On Linear Quadratic Approximations

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Abstract

We prove the generality of the methodology proposed in Benigno and Woodford (2006). We show that, even in the presence of a distorted steady state, it is always possible and relatively simple to obtain a purely quadratic approximation to the welfare measure. We also show that, in order to do so, the timeless perspective assumption is crucial.

JEL *classification*: C61, C63

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1 Introduction

It has been shown that the standard linear-quadratic approach does not usually lead to an accurate approximation of the optimal policy in the presence of a distorted non-stochastic steady state. This happens, for example, whenever the non-stochastic steady state is not efficient, due to the presence of some distortions like market power or distortionary taxes.[1]

Benigno and Woodford (2006), among others, show that a correct linear-quadratic approximation to an optimal policy problem is still possible even in the case of a distorted steady state, if the quadratic objective is chosen adequately. In this respect, the crucial step is to eliminate the linear terms in the second order approximation to the welfare measure, to obtain a purely quadratic expression. This is done through an appropriate linear combination of a second order approximation to the model structural equations. Such derivations are usually cumbersome and time consuming. Moreover, it has not been shown that such methodology is general.

This paper makes a contribution by proving the generality of the method proposed in Benigno and Woodford (2006). We indeed show that it is always possible to eliminate the linear terms in the second order approximation to the welfare measure. Moreover, we show that this procedure is quite simple, from an analytical point of view, and as tractable as the standard linear-quadratic approach (i.e. the non-distorted case). Finally, with our insight, it is possible to apply this methodology to a broad class of models through the use of standard numerical packages. In an independent work, Altissimo et al. (2005) propose a toolkit to solve linear-quadratic problems of the type proposed in Benigno and Woodford (2006).

All the papers that have followed this methodological approach have also made use of the *timeless perspective* concept. We show that, in the case of a distorted steady state, the *timeless perspective* assumption is crucial to perform such linear-quadratic approximations, since it allows to obtain an approximation to the welfare measure that is purely quadratic.

2 A Purely Quadratic Approximation

Consider a general problem with the following form

[1]For a discussion on this topic see e.g. Kim and Kim (2003) and Woodford (2003, Chapter 6).

$$\max_{\{x_t\}} E_0 \sum_{t=0}^{\infty} \beta^t u(x_{t-1}, x_t, s_t) \tag{1}$$
$$s.t. : E_t g(x_{t-1}, x_t, x_{t+1}, s_t, s_{t+1}) = 0, \ \ t \geq 0$$

where s_t is exogenously given and follows a Markov process, u is a standard utility function, g is a vector of constraints and E_t denotes rational expectations conditional on information available at time t. For our purposes, there is no need to distinguish among control and state variables. The problem is in a form such that all the first order conditions (FOCs) are taken with respect to the vector of variables x_t. Denoting the vector of lagrange multipliers associated with the constraints at time t by λ_t the FOCs for $t > 0$ can be written as

$$u_{2,t} + \beta u_{1,t+1} + \beta \lambda_{t+1} g_{1,t+1} + \lambda_t g_{2,t} + \beta^{-1} \lambda_{t-1} g_{3,t-1} = 0 \tag{2}$$
$$E_t g(x_{t-1}, x_t, x_{t+1}, s_t, s_{t+1}) = 0 \tag{3}$$

where we are using the short notation $u_{2,t}$ to denote the first derivative of the function $u(x_{t-1}, x_t, s_t)$ with respect to the second argument. Imposing a steady state relationship in the FOCs for $t > 0$ we obtain

$$u_2 + \beta u_1 + \beta \lambda g_1 + \lambda g_2 + \beta^{-1} \lambda g_3 = 0 \tag{4}$$
$$g = 0 \tag{5}$$

where all the functions are evaluated at steady state.

