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# Does Human Capital Risk Explain The Value Premium Puzzle?\*

Serginio Sylvain<sup>§</sup>

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## Abstract

Using a general equilibrium model with endogenous growth, I show that risk to human capital leads to a “Value” premium in equity returns. In particular, firms with relatively more firm-specific human capital or more positive covariance between asset growth and returns on human capital are less valuable (and hence have greater Book-to-Market Equity) and yield greater expected equity returns since human capital is more tied to the fate of said firms. Thus, I reproduce some of the results of Fama and French (1996) and show that in the model their HmL factor is a proxy for human capital risk as measured by macroeconomic and financial variables such as the covariance between human capital growth, or labor income growth, with the growth rate of firm assets. The model implies relatively lower investment-to-asset ratio and lower average asset growth for Value firms as observed in data and as argued in Zhang (2005). Furthermore, the model yields counter-cyclical Value premium and relative Book-to-Market Equity, greater long-run risk exposure for Value firms, and failure of the CAPM. Hence, it replicates several results from the related literature.

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# 1 Introduction

“Why is relative distress a state variable of special hedging concern to investors? One possible explanation is linked to human capital...” – Fama and French (1996)

I investigate the theoretical implications of investments in human capital for asset pricing. In particular, I investigate the possibility for risk in human capital to explain the Value premium. There is an extensive literature which documents the Value premium, an empirical feature of equity returns where securities with greater Book-to-Market ratio (BE/ME)<sup>1</sup> have greater average return. This excess return is not explained by standard measures of risk<sup>2</sup>. Fama and French (1992) were among the first to document the Value premium puzzle in the US stock markets. Fama and French (1996, 1997, 1998), Jagannathan, Kubota, and Takehara (1998), Davis, Fama, and French (1999), Rouwenhorst (1999) subsequently showed that the Value premium is a pervasive feature of equity returns in both US and international stock markets. Currently, the Value premium is simply considered a feature of the data which one needs to take into account in asset pricing<sup>3</sup>. There is an extensive literature that shows an empirical link between the Value premium and risk to human capital as measured by variability in labor income. Indeed, Jagannathan and Wang (1996), Jagannathan et al. (1998), Hansson (2004), and Santos and Veronesi (2005) show that variability in labor income explains part of the Value premium.

However, there are few theoretical models in which the Value premium arises endogenously. Santos and Veronesi (2005) show that variability and predictability of labor income leads to a value premium. But, the process for the labor income is exogenous. Bansal, Dittmar and Lundblad (2005), Kiku (2006) and Hansen, Heaton and Li (2008) show that if some securities are relatively more exposed to long-run risk (low-frequency fluctuations in aggregate consumption), then these securities returns will carry a Value premium. However, it is not clear why said securities are more exposed to long-run risk. In Zhang (2005), the author produces a model which replicates the Value premium but it relies on asymmetric adjustment costs and an exogenously specified countercyclical price of risk. Garleanu, Kogan and Panageas (2012) obtain a Value premium by assuming that Growth firms (firms with low BE/ME) have relatively more blueprints to produce intermediary goods (and are hence more innovative) than Value firms (firms with high BE/ME). On the other hand, there is already a well-established literature showing the importance of human capital for portfolio theory and asset pricing. In addition to some of the works already mentioned, this literature

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<sup>1</sup>The ratio of a firm’s total assets minus liabilities and total value of its publicly traded common equity. More formally, the Value premium is the difference in average returns between Value (high BE/ME) firms and Growth (low BE/ME) firms.

<sup>2</sup>Such as the “beta” coefficient from a regression of returns against the market returns (the value-weighted average returns on traded securities).

<sup>3</sup>For a example by including the Fama and French HmL factor (the average return of securities in the top 70th percentile of BE/ME minus that of securities from the bottom 30th percentile) in regressions. But, ignoring the reason why the Value premium arises can lead to identification strategies that are not valid.

includes Mayers (1972, 1973), Fama and Schwert (1977a), Landskroner (1977), Liberman (1980), Bodie, Merton, and Samuelson (1992), Campbell (1996), Glascock and Hung (2003), Palacios-Huerta (2003a, 2003b), and Lustig and Van Nieuwerburgh (2006). Thus, human capital risk may indeed explain the Value premium as Fama and French (1996) argue.

“[A] negative shock to a distressed firm more likely implies a negative shock to the value of specialized human capital... Thus, workers [may] avoid the stocks of [all] distressed firms.”– Fama and French (1996)

The authors interpret the BE/ME as a measure of firm relative distress. Human capital is of particular interest because agents provide it to firms as an input to their production process in return for a wage. However, the firms do not own the agent’s human capital; hence the value of a firm consists (mostly) of the value of its physical capital (and perhaps some intangible capital). Although the firm does not own the agent’s human capital, if the stock of human capital (or alternatively labor income and labor/leisure choice) is a state variable of concern to agents in the economy and there is significant co-variation between the shocks to the human capital and the shocks to firm physical capital (and hence the firm value and its security returns), in equilibrium human capital will have important effects on expected returns of claims to the firm’s profits. This is the essence of the explanation in the above quotes from Fama and French (1996).

I construct a general equilibrium model with endogenous growth and investments in both human and physical capital using elements from the recent literature on asset pricing in a production economy. This literature includes Jermann (1998), Hall (2000), Kogan (2001, 2003), Jones and Manuelli (2004), Uhlig (2007), Cochrane, Longstaff and Santa-Clara (2008), Eberly and Wang (2009, 2011), Brunnermeier and Sannikov (2011), and He and Krishnamurthy (2012). I find that risk to human capital does give rise to a Value premium endogenously; that is the firms with more human capital risk tend to have greater average equity returns and those greater returns are not explained by risk measures which do not take human capital into account. Thus, the CAPM<sup>4</sup> fails when we do not adjust for human capital returns. Furthermore, Value firms tend to have lower investment-to-asset ratio and lower asset growth than Growth firms. Lastly, the Value firms’ equity returns are, in the long run, more sensitive to changes in risk exposures as argued in the long-run risk models. I begin by describing the model in Section 2. Then, I discuss the asset pricing implications in Section 3, and the numerical results in Section 4. I explore the model’s testable implications and provide some empirical evidence in Section 5. Some of the model’s implications are contrary to those from

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<sup>4</sup>The Capital Asset Pricing Model (CAPM) is a popular asset pricing model introduced by Sharpe (1964), Lintner (1965) and Black (1972). According to the CAPM, (1) the expected return on a security is linear in the security’s beta (the slope coefficient from a regression) with the wealth portfolio, (2) the risk of the security is entirely reflected in the security’s beta with the wealth portfolio. I define the conditional CAPM as done in Jaganathan and Wang (1996). The conditional CAPM allows for time-varying betas. It implies that expected returns are linear in their conditional betas with the wealth portfolio and that their risk is measured by the aforementioned conditional betas.

Donangelo, Eiling and Palacios (2010), Belo and Lin (2012), and Donangelo (2013). I highlight these differences in Section 5 as well. I conclude in Section 6.

## 2 Model

I consider an economy populated with a continuum of identical agents with unit mass. At date zero, each agent is endowed with human capital,  $H_0$ . There are two types of firms,  $\{A, B\}$ , each with a continuum of firms with unit mass. Firms are endowed with physical capital  $K_0^i$  for  $i \in \{A, B\}$ . The agents have a recursive utility preference over consumption. In particular, I use the continuous-time formulation from Duffie and Epstein (1992a and 1992b) for the recursive preferences from Kreps and Porteus (1978), Epstein and Zin (1989), and Weil (1989). Each agent maximizes her value function

$$V_t = E_t \left( \int_t^\infty f \{C_s, V_s\} ds \right)$$

where  $C_t$  is the agent's consumption at date  $t$  and  $f \{C_t, V_t\}$  is the normalized aggregator function defined as

$$f \{C_t, V_t\} = \frac{\beta}{\rho} \left( \frac{C_t^\rho}{((1-\gamma)V_t)^{\frac{\rho}{1-\gamma}-1}} - (1-\gamma)V_t \right)$$

$\beta$  is the subjective discount factor,  $\gamma$  is the parameter which governs the agents' risk aversion (RA). The inter-temporal elasticity of substitution (IES) is associated with  $(1-\rho)^{-1}$ , where  $\rho < 1$  so that the IES ranges from zero to infinity. I consider three cases to verify that the results are robust, highlight the importance of human capital for asset pricing and explore the implications of recursive utility:

1.  $\gamma = (1-\rho) = 1$ : the normalized aggregator becomes

$$f \{C, V\} = \beta \ln(C) - \beta V$$

thus preferences collapse to the simple case of log-utility

$$V_t = E_t \left( \int_t^\infty e^{-\beta(s-t)} \beta \ln(C_s) ds \right)$$

2.  $\gamma \neq (1-\rho) = 1$ : the normalized aggregator becomes

$$f \{C, V\} = \beta(1-\gamma)V \left( \ln(C) - \frac{1}{1-\gamma} \ln((1-\gamma)V) \right)$$

3.  $\gamma > (1 - \rho) \neq 1$ : the agent has a preference for earlier resolution of uncertainty. I use this specification for all numerical results unless I say otherwise.

I derive the above implications in Section 7.1 of the Appendix. Firms of Type  $A$  use physical capital of Type  $A$  as well as human capital in their production process. For now, I will assume that the agent's human capital is productive only in firms of Type  $A$  and is thus specific to Type  $A$  firms. There is a fair amount of evidence that human capital can be tied to (or specific to) firms<sup>5</sup>. In such a case, an agent's human capital receives a wage in a particular firm or industry and cannot be readily transferred to another firm (or sector) because it is much less productive (if at all) in other sectors. An example for this may include low-skilled labor in the manufacturing sector or assembly lines in the automobile industry<sup>6</sup>.

Throughout the paper, I use the superscript  $i \in \{A, B\}$  to denote the firm type and the subscript  $j \in J$  to denote individual agents or firms; where  $J = [0, 1]$  is the set of agents or firms of each type. Type  $A$  firms output is linear<sup>7</sup>

$$Y_{j,t}^A = \mathcal{A}(K_{j,t}^A + H_{j,t})$$

$\mathcal{A}$  is the constant marginal product of capital. Firms of Type  $B$  use only physical capital,  $K^B$ , in their production process

$$Y_{j,t}^B = \mathcal{A}K_{j,t}^B$$

Aggregate production in this economy is simply

$$Y_t = \int_J (Y_{j,t}^A + Y_{j,t}^B) dj = Y_t^A + Y_t^B$$

The human and physical capital follow

$$\frac{dH_{j,t}}{H_{j,t}} = \Gamma \ln \left( 1 + \frac{I_{j,t}^H}{\theta H_{j,t}} \right) dt - \delta dt + \sigma_h dZ_t^A \quad (2.1)$$

$$\frac{dK_{j,t}^i}{K_{j,t}^i} = \Gamma \ln \left( 1 + \frac{I_{j,t}^i}{\theta K_{j,t}^i} \right) dt - \delta dt + \sigma dZ_t^i \quad \text{for } i \in \{A, B\} \quad (2.2)$$

Here,  $I_{j,t}$  is agent or firm  $j$ 's investment in capital on date  $t$ .  $\delta$  is the depreciation rate. The concave

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<sup>5</sup>See Parsons (1972), Jovanovic (1979b), Hashimoto (1981), Altonji and Shakotko (1987), Topel (1991), Neal (1995), Acemoglu and Pischke (1998), Parent (2000), Cappelli (2002).

<sup>6</sup>The value of their human capital is relatively more tied to the fate of the industry. Empirically, these workers have some influence on the asset allocation of their pension funds and hence equilibrium equity returns even if they may not be active investors.

<sup>7</sup>Using a Cobb-Douglas production function gives rise to some difficulties which I discuss in Section 7.2 of the Appendix.

function  $\Gamma \ln \left( 1 + \frac{I_{j,t}^H}{\theta H_{j,t}} \right)$  is from Eberly and Wang (2009)<sup>8</sup> and introduces an adjustment cost in the law of motion of capital. This function encompasses two special cases. First, if  $\Gamma \equiv \frac{\delta}{\ln(1+\frac{\delta}{\theta})}$  and  $\theta \rightarrow \infty$ , the adjustment cost goes to zero and we obtain the usual law of motion of capital with perfectly liquid capital ( $\frac{dH_t}{H_t} = \frac{I_t^H}{H_t} dt - \delta dt + \sigma_h dZ_t^A$ ) as in Cox, Ingersoll and Ross (1985). On the other hand, if  $\Gamma \equiv \frac{\delta}{\ln(1+\frac{\delta}{\theta})}$  and  $\theta \rightarrow 0$ , the law of motion collapses to the case of perfectly illiquid capital ( $\frac{dH_{j,t}}{H_{j,t}} = -\delta dt + \sigma_h dZ_t^A$ ) as in Cochrane, Longstaff and Santa-Clara (2008). The parameters  $\Gamma$  and  $\theta$  govern the adjustment cost of investment in capital. For convenience, these parameters are the same for both human and physical capital. In the context of human capital we can think of the adjustment cost as reflecting some opportunity cost of time spent on schooling (as a means of investing in human capital), psychic costs, or a reduction of the time spent on leisure activities that are valuable to agents. The adjustment cost for physical capital may reflect some frictions to capital reallocation, installation costs, or more general forms of capital illiquidity. Furthermore, to simplify the notation and following Cox, Ingersoll and Ross (1985), Eberly and Wang (2009, 2011), Brunnermeier and Sannikov (2011) and the literature on endogenous growth I incorporate technological shocks in the law of motion for capital. Thus  $H_{j,t}$  and  $K_{j,t}^i$  are effective units of capital which include the productivity shocks. In Theorem 2, I show that the current formulation of the model is equivalent to one in which the TFP shocks are modeled as separate variables.

$dZ_t = \{dZ_t^A, dZ_t^B\}'$  is a two-dimensional standard Brownian Motion increment<sup>9</sup> where  $dZ_t^A$  and  $dZ_t^B$  are independent. The growth rate of human capital,  $\frac{dH_{j,t}}{H_{j,t}}$ , and Type A physical capital,  $\frac{dK_{j,t}^A}{K_{j,t}^A}$ , are subject to the same total factor productivity shock,  $dZ_t^A$ . This is for convenience. I would get very similar results if the human capital were subject to shocks  $dZ_t^H$  where  $\text{Corr}(dZ_t^A, dZ_t^H) > 0$ . I have made the assumption that only firms of Type A use human capital. This is to create a stark contrast between the two types of firms where Type A firms have relatively more firm-specific human capital or are more human capital intensive. Nonetheless, the linear production processes make it easy to instead interpret the model as one where agents are free to work in firms of either type<sup>10</sup>. But, the human capital of every worker in the economy covaries more with the Type A physical capital,  $\text{Cov}\left(\frac{dH}{H}, \frac{dK^A}{K^A}\right) > \text{Cov}\left(\frac{dH}{H}, \frac{dK^B}{K^B}\right)$ . This is more in line with the explanation from Fama and French (1996) that Value firms are distressed firms and the value of human capital covaries more positively with the outcomes of such firms. I provide some evidence for this alternative interpretation of the model in Table 8 of Section 5.2. More generally, the qualitative

<sup>8</sup>Introducing the adjustment cost in the law of motion of capital traces back to Uzawa (1969).

<sup>9</sup>I fix a probability space  $(\Omega, \mathcal{F}, P)$  with filtration,  $\{\mathcal{F}_t\}_{t \geq 0}$ , generated by the two-dimensional standard Brownian Motion,  $Z_t$ . The filtration satisfies the usual conditions.

<sup>10</sup>We should note that for this alternative interpretation, without changing the current model specification, I could instead have that the production process for firms of Type  $i \in \{A, B\}$  is linear in the physical capital  $K^i$  and introduce a third type of competitive firms that rent human capital from agents and have a production process  $Y_{j,t}^H = \mathcal{A}H_{j,t}$ . This third type of firms make zero profits so the value of these firms is zero. We can then also think of “Value” firms as being comprised of a mix for Type A firms and the third type of firms; and “Growth” firms as being comprised of a mix for Type B firms and the third type of firms.

results would be the same if I allow Type  $B$  firms to also use firm-specific human capital in their production process so long as the human capital is relatively more tied to the Type  $A$  firms. That is,  $Cov\left(\frac{dH^A}{H^A}, \frac{dK^A}{K^A}\right) > Cov\left(\frac{dH^B}{H^B}, \frac{dK^B}{K^B}\right)$ . Lastly, in addition to the aggregate shocks ( $dZ_t$ ) I could include idiosyncratic shocks ( $dZ_{j,t}^H, dZ_{j,t}^B$ ) but this would not affect the results since there is continuum of agents and firms of each type. Indeed, by the Law of Large Numbers<sup>11</sup> the idiosyncratic shocks would integrate to zero in the aggregate. Throughout the paper, I will often refer to the relative covariance,  $Cov\left(\frac{dH}{H}, \frac{dK^A}{K^A}\right) > Cov\left(\frac{dH}{H}, \frac{dK^B}{K^B}\right)$ , as relative human capital risk.

## 2.1 Planner's problem

All agents and all firms of a given type are identical; thus following Eberly and Wang (2009, 2011), the planner simply chooses the optimal aggregate investments in each type of capital to maximize the Hamiltonian-Jacobi-Bellman (HJB) equation

$$\begin{aligned} 0 = & \max_{I^A, I^B, I^H} f\{C, V\} dt + V_H E(dH) \\ & + V_A E(dK^A) + V_B E(dK^B) \\ & + \frac{1}{2} \left( V_{AA} (dK^A)^2 + V_{BB} (dK^B)^2 + V_{HH} (dH)^2 + 2V_{AH} dH dK^A \right) \end{aligned} \quad (2.3)$$

with

$$C_t = Y_t - (I_t^A + I_t^B + I_t^H)$$

where  $E(dH)$ ,  $E(dK^A)$ , and  $E(dK^B)$ , denote the drifts of  $H$ ,  $K^A$ , and  $K^B$ , and  $V_l$  is the derivative of the value function with respect to physical ( $l \in \{A, B\}$ ) or human capital ( $l = H$ ).

### 2.1.1 Equilibrium

**Definition 1.** *An equilibrium in the planner's problem (2.3) consists of a set of adapted processes  $\{C_t, I_t^A, I_t^B, I_t^H\} \forall t$  such that*

1. *The processes  $\{C_t, I_t^A, I_t^B, I_t^H\}$  solve the HJB (2.3)*
2. *The resource constraint is satisfied:  $C_t + I_t^A + I_t^B + I_t^H = Y_t = \mathcal{A}(H_t + K_t^A + K_t^B)$*

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<sup>11</sup>I assume that the necessary technical conditions for the Law of Large Numbers to hold are satisfied. See Judd (1985).



3. *The laws of motion for aggregate human and physical capital are satisfied:*

$$\frac{dH_t}{H_t} = \Gamma \ln \left( 1 + \frac{I_t^H}{\theta H_t} \right) dt - \delta dt + \sigma_h dZ_t^A \quad (2.4)$$

$$\frac{dK_t^i}{K_t^i} = \Gamma \ln \left( 1 + \frac{I_t^i}{\theta K_t^i} \right) dt - \delta dt + \sigma dZ_t^i \quad \text{for } i \in \{A, B\} \quad (2.5)$$

I derive equations (2.4)-(2.5) in [Section 7.4](#) of the Appendix by integrating  $dH_{j,t}$  and  $dK_{j,t}^i$  across all  $j \in J$  and making use of the fact that the investment to capital ratios are independent of  $j$  since all agents are identical and all firms of each type are identical. Despite the linear production processes, the adjustment cost implies that the planner does not simply invest all resources in one type of capital<sup>12</sup>.

### 2.1.2 Solution

Following [Eberly and Wang \(2009\)](#) we conjecture (and later verify) that the value function can be written as

$$V \{ H + K^A + K^B, x^A, x^B \} = \frac{1}{1-\gamma} \left( (H + K^A + K^B) F \{ x^A, x^B \} \right)^{1-\gamma}$$

where  $x^i$ , for  $i \in \{A, B\}$ , denotes the Type  $i$  share of total capital.  $\{x^A, x^B\}$  are the two state variables of interest

$$x^i = \frac{K^i}{H + K^A + K^B}$$

The state variables follow

$$dx_t^i = \mu_{x,i,t} dt + \sigma'_{x,i,t} dZ_t$$

where  $\{\mu_{x,i,t}, \sigma_{x,i,t}\}$  for  $i \in \{A, B\}$  are defined in [Section 7.5](#) of the Appendix. Throughout the paper, I use  $F_i$  and  $F_{ij}$  for  $i, j \in \{A, B\}$  to denote the first derivative and the cross derivatives of  $F \{x^A, x^B\}$  with respect to  $\{x^A, x^B\}$ .

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<sup>12</sup>See [Section 7.4](#) of the Appendix.

