 1$, which leads to either $h_1[d_1] = 0$ or $h_2[d_2] = 0$, contradicting the assumption of the theorem. ■

Proof of theorem 6. Let $0 < \varepsilon < \min\left\{d_1 - h_1^{-1}[w_1], d_2 - h_2^{-1}[w_2]\right\}$. Shift of both d_1 and d_2 back by ε will increase both payoffs since

$$\begin{aligned}
& u_i[d_1 - \varepsilon, d_2 - \varepsilon] - u_i[d_1, d_2] \\
= & \int_{d_i}^{d_i - \varepsilon} h_i[s]\, ds - w_i E \max\{d_1 - \varepsilon + T_1, d_2 - \varepsilon + T_2\} \\
& + w_i E \max\{d_1 - \varepsilon + T_1, d_2 - \varepsilon + T_2\} \\
= & \int_{d_i}^{d_i - \varepsilon} h_i[s]\, ds + w_i \varepsilon \\
= & \int_{d_i - \varepsilon}^{d_i} (w_i - h_i[s])\, ds > 0.
\end{aligned}$$

This shows that (d_1, d_2) is not Pareto optimal. ■

Proof of Theorem 7. For u_i the hessian is

$$\begin{pmatrix} h_i'[d_i] - w_i f[d_1 - d_2] & w_i f[d_1 - d_2] \\ w_i f[d_1 - d_2] & -w_i f[d_1 - d_2] \end{pmatrix},$$

which is negative definite when $f[d_1 - d_2] > 0$ and negative semi-definite otherwise. Then payoff functions u_i are concave functions of (d_1, d_2) where $f[d_1 - d_2] > 0$ and weakly concave elsewhere.

Denote by $L_i[v] = \{(d_1, d_2) \,|\, u_i[d_1, d_2] \geq v\}$. By concavity of u_i, the level sets are convex.

If $d_i \to \infty$ or $d_i \to -\infty$, then $u_i \to -\infty$; this is easy to show using that F has compact support. Then any intersection of level sets $I(v_1, v_2) \equiv L_1(v_1) \cap L_2(v_2)$ is bounded. It is also closed and so it is compact.

It is immediate from the definition of Pareto optimality that a point (d_1, d_2) is a Pareto optimum iff there exists (v_1, v_2) such that the level sets $L_1(v_1)$ and $L_2(v_2)$ are tangent at (d_1, d_2).

Let (d_1^*, d_2^*) be the Nash equilibrium. By Theorems 1 and 2 it exists uniquely. Let $v_i^* = u_i(d_1^*, d_2^*)$ be the equilibrium payoffs. Let

$$\hat{v}_i = \max\{u_i(d_1, d_2) \mid (d_1, d_2) \in I(v_1^*, v_2^*)\}.$$

It follows immediately that $v_i^* \leq \hat{v}_i$. In fact we have $v_i^* < \hat{v}_i$, since otherwise the Nash equilibrium would also be a Pareto optimum, in contradiction of Theorem 5.

For any $v_1 \in [v_1^* \leq v_1 < \hat{v}_1[$, the level set intersection $I(v_1, v_2^*)$ is nonempty and the correspondence $v_1 \to I(v_1, v_2^*)$ is continuous. Let

$$\tilde{v}_2(v_1) = \max\{u_2(d_1, d_2) \mid (d_1, d_2) \in I(v_1, v_2^*)\}.$$

The argmax correspondence is single-valued by Lemma 1 below. The maximum is then attained at a single point $\tilde{d}(v_1) = \left(\tilde{d}_1(v_1), \tilde{d}_2(v_1)\right)$, which must be a Pareto optimum, since the level sets must be tangent at this point. By the Berge maximum theorem, the points $\tilde{d}(v_1)$, $v_1 \in [v_1^* \leq v_1 < \hat{v}_1[$, trace out a continuous curve. The curve is not a single point, since then the maxima $\hat{v}_1, \hat{v}_2$ would be attained at the same point, and such a point would be both Nash equilibrium and Pareto optimum.

For any $v_1 > v_1^*$, we have $\tilde{d}(v_1) \in L_1(v_1)$ and hence $u_1\left(\tilde{d}(v_1)\right) \geq v_1 > v_1^*$.

