Optimal Use of Put Options in a Stock Portfolio

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Abstract

In this paper I consider a portfolio optimization problem where an agent holds an endowment of stock and is allowed to buy some quantity of a put option on the stock. This basic question (how much insurance to buy?) has been addressed in insurance economics through the literature on rational insurance purchasing. However, in contrast to the rational purchasing literature that uses exact algebraic analysis with a binomial probability model of portfolio value, I use numerical techniques to explore this problem. Numerical techniques allow me to approximate continuous probability distributions for key variables. Using large sample, asymptotic analysis I identify the optimal quantity of put options for three types of preferences over the distribution of portfolio value. The location of the optimal quantity varies across preferences and provides examples of important concepts from the rational purchasing literature: coinsurance for log utility ($q^*<1$), full-insurance for quantile-based preferences ($q^*=1$), and over-insurance for mean-variance utility ($q^*>1$). I calculate the shape of the objective function and show the optimum is well defined for mean-variance utility and quantile-based preferences in an asymptotic setting. Using resampling, I show the optimal values are stable for the mean-variance utility and the quantile-based preferences but not the log utility. For the optimal value with mean-variance utility I show that the put option affects the probability distribution of portfolio value in an asymmetric way, which confirms that it is important to analyze the optimal use of derivatives in a continuous setting with numerical techniques.

Keywords: Portfolio, optimization, financial derivative, put option, quantity, expected utility, numerical analysis

Optimal Use of Put Options in a Stock Portfolio

This paper explores how to *use* financial derivatives in a simplified setting. I present a portfolio optimization problem where an agent who holds one unit of stock is allowed to buy a put option on the stock. What quantity of put options should they buy? I frame this research in context of a topic in insurance economics – rational choice of insurance coverage. The classic approach to rational insurance purchasing uses an exact, analytic solution to the portfolio optimization problem (Mossin, 1968). I update this approach with modern perspectives on financial derivatives and numerical analysis.

Literature Review

Jan Mossin is an influential scholar in economic theory of financial markets. His 1968 paper analyzes insurance from the buyer's perspective, which is referred to as rational insurance purchasing. The paper attempts to "illustrate the power of the expected utility approach to problems of risk taking" (1968, p. 553) by exploring the maximum premium an agent would pay to buy insurance, the optimal amount of insurance coverage at a given premium, and the optimal deductible amount. This second question, the optimal amount of insurance coverage, is the same topic that I address here – how best to use financial derivatives to mitigate risk of loss.

Mossin separates wealth into several different terms in (safe wealth, risky wealth, and size of loss). The separation imposes cumbersome notation, given in Equation (1), but it provides a parsimonious way to analyze how the optimal coverage changes with wealth (1968, p. 557). Furthermore, the model provides a way to move from assumptions about preferences to testable implications for behavior (1968, p. 564).

(1) $Y = A + L - X + (C/L)X - p\,C$

The key variable in Mossin's model is final wealth, defined in Equation (1) as follows: total wealth or portfolio value (Y), safe wealth (A), risky wealth (L), size of loss (X), price of insurance (p), and amount of coverage for loss (C). The loss (X) is the only random variable and it is non-negative ($X \geq 0$); therefore, the total portfolio value is composed of a random term ($A+L-X$) and the net payoff for a financial derivative ($(C/L)X - p\,C$). This separation of total value into two terms is a simple and powerful idea that I use in my optimization problem.

The objective in Mossin's model is to maximize expected utility over wealth (E(U(Y))). The agent's choice variable is the level of coverage (C), which is constrained because they cannot buy more coverage than the value of the asset ($0 \leq C \leq L$). This is where the similarities between Mossin and my model start to break down. For example, I do not constrain the quantity of put options that an agent buys the same as insurance. For another, Mossin solves the optimization problem algebraically. He shows that the first order conditions evaluated at the boundaries for the choice variable (C=0 or L) justify an interior solution (1968, p. 557), but he is not able to produce an exact formula for the solution in general.

To gain analytic traction, Mossin specifies a model that yields an analytic solution (log utility and binomial probability model for loss). He uses comparative statics to show how the optimal coverage changes with wealth, which provides deep insight into the propensity for an agent with risk-averse preferences to buy insurance (1968, p. 558). The binary model that Mossin uses is ubiquitous in insurance economics because it characterizes a situation where the agent suffers either no loss or the complete loss of an asset. The binomial model provides an exact solution, which is valuable in modern mathematical economics; however, the results are limited because they only consider losses of one size.