**Theorem 1.** *Let*

$$c = \frac{C}{H + K^A + K^B}$$

$$\iota^i = \frac{I^i}{K^i}$$

$$\iota^H = \frac{I^H}{H}$$

*The solution to the planner's problem is obtained by jointly solving the following system of partial differential equations for  $c, \iota^A, \iota^B, \iota^H$  and  $F\{x^A, x^B\}$*

$$c = \mathcal{A} - x^A \iota^A - x^B \iota^B - (1 - x^A - x^B) \iota^H \quad (2.6)$$

$$c = F \left( \frac{\Gamma(-x^A F_A - x^B F_B + F)}{\beta(\iota^H + \theta)} \right)^{\frac{1}{\rho-1}} \quad (2.7)$$

$$c = F \left( \frac{\Gamma(-(x^A - 1)F_A - x^B F_B + F)}{\beta(\iota^A + \theta)} \right)^{\frac{1}{\rho-1}} \quad (2.8)$$

$$c = F \left( \frac{\Gamma(-x^A F_A - (x^B - 1)F_B + F)}{\beta(\iota^B + \theta)} \right)^{\frac{1}{\rho-1}} \quad (2.9)$$

$$\begin{aligned} 0 = & \frac{\beta}{\rho} \left( \frac{c}{F} \right)^\rho + \phi\{x^A, x^B\} \\ & + \ln \left( \frac{\iota^H + \theta}{\theta} \right) (x^A + x^B - 1) \left( \frac{\Gamma x^A F_A}{F} + \frac{\Gamma x^B F_B}{F} - \Gamma \right) \\ & + \ln \left( \frac{\iota^B + \theta}{\theta} \right) x^B \left( -\frac{\Gamma x^A F_A}{F} - \frac{\Gamma(x^B - 1)F_B}{F} + \Gamma \right) \\ & + \ln \left( \frac{\iota^A + \theta}{\theta} \right) x^A \left( -\frac{\Gamma(x^A - 1)F_A}{F} - \frac{\Gamma x^B F_B}{F} + \Gamma \right) \end{aligned} \quad (2.10)$$

*with boundary conditions  $F\{1, 0\} = F\{0, 1\} = \bar{F}$  and  $F\{0, 0\} = \hat{F}$ .*

*Proof.* We obtain the first equation by normalizing the resource constraint. The next three equations are the first order conditions for investment in human and physical capital from the planner's problem. The last equation is the HJB from (Section 2.3) where we plug in the conjecture for the value function. I provide more details as well as the definition of the function  $\phi\{x^A, x^B\}$  and the boundary values  $\{\bar{F}, \hat{F}\}$  in Section 7.6 of the Appendix. I solve the above system of equations using high order projections as described in Judd (1998) and implemented in Kozak (2012)<sup>13</sup>.

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<sup>13</sup>See Section 7.21 of the Appendix

**Corollary 1.** *With  $(1 - \rho)^{-1} = 1$ , or equivalently  $\rho = 0$ , the consumption-to-total-capital ratio is constant*

$$c = \frac{A + \theta}{\Gamma + \beta} \beta$$

*and the consumption-to-wealth ratio is constant*

$$\frac{C}{Hp + K^A q^A + K^B q^B} = \beta$$

*Proof.* We obtain these results by combining equations (2.6)-(2.9) as done in Eberly and Wang (2009) and Kozak (2012). Thus in the case of log-utility and more generally when  $\rho = 0$ , the consumption level grows at the same rate as total aggregate capital in the economy,  $H + K^A + K^B$ .

**Theorem 2.** *The model with processes for effective units of capital*

$$\begin{aligned} \frac{dH_{j,t}}{H_{j,t}} &= \Gamma \ln \left( 1 + \frac{I_{j,t}^H}{\theta H_{j,t}} \right) dt - \delta dt + \sigma_h dZ_t^A \\ \frac{dK_{j,t}^i}{K_{j,t}^i} &= \Gamma \ln \left( 1 + \frac{I_{j,t}^i}{\theta K_{j,t}^i} \right) dt - \delta dt + \sigma dZ_t^i \quad \text{for } i \in \{A, B\} \end{aligned}$$

*is equivalent to one where the total factor productivity (TFP) shocks are modeled as separate variables,  $a_t^i$ , for  $i \in \{A, B\}$ . That is, the model can be re-written as*

$$\begin{aligned} \frac{dh_{j,t}}{h_{j,t}} &= \Gamma \ln \left( 1 + \frac{I_{j,t}^h}{\theta a_t^A h_{j,t}} \right) dt - \delta dt \\ \frac{dk_{j,t}^i}{k_{j,t}^i} &= \Gamma \ln \left( 1 + \frac{I_{j,t}^i}{\theta a_t^i k_{j,t}^i} \right) dt - \delta dt \\ da_t^i &= a_t^i \sigma dZ_t^i \quad \text{for } i \in \{A, B\} \end{aligned}$$

$$Y_t = \mathcal{A} (a_t^A h_t + a_t^A k_t^A + a_t^B k_t^B)$$

*where  $h_t = \int_J h_{j,t} dj$ ,  $k_t^i = \int_J k_{j,t}^i dj$  and the prices of human and physical capital are respectively defined as  $a_t^A p_t$  and  $a_t^i q_t^i$ . Due to scale invariance, the value function is unaffected.*

*Proof.* I obtain this result from Brunnermeier and Sannikov (2011) and I provide the derivation in Section 7.3 of the Appendix.

## 2.2 Competitive equilibrium

In order to derive the asset pricing implications, I now decentralize the planner's problem. The First and Second Welfare Theorems hold in this economy since there is no externality. Therefore, the solution to the planner's problem coincides with the solution to the agents' and firms' problems.

**Proposition 1.** *We can decentralize the planner's problem as follows.*

- Each agent is endowed with human capital  $H_0$ , takes the wage rate  $\omega_t$ , the price of human capital  $p_t$ , and her initial financial wealth  $\tilde{W}_0$  as given. The agent has access to a risk-less bond with return  $r_t$  and a risky claim on the sum of profits of all firms of each type. The risky security prices are  $S_t = \{S_t^A, S_t^B\}'$ ; where  $S_t^i$  is the price of the claim on the sum of profits of all firms of Type  $i$ .  $S_t^i$  follows

$$dS_t^i = (\mu_t^i S_t^i - D_t^i) dt + S_t^i \varsigma_t^i \cdot dZ_t \quad \text{for } i \in \{A, B\}$$

Stacking the above equation for  $i \in \{A, B\}$  yields

$$dS_t = \left( \mu_t' \text{diag}(S_t) - D_t \right) dt + \begin{pmatrix} S_t^A \varsigma_t^{A'} \\ S_t^B \varsigma_t^{B'} \end{pmatrix} dZ_t$$

where  $\varsigma_t^i$  is a two-dimensional vector which denotes the local volatility of security  $i$ 's return. The agent solves

$$\max_{\{C_{j,t}, I_{j,t}^H, H_{j,t}, \varpi_{j,t}\}_{t=0}^{\infty}} E_0 \int_0^{\infty} f\{C_{j,t}, V_{j,t}\} dt \quad \text{s.t.}:$$

$$d\tilde{W}_{j,t} = \left( \tilde{W}_{j,t} r_t + \varpi_{j,t} \cdot \tilde{W}_{j,t} (\mu_t - \mathbf{1} r_t) - C_{j,t} - I_{j,t}^H + \omega_t H_{j,t} \right) dt + \varpi_{j,t}' \begin{pmatrix} \tilde{W}_{j,t} \varsigma_t^{A'} \\ \tilde{W}_{j,t} \varsigma_t^{B'} \end{pmatrix} dZ_t$$

$$\frac{dH_{j,t}}{H_{j,t}} = \Gamma \ln \left( 1 + \frac{I_{j,t}^H}{\theta H_{j,t}} \right) dt - \delta dt + \sigma_h dZ_t^A$$

where  $\mathbf{1} = \{1, 1\}'$  and  $\varpi_{j,t} = \{\varpi_{j,t}^A, \varpi_{j,t}^B\}'$  is the fraction of financial wealth,  $\tilde{W}_{j,t}$ , that the agent invests in the risky securities  $\{A, B\}$ .

- Let  $\Lambda_t$  denote the state price density (SPD). Firms of types  $A$  and  $B$  (respectively) own physical capital  $K_j^i$  for  $i \in \{A, B\}$ , take the wage rate  $\omega_t$ , and the price of physical capital  $q_t^i$  as given and solve

$$\max_{K_{j,t}^A, H_{j,t}, I_{j,t}^A} \int_0^{\infty} \Lambda_t (\mathcal{A}(K_{j,t}^A + H_{j,t}) - \omega_t H_{j,t} - I_{j,t}^A) dt \quad \text{s.t.}:$$

$$\frac{dK_{j,t}^A}{K_{j,t}^A} = \Gamma \ln \left( 1 + \frac{I_{j,t}^A}{\theta K_{j,t}^A} \right) dt - \delta dt + \sigma dZ_t^A$$

$$\max_{K_{j,t}^B, I_{j,t}^B} \int_0^{\infty} \Lambda_t (\mathcal{A}K_{j,t}^B - I_{j,t}^B) dt \quad \text{s.t.}:$$

$$\frac{dK_{j,t}^B}{K_{j,t}^B} = \Gamma \ln \left( 1 + \frac{I_{j,t}^B}{\theta K_{j,t}^B} \right) dt - \delta dt + \sigma dZ_t^B$$

- The resource constraint and market clearing conditions are

$$C_t + I_t^A + I_t^B + I_t^H = Y_t \quad \text{where } Y_t = \mathcal{A}(H_t + K_t^A + K_t^B)$$

$$\varpi_t^A \tilde{W}_t + \varpi_t^B \tilde{W}_t = S_t^A + S_t^B \quad \text{market for risky securities clears}$$

$$1 - \varpi_t^A - \varpi_t^B = 0 \quad \text{zero net bond holdings}$$

$$\text{where } \varpi_t^i = \int_J \varpi_{j,t}^i dj, \quad C_t = \int_J C_{j,t} dj, \quad I_t^i = \int_J I_{j,t}^i dj, \quad I_t^H = \int_J I_{j,t}^H dj, \quad \tilde{W}_t = \int_J \tilde{W}_{j,t} dj.$$

**Corollary 2.** The price of human and physical capital (respectively) are

$$p = \frac{1}{\Gamma} (\iota^H + \theta) = \frac{1}{\beta} \left( \frac{c}{F} \right)^{1-\rho} (F - x^A F_A - x^B F_B)$$

$$q^A = \frac{1}{\Gamma} (\iota^A + \theta) = \frac{1}{\beta} \left( \frac{c}{F} \right)^{1-\rho} (F - (x^A - 1)F_A - x^B F_B)$$

$$q^B = \frac{1}{\Gamma} (\iota^B + \theta) = \frac{1}{\beta} \left( \frac{c}{F} \right)^{1-\rho} (F - x^A F_A - (x^B - 1)F_B)$$

Per the usual result from Q-Theory, the investment to capital ratio is linear in the marginal Q and positive if and only if the marginal Q is above a particular threshold

$$\iota^H > 0 \quad \text{iff} \quad p > \frac{\theta}{\Gamma}$$

$$\iota^i > 0 \quad \text{iff} \quad q^i > \frac{\theta}{\Gamma}$$

*Proof.* See Section 7.7 of the Appendix. In the equations above, I have made implicit the dependence of  $\{\iota^A, \iota^B, \iota^H, p, q^A, q^B\}$  on the state variables  $\{x^A, x^B\}$ . The drift and diffusion of the security prices,  $\{\mu_t^i + \frac{D_t^i}{S_t^i}, \varsigma_t^i\}$ , are also endogenous functions of the state variables. We should note that there is zero net holdings of the risk-free bond. Therefore, it is not needed in decentralizing the planner's problem. The two risky claims are necessary and sufficient to decentralize the planner's problem. Nonetheless, to maintain a structure that is fairly standard in theoretical asset pricing and to introduce the risk-free rate (which later plays an important role), I use the latter in defining a competitive equilibrium.

### 3 Asset Pricing Implications

**Proposition 2.** The state price density, the consumption growth and the value function follow

$$\frac{d\Lambda_t}{\Lambda_t} = -r_t dt - \sigma_{\Lambda,t} dZ_t \quad \text{with } \Lambda_0 = 1$$

$$\frac{dC_t}{C_t} = \mu_{c,t} dt + \sigma_{c,t} dZ_t$$

$$dV_t = -f\{C_t, V_t\}dt + V_t\sigma_{v,t}dZ_t$$

where

$$r_t = \beta + (1 - \rho)\mu_{c,t} - \frac{1}{2}(\rho - 2)(\rho - 1)\sigma_{c,t}^2 - \frac{(\rho - 1)(\gamma + \rho - 1)\sigma_{c,t}\sigma_{v,t}}{\gamma - 1} - \frac{\rho(\gamma + \rho - 1)\sigma_{v,t}^2}{2(\gamma - 1)^2}$$

$$\sigma_{\Lambda,t} = -(\rho - 1)\sigma_{c,t} - \left(\frac{\rho}{\gamma - 1} + 1\right)\sigma_{v,t}$$

*Proof.* See Section 7.8 of the Appendix for a derivation of the above as well as the subsequent results.

We can write the returns on the risky securities as

$$dR_t^i = \underbrace{\frac{\mathcal{A} - q_t^i\Gamma + \theta}{q_t^i}}_{\text{dividend yield}=D_t^i/S_t^i} dt + \underbrace{\frac{dq_t^i}{q_t^i} + \frac{dK_t^i}{K_t^i} + \frac{dK_t^i}{K_t^i} \times \frac{dq_t^i}{q_t^i}}_{\text{capital gains}=dS_t^i/S_t^i} = (r_t + \sigma_{\Lambda,t} \cdot \varsigma_t^i) dt + \varsigma_t^i \cdot dZ_t$$

$$\varsigma_t^i = \frac{1}{q_t^i} \frac{\partial q_t^i}{\partial x_t^A} \sigma_{x,A,t} + \frac{1}{q_t^i} \frac{\partial q_t^i}{\partial x_t^B} \sigma_{x,B,t} + \sigma \mathbf{1}_i \quad \text{for } i \in \{A, B\}$$

where  $S_t^i = K_t^i q_t^i$  is the value of security  $i \in \{A, B\}$  and also the value aggregate capital of Type  $i$ .  $\mathbf{1}_i$  is a coordinate vector in  $\mathbb{R}^2$ ; that is  $\mathbf{1}_A = \{1, 0\}'$  and  $\mathbf{1}_B = \{0, 1\}'$ .

Thus the expected return on security  $i$ , (that is  $E_t(dR_t^i)$ ) is the risk-free return minus the covariance of the risky security with the SPD. The realized return ( $dR_t^i$ ) is the expected return plus the innovation from the Brownian Motion increment with a local volatility  $\varsigma_t^i$ . We can similarly derive the return from human capital even though it is not explicitly traded. In particular, we can write the return on human capital as

$$dR_t^H = \underbrace{\frac{H_t\omega_t - I_t^H}{H_t p_t}}_{\text{dividend yield}=D_t^H/S_t^H} dt + \underbrace{\frac{d(H_t p_t)}{H_t p_t}}_{\text{capital gains}=dS_t^H/S_t^H} = (r_t + \sigma_{\Lambda,t} \cdot \varsigma_t^H) dt + \varsigma_t^H \cdot dZ_t$$

The return from human capital is composed of capital gains, which is due to appreciation or depreciation in the value of human capital ( $S_t^H = H_t p_t$ ), as well as a dividend yield. The latter is comprised of labor income ( $H_t\omega_t$  where  $\omega_t = \mathcal{A}$ ) less current investment in human capital. Although this is a continuous-time model we can think of the  $-I_t^H$  term as the real cost the agent faces at time  $t$  if she decides to invest  $I_t^H$  in human capital. Ignoring the general equilibrium effects for the moment, at time  $t + \Delta t$  the agent's skill level would on average be greater ( $H_{t+\Delta t} > H_t$ ) and she would receive some extra compensation for this additional skill,  $\omega_{t+\Delta t}(H_{t+\Delta t} - H_t)$ . Thus as Palacios-Huerta (2003a) points out, this capital gains and skill premium decomposition of the return on human capital is consistent with a fully specified model of endogenous labor supply and investment in schooling. Palacios-Huerta (2003a) shows that incorporating these two components of returns on human capital into the CAPM significantly improves the CAPM's ability to explain

the variability of observed security returns. The author finds lower pricing errors and greater  $R^2$ 's compared to what is obtained using the approach of Jaganathan and Wang (1996)<sup>14</sup> or Campbell (1996)<sup>15</sup>.

The market portfolio return is the value weighted return of the traded securities

$$dR_t^m = dR_t^A \frac{S_t^A}{S_t^m} + dR_t^B \frac{S_t^B}{S_t^m} = \mu_t^m dt + \varsigma_t^{m'} dZ_t$$

$$S_t^m = S_t^A + S_t^B$$

It is also convenient to derive the return on the total wealth portfolio. The total wealth is the sum to marketable wealth and human wealth,  $S^w = \underbrace{S^A + S^B}_{\text{marketable}} + \underbrace{S^H}_{\text{non-marketable}}$ .

$$dR_t^w = dR_t^A \frac{S_t^A}{S_t^w} + dR_t^B \frac{S_t^B}{S_t^w} + dR_t^H \frac{S_t^H}{S_t^w} = (r_t + \sigma_{\Lambda,t} \cdot \varsigma_t^w) dt + \varsigma_t^w \cdot dZ_t$$

Notice that the wedge between the return on the wealth portfolio and the return on the market portfolio is due entirely to the fact that market portfolio does not include returns on human capital. Stambaugh (1982) argues that this wedge is what causes the poor performance of the CAPM. Jaganathan and Wang (1996) and the many other empirical research which include the returns on human capital in a CAPM framework confirm this. In Sections 4.2.2 and 5.2, I use the CAPM to quantify the importance of human capital for asset pricing. We should also note that this wedge arises despite the fact that markets are dynamically complete in the model. I could easily introduce market incompleteness by having human capital subject to shocks  $dZ_t^H$  and not allowing agents to issue claims on the value of their human capital. As long as  $\text{Corr}(dZ_t^A, dZ_t^H)/dt = \varrho > 0$  all the qualitative results are exactly the same and I conjecture that the quantitative results would not be very different<sup>16</sup>.

Lastly, the Book-to-Market Equity (BE/ME) ratio is

$$\frac{K_t^i}{K_t^i q_t^i} = \frac{1}{q_t^i}$$

This definition of the BE/ME is identical to that from Eberly and Wang (2009, 2011) as well as Zhang (2005). This definition of the BE/ME ratio is not necessarily equivalent to that which we observe in data. This is because in the model the firms are entirely equity-financed whereas in data firms are both equity and debt-financed. Thus,  $1/q$  omits the effect of liabilities. On the other

<sup>14</sup>Jaganathan and Wang (1996) use the growth in labor income as a proxy for the returns on human capital.

<sup>15</sup>Campbell (1996) uses a more general approach but still ignores endogenous labor supply and investment in human capital.

<sup>16</sup>See Section 7.9 of the Appendix.

hand, the Value premium is also present in data if instead of the Book-to-Market ratio we use a different price ratio<sup>17</sup>. Therefore, the fact that  $1/q$  is not equivalent to the Book-to-Market ratio is not worrisome and can still reflect the Value premium at least qualitatively. Furthermore, so long as the Modigliani-Miller Theorem holds (that is the case in the model), a firm's value is unaffected by its financing. Hence, if a firm of Type  $i$  finances a fraction  $d_t^i$  of its value with debt then  $1/q$  is indeed the Book-to-Market ratio

$$BE/ME = \frac{Assets - Liabilities}{Equity} = \frac{K_t^i - d_t^i S_t^i / q_t^i}{S_t^i - d_t^i S_t^i} = \frac{1}{q_t^i}$$

I could modify the model to allow for endogenous debt financing by using the approach from Brunnermeier and Sannikov (2011) and replace firms with managers who finance their net worth by issuing equity and borrowing from households. However, this approach would reduce the model's tractability, would not affect the qualitative results much and hence would not be very beneficial. Alternatively, I can use the approach from He and Krishnamurthy (2012) and assume that the households hold an amount  $d^i S_t^i$  of debt from firms of Type  $i$  which pays the risk-free rate. In Section 7.10 of the appendix, I show that with this assumption, the BE/ME ratio would indeed be  $1/q^i$  and the Type  $A$  security expected excess return would not be affected much

$$E \left( d\tilde{R}_t^A - d\tilde{R}_t^B \right) \approx -0.0005dt + E \left( dR_t^A - dR_t^B \right)$$

However, this approach makes the share of debt (or the leverage),  $d^i$ , exogenous. It also imposes that the firms debt pays the risk-free return. Lastly, I could instead introduce a risk-less capital (as in Kozak; 2012) which is owned by the two types of firms. The firms could then issue claims to this risk-less capital in the form of a risky bond<sup>18</sup>. However, this introduces one more state variable to the model. Consequently, I do not allow for net bond holdings nor do I allow for any difference between firms of Types  $A$  and  $B$  other than the Type  $A$  relative human capital risk. Allowing for some heterogeneity would strengthen the results since in data Value firms are more volatile than Growth firms. I instead focus entirely on the implications of human capital risk.