In the standard linear-quadratic approach, the original problem is transformed such that the objective function is a local quadratic approximation to the exact objective and the constraints are local linear approximations to the exact constraints. The second order Taylor approximation to utility around the steady state found above is

$$\sum_{t=0}^{\infty} \beta^t u(x_{t-1}, x_t, s_t) \simeq \sum_{t=0}^{\infty} \beta^t (u_2(x_t - x) + \beta u_1(x_t - x)) + s.o.t. + t.i.p. \tag{6}$$

where $s.o.t.$ and $t.i.p.$ denote second order terms and terms independent of policy respectively.[2] Here we do not explicitly derive the $s.o.t.$, as they do not matter for our considerations.

[2]The term x_{-1} is also $t.i.p.$

As it is known, in the case of a distorted steady state, the standard linear-quadratic approach would lead to incorrect conclusions. The way proposed by Benigno and Woodford (2006) to overcome this problem is to substitute for the linear terms in (6), thus obtaining a purely quadratic expression. These derivations are sometimes hard to follow and cumbersome. In what follows we show that this procedure can be applied to a broad class of model and that the necessary calculations are relatively inexpensive from an analytical point of view.

Consider that the constraint in Eq. (3) is also present at $t = -1$. After writing a second order Taylor approximation to the constraints and summing over time, we obtain

$$\sum_{t=0}^{\infty} \beta^t g(x_{t-1}, x_t, x_{t+1}, s_t, s_{t+1}) \simeq \sum_{t=0}^{\infty} \beta^t (\beta g_1(x_t - x) + g_2(x_t - x) + \beta^{-1} g_3(x_t - x)) + s.o.t. + t.i.p. \tag{7}$$

Now we can multiply Eq. (7) by λ and sum the resulting expression with Eq. (6).[3] It is easy to see that all the linear terms drop out due to the steady state relationship described in Eq. (4).

This technique shows that one can always eliminate the linear terms from a second order approximation to the welfare measure. This result holds no matter what is the number of linear terms to eliminate and the number of constraints in the problem.[4] Indeed, it suffices to take a linear combination of the second order approximation to the constraints, using as weights the steady state values of the lagrange multipliers associated with the constraints, and sum the resulting expression to the approximated welfare measure.

Such calculations can be performed with relatively low effort. This implies that, from an analytical point of view, the case of a distorted steady state is as tractable as the standard linear-quadratic approximation. Moreover, our result implies that these substitutions can be performed automatically with a mathematical package. The resulting formulation of the problem can then be solved and simulated with any standard toolkit.

[3]Eq. (7) equals zero (up to third order terms). Therefore, we can sum this expression to Eq. (6) without altering the approximated welfare measure.

[4]In some cases it might seem fortuitous that it is possible to eliminate n linear terms with a linear combination of less than n equations. On the contrary, this is a direct consequence of the optimality conditions at the non-stochastic steady state.

We would like to emphasize some key features of this procedure. Note that the FOCs w.r.t. x_0 of the original problem are different. As a consequence, it would not be possible to eliminate the linear terms in x_0. However, this is possible if one considers the problem only for $t > 0$ or if one assumes that the constraints were present also at $t = -1$. In other words, it is crucial to impose a proper value of initial commitments.[5] Therefore, we can conclude that the *timeless perspective* assumption, beside other considerations, is crucial to obtain an approximation to the welfare measure that is purely quadratic.

3 Conclusions

We have shown that the linear quadratic approach in the case of a distorted steady state proposed in Benigno and Woodford (2006) can be applied to any model. To do so, it is crucial to consider the policy problem from a *timeless perspective*. We have shown that the procedure to eliminate the linear terms in the second order approximation to the welfare measure is relatively inexpensive from an analytical point of view. Solving economic models with this methodology can be done through slight extensions to available numerical packages.

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[5] Marcet and Marimon (1998) describe that past commitments are summarized in the lagrange multipliers associated with forward looking constraints. If one considers the original problem, where at $t = 0$ no commitments have been made, then the lagrange multipliers should be initialized at the value of zero.

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