Mossin's 1968 paper was influential. It was extended by Razin (1976) to consider the minimax regret function from Leonard Savage's decision theory and again by Briys & Loubergé (1985) to consider bounded rationality, an important extension to classical rational choice theory. To recognize the value of this subsequent research, I use several types of preferences for the objective function. I even introduce an objective based on Value at Risk: the 5% quantile from the distribution of total portfolio value. Although this quantile is not a utility function, it is a useful criteria for risk management.

So far I have noted several features of Mossin's model that I intend to include in my own: the research question, the definition of portfolio value as a random term plus the net payoff of a derivative, and the expected utility optimization problem. However, I make two substantial changes. First, I use language associated with financial derivatives rather than insurance. This means I replace terms like deductible, coverage, and wealth with ones like strike price, quantity, and portfolio value. It also means that I drop the upper bound on the choice variable, as discussed above. The second difference is the analytic approach: I use numerical techniques, which allow me to approximate the continuous probability distributions of key variables with simulation and generate results that are not possible with algebraic analysis.

Model Setup

To develop my model, I briefly discuss assumptions about the agent and their portfolio. I assume the agent cares only about the portfolio value when the derivative expires. The portfolio value at expiry is random, but I assume the agent knows the probability distribution for the value. The agent knows with certainty how the derivative affects the distribution of portfolio value, hence they use their preferences over the distribution to rank different quantities of put options. This is classical decision theory – portfolio optimization with known probabilities.

I assume the portfolio is composed of one unit of stock and some quantity of European put options. The agent knows the initial value of the stock and the distribution of the future value. The put option is infinitely divisible, the strike price is equal to the initial stock price (at the money), and the stock expires in one time step (one year). Equation (2) represents the agent's portfolio optimization problem in this simple setting.

(2) $\max_{q>0} V(W(q)) \quad \text{s.t.} \quad W(q) = S + N(S,q)$

The choice variable in Equation (2) is the quantity of derivatives (q). The objective function is denoted as V(), which can be thought of as expected utility. For robustness, I use three different forms for V() that represent important preferences in the literature (expected log utility, mean-variance utility[1], and 5% quantile[2]). The value of the portfolio at expiry is denoted W(q), which is a random variable. The value of the stock at expiry is denoted S and net payoff for the derivative is N(S,q).

(3) $N(S,q) = q\,[\,(K-S)^{+} - O\,]$

Equation (3) defines the net payoff for a put option. The quantity (q) appears as a linear, multiplicative term. The term $(K-S)^{+}$ is the intrinsic value of the put at expiry, which is non-negative. As above, the strike (K) is equal to the stock price when the agent makes the initial

[1] I use a standard value for the risk aversion coefficient (λ=0.1) for the mean-variance utility ($U(X)=\mu-(\lambda/2)\sigma^2$).
[2] The 5% quantile is taken from the distribution of portfolio value (W(q)); it is *w* such that: $\Pr(W(q)<w)=0.05$. The quantile is not a utility function but it is useful for risk management. It is based on Value at Risk, which is the 5% quantile for the loss distribution. A larger VaR is bad, but a larger 5% quantile for portfolio is good.

decision for the quantity (q). I calculate the option price (O) using Black-Scholes formula because the stock price is log-normal. When the agent picks q, they do not know the value for N(S,q) because it depends on the future value of the stock (S), which is random. However, the agent knows the distribution of S and uses it to calculate distribution of portfolio value (W(q)) for any value of q.

Numerical Analysis

For readers interested in the specifics of my statistical model, I provide an appendix that contains code for Matlab to reproduce all of my results. I assume the initial price of the stock is $100 and the returns are normally distributed with 0% average and 10% volatility for one time step. This model for asset returns satisfies the random walk hypothesis and provides a basis to price the put option with Black-Scholes.

Asymptotic Analysis

For each value of q in an interval ([0.00,2.00] with step size 0.01), I simulate the stock price a large number of times (1,000,000) to estimate the distribution of portfolio value for that quantity. I calculate utility over the distribution and analyze it in several different ways. To begin, I report the optimal quantity (q*) for each type of preferences in Table 1.