## 4 Results

### 4.1 Calibration, model fit and descriptive statistics

Table 1 shows the calibrated and assumed parameters of the model. All parameters and moments are in annual units. I set the subjective discount rate to  $(\beta)$  to 7% and for simplicity, I set the depreciation rate to zero<sup>19</sup> ( $\delta = 0$ ), restrict the adjustment cost parameters with  $\Gamma = \theta$ , equalize the

<sup>17</sup>For example the Value premium is also present if we sort securities by their Price-to-Earnings ratio.

<sup>18</sup>The bond would be risky because of discount rate risk.

<sup>19</sup>With  $\delta = 0$ , we need  $\Gamma = \theta$  to guarantee that the expected growth rate of capital as a function of the investment-



local volatilities of capital ( $\sigma = \sigma_h$ ) and fix the investment-to-capital ratio to 2% in an economy with one capital type<sup>20</sup>. Let,  $c^*$  and  $F^*$  denote the consumption-to-total-capital ratio and the normalized value function in an economy with one capital type. Hence,  $c^*$  and  $F^*$  are scalars and because  $\sigma = \sigma_h$  we have  $F^* = \bar{F} = \hat{F}$ . I search for a combination of  $\{\mathcal{A}, c^*, F^*, \gamma, \theta, \rho, \sigma\}$  over the corresponding seven-dimensional space to satisfy seven conditions: 1) a risk-free rate of 0.90%, 2) a volatility of returns on the market portfolio of 16%, and the next five equations from an economy with one type of capital; 3) an expected output growth of 2%, 4) a consumption-to-output ratio of 90%, 5) the resource constraint, 6) the first-order condition for investment, 7) the HJB<sup>21</sup>. A consumption-to-output ratio of 90% in a one-capital economy ( $C/Y = c^*/\mathcal{A} = 0.90$ ) yields a consumption-to-output ratio of 86% on average in the full model. In US data,  $C/Y$  is 67% but the government expenditures-to-output ratio is 27%. Since there is no government in the model, for the resource constraint to be satisfied I allow for a  $C/Y$  that is greater than that in data. In the model, the total investment-to-output ratio  $((Y - C)/Y)$  is 14% and is also somewhat greater than it is in data (11%). There is substantial controversy about whether the parameter which governs the IES is larger<sup>22</sup> or smaller<sup>23</sup> than one. Following the more recent literature, I find that an implied IES parameter greater than one is needed to better match some moments of the data  $((1 - \rho)^{-1} = 2.0025)$ . All the qualitative results hold with different values for the parameters as long as  $\gamma > 0$  and  $\Gamma = \theta < \infty$ <sup>24</sup>.

Variable	Name/Calculation	Value
$\mathcal{A}$	Marginal Product of Capital	20.63%
$\beta$	Subjective Discount Rate	7.00%
$c^*$	$\frac{\text{Consumption}}{\text{Capital}}$ in a one-capital economy	18.63%
$\delta$	Depreciation	0.00%
$F^*$	Normalized Value Function, $F$ , in a one-capital economy	0.0791
$\gamma$	Risk Aversion Parameter	3.97
$\Gamma = \theta$	Adjustment Cost Parameters	2.73%
$\rho$	Implied IES Parameter = 2.0025	0.5006
$\sigma = \sigma_h$	Standard deviation of Capital Growth	21.27%

Table 1: Parameters of the model

In Figure 1, I plot the policy functions from the planner's problem. I henceforth label firms of to-capital ratio is a concave function which always lies below the 45-degree line. That is, for each positive unit of investment the expected capital growth increases by less than one unit.

<sup>20</sup>An economy with one capital type is a boundary case of the model with  $x^A = 1$  (the case with only Type A physical capital), or  $x^B = 1$  (the case with only Type B physical capital), or  $1 - x^A - x^B = 1$  (the case with only human capital). With  $\sigma = \sigma_h$ , these boundary cases have the same solution.

<sup>21</sup>I fix the state variables at their average value and then look for the seven unknown scalars  $\{\mathcal{A}, c^*, F^*, \gamma, \theta, \rho, \sigma\}$  that solve the seven non-linear equations described above. Equations 4) and 5) determine  $\mathcal{A}$  and  $c^*$ . Equations 1), 6) and 7) determine  $F^*$ ,  $\gamma$ , and  $\rho$ . Equation 3) determines  $\theta$  and equation 2) determines  $\sigma$ .

<sup>22</sup>Attanasio and Vissing-Jørgensen (2003), Bansal and Yaron (2004), Hansen, Heaton and Li (2008), Van Binsbergen et al. (2010), Constantinides and Ghosh (2013)

<sup>23</sup>Hansen and Singleton (1982), Hall (1988)

<sup>24</sup>As  $\Gamma = \theta \rightarrow \infty$  the adjustment cost goes to zero. In Section 7.11 of the Appendix I show that without adjustment cost the expected excess return of Security A is zero,  $E(dR^A) - E(dR^B) = 0$

Type A, Value Firms, and firms of Type B, Growth firms. Since the state variables are  $x^A \in [0, 1]$  and  $x^B \in [0, 1]$ , the state space is triangular. The investment-to-physical-capital ratio for Value firms ( $\iota^A$ ) and the investment-to-human-capital ratio ( $\iota^H$ ) are identical because the production processes are linear,  $\sigma = \sigma_h$ , and the productivity shocks to human capital and Value firms physical capital are perfectly correlated. Over much of the state space the investment to capital ratio is larger for Growth firms; that is  $\iota^A < \iota^B$ . The model allows for negative investments however due to the choice of parameters the investment-to-capital ratios are positive over the entire state space.

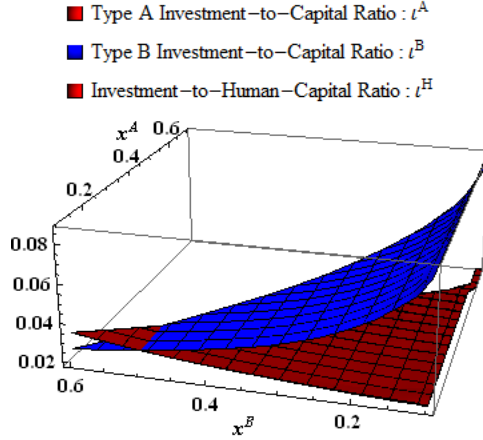


Figure 1: Plot of the three policy functions:  $\iota^A \{x^A, x^B\}$ ,  $\iota^B \{x^A, x^B\}$ ,  $\iota^H \{x^A, x^B\}$ . With the parameter choice  $\sigma = \sigma_h$  we have  $\iota^A = \iota^H$ .

To calculate the empirical returns for the Value and Growth firms I merge monthly returns data from CRSP with fundamentals data from Compustat for the years 1963-2012. I restrict the data to securities traded on the NYSE, NASDAQ, and the AMEX. Following the approach of Fama and French (1993, 1996), I form Value and Growth portfolios using the top thirtieth and the bottom thirtieth percentiles of BE/ME distributions with the BE/ME cut-offs from the Kenneth R. French Data Library<sup>25</sup>. BE is the sum of book equity, deferred taxes, and investment tax credit, minus the book value of preferred stock for fiscal year  $t - 1$ . ME is the value of common equity at the end of year  $t - 1$ . I then calculate returns from July of year  $t$  through June of year  $t + 1$ . The mean return shown in Table 2 for the Value and Growth portfolios is the annualized<sup>26</sup> average monthly returns. The market return is the value-weighted sum of returns on the traded securities. I adjust the security returns for inflation using the GDP deflator<sup>27</sup>. The model is not be able to simultaneously match the volatility of the market portfolio returns and the volatility of output. That is because there is no exogenous variation to increase the volatility of the Tobin Q's independently of the volatility of output. So, the local volatility of capital ( $\sigma$ ) determines both the volatility of output and the

<sup>25</sup>The empirical results which follow are similar if we exclude financial sector securities from the portfolios.

<sup>26</sup>I multiply the average monthly return by twelve and the standard deviation by the square-root of twelve.

<sup>27</sup>I use a two-year rolling geometric average of the GDP deflator. I do so because in constructing BE/ME and Value-Growth portfolios returns, I use the BE and ME from the previous year ( $t - 1$ ).

Moments	Data/Targets (%)	Model (%)
Mean Consumption Growth	2.00	1.91
Mean Output Growth*	2.00	1.92
Mean Risk-free Return*	0.90	0.22
Standard Deviation of Risk-free Return	2.00	0.68
Mean Return of Value Stocks	10.32	10.85
Standard Deviation of Value Stocks	16.73	18.78
Sharpe Ratio of Value Stocks	56.31	56.60
Mean Return of Growth Stocks	6.24	8.79
Standard Deviation of Growth Stocks	16.62	17.11
Sharpe Ratio of Growth Stocks	32.13	50.09
Mean Value premium	4.08	[2.06, 3.12]
Sharpe Ratio of Value premium	38.50	10.80
Mean Market Return	7.16	10.09
Standard Deviation of Market Return*	15.45	15.83

Table 2: The parameters are set to match some annual moments. The moments with \* are some of the moments that I target in the calibration. To obtain the model's moments, I simulate 25 years of observations 10,000 times with a monthly frequency. I then take the sample mean (or standard deviation) over time and then its average across simulations. All values are in annual units.

volatility of market returns. The above calibration yields a value of about 16% for the volatility of consumption and output growth since  $\sigma \approx 21\%$ . Again, that is because the model is a simple model with three identical risky sectors<sup>28</sup> and no exogenous discount rate shocks. I obtain similar results for key moments of the model (e.g: the Value premium) if instead of targeting the volatility of the market portfolio returns I target the volatility of output. This alternative calibration yields a volatility of consumption of 4% but the volatility of the market portfolio returns is only 4%. I provide the model's moments from this alternative calibration in [Section 7.12](#) of the Appendix. For all the results that will follow, I use the first calibration as described above (where I target the volatility of the market portfolio returns) since I am more interested in the asset pricing implications of the model and since the results are very similar with the alternative calibration (where I target the volatility of output).

I generate ten thousand simulations of the model each with twenty-five years of observation<sup>29</sup>. Using each time series I calculate the moment of interest and then take the average across simulations. The model's estimates for the average output and consumption growth are 1.9% and are close their

<sup>28</sup>However, if I allow for additional types of capital with no risk or negatively correlated shocks, I could potentially match both the volatility of output and the volatility of returns. Alternatively, I could allow for heterogeneity between firms or agents. Such heterogeneity has been shown to help simultaneously match financial and macroeconomic moments ([Constantinides and Duffie; 1996](#), [Panageas and Garleanu; 2010](#), [Gomes and Michaelides; 2008](#)). I could also allow for liquidity shocks or discount rate shocks by making  $\theta$  or  $\gamma$  (respectively) stochastic. This would increase the volatility of the SPD and the volatility of returns without affecting the volatility of output since the latter is a weighted average sum of the volatilities of effective capital.

<sup>29</sup>I use a monthly frequency. The results are almost identical when I use a frequency of one hundredth of a year (or roughly four days).

2.0% targeted values. The model yields a Value premium of 2.06% to 3.12%; that is fifty to seventy percent of the observed value premium, 4.08%. Recall that I did not target the Value premium in the calibration. But, if I so desired, I could have targeted (and exactly matched) the Value premium. Indeed, by having  $\sigma_h \neq \sigma$  and/or by making the human capital adjustment cost parameters different from those for physical capital I would obtain additional degrees of freedom to match more moments. Furthermore, the empirical Value premium is calculated as the difference in average return in the top and bottom thirtieth percentiles of BE/ME. Using the fiftieth percentile would yield an empirical Value premium closer to 2%. The lower bound in the model Value premium is a conservative estimate from the difference in mean returns,  $E(dR^A) - E(dR^B)$ , and the upper bound is calculated as done in the empirical literature by relabeling the Value and Growth stock each year using the BE/ME ( $1/q$ ). We should also note that in the model I focus on firms and not necessarily on industries or sectors. Nonetheless, in the model we could think of industries or sectors as containing firms of both Type  $A$  and  $B$ . Thus, the Value premium would also be present within industries as we observe in data.

The model matches the mean risk-free rate, the market premium, the returns and volatility of the Value, Growth and market portfolio fairly well. More importantly, the model yields a greater Sharpe Ratio for Value firms (0.57) than Growth firms (0.50); as we indeed observe in data. The model's volatility of the risk-free rate is much smaller than observed in data. However, this is not very worrisome since usually asset pricing models yield high values for the volatility of the risk-free rate and struggle to reduce this volatility. The Sharpe ratio of the Value premium in the model (0.11) is somewhat lower than that in data (0.39). That is because, in the model the covariance of the two risky securities is only slightly positive. In [Section 7.20](#) of the Appendix, I propose a possible extension of the model which would increase the model's fit<sup>30</sup>. The model also yields greater average dividend yields for Value stocks (0.09) than Growth stocks (0.07) as observed in data. However, the model does not match the level of dividend yields since this moment is not targeted in the calibration.

By pooling all observations across time and simulations, I produce plots of the distributions for several key variables of the model in [Figures 2a-2b](#). Thus, in equilibrium human capital and physical capital of both types co-exists. In particular, on average the shares total capital comprised by physical capital of Type  $i$  (that is  $x^i$ ) are  $E(\frac{K^A}{H+K^A+K^B}) = E(x^A) = 0.313$  and  $E(\frac{K^B}{H+K^A+K^B}) = E(x^B) = 0.374$ ; hence  $E(\frac{H}{H+K^A+K^B}) = 1 - E(x^A) - E(x^B) = 0.313$ . From [Figure 5b](#) we also find that the distribution for the value of human capital,  $p_t$ , and the value of physical capital of Type  $A$ ,  $q_t^A$ , are identical (up to minor numerical errors). That is because the Marginal Q's are linear in the investment-to-capital ratios and  $\iota^A = \iota^H$  due to the parameter restrictions (discussed earlier).

<sup>30</sup>In particular, by making the parameter which governs the risk aversion ( $\gamma_t$ ) or the adjustment cost parameter ( $\Gamma_t = \theta_t$ ) stochastic we would introduce additional covariation between the risky securities without affecting the Value premium much. This would increase the Sharpe ratio of the Value premium while also providing a way of increasing the volatility of returns independently from the volatility of output. Thus in this extension of the model we could potentially match both the volatility of output and the volatility of market returns.

In Figures 2c-2d, I plot the distribution of the state variables across simulations at five different

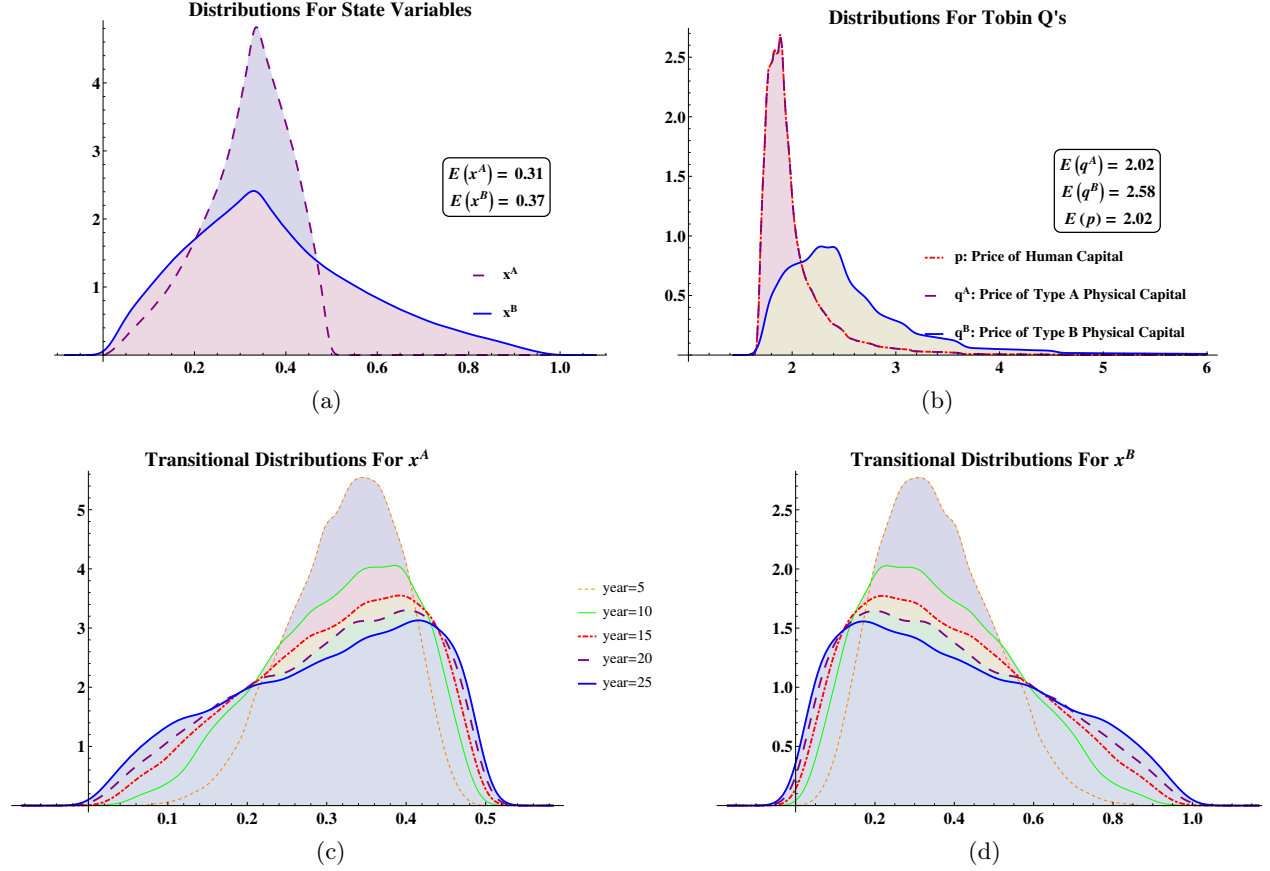


Figure 2: I pool all observations (across time and simulations) from the model for  $\{x^A, x^B, p, q^A, q^B\}$  to produce the distributions in Figures 2a-2b. In Figures 2c-2d, I plot the distribution of the state variables across simulations at five different points in time.

points in time. These plots confirm that twenty-five years are sufficient for the distribution of state variables to converge to a stationary distribution. Indeed, the distributions of the state variables across simulations are very similar at years fifteen, twenty and twenty-five.

Notice that in the model other than the relative human capital risk, the two firm types are identical. Thus, all the results are driven by the fact that human capital is more tied to one firm type than the other. More specifically,  $Cov\left(\frac{dH}{H}, \frac{dK^A}{K^A}\right) > Cov\left(\frac{dH}{H}, \frac{dK^B}{K^B}\right)$  along with positive adjustment costs give rise to the Value premium. Allowing for some modest heterogeneity<sup>31</sup>, for example by making  $Vol\left(\frac{dK^A}{K^A}\right) > Vol\left(\frac{dK^B}{K^B}\right)$  or making the adjustment cost greater for Value firms ( $\Gamma^A = \theta^A < \Gamma^B = \theta^B$ ) as implied by Zhang (2005), would amplify the effect of the human capital risk. To focus entirely on the effect of human capital on asset pricing I do not allow for such firm heterogeneity.

<sup>31</sup>This is indeed observed in data.