For v_1 close to v_1^* we have $u_2\left(\tilde{d}\left(v_1\right)\right)$ is close to $\hat{v}_2$, and $\hat{v}_2 > v_2^*$. Hence there is a continuum of Pareto optima where both players gain strictly relative to Nash equilibrium. ■

Lemma 1 *The argmax correspondence in the proof of Theorem 7 is single-valued.*

Proof of Lemma 1. Assume on the contrary that $(d_1^1, d_2^1) \neq (d_1^2, d_2^2)$ and $u_2\left(d_1^1, d_2^1\right) = u_2\left(d_1^2, d_2^2\right) = \tilde{v}_2\left(v_1\right)$. Let $d\left(\lambda\right) = \left(d_1\left(\lambda\right), d_2\left(\lambda\right)\right) = \left(d_1^1, d_2^1\right) + \lambda\left(\left(d_1^2, d_2^2\right) - \left(d_1^1, d_2^1\right)\right)$. By weak concavity of u_2, $u_2\left(d\left(\lambda\right)\right) = \tilde{v}_2\left(v_1\right)$. Differentiating twice with respect to λ leads to

$$\begin{aligned} 0 &= h_2\left[d_2\left(\lambda\right)\right]\left(d_2^2 - d_2^1\right) - F\left[d_1\left(\lambda\right) - d_2\left(\lambda\right)\right]\left(d_1^2 - d_1^1\right) - \left(1 - F\left[d_1\left(\lambda\right) - d_2\left(\lambda\right)\right]\right)\left(d_2^2 - d_2^1\right) \\ 0 &= h_2'\left[d_2\left(\lambda\right)\right]\left(d_2^2 - d_2^1\right)^2 - f\left[d_1\left(\lambda\right) - d_2\left(\lambda\right)\right]\left(d_1^2 - d_1^1\right)\left(\left(d_1^2 - d_1^1\right) - \left(d_2^2 - d_2^1\right)\right)^2, \end{aligned}$$

which is a contradiction. ■

Proof of theorem 8. Denote $\alpha = \beta_1$ and let $\beta_2 = \alpha^{-1}$. Payoffs in the game with penalties are

$$\begin{aligned} \breve{u}_1\left[d_1, d_2\right] &= \int_0^{d_1} h_1\left[s\right] ds - w_1 E \max\left\{d_1 + T_1, d_2 + T_2\right\} - \alpha w_1 E \max\left\{A_1 - A_2, 0\right\} \\ &= \int_0^{d_1} h_1\left[s\right] ds - \left(1 + \alpha\right) w_1 E \max\left\{d_1 + T_1, d_2 + T_2\right\} + \alpha w_1\left(d_2 + ET_2\right) \end{aligned}$$

and

$$\breve{u}_2\left[d_1, d_2\right] = \int_0^{d_2} h_2\left[s\right] ds - \left(1 + \alpha^{-1}\right) w_2 E \max\left\{d_1 + T_1, d_2 + T_2\right\} + \alpha^{-1} w_2\left(d_1 + ET_1\right).$$

Therefore the first-order conditions for the Nash equilibrium (NE) are

$$\begin{cases} h_1\left[d_1\right] - \left(1 + \alpha\right) w_1 F\left[d_1 - d_2\right] = 0 \\ h_2\left[d_2\right] - \left(1 + \alpha^{-1}\right) w_2\left(1 - F\left[d_1 - d_2\right]\right) = 0 \end{cases}. \tag{16}$$

Since both left hand sides in (16) are decreasing, (16) is also a sufficient condition for Nash equilibrium. Define

$$G[S] = S - h_1^{-1}[(1+\alpha) w_1 F[S]] + h_2^{-1}\left[\left(1+\alpha^{-1}\right) w_2 (1 - F[S])\right], \quad (17)$$

noting that the inverses exist. Then note that (16) implies $G[S] = 0$. We have to show existence and uniqueness of solution this equation. First note that G has non-empty domain. Indeed, since F is continuous there exists S_0 such that $F[S_0] = (1+\alpha)^{-1}$ and this gives $G[S_0] = S_0 - h_1^{-1}[w_1] + h_2^{-1}[w_2]$ which is well-defined because h_1 and h_2 range across $\mathbb{R}^+$. Second, the domain of G is an open interval since the ranges of h_1 and h_2 are open intervals. Third, all three terms in (17) are non-decreasing. When S approaches the left end of the domain, there are two cases: either $h_1^{-1}[(1+\alpha) w_1 F[S]] \to \infty$ and $h_2^{-1}[(1+\alpha^{-1}) w_2 (1 - F[S])]$ does not increase, or $h_2^{-1}[(1+\alpha^{-1}) w_2 (1 - F[S])] \to -\infty$ and $h_1^{-1}[(1+\alpha) w_1 F[S]]$ does not decrease. In both cases we have $G[S] \to -\infty$. Similarly, $G[S] \to \infty$ when S approaches the right end of the domain. Since G is continuous and increasing it follows that equation (17) has a unique solution.