Table 1: Optimal values for choice variable across Preferences

Preferences	Optimal Quantity (q*)
Log Utility	0.63
Mean-Variance Utility	1.57
5% Quantile	1.00

Table 1 shows that the optimal quantity depends on preferences. The optimal quantity for the 5% quantile preferences is noticeable (q*=1.00) because it is a very specific value with important economic meaning – it is akin to full insurance in the rational purchasing literature. The standard question for insurance has been whether it is better to have full insurance (q*=1) or coinsurance (q*<1) and the results in Table 1 show that either choice is optimal under different preferences in my model with financial derivatives. However, Table 1 shows that the optimal quantity for mean-variance utility is well above one (q*>1), which is inadmissible in the classical model of rational insurance purchasing. Therefore, the results in Table 1 suggest that my modelling framework can generate results that match and extend the classic results from a rational insurance purchasing model.

Now that I have identified the optimal choice across different preferences, I will briefly characterize the character of each optimum. I do this by estimating the shape of the objective function over a range of values for the choice variable. This is straightforward because the objective function and choice variable are each 1-dimensional in this case. In Table 2 I report a money-metric associated with the expected utility for each value of the choice variable and utility measure. For the log and mean-variance utility, this money-metric is the certainty equivalent. For the 5% quantile objective function, the money-metric is the 5% quantile from the distribution of portfolio value. In each case, a higher value is preferred to a lower value.

Table 2: Shape of objective function for plausible values of choice variable

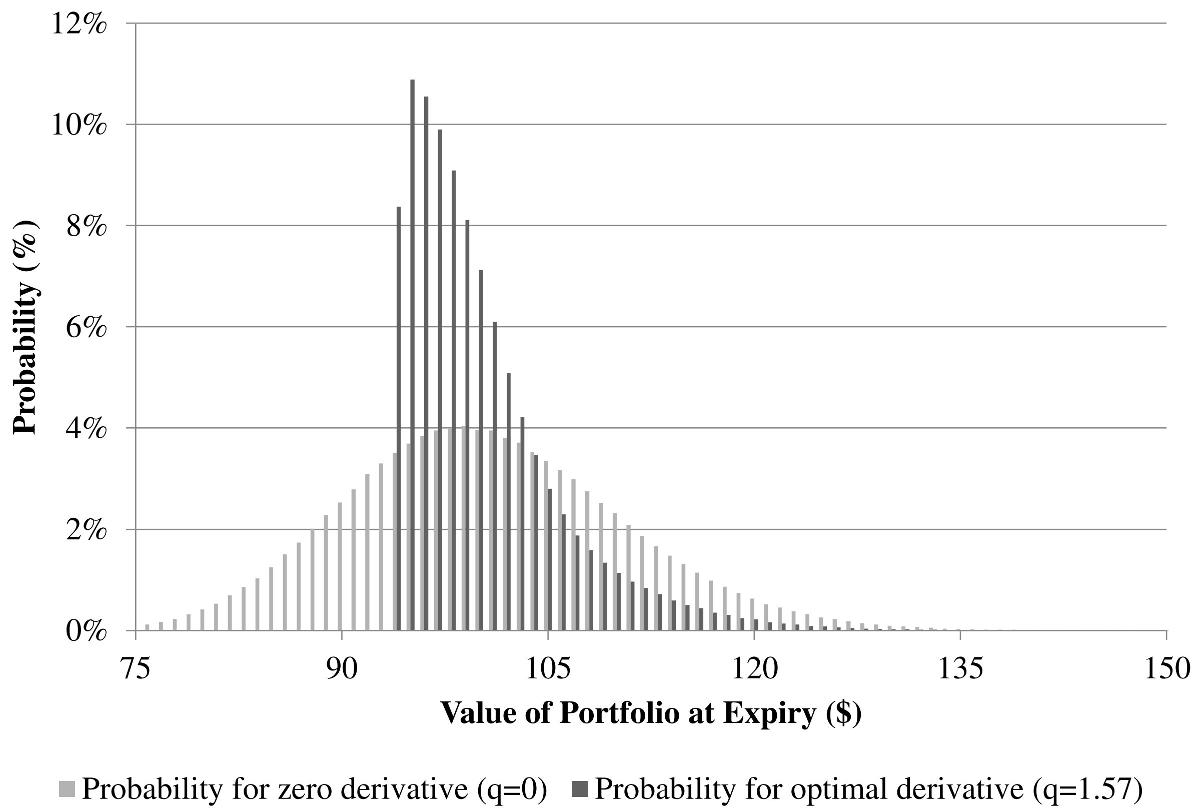
	0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
Log Utility	100	100	100	***100***	***100***	***100***	100	100	100	100	100
Mean-Variance Utility	50	58	65	71	76	80	82	***84***	***84***	***83***	81
5% Quantile	85	87	89	92	***94***	***96***	***95***	95	94	93	93