## 4.2 Results and dynamics

### 4.2.1 Reproducing and explaining the results from Fama and French (1996)

The Value premium is

$$E_t(dR_t^A) - E_t(dR_t^B) = Cov_t\left(-\frac{d\Lambda_t}{\Lambda_t}, \frac{dq_t^A}{q_t^A}\right) - Cov_t\left(-\frac{d\Lambda_t}{\Lambda_t}, \frac{dq_t^B}{q_t^B}\right) + \sigma_{\Lambda,t} \cdot \{1, -1\}' \sigma dt$$

In equilibrium the instantaneous covariance between the SPD and the price of physical capital of Value firms (Type  $A$ ) is on average smaller than that between the SPD and the price of physical capital of Growth firms (Type  $B$ ). In particular,  $\sigma_{\Lambda,t} \cdot \{1, -1\}' \sigma dt > Cov_t\left(-\frac{d\Lambda_t}{\Lambda_t}, \frac{dq_t^A}{q_t^A}\right) - Cov_t\left(-\frac{d\Lambda_t}{\Lambda_t}, \frac{dq_t^B}{q_t^B}\right) > 0$  on average. Negative shocks to Value firms are costlier to the economy since they coincide with negative shocks to the human capital; hence the price of risk is larger for the shock  $dZ^A$  than the shock  $dZ^B$  (that is,  $\sigma_{\Lambda,t}^{(1)} > \sigma_{\Lambda,t}^{(2)}$ ) and hence the premium of security  $A$  relative to security  $B$  (as shown in Table 2). Notice that with log-utility, the price of risk is  $\sigma_{\Lambda,t} = \sigma_{c,t}$  therefore the Value stock returns covary more with consumption growth than the returns on Growth stock do (see Appendix Section 7.16.3). This also holds in the general model (with  $\gamma > (1 - \rho) \neq 1$ ). As a result, the model reproduces the long-run risk explanations for the Value premium. More specifically, in Section 7.16 of the Appendix I follow Hansen (2011) and Borovicka et al. (2011) and produce risk-price elasticities for securities  $A$  and  $B$ . That is

$$\pi^i\{x, t\} = \frac{1}{t} \frac{d}{d\epsilon} \ln \{E(S_t^i \zeta_t \{\epsilon\} | x_0 = x)\} \Big|_{\epsilon=0} - \frac{1}{t} \frac{d}{d\epsilon} \ln \{E(\Lambda_t S_t^i \zeta_t \{\epsilon\} | x_0 = x)\} \Big|_{\epsilon=0}$$

for  $i \in \{A, B\}$

where  $x$  is the vector of state variables ( $x = \{x^A, x^B\}$ ) and  $\zeta_t \{\epsilon\}$  is a perturbation process

$$\ln(\zeta_t \{\epsilon\}) = \int_0^t -\frac{1}{2} \epsilon^2 \alpha \cdot \alpha ds + \int_0^t \epsilon \alpha \cdot dZ_s \quad \text{and} \quad \alpha = \left\{ \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\}$$

The risk-price elasticity,  $\pi^i\{x, t\}$ , captures the sensitivity of a date  $t$  expected return on security  $i$  to a date zero change in risk exposure. I quantify the long-run risk as  $\pi^i\{x, \infty\}$  and find that  $\pi^A\{\bar{x}, \infty\} > \pi^B\{\bar{x}, \infty\}$ , where  $\bar{x}$  is the mean of the vector of state variables. Thus, Value stocks (security  $A$ ) are (is) relatively more exposed to long-run risk. In the model, this is entirely due the fact that human capital is more tied to firms of Type  $A$  than Type  $B$ . Of course, it is tautological to say that the expected returns of Security  $A$  is larger than that of Security  $B$  because the returns on Security  $A$  covary more negatively with the SPD. However, that this is entirely driven by the human capital risk that is more tied to Security  $A$  is interesting.

The fact that negative  $dZ^A$  shocks are more costly than  $dZ^B$  shocks also implies that physical capital of Value firms are less valuable than that of Growth firms (on average  $q^A < q^B$  and hence

$\frac{1}{q^A} > \frac{1}{q^B}$ ). Thus, the model yields the result from Fama and French (1996). In particular, Table

BE/ME Quintiles				
Low	2	3	4	High
Expected Return $E(dR_t)$				
6.36	6.84	8.40	9.12	10.44

(a) This table uses data from CRSP and Compustat.

BE/ME ( $\frac{1}{q}$ ) Quintiles				
Low	2	3	4	High
Expected Return $E(dR_t)$				
6.72	8.81	10.23	11.29	11.97

(b) This table is constructed using data from the simulations of the model.

	BE/ME	Mean (%) $E(dR_t)$	Std. Dev.(%) $\sigma(dR_t)$	Sharpe Ratio $\frac{E(dR_t - r_t)}{\sigma(dR_t)}$
Value Stocks	1.38	10.32	16.73	0.56
Growth Stocks	0.31	6.24	16.62	0.32

(c) This table uses data from CRSP and Compustat. It shows the (annualized) average monthly returns, standard deviations and Sharpe Ratios for Value and Growth Stocks.

	Marginal Q $E(q_t^i)$	BE/ME $E(1/q_t^i)$	Mean (%) $E(dR_t^i)$	Std. Dev. (%) $E \varsigma_t^i $	Sharpe Ratio (%) $\frac{E(dR_t^i - r_t)}{E \varsigma_t^i }$
Security A	2.02	0.51	10.85	18.78	0.57
Security B	2.58	0.41	8.79	17.11	0.50

(d) In this panel I report the corresponding objects from the model

Table 3: Table 3a reproduces the results from Table 1 of Fama and French (1996). It shows the (annualized) average monthly returns and standard deviations of securities by BE/ME quintiles. I calculate the data from Table 3b as follows. At each point in time I pool the expected returns on Securities A and B across all simulations. I then group these returns into five quintiles of Book-to-Market Equity,  $\frac{1}{q_t}$ . I then calculate the mean of the expected return in each quintile and then take the average over time.

3a reproduces the results from Fama and French (1996) and highlights the positive (and monotone) relationship between equity returns and their BE/ME. Securities with greater BE/ME have greater average returns. This relationship holds if we consider the Sharpe ratios<sup>32</sup> of Value and Growth stocks. Indeed, Table 3c shows that Value stocks have a more attractive risk-return profile, as measured by the Sharpe Ratio, compared to Growth Stocks. This is the essence of the Value premium puzzle. In Table 3d, I provide the model's equivalent results. The Value firms (Type A) have greater expected returns (10.85%), BE/ME (0.51) and Sharpe Ratio (0.57) compared to Growth firms (Type B).

I could reproduce Table 3a by introducing additional types of physical capital with different degrees of covariance with human capital growth. This would require additional state variables.

<sup>32</sup>A measure of risk adjusted returns which we obtain by dividing the excess return on the risk-free returns by the standard deviation of stock returns.

Instead, in Table 3b, at each point in time I pool the expected returns on Securities  $A$  and  $B$  across all simulations. I then group these returns into five quintiles of Book-to-Market Equity,  $\frac{1}{q_t}$ . Lastly, I calculate the mean of the expected return in each quintile and then take the average over time. I find a similar positive relationship between expected returns and the BE/ME ratio. Figures 3a and 4a further illustrates this result. This pattern is due to the fact that Security  $A$  tends to have greater expected return and BE/ME<sup>33</sup> than Security  $B$  and the fact that this expected excess return,  $E_t(dR_t^A) - E_t(dR_t^B)$ , is increasing in the Security  $A$ 's excess BE/ME,  $\frac{1}{q_t^A} - \frac{1}{q_t^B}$ . Figures 3b and 4b confirm this result. Furthermore, Figures 4a and 4b shows that the model also reproduces the following two results from Zhang (2005):

1. Industry equity returns are increasing in the industry Book-to-Market ratio.
2. The Value spread, the relative Book-to-Market ratio, is a powerful predictor of the future Value premium,  $E_t(dR_t^A) - E_t(dR_t^B)$ .

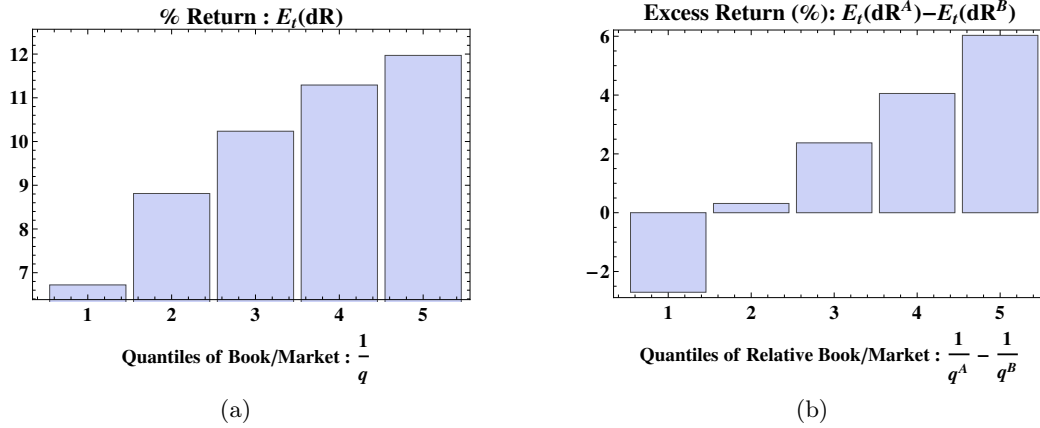


Figure 3: I calculate the data in the above figures as follows. At each point in time I pool the expected returns (excess returns) across all simulations. I then group these returns into five quintiles of Book-to-Market Equity,  $\frac{1}{q_t}$  (excess Book-to-Market Equity,  $\frac{1}{q_t^A} - \frac{1}{q_t^B}$ ). I then calculate the mean of the expected returns (expected excess return) in each quintile and then take the average over time.

<sup>33</sup>Instead of pooling both security returns, I can do this exercise for securities  $A$  and  $B$  separately. The positive relationship between BE/ME and expected returns is present but much weaker since the spread of expected return and BE/ME are smaller.



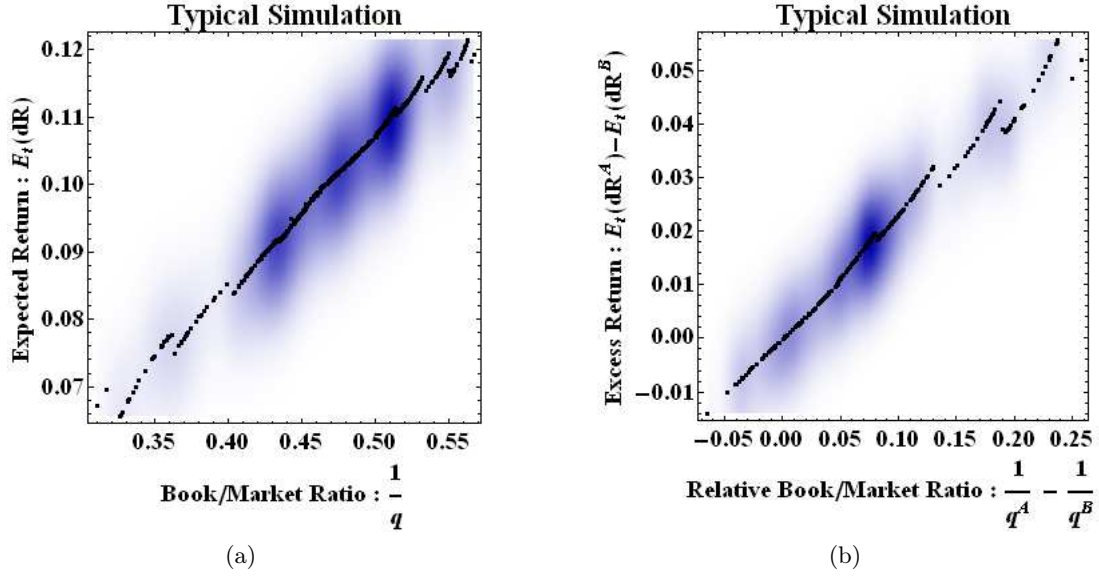


Figure 4: Here, I show a scatter plot of the expected return (expected excess return) against the BE/ME ratio (relative BE/ME ratio) for a given simulation. The dark shading shows where the observations are relatively more concentrated.

#### 4.2.2 The conditional CAPM

With log-utility ( $\gamma = (1 - \rho) = 1$ ) the price of risk is  $\sigma_{\Lambda,t} = \sigma_{c,t}$ . Furthermore, from **Corollary 1** since the consumption-to-total-wealth ratio is constant we can easily show that  $\sigma_{\Lambda,t} = \sigma_{c,t} = \varsigma_t^w$ . This implies that the conditional CAPM holds. In particular

$$\begin{aligned}
 E_t(dR_t^i) &= (r_t + \varsigma_t^w \cdot \varsigma_t^i) dt \\
 \Rightarrow E_t(dR_t^i) &= (r_t + \varsigma_t^w \cdot \varsigma_t^w \times \beta^{i,w}) dt
 \end{aligned} \tag{4.1}$$

$$\beta^{i,w} = \frac{\text{cov}_t(dR_t^i, dR_t^w)}{\varsigma_t^w \cdot \varsigma_t^w} = \frac{\varsigma_t^i \cdot \varsigma_t^w}{\varsigma_t^w \cdot \varsigma_t^w} \tag{4.2}$$

Thus, with log-utility the expected return on the traded securities is linear in their beta with the total wealth portfolio. In particular, if we regress the expected excess return of Security *A* against that security's excess beta on the total wealth portfolio scaled by the volatility of the wealth portfolio ( $\varsigma_t^w \cdot \varsigma_t^w (\beta_t^{A,w} - \beta_t^{B,w})$ ) we obtain a regression slope equal to one and an intercept that is not statistically different from zero. With log-utility this is purely mechanical. In column (1) of the first panel of **Table 4**, I show the results from running this regression. I calculate the regression coefficients and their standard errors (not shown) by taking the average and the standard deviation of

Conditional CAPM Regressions									
(1) Wealth Portfolio: $E_t(dR_t^A) - E_t(dR_t^B) = \underbrace{\alpha_0}_{\text{Pricing Error}} + \underbrace{\alpha_1}_{\text{Slope}} \times \varsigma_t^w \cdot \varsigma_t^w (\beta_t^{A,w} - \beta_t^{B,w}) dt$									
(2) Market Portfolio: $E_t(dR_t^A) - E_t(dR_t^B) = \underbrace{\alpha_0}_{\text{Pricing Error}} + \underbrace{\alpha_1}_{\text{Slope}} \times \varsigma_t^m \cdot \varsigma_t^m (\beta_t^{A,m} - \beta_t^{B,m}) dt$									
$\gamma = (1 - \rho) = 1$			$\gamma \neq (1 - \rho) = 1$			$\gamma > (1 - \rho) \neq 1$			
	(1)	(2)	Diff.	(1)	(2)	Diff.	(1)	(2)	Diff.
$\alpha_0$ (%)	0.00	1.1	-1.1	-0.12	1.4	-1.5	-0.06	1.9	-2.
t-stat	0.38	290.	-290.	-6.9	110.	-120.	-2.9	100.	-110.
$\alpha_1$	1.	0.97	0.04	3.	3.	0.01	3.4	0.09	3.3
t-stat	$1.3 \times 10^{16}$	290.	$1.3 \times 10^{16}$	790.	170.	620.	370.	120.	260.
$R^2$	1.	1.	0.00	1.	0.97	0.02	1.	0.97	0.03
t-stat	7000.	330.	1.5	330.	31.	0.78	410.	33.	0.95

Table 4: Each panel shows the regression results for three different specifications for preferences. To obtain the regression coefficients I simulate the model 10,000 times and run a regression for each simulation. The coefficients displayed are the average coefficients across simulations. I then calculate the standard errors as the standard deviation of the statistics across simulations. The t-stats are the ratio of the average statistic and their standard error. In the columns labelled (1), I include the results from the CAPM regressions of excess returns against excess betas on the wealth portfolio. In the columns labelled (2), I include the results from the CAPM regressions of excess returns against excess betas on the market portfolio. The column labelled *Diff* shows the difference between the statistics from regressions (1) and (2) as well as the corresponding t-stats.

the coefficients across simulations. The t-stat is the ratio of these two statistics. Column (2) shows the results from using the market portfolio rather than the total wealth portfolio. This allows us to quantify the effect of using the wrong proxy for the total wealth portfolio. As argued in [Stambaugh \(1982\)](#) and [Jaganathan and Wang \(1996\)](#) using the market portfolio as a proxy for the total wealth portfolio leads to concluding that the CAPM does not hold even when it does hold. This is because the market portfolio does not take into account the component of total wealth which is comprised by human capital and which is not traded. As we can see in the column labelled *Diff* from [Table 4](#), the pricing errors are significantly larger (-1.1% per year) when we do not take human capital into account.

With the second specification for preferences,  $\gamma \neq (1 - \rho)$  with  $\rho \rightarrow 0$  and  $f\{C, V\} = \beta(1 - \gamma)V \left( \ln(C) - \frac{1}{1-\gamma} \ln((1 - \gamma)V) \right)$ , the conditional CAPM does not hold ( $\sigma_{\Lambda,t} = \sigma_{c,t} - \sigma_v \neq \varsigma_t^w$ ). Nonetheless, when we run CAPM regressions with the total wealth portfolio the  $R^2$  is large, 100%, and the pricing error is not economically significant although it is statistically significantly different from zero. Thus, if we start from log-utility and increase the risk aversion the conditional CAPM still works very well if we use the total wealth portfolio. Using the market portfolio yields a lower  $R^2$  of 97% and a larger pricing error of 1.4%. Lastly, if agents have preference for earlier resolution of uncertainty,  $\gamma \neq (1 - \rho)$  and  $\rho > 0$ , the conditional CAPM does not hold and performs worse. When we use the market portfolio the pricing error is largest, 1.9% per year, and the  $R^2$  is 97%. With the

total wealth portfolio the pricing error is not economically significant and the  $R^2$  is very large. The  $R^2$ 's are overall fairly large because in running these regressions in the model  $\beta_t^{i,m}$ ,  $\beta_t^{i,w}$ ,  $\varsigma_t^m$ , and  $\varsigma_t^w$  are perfectly estimated. However if these variables were measured with error (as is indeed the case empirically), the  $R^2$  would be lower and the difference between the  $R^2$  from regressions (1) and (2) would be larger (See Section 7.13 of the Appendix). The results from Table 4 also imply that empirically any measure of risk which does not take human capital into account cannot explain the Value premium. Indeed, from Table 5 we can see that in data the market betas are not monotonically

BE/ME Quintiles				
Low	2	3	4	High
Univariate betas with Market Portfolio				
1.04	0.98	0.92	0.89	0.97

Table 5: The this table uses data from CRSP and Compustat and shows the coefficients from univariate regressions of monthly BE/ME quintile portfolio returns against the market portfolio returns.

increasing with the BE/ME quintiles<sup>34</sup>. Similarly, from the model the difference in the market betas is fairly small and hence cannot explain the Value premium. But the difference in total-wealth betas is much larger because it takes the difference in human capital risk into account (see Table 6). The difference in total-wealth betas is large enough to explain the Value premium entirely (log-utility) or almost entirely (stochastic differential utility). In practice, we use the unconditional CAPM

	Market Betas			Total-Wealth Betas		
	$\beta_t^{A,m}$	$\beta_t^{B,m}$	$\beta_t^{A,m} - \beta_t^{B,m}$	$\beta_t^{A,w}$	$\beta_t^{B,w}$	$\beta_t^{A,w} - \beta_t^{B,w}$
Mean	0.97	0.95	0.02	1.05	0.83	0.21

Table 6: In the table above, I show the market betas, the total-wealth betas as well as the relative betas of the Type A and Type B securities for preferences with  $\gamma > (1 - \rho) \neq 1$ .

regressions more often than the conditional CAPM. Jaganathan and Wang (1996) show that the conditional CAPM implies an unconditional CAPM with an additional factor. Thus the model's implications for the conditional CAPM should also apply to the corresponding unconditional CAPM with that additional factor<sup>35</sup>.

There is an additional interpretation of the model related to the results from Eberly and Wang (2009). In particular, with  $\sigma = \sigma_h$ ,  $\text{Corr}(\frac{dH}{H}, \frac{dK^A}{K^A})/dt = 1$ , and linear production we have that  $H$  and  $K^A$  are identical. Thus the key friction of the model is due to the concave function for capital growth (the adjustment cost). The expected growth rate of capital and the investment-to-capital

<sup>34</sup>The correct comparison should be done with a cross-sectional regression rather than the time-series regressions of Table 4. Nonetheless, by taking the average of both sides of the regressions in Table 4 we find that a large pricing error in time-series regressions,  $E_t(dR_t^i) = \alpha_0 + \alpha_1 \times \varsigma_t^m \cdot \varsigma_t^m (\beta_t^{i,m}) dt$  for  $i \in \{A, B\}$ , implies a large pricing error in the cross-sectional regression,  $E(dR^i) = \bar{\alpha}_0 + \bar{\alpha}_1 \times \varsigma^m \cdot \varsigma^m (\beta^{i,m}) dt$ .

<sup>35</sup>I discuss this implication in Section 5.2.

ratio are lower in Type  $A$  firms than Type  $B$  firms because both  $H$  and  $K^A$  incur adjustment costs and hence production in Type  $A$  firms ( $Y^A = \mathcal{A}(H + K^A)$ ) is less efficient<sup>36</sup> than in Type  $B$  firms ( $Y^B = \mathcal{A}K^B$ ). As a result, the capital stocks  $H$  and  $K^A$  are on average less valuable than  $K^B$ ; that is  $E(q^A) = E(p) < E(q^B)$ . This then implies all of the aforementioned results with the exception of the results concerning the conditional CAPM. In particular, with this interpretation of the model  $H$  is not human capital but rather any additional stock of capital with a separate accumulation process. As a result, we may no longer argue the returns on  $H$  are not taken into account in the market portfolio returns. Hence  $dR^m = dR^w$  and the conditional CAPM would work very well (and hold exactly with log-utility) in this interpretation of the model<sup>37</sup>.