It is easy to check that any solution to (16) satisfies the first equation in (9). Since h_1 and h_2 are always positive, $0 < F[d_1 - d_2] < 1$, thus the remaining equations in (9) are satisfied as well and we have a Pareto optimum.

By (7), for any $0 < \lambda < 1$, the first-order conditions for maximization of $v[d_1, d_2] = \lambda u_1[d_1, d_2] + (1-\lambda) u_2[d_1, d_2]$ are

$$\begin{aligned} \lambda [h_1[d_1] - w_1 F[d_1 - d_2]] - (1-\lambda) w_2 F[d_1 - d_2] &= 0 \\ -\lambda [w_1 (1 - F[d_1 - d_2])] + (1-\lambda)[h_2[d_2] - w_2 F[d_1 - d_2] + 1] &= 0, \end{aligned}$$

which is the same as (16) with $\lambda = (1+\alpha)^{-1}$. By Lemma 2, the payoff u_2 is

non-decreasing and the payoff u_1 is non-increasing as functions of parameter α. It follows from the above that $S = d_1 - d_2$ is non-increasing in α since function G defined by (17) is non-increasing in α.

The paragraph preceding the theorem shows that Pareto optimum does not result if $\beta_1\beta_2 \neq 1$. ■

Lemma 2 *Let u_1 and u_2 be real valued functions and $d^*[\lambda]$ be a unique solution to $\max_d \{\lambda u_1[d] + (1-\lambda) u_2[d]\}$ for any $\lambda \in [0,1]$. Then $u_1[d^*[\lambda]]$ is non-decreasing and $u_2[d^*[\lambda]]$ is non-increasing in λ.*

Proof of lemma 2. Let $\lambda_1 < \lambda_2$. It follows from the conditions that

$$\begin{aligned}
\lambda_1 u_1[d^*[\lambda_1]] + (1-\lambda_1) u_2[d^*[\lambda_1]] &\geq \lambda_1 u_1[d^*[\lambda_2]] + (1-\lambda_1) u_2[d^*[\lambda_2]] \\
\lambda_2 u_1[d^*[\lambda_2]] + (1-\lambda_2) u_2[d^*[\lambda_2]] &\geq \lambda_2 u_1[d^*[\lambda_1]] + (1-\lambda_2) u_2[d^*[\lambda_1]]
\end{aligned}$$

Adding these inequalities and rearranging the terms gives

$$(\lambda_1 - \lambda_2)(u_1[d^*[\lambda_1]] - u_1[d^*[\lambda_2]]) \geq (\lambda_2 - \lambda_1)(u_2[d^*[\lambda_2]] - u_2[d^*[\lambda_1]]),$$

which implies

$$u_1[d^*[\lambda_2]] - u_1[d^*[\lambda_1]] \geq u_2[d^*[\lambda_2]] - u_2[d^*[\lambda_1]].$$

Assuming on the contrary that $0 > u_1[d^*[\lambda_2]] - u_1[d^*[\lambda_1]]$ yields also $0 > u_2[d^*[\lambda_2]] - u_2[d^*[\lambda_1]]$. Multiplying the two last inequalities by λ_2 and $(1-\lambda_2)$ respectively, adding them and rearranging the terms leads to

$$\lambda_2 u_1[d^*[\lambda_1]] + (1-\lambda_2) u_2[d^*[\lambda_1]] > \lambda_2 u_1[d^*[\lambda_2]] + (1-\lambda_2) u_2[d^*[\lambda_2]],$$

which is a contradiction. Hence u_1 is non-decreasing. The fact that u_2 is non-increasing can be proved in a similar way. ■

Proof of Theorem 9. Denote by $Z(d, \alpha)$ the vector of left-hand sides in equations (11), i.e. the partial derivatives of the modified payoffs in the game with compensations. At $\alpha = 1$, the Jacobian of Z with respect to d is $diag\left(h_1'[d_1], h_2'[d_2]\right)$, which is an invertible matrix since $h_i'[d_i] < 0$. By the implicit function theorem there is a unique solution $d^*[\alpha]$ to (11), continuous in α for α in a neighborhood of 1, which means that the first-order condition for Nash equilibrium is satisfied. It follows by continuity that the second-order condition remains satisfied for α near 1 such that $d^*[\alpha]$ is a Nash equilibrium. ■