I use Table 2 to assess whether the optimal quantity is well defined. For each type of preferences, I highlight the location of the optimum in ***bold and italics*** in order to show that the objective function is concave around the optimum. I use sparse sampling points for the choice variable to give a basic sense of the shape. Table 2 shows that the optimal quantity for the mean-variance utility and 5% quantile objective functions are both well-defined because the objective function is concave around the optimum. In contrast, the results for the log utility raise concerns because there is little variation in the agent's valuation of different portfolios (always equal to $100). The results suggest that optimal quantity for log utility may not be robust to sample selection. Although Table 2 uses a very low sampling rate, it gives confidence in the mean-variance utility and the 5% quantile preferences, if not the log utility. It is possible to enhance the results by using higher sampling rates.

When an agent with mean-variance utility holds zero put options (q=0), Table 2 shows that they would trade the portfolio for $50. Please note that the initial value of the stock is $100. The large difference between valuations speaks to the negative affect of risk on a risk-averse agent. If the same agent is allowed to buy a close approximation to their optimal quantity of put options (q*=1.6), then they would be much better off and would trade the portfolio for $84. This is a significant increase that shows the value of risk management in a basic portfolio context.

To follow with this sort of comparison, I show how a derivatives changes the distribution of portfolio values in Figure 1. The figure shows the probability distribution for portfolio value with zero put options (q=0) and the optimal quantity of put options (q*=1.57 for mean-variance utility). The figure makes clear that derivatives can have a complex effect on the shape of the distribution, which confirms that it is important to investigate these decisions in a continuous setting with numerical techniques.

Figure 1: Histogram for No Put and Optimal Put



I will point out two important features in Figure 1. First, the probability of low values for the portfolio is less when the agent buys put options; this is because the put option is designed to offset losses. Notice how the value of the portfolio with derivatives is never less than $90. Second, the probability of high values for the portfolio is also less when the agent buys put options; this is because the agent has to pay for the put options in the good times. Notice the large difference between the distributions for wealth above $105. These two features show that the cumulative effect of a derivative reduces the frequency and severity of both high and low values for the portfolio, which reduces the variance of the portfolio value at both ends.

Robustness to Small Sample

The results so far use large sample sizes to estimate continuous distributions. However, these large samples may hide variability in the results. To reveal this variability I conduct further simulations with resampling. Basically, I repeat the analysis presented above to estimate the distribution for the optimal value. I use a smaller sample size (1,000 draws of S) to calculate the optimum and then loop the calculation many times (10,000 estimates of q*). I present the distribution of the optimal values in Table 3.

Table 3: Distribution of location of optimal value (q*) across preferences

	0-0.2	0.2-0.4	0.4-0.6	0.6-0.8	0.8-1.0	1.0-1.2	1.2-1.4	1.4-1.6	1.6-1.8	1.8-2.0
Log-Utility	18%	29%	22%	15%	9%	4%	2%	1%	0%	0%
Mean-Variance Utility	0%	0%	0%	0%	0%	0%	3%	43%	49%	5%
5% Quantile	0%	0%	0%	0%	0%	100%	0%	0%	0%	0%

Table 3 shows the distribution of the optimal quantity across each objective function. The columns in Table 3 are closed on the lower side and open on the upper side (0-0.2 denotes the interval [0,0.2]). Table 3 shows that the distribution of the optimal value varies greatly across preferences. Again, the results for the 5% quantile are noticeable because the optimum is always located at the special point of full insurance. The results for the mean-variance utility are encouraging because they have a symmetric, narrow distribution around the optimal value. The optimal choices under the quantile and mean-variance preferences are, in some sense, well behaved. In contrast, the optimal values for log utility are highly dispersed; thus, the optimal quantity for log utility is not robust to sample selection. This echoes the potential problems noted in Table 2, where little variation in valuation across portfolios under log utility raised potential risks about the nature of the optimal quantity. Please note, however, that the optimal quantity for log utility is lower than 1.0, which means coinsurance is still optimal – as was argued in the classic literature (Mossin, 1968, p. 558).