#### 4.2.3 Fama and French (1997)

Fama and French (1997) show substantial variation in the Value premium both in aggregate and at the industry level. The model reproduces this results. Indeed in the distribution of expected returns in Figure 5a there is a significant region where the expected excess return of security  $A$  is negative. There is a 24.07% probability that  $E_t(dR_t^A) - E_t(dR_t^B) < 0$ . Similarly, there is a lot of variation in the BE/ME. And, from Figure 5b we can see that there is a non-trivial area in the distribution of excess BE/ME ( $\frac{1}{q_t^A} - \frac{1}{q_t^B}$ ) where firms of Type  $A$  behave more like Growth firms and have lower BE/ME. I plot the joint distribution for  $\{E_t(dR_t^A) - E_t(dR_t^B), \frac{1}{q_t^A} - \frac{1}{q_t^B}\}$  in Figure 5c. This figure confirms the strong positive correlation between the Value premium and the excess BE/ME (notice the elliptic contour plots). It also depicts the high degree of time-variation in the Value premium. There is a 24% probability that both  $E_t(dR_t^A) - E_t(dR_t^B) < 0$  and  $\frac{1}{q_t^A} - \frac{1}{q_t^B} < 0$ .

Figure 5 also implies that simply labeling securities  $A$  and  $B$  as Value and Growth (respectively) as done in Tables 2 and 3d yields conservatives estimates. That is because I do not calculate these moments as done in the empirical literature by first relabeling the Value and Growth stock each year using the BE/ME (portfolio formation). I provide the conservative values for more clarity and because they are not much different from the values calculated with annual portfolio formation. The exceptions are the Value premium for which I provide both the conservative estimate and the estimate with portfolio formation in Table 2 and the shares of total capital, which I discuss next. The model yields  $E(x^A) < E(x^B)$  but if instead I allow portfolio formation, I find that Value firms have

<sup>36</sup>If instead we use a Cobb-Douglas production for firms of Type  $A$ , the same mechanism persists and I conjecture that it is strengthened since  $Y^A = \mathcal{A}(K^A)^\alpha (H^A)^{1-\alpha}$  implies that  $K^A$  is less effective in contributing to total output ( $Y = Y^A + \mathcal{A}K^B$ ) compared to  $K^B$ . I also conjecture that the qualitative results would be unchanged if Type  $B$  firms also have a Cobb-Douglas production in two inputs,  $Y^B = \mathcal{A}(K^B)^\alpha (H^B)^{1-\alpha}$ , so long as  $cov\left(\frac{dH^A}{H^A}, \frac{dK^A}{K^A}\right) > cov_t\left(\frac{dH^B}{H^B}, \frac{dK^B}{K^B}\right)$ . This interpretation of the model is reminiscent of Tuzel (2010) where firms use structures and equipments in their production process and both inputs are subject to adjustment costs.

<sup>37</sup>There is some research which argues that the Value premium can be explained by intangible capital. However, intangible capital can be measured on the firms balance sheet and income statement (Goodwill or expenditures on R&D) or inferred from IPO's and M&A activities. Furthermore, in data Value firms tend to have less intangible capital (as measured by Goodwill or R&D) both in levels and relative to total assets compared to Growth firms.

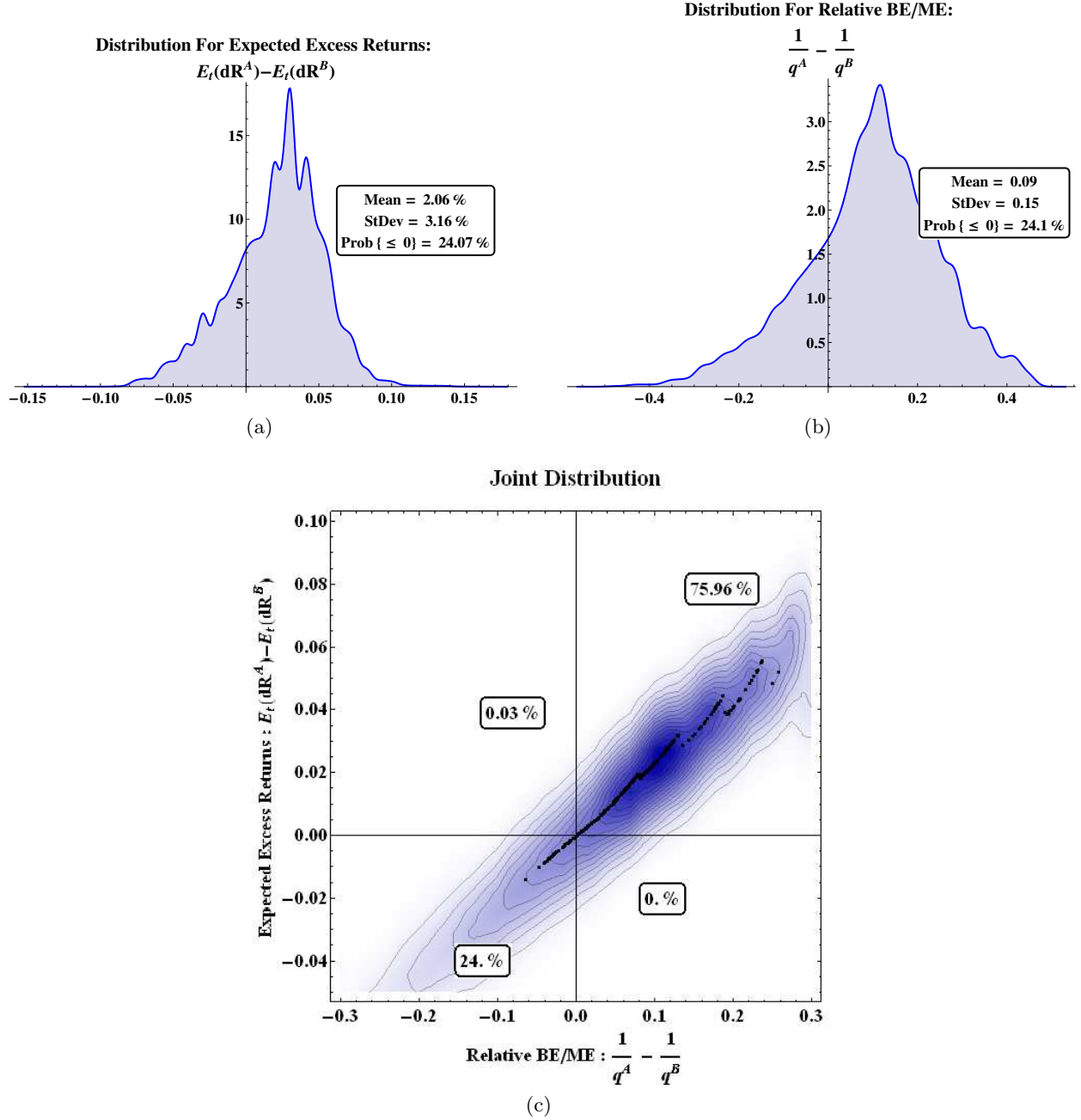


Figure 5: I pool all observations (across time and simulations) from the model for the expected excess return ( $E_t(dR_t^A) - E_t(dR_t^B)$ ) and the excess BE/ME ( $\frac{1}{q_t^A} - \frac{1}{q_t^B}$ ) to produce the distributions. In Figure 5c, I also include the scatter plot from one of the simulations as well as the probability associated with each quadrant of the joint distribution.

a larger share of total capital than Growth firms ( $E(x^V) = 0.43 > E(x^G) = 0.27$ ). This is akin to the empirical result from [Hansson \(2004\)](#) which shows that Value firms have greater levels of assets than Growth firms. In U.S. data, I find that the total-asset-to-nominal-GDP ratio for value firms

is on average 0.41 and for Growth firms it is 0.34. The discrepancy between  $E(x^A) < E(x^B)$  and  $E(x^V) > E(x^G)$  is due to endogenous reallocation (diversification incentive). As  $x^B \uparrow 1$  (and  $x^A \downarrow 0$ ), Type  $B$  physical capital becomes relatively abundant and hence less valuable,  $q^B \downarrow$ . Eventually if  $x^B$  is large enough ( $x^B \gg x^A$ ), Type  $B$  capital becomes less valuable than Type  $A$  capital,  $q^B < q^A$ . As a result,  $\frac{1}{q_t^A} - \frac{1}{q_t^B} < 0$  and  $E_t(dR_t^A) - E_t(dR_t^B) < 0$  as shown in Figure 5c and if we define Value and Growth stocks using BE/ME ( $1/q$ ), Security  $B$  is the Value stock. This also implies that the asset share of Value firms must be greater when we allow for portfolio formation than when we simply call Security  $A$  the Value stock ( $E(x^V) > E(x^A)$ ). That is simply because Security  $B$  is re-labelled Value when  $x^B \gg x^A$ . The investment-to-capital ratio for Value firms is lower than that of Growth firms even when Security  $B$  is the Value stock because  $\frac{1}{q^A} - \frac{1}{q^B} < 0 \Rightarrow \iota^B - \iota^A < 0$ . Lastly, we should note that without human capital risk  $E(x^A) = E(x^B)$  and although the realized capital shares in general are not equal ( $x_t^A \neq x_t^B$ ) they tend to be similar ( $x_t^A \approx x_t^B$ ) due to symmetry. Hence the portfolio formation mechanism by itself would yield a Value premium that is only slightly positive and close to zero.

#### 4.2.4 Impulse responses

**Theorem 3.** *The deterministic formulation of the model (with  $\sigma = \sigma_h = 0$ ) admits a steady state ( $dx_t^A = dx_t^B = 0 \quad \forall t$ ) where the prices of capital are equalized*

$$p = q^A = q^B$$

*the security returns are equal to the risk-free rate*

$$dR^A = dR^B = rdt = ((1 - \rho)\mu_c + \beta) dt$$

*the local growth rates of consumption, human capital, physical capital of types  $A$  and  $B$ , and aggregate output are equalized*

$$\frac{dC}{C} = \frac{dH}{H} = \frac{dK^A}{K^A} = \frac{dK^B}{K^B} = \frac{dY}{Y} = \mu_c dt = \left( \Gamma \ln \left( p \frac{\Gamma}{\theta} \right) - \delta \right) dt$$

*the capital shares,  $x^A$  and  $x^B$ , are indeterminate.  $p$  and  $F$  are constant and jointly solve*

$$\begin{aligned} p &= \frac{1}{\beta} \left( \frac{\mathcal{A} - \Gamma p + \theta}{F} \right)^{1-\rho} F \\ 0 &= \frac{\beta}{\rho} \left( \frac{\mathcal{A} - \Gamma p + \theta}{F} \right)^{\rho} - \frac{\beta}{\rho} + \Gamma \ln \left( p \frac{\Gamma}{\theta} \right) - \delta \end{aligned}$$

*The investment-to-capital ratios are equalized*

$$\iota^A = \iota^B = \iota^H = \Gamma p - \theta$$

*Proof.* I derive the above results in Section 7.14 of the Appendix by setting  $\sigma = \sigma_h = 0$  and  $dx_t^A = dx_t^B = 0$ . Thus, we need a stochastic model to generate greater average returns and greater book-to-market equity for Value firms. Furthermore,  $x^A$  and  $x^B$  are indeterminate in the deterministic steady state because of linear production and because  $H$ ,  $K^A$  and  $K^B$  are risk-less and have the same productivity,  $\mathcal{A}$ . Thus, it is possible to have that the share of total assets comprised by Value firms is zero in equilibrium ( $x^A = \frac{K^A}{H+K^A+K^B} = 0$ ). Therefore, the hedging motive (or diversification incentive) from the stochastic model along with the adjustment cost are what guarantee that the share of total assets comprised by Value firms does not go to zero on average. Indeed, I find  $E(x^A) = 0.313$  and  $E(x^B) = 0.374$  (see Figure 2a as well as the joint distribution for  $\{x^A, x^B\}$  in Figure 11 of Appendix 7.15 ).

To explore the diversification incentive and further understand the dynamics of the model and its implications for the cyclicity of some of the endogenous variables I produce some impulse responses in Figures 6 and 7. In particular, let  $\chi_t$  denote a given endogenous variable of the model. Following Koop, Pesaran and Potter (1996) and Kozak (2012), I construct two non-linear impulse responses for  $\chi_t$  using

$$IRF_{\chi} \{x_t^A, x_t^B\} = E \left( \chi_t \middle| x_0^A = \bar{x}^A, x_0^B = \bar{x}^B, dZ_0 = \mathbf{1}_i \right)$$

$$\mathbf{1}_i \in \left\{ \{1, 0\}', \{0, 1\}' \right\}$$

Thus, starting at the mean values for the state variables  $\{\bar{x}^A, \bar{x}^B\}$ , I introduce one-standard-deviation positive orthogonal shocks (through  $dZ^A$  or  $dZ^B$ ) at date zero and simulate twenty-five years of observations one hundred thousand times. I condition on a particular history following the date zero shocks as done in equation (6) of Koop, Pesaran and Potter (1996). I then compute the average across simulations. I define the impulse response relative to a baseline with no shock at date zero. As Koop, Pesaran and Potter (1996) and Kozak (2012) point out, using such generalized non-linear impulse responses is a correct way of constructing impulse responses for highly non-linear, state (and shock) dependent models. Using the traditional impulse response where we hit the economy with a shock at date zero and turn off all future shocks would yield spurious results.

From Figure 6, the investment-to-capital ratios  $\iota^A$  and  $\iota^H$  display a counter-cyclical evolution following  $dZ^A$  shocks. This is because a positive  $dZ^A$  shock increases the effective human capital and Type A physical capital. Hence, the value of capital ( $q^A$  and  $p$ ) and the investment-to-capital ratios ( $\iota^A = \Gamma q^A - \theta$  and  $\iota^H = \Gamma p - \theta$ ) decrease ceteris paribus. Similarly,  $\iota^B$  decreases following a positive  $dZ^B$  shock. The impulse responses in Figures 6a-6c also highlight the endogenous re-allocation due to the diversification incentive. That is,  $\iota^A$  and  $\iota^H$  increase following a positive  $dZ^B$  shock, and  $\iota^B$  increases following a positive  $dZ^A$  shock. This diversification incentive ensures that Value firms and Growth firms coexist in equilibrium.



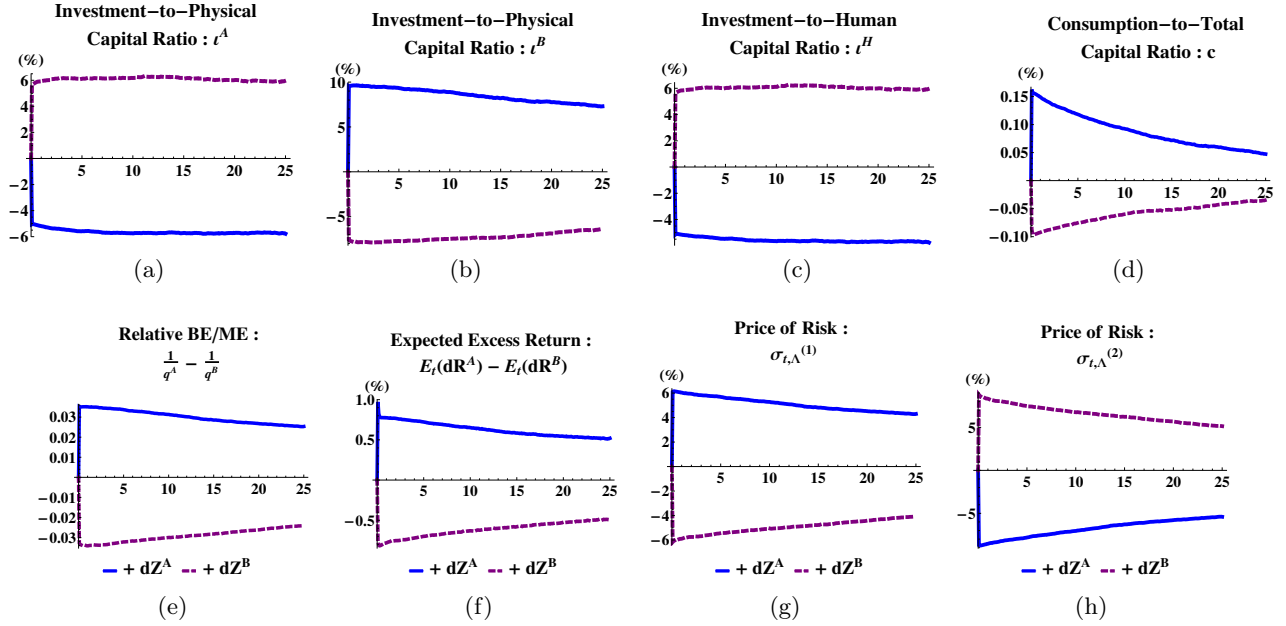


Figure 6: Starting at the mean values for the state variables  $\{\bar{x}^A, \bar{x}^B\}$ , I introduce a one-standard-deviation positive shock through  $dZ^A$  or  $dZ^B$  at date zero and simulate twenty-five years of observations 100,000 times. I then compute the average across simulations. I define the impulse response relative to a baseline with no shock at date zero.

A positive  $dZ^A$  shock leads to a relative decrease in the value of Type A physical capital ( $q^A$ ) since this capital is now relatively more abundant. Thus  $q^A - q^B$  decreases, hence the relative BE/ME ( $\frac{1}{q^A} - \frac{1}{q^B}$ ) increases (Figure 6e). The investment to capital ratios are linear in the Marginal Q's therefore, the relative investment to physical capital ratios,  $\iota^A - \iota^B$ , decreases. As shown before, an increase in the relative BE/ME is strongly associated with an increase in the expected excess return,  $E_t(dR_t^A) - E_t(dR_t^B)$ ; hence the results in Figure 6f. Hypothesis 2 of Zhang (2005) states that the Value premium, the relative BE/ME, and the price of risk are counter-cyclical<sup>38</sup>. To see if these hold in the model, I define a boom as a positive shock in both  $dZ^A$  and  $dZ^B$  and produce the impulse responses in Figure 7. Since the  $dZ^A$  shock enters both the law of motion of human capital and that of the Type A physical capital, it is twice as impactful as the  $dZ^B$  shock. Thus, I scale the  $dZ^A$  shock<sup>39</sup> by  $\alpha = \frac{1}{2}$ . The results are qualitatively identical for any  $\alpha < 1$ . I find that the Value premium, the relative BE/ME and the norm of the price of risk are indeed counter-cyclical in the model. In particular, the price of risk associated with  $dZ^A$  (that is the first element of the  $\sigma_{\Lambda,t}$  vector, the local volatility of the SPD) is counter-cyclical while the price of risk associated with  $dZ^B$ , that is  $\sigma_{\Lambda,t}^{(2)}$ , is pro-cyclical.  $\sigma_{\Lambda,t}^{(1)}$  is on average much larger than  $\sigma_{\Lambda,t}^{(2)}$  as I explain in Section 4.2.1.

<sup>38</sup>Zhang (2005) mentions that this hypothesis is consistent with the limited available evidence. It is difficult to test such hypotheses. Nonetheless, given the evidence for countercyclical risk-premia from Fama and French (1989), it should indeed be the case that the Value premium is countercyclical.

<sup>39</sup>The correct approach is to use a third Brownian Motion increment,  $dZ^H$ , in the law of motion of human capital with  $\text{Corr}(dZ^A, dZ^H)/dt \in (0, 1)$ . Then there would be no need to scale the  $dZ^A$  shock.