Proof of Theorem 10. Define

$$G(S) = S - h_1^{-1}\left[\alpha w_1 + (1-\alpha) w_1 F[S]\right] + h_2^{-1}\left[\alpha^{-1} w_2 + \left(1-\alpha^{-1}\right) w_2 (1 - F(S))\right] \tag{18}$$

noting that the inverses exist for any real S due to the inequalities assumed in the theorem. Then note that (13) is equivalent to $G(S) = 0$. Since G is continuous, existence of a solution to this equation follows from the fact that

$$\begin{aligned} \lim_{S\to-\infty} G(S) &= -\infty - h_1^{-1}[\alpha w_1] + h_2^{-1}[w_2] < 0, \\ \lim_{S\to\infty} G(S) &= \infty - h_1^{-1}[w_1] + h_2^{-1}\left[\alpha^{-1} w_2\right] > 0. \end{aligned}$$

■

Proof of Theorem 11. Function arguments are omitted for brevity. Consider again the function G defined in (18) in the proof of Theorem 10. The equilibrium value of $S[\alpha] = d_1^*[\alpha] - d_2^*[\alpha]$ is defined implicitly by the equation $G(S) = 0$.

We differentiate this with respect to $\ln \alpha$ and aim to show that $S' < 0$. Rearrange slightly to find that

$$h_2' \alpha w_1 (1-F) + h_1' \alpha^{-1} w_2 F = \left(h_1' h_2' - h_2' (1-\alpha) w_1 f - h_1' \left(1-\alpha^{-1}\right) w_2 f \right) S'.$$

The LHS of this equation is negative for any value of α, so we just need to show that the parenthesis multiplying S' is positive. Suppose $\alpha \geq 1$ (the other case is treated symmetrically), then

$$\begin{aligned} h_1' h_2' - h_2' (1-\alpha) w_1 f - h_1' \left(1-\alpha^{-1}\right) w_2 f &= \\ h_2' \left(h_1' - (1-\alpha) w_1 f\right) - h_1' \left(1-\alpha^{-1}\right) w_2 f &> 0, \end{aligned}$$

where the last inequality follows from noting that the second-order condition for player 1 is that

$$h_1' - w_1 (1-\alpha) f \leq 0.$$

■

C Travel time variability illustration

This section presents some travel time data collected in Stockholm, where an automatic license plate recognition system is in operation. The system uses cameras installed on major arterial roadways throughout the Stockholm urban area. For each pair of cameras, matched travel time observations are aggregated into 15-minute averages for navigating any path between the two cameras. We identified nine pairs of such paths, such that in each pair, both segments carry traffic toward the city center and both share the same downstream location but have clearly dif-

ferent upstream locations.

We use data from the peak periods, 7:00 a.m. to 9:00 p.m., and 3:30 p.m. to 6:00 p.m., and we include only weekdays during the months of September and October, 2005 to 2007.

Table 1 shows the mean travel times (in seconds), standard deviations, and coefficients of variation (CV) for the two paths in each pair, followed by the correlation coefficient between the two paths. The standard deviation and the correlation are based on residual travel times computed by taking out the average for each 15 minute period. This takes into account systematic variation in travel time over the peak.

Table 1: Variations and Correlations in Travel Times for Selected Pairs of Convergent Paths, AM and PM Peak Periods

Case	Peak Period	μ_1 (min)	μ_2 (min)	σ_1 (min)	σ_2 (min)	CV_1 (%)	CV_2 (%)	ρ_{12}
A	a.m.	4.38	4.76	1.78	2.67	40.6	56.0	0.116
	p.m.	4.25	4.76	3.01	3.57	70.6	74.9	0.295
B	a.m.	2.95	6.03	1.24	0.92	42.0	15.2	-0.425
	p.m.	3.06	5.44	1.22	0.35	39.8	6.5	-0.133
C	a.m.	2.73	3.35	0.71	1.50	26.2	44.9	0.158
	p.m.	2.45	3.68	0.83	1.84	33.8	49.9	0.119
D	a.m.	6.82	7.49	1.45	1.64	21.3	21.9	0.165
	p.m.	5.49	4.81	1.17	0.50	21.3	10.3	0.067
E	a.m.	7.26	4.88	1.34	0.79	18.4	16.1	-0.122
	p.m.	8.86	6.29	2.71	1.15	30.6	18.3	0.121
F	a.m.	3.55	3.63	0.70	0.32	19.8	8.7	-0.123
	p.m.	5.48	3.66	2.21	0.58	40.4	15.7	-0.009
G	a.m.	9.41	3.65	0.63	0.37	6.6	10.0	0.107
	p.m.	9.56	3.76	0.56	0.64	5.9	17.0	0.055
H	a.m.	3.39	9.47	0.76	0.70	22.3	7.4	0.285
	p.m.	4.65	9.55	1.60	0.50	34.5	5.2	0.221
I	a.m.	5.56	6.45	1.27	1.13	22.9	17.5	0.430
	p.m.	6.21	8.79	1.51	2.43	25.0	27.7	0.086