Conclusion

This paper analyzes the portfolio optimization problem where an agent holds one unit of stock and is allowed to buy put options on the stock. This is a specific example of the general situation where an agent endowed with an asset is allowed to trade derivatives on the asset. I position this research in relation to the question of optimal coverage in the literature on rational insurance purchasing. The rational insurance purchasing literature uses algebraic analysis to identify an exact solution under a binomial probability model for asset values (Mossin, 1968). In contrast, I use numerical analysis to approximate a solution under a log-normal probability model. This different analytic perspective allows me to develop richer insight into the optimization problem by approximating continuous variables of interest.

The results show that the nature of the optimal quantity of put options depends on the agent's preferences over the distribution of portfolio values. For log utility, the optimum quantity is not stable to resampling – it takes values in a wide range from 0.0 to 0.8. In contrast, the optimum quantity for the quantile-based preferences is always equal to 1.0 – a striking, knife-edge result. I discuss how each of these optimal quantities correspond to an important concept in the classical insurance models (coinsurance, full insurance).

The rational insurance purchasing literature explicitly disallows over-insurance (Mossin, 1968, p. 557). In contrast, my results show that over-insurance is the optimal choice for the mean-variance utility function. This means over-insurance is optimal for some risk-averse agents. To further analyze this result, I compare the probability distribution of portfolio value for zero insurance against over-insurance. Over-insurance decreases the severity and frequency of low values for the portfolio because the agent receives the option payoff in bad times, and decreases the severity and frequency of high values for the portfolio because the agent pays the option premium in good times. Thus, over-insurance reduces the variance of the portfolio value, which is desirable for an agent with mean-variance utility. The results show that put options have an asymmetric effect on the distribution of wealth and it is important to analyze these effects with numerical techniques that can approximate the continuous distribution of wealth.

To conclude I briefly discuss limitations of the research. By using numerical techniques, I have picked arbitrary values for many parameters and it is possible that different values for parameters may change the qualitative nature of the results. Interested readers could investigate the parameters in the stock price, the level of the risk aversion, or the percentage in the quantile-based preferences. Another limitation is my simple assumptions about the agent's portfolio – it is possible that different values for the strike price of the option or the timing of cash flows could change the results further. The model also takes a simple view of randomness. I use known randomness, not Knightian uncertainty, and it may be possible to extend the analysis to a Bayesian setting with subjective beliefs about probability distributions and preferences. This may translate nicely into experiments with human subjects. I would also like to recognize Ole Peters work (2011) that addresses possible misconceptions at the heart of expected utility: all of my analysis has used ensemble averages and the results may be very different with time averages.