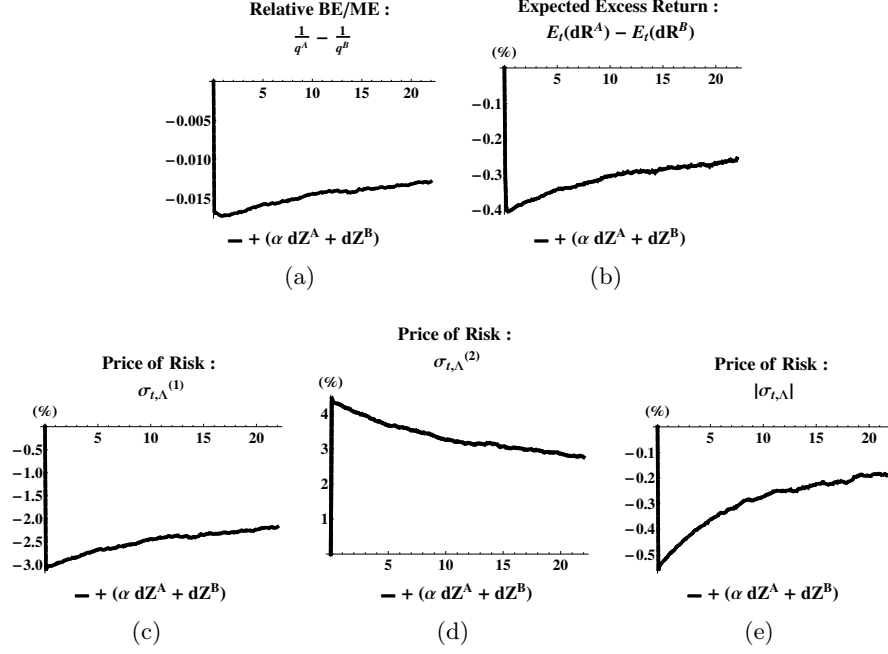


Figure 7: Impulse responses following a positive shock to both  $dZ^A$  and  $dZ^B$ .

Thus,  $|\sigma_{\Lambda,t}|$  (the norm of the local volatility of the SPD), which is a more comprehensive measure of the price of risk, is counter-cyclical (see Figure 7e). That is, in good times agents are willing to bear more risk by postponing current consumption and investing relatively more in the Value firm ( $\iota^A - \iota^B > 0$ ). The pro-cyclical relative investment-to-asset ratio implies that the relative BE/ME is countercyclical ( $\iota^A - \iota^B > 0 \Rightarrow \frac{1}{q^A} - \frac{1}{q^B} = \frac{\Gamma}{\iota^A + \theta} - \frac{\Gamma}{\iota^B + \theta} < 0$ ). Since relatively more capital flows towards Value firms in good times the excess equity return of Value firms (the Value premium) must be countercyclical. In the model, these result are endogenous whereas in Zhang (2005) they rely on an exogenously specified counter-cyclical stochastic discount factor.

## 5 Empirical Evidence and Testable Implications

There is a large literature which documents the strong link between human capital risk or returns on human capital (as measured by labor income) and the Value premium. In particular, Jaganathan and Wang (1996) show that controlling for returns on human capital helps explain the Value premium in CAPM regressions with time-varying betas. The authors define returns on human capital as the growth rate in per-capita labor income. Palacios-Huerta (2003a) uses a rich set of empirical variables to capture human capital and provides further evidence that taking returns on human capital into account explains the Value premium. Hansson (2004) uses Swedish data and shows that differences in wage growth explains much of the Value premium. Santos and Veronesi (2005) show that the ratio

of labor income to consumption explains a large portion of the Value premium. Below, I reproduce some of the results from this empirical literature while focusing on the predictions of the model.

## 5.1 Value firms, Growth firms and labor income risk

The key mechanism which gives rise to the Value premium in the model is the fact that there is relatively more human capital risk associated with Value firms,  $Cov\left(\frac{dH}{H}, \frac{dK^A}{K^A}\right) > Cov\left(\frac{dH}{H}, \frac{dK^B}{K^B}\right)$ . This assumption of the model implies that labor income growth covaries more with the asset growth of Value firms,  $Cov\left(\frac{d(wH)}{wH}, \frac{dK^A}{K^A}\right) > Cov\left(\frac{d(wH)}{wH}, \frac{dK^B}{K^B}\right)$ ; which itself implies that the equity returns of Value firms covary more with labor income growth,  $Cov\left(\frac{d(wH)}{wH}, dR^A\right) > Cov\left(\frac{d(wH)}{wH}, dR^B\right)$ . The model is flexible enough to allow me to re-formulate these two predictions with the following hypotheses; for which I provide some empirical support in [Tables 7-14](#).

**Hypothesis 1.** *Within Value firms the human capital of workers covary more the firms' assets and equity returns compared to Growth firms because Value firms have more firm-specific human capital and are hence more burdened by their wage bill.*

**Hypothesis 2.** *The human capital of all agents in the economy (regardless of their place of work) covaries more positively with the assets and equity returns of Value firms than with those of Growth firms because Value firms are relatively more distressed (as posited by [Fama and French; 1996](#)) and negative productivity shocks to such firms' assets tend to coincide with negative productivity shocks to human capital.*

In [Table 7a](#), I show the results from regressing the natural logarithm of firm assets against the natural logarithm of the wage bill as well as the wage bill times an indicator variable for Value firms. The data is in annual frequency and the wage bill is the sum of salaries, wages, pension costs, profit sharing and incentive compensation, payroll taxes and other employee benefits<sup>40</sup>. We can see that the coefficient on the interacted regressor is positive and highly significant. In particular, for Value firms log-wages covary more positively with log-assets than for Growth firms. A one percent increase in log-wages implies an additional 26.0 basis point increase in firm assets. I also present the results from regressing log-growth in assets against log-growth of the wage bill as well as the interaction of the latter with the Value firm indicator. Again we find a relatively more positive covariance between the asset growth and wage growth for value firms. A one percent increase in the wage growth coincides with an additional 11 basis point increase in asset growth for Value firms relative to Growth firms. The  $R^2$  is much smaller (0.29) since asset growth is a lot more volatile than wage growth. Thus, the equity return is increasing in the covariance between firm asset growth and human capital growth. This is akin to the results from [Mayers \(1972, 1973\)](#) and [Lieberman \(1980\)](#)

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<sup>40</sup>As measured by the *XLR* variable from the firm income statement in Compustat.

				$R^2$	#Obs.
(1)	$\ln (Asset)_{i,t} = a_1 \ln (Wage)_{i,t} + a_2 \ln (Wage)_{i,t} \times ValueDummy_{i,t} + \varepsilon_{i,t}$			0.89	16,998
(2)	$\Delta \ln (Asset)_{i,t} = a_1 \Delta \ln (Wage)_{i,t} + a_2 \Delta \ln (Wage)_{i,t} \times ValueDummy_{i,t} + \varepsilon_{i,t}$			0.29	14,798
		Coef.	Std.Err.	T-stat	P-value
(1)	$\ln (Wage)$	1.33	0.01	246.71	0.00
	$\ln (Wage) \times ValueDummy$	0.26	0.01	33.80	0.00
(2)	$\Delta \ln (Wage)$	0.48	0.01	53.83	0.00
	$\Delta \ln (Wage) \times ValueDummy$	0.11	0.01	7.75	0.00

(a) Panel (1) shows the results from regressing the natural logarithm of firm assets against the natural logarithm of the wage bill as well as the wage bill times an indicator variable for Value firms. Panel (2) shows the results from regressing the annual change in the natural logarithm of assets against the change in the natural logarithm of the wage bill as well as the interaction of the latter with the Value firm indicator. The wage bill is the sum of salaries, wages, pension costs, profit sharing and incentive compensation, payroll taxes and other employee benefits (As measured by the *XLR* variable from the firm income statement in Compustat).

	Labor Share ( $\frac{LaborExpense}{LaborExpense+Profits}$ )			Wage Growth ( $\Delta \ln (Wage)$ )		
	Coef.	T-stat	P-val	Coef.	T-stat	P-val
Value Firms - Growth Firms	0.16	1.71	0.09	-0.04	-12.22	0.00

(b) I calculate the difference in labor share (and wage growth) between Value and Growth firms by regressing the labor share (and wage growth) on a Value firm indicator variable and a constant. The labor share is firm labor expense divided by labor expense plus firm profits. Labor expense is the number of employees from Compustat times the average industry wage from the firm's industry. The industry wage is the employee compensation from NIPA Tables 6.2B and 6.2C divided by the number of full-time and part-time employees from NIPA Tables 6.4B and 6.4C. Firm profits are sales minus cost of goods sold and selling, general and administrative expenses. In the second panel, I define the wage growth as done Table 7a.

Table 7: All data is in annual frequency.

where human capital matters for asset pricing only to the extent that it covaries with returns from marketable securities<sup>41</sup>. Since Growth firms tend to hire more employees than Value firms, in Table 21 of Section 7.17 of the Appendix, I reproduce the regressions from Table 7a while controlling for the level (or growth rate) of employment. I again find that the coefficient on the interaction term is positive and highly significant. We can confirm that Value firms are indeed more exposed to human capital risk than Growth firms by comparing their labor share of profits<sup>42</sup>. In Table 7b, I follow the approach of Favilukis and Lin (2013a) and show that Value firms have greater labor share,  $\frac{LaborExpense}{LaborExpense+Profits}$ , than Growth firms. The difference in labor share is 16% and is significantly positively (the p-value is 0.09). This implies that Value firms are more burdened by their wage bill and are more labor (or human capital) intensive than Growth firms; as I posit in Hypothesis 1. From

<sup>41</sup>See equation (2) of Landskroner (1977).

<sup>42</sup>The labor share is defined as in Favilukis and Lin (2013a); firm labor expense divided by labor expense plus firm profits. Labor expense is the number of employees from Compustat times the average industry wage from the firm's industry. The industry wage is the employee compensation from NIPA Tables 6.2B and 6.2C divided by the number of full-time and part-time employees from NIPA Tables 6.4B and 6.4C. Firm profits are sales (*SALE*) minus cost of goods sold (*COGS*) and selling, general and administrative expenses (*XSGA*). I obtain the NIPA tables from EconStats.

Table 7b, we also find that Value firms have an average wage growth that is 4% lower than that of Growth firms.

The model predicts that the relative instantaneous covariance between firm assets and human capital (or labor income) growth causes the Value premium. Therefore in theory, if we could estimate this instantaneous covariance we could perfectly predict the Value premium. We could also replace the Fama-French HmL factor<sup>43</sup> with a different signal built from sorting securities by the instantaneous covariance between firm assets and human capital (or labor income). This approach should work at least as well as a sort based on BE/ME (as done when constructing the Fama-French HmL factor). Furthermore, this instantaneous covariance should perfectly predict the HmL factor (verifying that HmL is a proxy for human capital risk). In reality however, it is not possible to estimate this instantaneous covariance. In particular, data is not accurate and is available at low frequency. Data on publicly traded firms' expenses on employees as well as total assets is readily available on Compustat but only at an annual frequency. Thus, even though we could use a rolling ten-year window (that is ten data points) to calculate the covariance of firm assets with labor income the estimate we would obtain would be highly inaccurate. As a result, we would be using a low frequency and spurious factor to predict high frequency returns. This would not be a good way of predicting the Value premium. In Figures 4a and 4b, the model shows a better approach which can be done empirically with more accuracy and higher frequency. That is, using the relative Book-to-Market Equity: the Fama and French approach.

Nonetheless, if Value firms are indeed more exposed to human capital risk and the BE/ME is a proxy for human capital risk, we should be able to explain away the monotone increasing relationship between BE/ME and average equity returns with a dual sort. Before attempting this, I will first define the human capital returns,  $R_t^h$ , as done in Jaganathan and Wang (1996). I calculate the real monthly growth in labor income, using the aggregate labor income per capita less dividends from the Bureau of Economic Analysis NIPA Table 2.6 adjusted by the GDP deflator.  $R_t^h$  is the two-month moving average of the growth in labor income ( $L_t$ )

$$R_t^h = 100 \times \left( \frac{L_{t-1} + L_{t-2}}{L_{t-2} + L_{t-3}} - 1 \right) \quad (5.1)$$

This definition of the returns on human capital is common in the related empirical literature and reflects the dividend yield component of returns on human capital but does not fully reflect the capital gains component. This does lead to some bias. However, correcting for this bias would leave the qualitative results unchanged. Indeed, in the context of the model because  $Cov\left(\frac{dH}{H}, \frac{dK^A}{K^A}\right) > Cov\left(\frac{dH}{H}, \frac{dK^B}{K^B}\right)$ , we automatically have that the capital gains component of human capital returns covary more with the assets growth or equity returns of Value firms than Growth firms. Furthermore, in data (as I show below), because income per-capita covary more with assets (and equity returns) of Value firms and because the value of human capital is a present-discounted value of future income flows we should expect that Value firms returns covary more positively with the capital gains component of human capital than for Growth firms. Palacios-Huerta (2003a) shows

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<sup>43</sup>The excess equity return of Value relative to Growth firms.

that incorporating these two components<sup>44</sup> of returns on human capital into the CAPM strengthens the results (lower pricing errors and greater  $R^2$ 's). Lastly, since the model predicts that equity returns of Value firms covary more with labor income growth, the empirical proxy for returns on human capital as defined above is the correct object to use when testing this prediction of the model.

	Portfolio Level Human Capital		Aggregate Human Capital		~Obs
	$\beta_{i,h}$	P-val	$\beta_{i,h}$	P-val	
Value Portfolio	2.24	0.10	1.96	0.03	50
Growth Portfolio	1.26	0.53	0.82	0.38	50

	Firm Level Human Capital		Aggregate Human Capital		~Obs
	$\beta_{i,h}$	P-val	$\beta_{i,h}$	P-val	
Value Firms	0.01	0.78	0.95	0.00	38,000
Growth Firms	0.01	0.87	0.41	0.00	38,000

Table 8: In the first panel I show the univariate betas and p-values from regressions of annual returns from Value and Growth portfolios against annual portfolio-level human capital returns (value-weighted average of the annual growth in wage-per-employee for firms in the Value and Growth portfolios) as well as annual aggregate human capital returns (annual growth in aggregate labor income per capita). In The second panel I run similar regressions with annual returns from Value firms and Growth firms. The column labelled “~Obs” lists the approximate number of observations.

Table 8 provides additional evidence for [Hypotheses 1-2](#). The first panel shows the univariate betas and the associated p-values from regressions of annual returns<sup>45</sup> from Value and Growth portfolios against both portfolio-level human capital returns (value-weighted average of the annual growth in wage-per-employee for firms in the Value and Growth portfolios) and aggregate human capital returns (annual growth in aggregate labor income per capita less dividends). I find that Value firms are indeed exposed to relatively more portfolio-level ([Hypothesis 1](#)) and aggregate ([Hypothesis 2](#)) human capital risk since  $\beta_{i,h}$  is much larger ( $\beta_{i,h} \in \{2.24, 1.96\}$ ) and significantly positive (p-value less than 5%) for the Value portfolio. The corresponding values for the Growth portfolio are much smaller ( $\beta_{i,h} \in \{1.26, 0.82\}$  with p-values greater than 30%). In the second panel, I produce similar statistics from regressions of annual returns on firm-level equity returns against firm-level and aggregate human capital returns. Again, I find that Value firms have greater human capital betas than Growth firms. These firm-level regressions provide additional evidence in support of the second interpretation of the model from [Hypothesis 2](#). Indeed, in the regressions with firm-level human capital, the betas are not significantly different from zero for both Value and Growth firms. On the other hand, the aggregate human capital betas are more precisely estimated and much larger for Value than for Growth firms (0.95 and 0.41 respectively). This is evidence that the human capital of all agents in the economy (regardless of their place of work) covaries more with the outcomes of Value firms than with those of Growth firms. This may be why the Value premium can be empirically

<sup>44</sup>Dividend yields and capital gains.

<sup>45</sup>I obtain similar results with regressions of monthly equity returns against monthly human capital returns. I show these results in [Section 7.18](#) of the Appendix.

difficult to explain with firm or industry characteristics. It may also be why Jaganathan and Wang (1996) obtain strong results by using aggregate human capital returns in explaining equity returns and expected returns.

$\beta_h$ Quintiles					BE/ME Quintiles					
Low	2	3	4	High	$\beta_h$ Quintiles	Low	2	3	4	High
Expected Return					Low	-0.34	3.30	6.14	4.70	7.45
$E(dR_t)$					2	1.06	3.25	5.66	3.68	8.75
					3	3.07	3.10	4.06	6.71	4.63
					4	4.57	3.40	4.87	6.70	9.54
					High	9.33	6.49	5.95	7.48	8.21

(a) Mean annualized real monthly returns for  $\beta_h$ -quintile portfolios.

(b) This table shows the annualized average real monthly return of 25 dual-sort portfolios. The dual-sort portfolios are from groupings of securities by BE/ME quintiles and  $\beta_h$  quintiles. The shaded cells show where the monotone increasing relationship between BE/ME and average equity returns breaks down.

Table 9:  $\beta_h$  is the slope from a rolling 12 month univariate regression of equity returns against human capital returns.

Table 9 provides some evidence that BE/ME is indeed a proxy for human capital risk. Table 9a shows the mean annualized real monthly returns for  $\beta_h$ -quintile portfolios.  $\beta_h$  is the slope from a rolling twelve-month univariate regression of monthly equity returns against monthly human capital returns,  $R_t^h$  (as defined above), and the underlying  $\beta_h$ 's are updated monthly<sup>46</sup>. I find a monotone increasing relationship between equity returns and the  $\beta_h$ -quintiles. This is akin to the relationship between equity returns and BE/ME quintiles from Table 3a. In Table 9b, I show the annualized average real monthly return of twenty-five dual-sort portfolios. The dual-sort portfolios are from groupings of securities by BE/ME quintiles and  $\beta_h$  quintiles. In each row of the table, I shade in the cells where the monotone increasing relationship between BE/ME and average equity returns breaks down. Instead, this monotone relationship between returns and quintiles appears (and is very strong) along the diagonal of the table. Thus, there is a strong relationship between BE/ME quintiles and average equity returns in Table 3a because securities with relatively more human capital risk (as measured by  $\beta_h$ ) tend to have greater equity returns. This confirms that BE/ME may indeed be a proxy for human capital risk since high BE/ME firms tend to have greater  $\beta_h$  (see Tables 8 and 14).

### 5.1.1 Modified Value-Growth portfolio

The above results hint at a more precise way of constructing a portfolio that is long Value stocks and short Growth stocks (that is the HmL portfolio; high BE/ME minus low BE/ME). This may be useful to obtain a more precise Fama-French HmL factor or to obtain a portfolio which yields a

<sup>46</sup>I find similar results when I update the betas annually (see Section 7.19). Using a larger window for the rolling regressions is problematic because securities tend to switch BE/ME quintiles as we increase the window.

greater Value premium. Indeed, the usual definition of Value and Growth is based solely on BE/ME. However, BE/ME is available at best at an annual frequency hence the HmL portfolio is rebalanced annually. From Table 9b, we could instead construct two modified HmL portfolios which we can rebalance monthly since the human capital  $\beta$ 's are calculated monthly. In particular, Modified Portfolio 1 is short securities in the two lowest BE/ME and  $\beta_h$  quintiles and long securities in the two highest BE/ME and  $\beta_h$  quintiles. Modified Portfolio 2 is defined similarly but contains only securities on the diagonal of Table 9b. Table 10 provides an illustration of the portfolio construction. In Table 11, I show some descriptive characteristics of the modified and the original HmL portfolios.

$\beta_h$ Quintiles	BE/ME Quintiles				
	Low	2	3	4	High
Low	Short	Short			
2	Short	Short			
3					
4				Long	Long
High				Long	Long

Table 10: The modified HmL portfolios are long the Value portfolio and short the Growth portfolio. The Value portfolio is a value-weighted sum of securities in the cells labelled “Long”. The Growth portfolio is a value-weighted sum of securities in the cells labelled “Short”. Modified Portfolio 1 contains securities from all cells labelled “Long” or “Short”. Modified Portfolio 2 contains only securities from the shaded cells.

The modified portfolios hold relatively fewer securities since their dual-sort criteria is more stringent than the single-sort criteria for the original HmL portfolio. Nonetheless, both the long and short side of all three portfolios hold securities of similar size.

	Portfolio Components					
	Long Value			Short Growth		
	#Securities	%Small	%Big	#Securities	%Small	%Big
Port. 1	515.76	84.46	15.54	619.72	65.91	34.09
Port. 2	267.27	84.18	15.82	324.10	67.15	32.85
Orig.	973.08	84.69	15.31	1125.26	63.85	36.15

Table 11: The column labelled “#Securities” respectively show the average number of securities every month in the long and short legs of the corresponding portfolio. “%Small” shows the percentage of these securities that are small stocks (below median market value). “%Big” shows the percentage of these securities that are large stocks (above median market value).

Table 12 shows the returns characteristics of the modified portfolios as well as those of the original HmL portfolio and the market portfolio to facilitate comparisons. I use monthly equity returns in all calculations. Modified Portfolio 2 has the greatest average return (5.80% per year) and Sharpe Ratio (0.43). Modified Portfolio 1 has a similar performance with an average return of 5.59% and a Sharpe Ratio of 0.42. The Sharpe Ratios are greater than the corresponding values from the original HmL portfolio as well as the market portfolio. The columns labelled “CAPM Results” show the intercept ( $\alpha$ ) and slope coefficients ( $\beta$ ) from regressions of the portfolio returns



against the market. The market  $\beta$ 's are economically insignificant although they are statistically significant for the original portfolio as well as Modified Portfolio 2. Modified Portfolio 1 has the market  $\beta$  with the smallest magnitude ( $-0.03$ ) and it is not statistically significant. As expected, Modified Portfolios 1 and 2 have the largest  $\alpha$ 's ( $0.48\%$  and  $0.51\%$  monthly, respectively). The last panel of the table shows correlations between all four portfolios. Modified Portfolios 1 and 2 are not highly correlated with the original portfolio (these correlations are  $0.54$  and  $0.45$  respectively). This highlights the fact that these portfolios offer notable diversification potential. All correlations with the market portfolio are fairly small, as implied by the aforementioned market  $\beta$ 's. The two modified HmL portfolios have a high correlation of  $0.89$ .

Portfolio Characteristics From Monthly Returns									
	Annualized (%)			CAPM Results		Correlations			
	$E(R)$	$\sigma(R)$	S.R.	$\alpha$	$\beta$	Port. 1	Port. 2	Orig.	Mkt.
Port. 1	5.59	13.30	42.03	0.48 [3.00]	$-0.03$ [ $-0.84$ ]	1.00	0.89	0.54	$-0.04$
Port. 2	5.70	13.37	42.63	0.51 [3.20]	$-0.08$ [ $-2.29$ ]		1.00	0.45	$-0.10$
Orig.	4.08	10.60	38.49	0.38 [3.02]	$-0.09$ [ $-3.23$ ]			1.00	$-0.13$
Mkt.	7.16	15.45	40.52	—	—				1.00

Table 12: Calculations are done with monthly portfolio returns. Returns are annualized by multiplying the mean by 12 and the standard deviations by  $\sqrt{12}$ .

Table 13 reproduces the calculations from Table 12 using annual returns. The market  $\beta$ 's are a little more negative but nonetheless not economically important. The Sharpe Ratios from the modified HmL portfolios are larger while those of the original HmL portfolio and the market portfolio are smaller. Figure 8 compares the returns of the original portfolio to Modified Portfolios 1 and 2. The modified portfolios tend to have lower down-swings particularly around the years 1980 and 2000. Figure 8c shows that the cumulative performance of the modified HmL portfolios has been much more attractive than that of the original HmL portfolio.



Portfolio Characteristics From Cumulative Annual Returns									
	( $\%$ )			CAPM Results		Correlations			
	$E(R^e)$	$\sigma(R^e)$	S.R.	$\alpha$	$\beta$	Port. 1	Port. 2	Orig.	Mkt.
Port. 1	5.83	12.85	45.37	6.56	-0.14	1.00	0.83	0.62	-0.13
				[10.82]	[-4.00]				
Port. 2	5.94	13.13	45.24	7.39	-0.27		1.00	0.54	-0.31
				[12.86]	[-8.28]				
Orig.	4.70	14.15	33.22	5.83	-0.25			1.00	-0.29
				[10.27]	[-7.87]				
Mkt.	7.56	16.74	39.79	—	—				1.00

Table 13: Calculations are done with (twelve-month cumulative) annual portfolio returns.

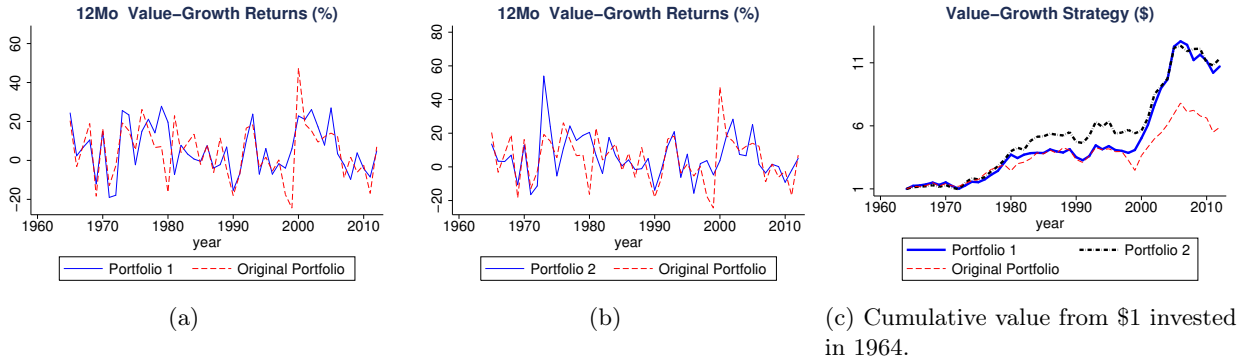


Figure 8: Portfolio Performance

## 5.2 Conditional and unconditional CAPM

From Section 4.2.2, the model implies that conditional CAPM regressions should work well in describing the returns of Value and Growth stocks once we take returns on human capital into account. To verify this implication in data, I first construct twenty portfolios using the twenty quantiles of Book-to-Market Equity from the Kenneth French Data Library. I use the standard approach (described in Section 4.1) to construct these portfolios. First, I consider the static (unconditional) CAPM. Using the Fama-MacBeth two-step approach, I regress each of the twenty test portfolios returns,  $R_{i,t}$ , against the market return,  $R_t^m$ , and collect the slope coefficients,  $\beta_{i,m}$  for  $i \in \{1, \dots, 20\}$ . I then regress the mean returns of the portfolios against the betas with the market.

$$E(R_i) = \gamma_0 + \gamma_m \beta_{i,m} \quad (5.2)$$

My goal is to test whether the market portfolio returns are a good proxy for the wealth portfolio returns or whether the market portfolio returns need to be augmented with returns on human capital to obtain a good proxy for the wealth portfolio. Recall from [Section 4.2.2](#) when the conditional CAPM holds (with log-utility) we have

$$\begin{aligned} E_t(dR_t^i) &= (r_t + \varsigma_t^w \cdot \varsigma_t^w \times \beta^{i,w}) dt \\ \Rightarrow E_t(dR_t^i) &= (r_t + \beta^{i,w} \times E_t(dR_t^w/dt - r_t)) dt \end{aligned} \quad (5.3)$$

Thus, if the unconditional CAPM holds we can test the validity of the proxy for the wealth portfolio by verifying that  $\gamma_0$  equals the risk-free rate and that  $\gamma_m$  equals the market premium. Using the definition of  $\beta^{i,w}$  from (4.2) and  $dR_t^w$  from [Section 3](#), we can re-write (5.3) as a Human Capital Asset Pricing Model (HCAPM):

$$E_t(dR_t^i) = \left( r_t + \beta^{i,m} \times \frac{S_t^m}{S_t^w} E_t(dR_t^m/dt - r_t) + \beta^{i,h} \times \frac{S_t^H}{S_t^w} E_t(dR_t^H/dt - r_t) \right) dt \quad (5.4)$$

This leads me to instead implement a weaker test using the steps below

1. Run the regression in (5.2) as well as

$$E(R_i) = \gamma_0 + \gamma_m \beta_{i,m} + \gamma_h \beta_{i,h} \quad (5.5)$$

where  $\beta_{i,h}$  is the coefficient from a univariate regression of portfolio  $i$ 's returns on  $R_t^h$  from (5.1)

2. Determine which of the regression specifications yields a  $\gamma_0$  that is not statistically different from the average three-month T-Bill rate<sup>47</sup>,  $R_f$
3. Determine whether both slope coefficients  $\gamma_m$  and  $\gamma_h$  in (5.5) are significant and positive or whether  $\beta_{i,m}$  is enough to explain the cross section of average returns
4. Determine which of the regressions (5.2 or 5.5) yields the largest coefficient of determination,  $R^2$

[Jaganathan and Wang \(1996\)](#) show that conditional CAPM implies an unconditional CAPM with an additional factor. In particular, the authors suggest the following cross-sectional regression for the conditional CAPM

$$E(R_i) = \gamma_0 + \gamma_m \beta_{i,m} + \gamma_{prem} \beta_{i,prem} \quad (5.6)$$

where  $\beta_{i,prem}$  is the coefficient from a univariate regression of portfolio  $i$ 's returns on the corporate spread. The latter is difference in the average monthly returns between corporate bonds with a

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<sup>47</sup>I obtain this data from the Kenneth French Data Library.

BAA- rating and those with a AAA ratings as provided by the Board of Governors of the Federal Reserve System. Assuming that the conditional CAPM holds, I again repeat the steps above and compare the regression in (5.6) to

$$E(R_i) = \gamma_0 + \gamma_m \beta_{i,m} + \gamma_{prem} \beta_{i,prem} + \gamma_h \beta_{i,h} \quad (5.7)$$

in order to determine whether using returns on human capital yields a better proxy for the wealth portfolio. From the two panels with non-log-utility preferences in Table 4, even if the conditional CAPM only holds approximately we may still use the aforementioned test to determine whether human capital returns allows us to better explain returns and expected returns on BE/ME sorted portfolios.

Quantiles of BE/ME										
	Low	2	3	4	5	6	7	8	9	10
$\beta_{i,h}$	0.04	0.01	0.23	-0.07	0.09	0.03	0.32	0.70	0.24	0.57
$\beta_{i,m}$	1.08	1.00	1.05	1.09	1.02	0.99	0.98	0.97	1.00	0.91
$\beta_{i,prem}$	0.39	0.51	0.61	0.58	0.60	0.49	0.58	0.54	0.63	0.66
$E(R_i)$	0.46	0.53	0.64	0.58	0.59	0.53	0.61	0.61	0.69	0.72
	11	12	13	14	15	16	17	18	19	High
$\beta_{i,h}$	0.68	0.27	0.31	0.82	0.98	0.75	1.12	0.71	0.84	1.11
$\beta_{i,m}$	0.92	0.91	0.93	0.91	0.94	0.95	1.01	0.93	0.98	1.12
$\beta_{i,prem}$	0.72	0.61	0.65	0.61	0.70	0.64	0.81	0.82	0.67	1.11
$E(R_i)$	0.83	0.66	0.69	0.73	0.84	0.81	1.00	0.87	0.78	1.29

Table 14: The first three rows show the coefficients from univariate regressions of the returns on the corresponding BE/ME quantile portfolio against the returns on human capital, the market or the corporate spread (respectively). The last row shows the average return of the portfolios. All returns are in real monthly units.

In Table 14, I show some characteristics of the twenty test portfolios. In particular, I show their human capital betas ( $\beta_{i,h}$ ), market betas ( $\beta_{i,m}$ ), corporate spread betas ( $\beta_{i,prem}$ ) from univariate regressions, as well as the their average real monthly return. We can first notice that human capital betas tend to be relatively greater for the high BE/ME portfolios as the model predicts. Indeed, this is empirical evidence that Value firms are relatively more exposed to aggregate human capital risk as implied by Hypothesis 2. There is a wide spread in these betas,  $\beta_{i,h} \in [-0.07, 1.12]$ . Similarly, there is a fairly wide spread in the betas with the corporate spread. Indeed,  $\beta_{i,prem}$  ranges from 0.39 to 1.11. However, the betas with the market,  $\beta_{i,m}$ , remain fairly flat around 1. As a result, the market beta by itself is not able to explain the spread in average returns on the test portfolios. These average returns are on the last row of Table 14 and tend to be greater for higher BE/ME quantiles.

In Table 15, I report the CAPM regressions results. The top two panels show the regression re-

sults for the unconditional CAPM with and without including returns on human capital. The bottom

Unconditional CAPM									
$E(R_i) = \gamma_0 + \gamma_m \beta_{i,m} + \gamma_h \beta_{i,h}$									
	Coef	T-stat	P-val	Bootstrap		Clustered Bootstrap		$Z(\gamma_0 - R_f)$	$R^2$
				Z-stat	P-val	Z-stat	P-val	0.61	0.00
$\gamma_0$	0.57	0.83	0.42	0.77	0.44	0.66	0.50		
$\gamma_m$	0.15	0.22	0.83	0.20	0.84	0.17	0.87		
	Coef	T-stat	P-val	Z-stat	P-val	Z-stat	P-val	−0.80	0.77
$\gamma_0$	−0.30	−0.85	0.41	−0.57	0.57	−0.62	0.54		
$\gamma_m$	0.82	2.32	0.03	1.50	0.13	1.57	0.12		
$\gamma_h$	0.44	7.63	0.00	4.16	0.00	4.21	0.00		
Conditional CAPM									
$E(R_i) = \gamma_0 + \gamma_m \beta_{i,m} + \gamma_{prem} \beta_{i,prem} + \gamma_h \beta_{i,h}$									
	Coef	T-stat	P-val	Bootstrap		Clustered Bootstrap		$Z(\gamma_0 - R_f)$	$R^2$
				Z-stat	P-val	Z-stat	P-val	−0.40	0.93
$\gamma_0$	0.03	0.17	0.87	0.13	0.95	0.16	0.88		
$\gamma_m$	−0.12	−0.65	0.52	−0.46	0.64	−0.47	0.64		
$\gamma_{prem}$	1.25	16.51	0.00	9.28	0.00	8.05	0.00		
	Coef	T-stat	P-val	Z-stat	P-val	Z-stat	P-val	−1.58	0.98
$\gamma_0$	−0.15	−1.50	0.15	−0.85	0.40	−0.96	0.34		
$\gamma_m$	0.20	1.83	0.09	1.00	0.32	1.00	0.32		
$\gamma_{prem}$	0.93	14.11	0.00	7.94	0.00	6.57	0.00		
$\gamma_h$	0.16	6.32	0.00	4.43	0.00	3.77	0.00		

Table 15: The top two panels show the regression results for the unconditional CAPM with and without including returns on human capital. The bottom two panels show the results for conditional CAPM regressions. I include the usual t-distributed test statistics and p-values as well the bootstrap normally-distributed test statistics and p-values. I calculate the bootstrap results using 10,000 repetitions with and without clustering at the year level. The column labelled  $Z(\gamma_0 - R_f)$  shows the normally-distributed bootstrap statistic for testing if  $\gamma_0$  is significantly different from the three-month T-Bill rate,  $R_f$ .

two panels show the results for conditional CAPM regressions. I include the usual t-distributed test statistics and p-values. In order to correct for bias due to estimation errors from the Fama-Macbeth two-step approach, I provide the bootstrap normally-distributed test statistics and corresponding p-values. I calculate the bootstrap results using ten-thousand repetitions. It is well documented that equity returns display some autocorrelation (momentum and reversal). As a result, I also provide the statistics from a bootstrap with clustering at the year level. The column labelled  $Z(\gamma_0 - R_f)$  shows the normally-distributed bootstrap statistic for testing if  $\gamma_0$  is significantly different from the three-month T-Bill rate,  $R_f$ . First, we note that  $\gamma_0 - R_f$  is not significantly different from zero for

all four regression specifications. Nonetheless, the unconditional CAPM regression without human capital performs worst. Indeed, the  $R^2$  is less than one percent. Thus the market betas alone cannot explain the spread in average returns. The slope coefficient,  $\gamma_m$ , has a test statistic of 0.22 and is slightly lower with the bootstrap approach. Hence,  $\gamma_m$  is not significantly different from zero. In the second panel where I include the betas on human capital as an additional regressor the unconditional CAPM performs a lot better. The  $R^2$  is much larger ( $R^2 = 0.77$ ).  $\beta_{i,m}$  plays a relatively more important role in explaining the average returns since  $\gamma_m$  now has p-values ranging from 0.03 to 0.13. Furthermore,  $\gamma_m$  has the correct sign ( $\gamma_m > 0$ ). The regression coefficient for the human

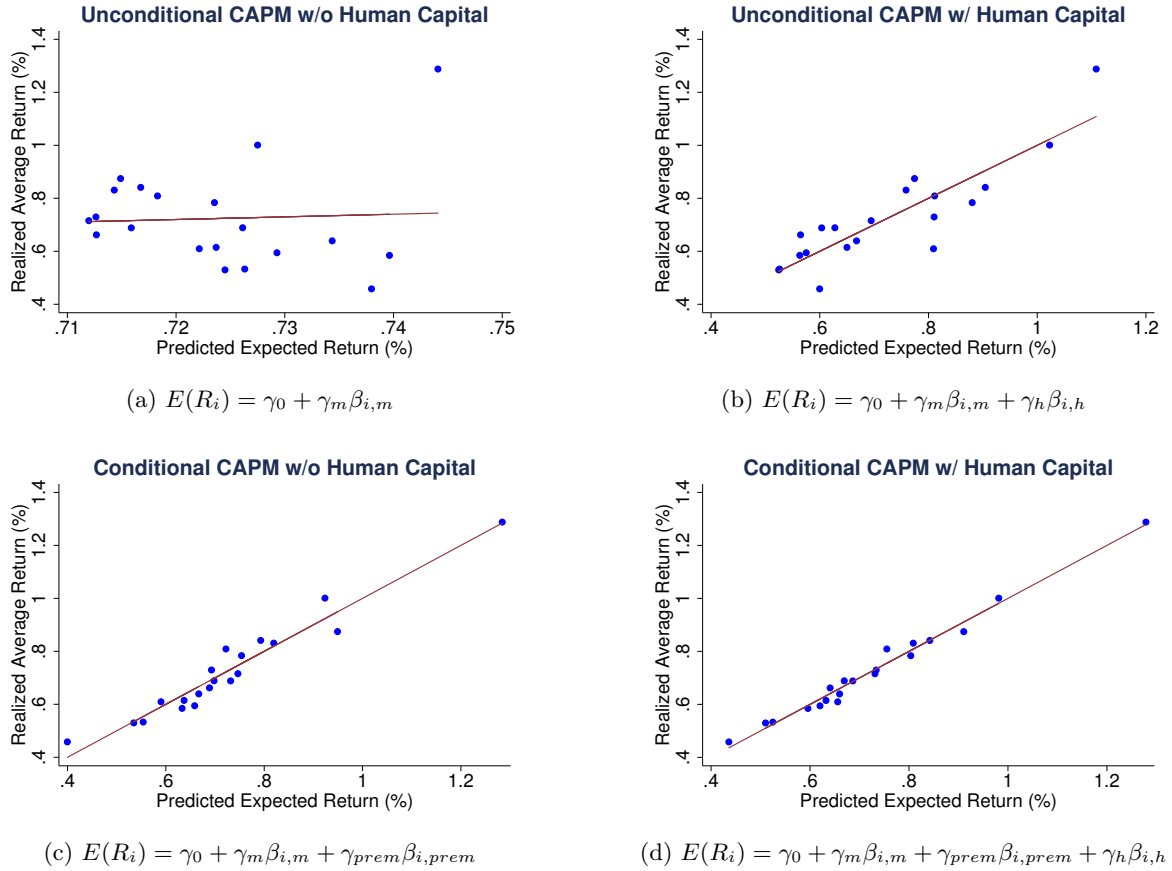


Figure 9: Each graph displays a scatter plot of the realized average monthly return (in real units) on the twenty test portfolios versus the predicted returns from the corresponding cross-sectional regression.

capital betas,  $\gamma_h$ , is highly significant and positive even with the bootstrap and clustered bootstrap corrections. In Figures 9a-9b, I provide scatter plots of the average portfolios returns and the predicted returns from the unconditional CAPM regressions. These figures confirm that we indeed need to take returns on human capital into account to obtain a good proxy for the wealth portfolio and predict returns. The conditional CAPM regressions have greater  $R^2$  than the unconditional CAPM regressions. This is partly because they include one more regressor. However, the adjusted  $R^2$ 's (not

shown) are also greater. Thus the conditional CAPM regressions provide a relatively better description of returns and average returns than the unconditional CAPM. Jaganathan and Wang (1996) reach a similar conclusion. The conditional CAPM regression without human capital works well. However, when we include the human capital betas, the market betas have a coefficient that is more significant and has the correct sign ( $\gamma_m > 0$ ). Indeed,  $\gamma_m$  has a t-stat and p-value of 1.83 and 0.09 (respectively) compared to  $-0.65$  and  $0.52$  when we omit returns on human capital. The bootstrap test statistics are much lower in both regressions and their p-values are much larger (0.32). More importantly, the coefficient on the human capital betas,  $\gamma_h = 0.16$ , is highly significant and positive in the conditional CAPM regression even with the bootstrap correction. Its p-values are much lower than one-percent. This again confirms the model’s implications from Table 4 that human capital returns helps predict security returns and expected returns above and beyond the contribution of the market portfolio returns. Lastly, by comparing Figures 9c and 9d we see that including human capital returns in the conditional CAPM regressions yields a slightly better fit.