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Appendix

```
%%  Code Appendix -- Optimal Use of Derivatives
%   © Peter Bell, March 10 2014
%   Written for Matlab to produce all results used in working paper.
%
%%  Section 1:  Global Parameters
%   Set random number generator
clear all
stream = RandStream('mt19937ar','Seed',12);
RandStream.setDefaultStream(stream);

%   Simulate price for stock
numPrice = 10^6;    S0 = 100;   sigma=0.1;

%   Simulate option price
K = 100;    r = 0;
d1 = (1/sigma)*(log(S0/K)+r+sigma^2/2);     d2 = d1 - sigma;
O = cdf('norm',-d2,0,1)*K - cdf('norm',-d1,0,1)*S0;

%   Agent Utility
lambda = 0.5; %

%   Simulations in Section 3 for Asymptotic Setting
numSimOne = 201;  qScaleOne=100;  qStep=1/qScaleOne;
resultTable = zeros(3,numSimOne);
%   numSimOne represents # points for choice variable
%   qScaleOne is parameter to make so that consider q in [0,2]
%   Each simulation has length numPrice (10^6), specified above


%   Simulations in Section 4 for small samples
numSimTwo= 10^4;    numPriceSmall = 1000;
%   numSimTwo represents # of times that identify optimal quantity (q*)
%   numPriceSmall is length of time series, which replaces numPrice


%%  Section 2:  Demo with single value for choice variable
%   Goal: demonstrate how particular quantity affects utility
q = 0.5;
S = S0*exp(randn(numPrice,1)*sigma);
W = S + q*(max(K-S,0)-O);

%   Log Utility
exp(mean(log(S)))
exp(mean(log(W)))

%   Mean-Variance Utility
mean(S) - lambda*var(S)
mean(W) - lambda*var(W)

%   5% Quantile for distribution
temp1 = sort(S);
```

```
temp1(length(temp1)*5/100)
temp2 = sort(W);
temp2(length(temp1)*5/100)


%%  Section 3:  Analyze shape of objective function in asymptotic setting
%   Goal: Calculate material for Table 1, 2, and Figure 1.
%
for numChoice = 1:numSimOne
    qLoop = (numChoice-1)/qScaleOne
    %   Simulate large number of prices for each q, calculate objective
    S = zeros(1,1);     W = zeros(1,1);
    S = S0*exp(randn(numPrice,1)*sigma);
    W = S + qLoop*(max(K-S,0)-O);

    %   Log Utility
    resultTable(1,numChoice) = exp(mean(log(W)));

    %   Mean-Variance Utility
    resultTable(2,numChoice) = mean(W) - lambda*var(W);

    %   5% Quantile for distribution
    temp2 = sort(W);
    resultTable(3,numChoice) = temp2(length(temp2)*5/100);
    resultTable(4,numChoice) = qLoop;
end

%   Table 1:
qTemp = 1:20:220;
tableOne = [(qTemp-1)*qStep;resultTable(1:3, qTemp)];

%   Table 2:    Optimal Choice by Utility
[uMaxLog iMaxLog] = max(resultTable(1,:));
[uMaxMeanVar iMaxMeanVar] = max(resultTable(2,:));
[uMaxQuantile iMaxQuantile] = max(resultTable(3,:));

qStarLog = (iMaxLog-1)*qStep;
qStarMeanVar = (iMaxMeanVar-1)*qStep;
qStarQuantile = (iMaxQuantile-1)*qStep;

tableTwo = [qStarLog qStarMeanVar qStarQuantile ]

%   Figure 1:    Calculate histogram for wealth with optimal derivative
WStarLog = S + qStarLog*(max(K-S,0)-O);
WStarMeanVar = S + qStarMeanVar*(max(K-S,0)-O);

histIndex = 75:1:150;
[nZeroPut xOutOne] = hist(S, histIndex);
[nOptimalPut xOutTwo] = hist(WStarMeanVar, histIndex);

figureOne = [xOutOne' (nZeroPut./numPrice)' (nOptimalPut./numPrice)'];


%%  Section 4:  Robustness of results to resampling with small samples
```

```
%   Goal: Build Table 3 in paper (histogram of q* for each utility)
%
for simCount = 1:numSimTwo
    simCount
    resultTable = zeros(3,numSimOne);
    for numChoice = 1:numSimOne
        qLoop = (numChoice-1)/qScaleOne;
        S = S0*exp(randn(numPriceSmall,1)*sigma);
        W = S + qLoop*(max(K-S,0)-O);

        %   Log Utility
        resultTable(1,numChoice) = exp(mean(log(W)));
        %   Mean-Variance Utility
        resultTable(2,numChoice) = mean(W) - lambda*var(W);
        %   5% Quantile for distribution
        temp2 = sort(W);
        resultTable(3,numChoice) = temp2(length(temp2)*5/100);
    end

    %   Optimal Choice by Utility
    [uMaxLog iMaxLog] = max(resultTable(1,:));
    [uMaxMeanVar iMaxMeanVar] = max(resultTable(2,:));
    [uMaxQuantile iMaxQuantile] = max(resultTable(3,:));

    %   Collect optimal choice (q*) for each run in loop
    qStarLoop(simCount,1) = (iMaxLog-1)*qStep;
    qStarLoop(simCount,2) = (iMaxMeanVar-1)*qStep;
    qStarLoop(simCount,3) = (iMaxQuantile-1)*qStep;
end


%   Calculate histogram for optimal choice q* across resampling
histIndexTwo = 0:0.2:2;
[qStarHistLog xOut] = hist(qStarLoop(:,1), histIndexTwo);
[qStarHistMeanVar xOut] = hist(qStarLoop(:,2), histIndexTwo);
[qStarHistQuantile xOut] = hist(qStarLoop(:,3), histIndexTwo);

tableThree = [(qStarHistLog./numSimTwo); ...
   (qStarHistMeanVar./numSimTwo); (qStarHistQuantile./numSimTwo)];

%%
```