### 5.3 Optimal production choices

One additional implication of the model is that on average Value firms have lower investment-to-capital ratio than Growth firms. Indeed, on average  $q^A < q^B$  therefore  $\iota^A = \Gamma q^A - \theta < \iota^B = \Gamma q^B - \theta$ . In particular, the model yields  $E(\iota^A) = 2.78\%$  and  $E(\iota^B) = 4.30\%$ . On Table 16, I report the mean annual investment-to-asset ratio for firms grouped in BE/ME quintiles as well as the difference between Value and Growth<sup>48</sup>. This table confirms that Value firms indeed tend to have relatively

	BE/ME Quintiles					V-G
	Low	2	3	4	High	
Acquisitions	0.023	0.024	0.020	0.015	0.010	-0.012
CapEx	0.077	0.071	0.064	0.059	0.050	-0.024
Net Investments	0.097	0.085	0.074	0.062	0.041	-0.048
$\Delta \text{Ln}(Asset)$	0.151	0.111	0.090	0.072	0.040	-0.092

Table 16: The first five columns shows conditional means for each of the BE/ME quintiles. The last column shows the mean differences between Value and Growth firms which I calculate as the slope coefficient from a regression of the variable of interest on a constant and an indicator variable for Value firms. The first four rows show the results for investment to capital ratios. I define the investment-to-capital ratio as the ratio of firm Acquisitions to total assets, the ratio of capital expenditures (CapEx) to total assets, and lastly, the ratio of net cash flows from all investments activities (where decreases in cash are treated as positive numbers) to total assets. The last row show results for the growth rate of total assets (annual difference in the natural logarithm of total assets). All data is in annual units and all values are significantly different from zero at a one percent level.

lower investment-to-capital ratios. In particular, from the last column, the total acquisitions-to-asset ratio and capital expenditures-to-asset ratio are 1.2% and 2.4% lower for Value firms. Using the more

<sup>48</sup>This table was inspired by Table 2 of Bazdresch, Belo and Lin (2013).

comprehensive measure<sup>49</sup>, net investment-to-capital ratio, shows that investment-to-capital ratio is 4.8% lower for Value firms. The first five columns of the table shows that there is a decreasing and monotone relationship between the investment-to-asset ratio or asset growth and the Book-to-Market Equity. These data are all significantly different from zero (at a one percent level). In the last row of the table, I show that Value firms have significantly lower asset growth than Growth firms. Zhang (2005) has similar implications. His results rely on asymmetric adjustment costs (it is relatively more costly to dis-invest), firm heterogeneity and an exogenously specified counter-cyclical stochastic discount factor. In the model that I present, other than the positive covariance  $Cov\left(\frac{dH}{H}, \frac{dK^A}{K^A}\right)$ , the two types of firms are identical. Thus, I can reproduce some of the results of Zhang (2005) with a fully endogenous model with human capital risk. Figure 10 shows scatter plots for a given simulation of the model. It further emphasizes that the model replicates the important

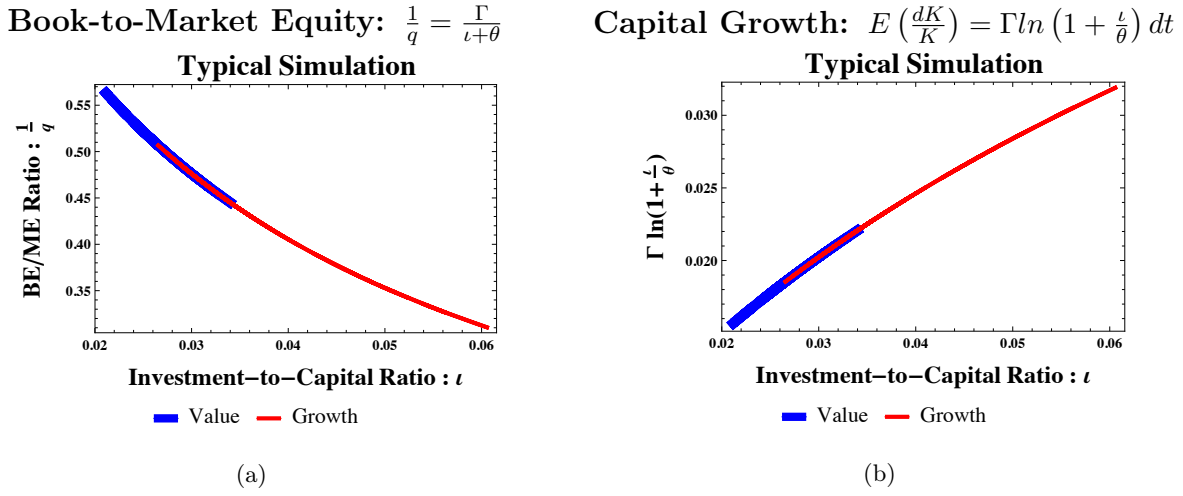


Figure 10: In this figure, I show a scatter plot of the book-to-market ratio and expected growth ( $\Gamma \ln\left(1 + \frac{\iota}{\theta}\right)$ ) in physical capital for Value and Growth firms versus the investment-to-capital ratio.

differences in optimal production choices for Value and Growth firms that we observe in data (See Table 16). Value firms tend to have lower investment-to-asset ratio, greater Book-to-Market Equity, and lower asset growth than Growth firms. The scatter plots trace out exact functions relating the BE/ME or the expected capital growth to the investment-to-capital ratio; that is  $\frac{1}{q} = \frac{\Gamma}{\iota + \theta}$  and  $E\left(\frac{dK}{K}\right) = \Gamma \ln\left(1 + \frac{\iota}{\theta}\right) dt$  (respectively). The model yields an average asset growth of 1.92% and 2.58% for the Type A and B firms respectively. We would obtain the same qualitative results with any other specification for the adjustment cost. Zhang (2005) produces similar results but he relies heavily on firm heterogeneity and asymmetric adjustment costs.

<sup>49</sup>The Net-Investment is the negative of the Investing Activities Net Cash Flow variable (ICNCF) from the Cash Flow items in Compustat. Net-Investment = Increase in Investments (IVCH) + Capital Expenditures (CAPX) + Acquisitions (AQC) - Sale of Investments (SIV) - Short Term Investments Change (IVSTCH) - Sale of Property (SPPE) - Investing Activities Other (IVACO).

## 5.4 Some related research

One of the implications of the model is that firms with relatively more firm specific human capital tend to have greater equity returns. This goes against some related theoretical results in recent research. First, [Belo and Lin \(2012\)](#) argue that firms in skill-intensive industries tend to have greater equity returns. In their model, it is more costly for firms to invest in (hire) high-skilled workers due to training cost, etc. As a result, firms in skill-intensive industries face additional risk; hence their excess equity return. However, this skill-intensive return spread is statistically significant only for small firms. Furthermore, it is not clear whether high-skill or low-skill workers' human capital tend to be more specific to firms. A custodian or a computer programmer may easily switch industry but a low-skilled worker in an assembly line or a nuclear physicist may not. Thus, the relative labor skill intensity may not be the appropriate way of measuring the relative firm specificity of human capital which is in my model.

On the other hand, [Donangelo, Eiling and Palacios \(2010\)](#) and [Donangelo \(2013\)](#) argue that firms with relatively less labor mobility tend to have lower equity returns. Firm specific human capital tend to be less mobile therefore the theoretical results in these papers is contrary to that of my model. These authors argue that labor mobility is an additional risk to firms because it allows workers to switch industries following negative shocks; which exacerbates the impact of these negative shocks on said industries. However, [Donangelo \(2013\)](#) uses a partial equilibrium model with exogenously specified stochastic discount factor, labor supply and wage processes. Furthermore, the conditional CAPM holds in [Donangelo \(2013\)](#). This is counterfactual per the results from [Jaganathan and Wang \(1996\)](#) and [Palacios-Huerta \(2003a\)](#). Although [Donangelo, Eiling and Palacios \(2010\)](#) use a general equilibrium model, labor is exogenously and inelastically supplied. The productivity growth (and hence the state variable) is exogenous as well. The labor supply side plays a similarly minimal role in [Belo and Lin \(2012\)](#). In my model, human capital is endogenously supplied by agents who dynamically choose how much to invest in their stock of human capital. Thus, although some of the theoretical results of my model go against those from the aforementioned research it is not clear whether such a comparison is appropriate since [Donangelo, Eiling and Palacios \(2010\)](#), [Belo and Lin \(2012\)](#) and [Donangelo \(2013\)](#) focus more on the production side of the economy and make assumptions that are very different from those in my model. My results are somewhat similar to those from [Garleanu, Kogan and Panageas \(2012\)](#) where Value firms are relatively more exposed to innovation risk. In [Garleanu, Kogan and Panageas \(2012\)](#), Value firms are less able to acquire new blueprints<sup>50</sup> compared to Growth firms. As a results, Growth firms provide a hedge against future innovations. These innovations tend to lead to a displacement of older workers as they are less able to learn and work with newer technologies than younger workers. Thus, Value firms are relatively more exposed to displacement risk; hence a Value premium in equity returns.

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<sup>50</sup> The right to produce new intermediary goods.



Lastly, some of the recent empirical evidence from [Betermier, Calvet, and Sodini \(2013\)](#) support the results from my model. In particular, these authors use a large administrative panel dataset on Swedish investors and find that investors with relatively more wealth and less income (human capital) risk tend to invest relatively more in Value stocks. On the other hand, investors that are less wealthy, less educated and have more income (human capital) risk hold relatively more Growth stocks in their portfolio.

## 6 Conclusion

There is increasing interest in studying the asset pricing implications of investment in physical capital. I complement the existing literature by investigating the joint effects of investment in physical and human capital on asset pricing. I produce a general equilibrium model with endogenous growth where human capital risk explains the Value Premium. Thus the Fama-French HmL factor is a proxy for some macro and financial variables reflecting human capital risk. The model allows us to identify some of these variables as the covariance between Human capital, or labor income growth with the growth rate of firm assets. More specifically, in the model the Value premium arises from the fact that Value firms are associated with more risk to human capital. This is because Value firms have relatively more firm-specific human capital and are hence more burdened by their wage bill and also because human capital (both aggregate and firm-level) is more dependent on the fate of Value firms and hence covary more positively with the assets of Value firms. As a result, a negative productivity shock to Value firms' assets implies a negative productivity shock to human capital. Consequently, these firms are less valuable, have greater Book-to-Market Equity and greater expected equity returns. I provide empirical evidence in support of these model implications.

The model also predicts that for Value firms wage growth is relatively more positively correlated with asset growth (and equity returns) and that Value firms have relatively lower investment-to-asset ratio and lower asset growth. I verify that these are indeed present in data. [Zhang \(2005\)](#) has similar results. However, unlike [Zhang \(2005\)](#), these arise endogenously in the model and do not require firm heterogeneity, asymmetric adjustment costs or exogenously specified countercyclical price of risk. Furthermore, the model endogenously produces a counter-cyclical Value premium, relative Book-to-Market Equity and price of risk. The model confirms that taking returns on human capital into account in conditional and unconditional CAPM regressions helps to better explain returns and expected returns and is necessary to obtain a good proxy for the total wealth portfolio; as [Jaganathan and Wang \(1996\)](#) argue. The excess covariance of asset growth with human capital growth for Value firms implies that Value stocks are relatively more exposed to long-run risk. This is often assumed in long-run risk models but my model provides a rationale for it. Lastly, the model implies substantial variation of the Value premium as shown empirically by [Fama and French \(1997\)](#). Thus human capital (or labor income) risk not only explains the Value premium but also yields a

broad range of implications for the characteristics of firms and security returns which we indeed observe in data.

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## 7 Appendix

### 7.1 Limiting cases of SDU

$$f\{C, V\} = \frac{\beta}{\rho} \left( \frac{C^\rho}{((1-\gamma)V)^{\frac{\rho}{1-\gamma}-1}} - (1-\gamma)V \right)$$

#### 7.1.1 $\gamma = (1-\rho) = 1$

If  $\gamma = (1-\rho)$  we have

$$f\{C, V\} = \frac{\beta}{\rho} (C^\rho - \rho V)$$

As seen in [Duffie and Skiadas \(1994\)](#) we obtain the usual time-separable utility when  $\gamma = (1-\rho)$ . Therefore with  $\gamma = (1-\rho) \rightarrow 1$  we obtain

$$\lim_{\rho \rightarrow 0} f\{C, V\} = \beta \ln(C) - \beta V$$

Now, recall that  $dV = -f dt + V \sigma_v dZ$ . With  $\gamma = 1$  we have  $\sigma_v = 0$

$$dV/dt = -\beta \ln(C) + \beta V$$

$$\mathcal{D}(V e^{-\beta t}) = -e^{-\beta t} \beta \ln(C)$$

$$V\{t\} = - \int_0^t e^{-\beta(s-t)} \beta \ln(C_s) ds$$

The transversality condition is  $\lim_{T \rightarrow \infty} E_t(V_T) = 0$ , therefore

$$E_t(V_T - V_t) = -E_t \left[ \int_t^T e^{-\beta(s-t)} \beta \ln(C_s) ds \right]$$

Taking the limit as  $T \rightarrow \infty$

$$V_t = E_t \left[ \int_t^\infty e^{-\beta(s-t)} \beta \ln(C_s) ds \right]$$

**7.1.2**  $\gamma \neq (1 - \rho)$  and  $\rho \rightarrow 0$

$$\lim_{\rho \rightarrow 0} f\{C, V\} = \beta(1 - \gamma)V \lim_{\rho \rightarrow 0} \frac{1}{\rho} \left( \frac{C^\rho - ((1 - \gamma)V)^{\frac{\rho}{1-\gamma}}}{((1 - \gamma)V)^{\frac{\rho}{1-\gamma}}} \right)$$

As in Section 2.4 of [Duffie and Epstein \(1992a\)](#)

$$\lim_{\rho \rightarrow 0} f\{C, V\} = \beta(1 - \gamma)V \left( \ln(C) - \frac{1}{1 - \gamma} \ln((1 - \gamma)V) \right)$$

## 7.2 Difficulties with non-linear production

I have also considered using a Cobb-Douglas production function. With linear production the model yields natural boundary conditions necessary for solving the PDE we obtain from the Hamiltonian-Jacobi-Bellman equation. We obtain these boundary conditions by solving the model with a single capital type (human capital, Type  $A$  physical capital, or Type  $B$  physical capital). With any CES production function,  $Y^A = \mathcal{A} \left( \alpha H^{\frac{\eta-1}{\eta}} + (1 - \alpha) (K^A)^{\frac{\eta-1}{\eta}} \right)^{\frac{\eta}{\eta-1}}$ , with elasticity of substitution  $\eta \leq 1$  as found in empirical estimates (and of which Cobb-Douglas production is a special case) and  $\alpha \in (0, 1)$  the boundary value  $F\{0, 0\}$  is not well defined. In particular, the boundary case with solely human capital would imply

$$\lim_{K^A \rightarrow 0} Y = \lim_{K^A \rightarrow 0} Y^A = \lim_{K^A \rightarrow 0} \left( \alpha (H^A)^{\frac{\eta-1}{\eta}} + (1 - \alpha) (K^A)^{\frac{\eta-1}{\eta}} \right)^{\frac{\eta}{\eta-1}} = 0$$

hence the HJB is not well-defined and  $F\{0, 0\}$  is indeterminate. Thus to solve the model with Cobb-Douglas production (or CES with  $\eta \leq 1$ ) one would need to impose an ad hoc boundary condition. I have thus far found that imposing such a boundary condition and solving the model with Cobb-Douglas production is not possible. Furthermore, even with an ad hoc boundary condition the problem would still be ill-defined at the bound  $x^A = x^B = 0$ . The state variables can visit this bound. Indeed the state variables can be absorbed at the boundaries because from [Equation 7.5](#) we have

$$\mu_{x,A,t} \{0, x_t^B\} = \mu_{x,A,t} \{1, x_t^B\} = 0 \quad \& \quad \sigma_{x,A,t} \{0, x_t^B\} = \sigma_{x,A,t} \{1, x_t^B\} = 0$$

$$\mu_{x,B,t} \{x_t^A, 0\} = \mu_{x,B,t} \{x_t^A, 1\} = 0 \quad \& \quad \sigma_{x,B,t} \{x_t^A, 0\} = \sigma_{x,B,t} \{x_t^A, 1\} = 0$$

[Jones and Manuelli \(2004\)](#) show how to solve the model with Cobb-Douglas production but without adjustment cost. They also discuss the asset pricing implications of human capital in

such a context. Without adjustment costs solving the model no longer requires solving a PDE; we need only solve a system of non-linear equations for the now constant shares of capital  $x^A$  and  $x^B$ . Without the adjustment cost the model loses much of the interesting dynamics since the Marginal Q's are equal to 1 (so BE/ME and expected equity returns are the same for the two types of securities, see [Section 7.11](#) of the Appendix). Furthermore, without adjustment costs the model requires very specific values for the parameters to ensure that all types of capital co-exist in equilibrium. [Palacios \(2013\)](#) solves a similar model with human capital, one type of physical capital and CES production. However, with one type of physical capital there is only one state variable and hence his model is much simpler than mine. Furthermore, the author assumes that the stochastic component of output comes solely from physical capital,  $dY = dK = \mathcal{A} \left( \alpha H^{\frac{\eta-1}{\eta}} + (1-\alpha) (K)^{\frac{\eta-1}{\eta}} \right)^{\frac{\eta}{\eta-1}} dt + \sigma K dZ^k$  rather than  $Y = \mathcal{A} \left( \alpha H^{\frac{\eta-1}{\eta}} + (1-\alpha) (K)^{\frac{\eta-1}{\eta}} \right)^{\frac{\eta}{\eta-1}} \Rightarrow dY = \frac{\partial Y}{\partial H} dH + \frac{\partial Y}{\partial K} dK + \frac{1}{2} \left( \frac{\partial^2 Y}{\partial H^2} dH^2 + \frac{\partial^2 Y}{\partial K^2} dK^2 + 2 \frac{\partial^2 Y}{\partial H \partial K} dH dK \right)$ . This ensures that the local volatility is constant and avoids some additional complexity which is present in my model. For tractability, the author also makes the simplifying assumptions that the production of human capital requires only human capital as input.

A discrete-time version of the model would not be easier to solve. Indeed, as argued in [Kaltenbrunner and Lochstoer \(2008\)](#) and [Gourio \(2013\)](#) to make quantitative statements we would need to solve the model globally rather than solving some first or second-order local approximations. The more common approaches to solving discrete-time models globally include function iterations and value function approximation with high order polynomials (as described in [Judd; 1998](#) and [Miranda and Fackler; 2002](#)). In a discrete-time model, regardless of the solution method used we would need to carefully compute the expectation of the continuation value. Since the two state variables are endogenous, stochastic and not independently distributed, it is not trivial to compute this expectation. We avoid this issue with a continuous-time model. Obtaining a global solution to the discrete-time model would also yield numerical difficulties at the bounds of the state variables if we use a CES production function with an elasticity of substitution  $\eta \leq 1$ .

### 7.3 Total factor productivity

The following derivation is based on the results of [Brunnermeier and Sannikov \(2011\)](#). Let  $h_t, k_t^i$  denote the levels of human and physical capital of Type  $i$

$$dk_t^i = k_t^i \left( \Gamma \ln \left( 1 + \frac{I_t^i}{\theta a_t k_t^i} \right) - \delta \right) dt \quad \text{with a similar process for } h_t$$

The TFP shocks follows

$$da_t^i = a_t^i (\bar{a}dt + \sigma dZ_t^i)$$

In endogenous growth models,  $\bar{a} = 0$ , thus

the TFP is Martingale and any growth will be endogenous.

The production process follows

$$Y_t = \mathcal{A} (a_t^A h_t + a_t^A k_t^A + a_t^B k_t^B) = \mathcal{A} (H_t + K_t^A + K_t^B)$$

The price of human and physical capital are now defined as  $a_t^A p_t$  and  $a_t^i q_t^i$ . Introducing the TFP shocks in the law of motion of capital traces back to Cox, Ingersoll and Ross (1985) and Jones and Manuelli (2004). As argued in Brunnermeier and Sannikov (2011), for scale invariance the adjustment function has to be a function of effective capital.

*“The Brownian shocks  $dZ_t$  reflect the fact that one learns over time how “effective” the capital stock is. That is, the shocks  $dZ_t$  capture changes in expectations about the future productivity of capital, and  $[K_t]$  reflects the “efficiency units” of capital, measured in expected future output rather than in simple units of physical capital (number of machines). For example, when a company reports current earnings it not only reveals information about current but also future expected cash flow. In this sense our model is also linked to the literature on news to business cycles, see e.g. Jaimovich and Rebelo (2009).[...] To preserve the tractable scale invariance property one has to modify the adjustment cost function  $(\Phi(\iota_t/a_t))$ . The fact that adjustment costs are higher for high  $a_t$  can be justified by the fact that high TFP economies are more specialized.”* – Brunnermeier and Sannikov (2011)

Notice that the value function is unchanged and  $a_t$  is not a state variable since  $\{Y, C\}$  scale with  $(a_t^A h_t + a_t^A k_t^A + a_t^B k_t^B)$

$$\begin{aligned} 0 = & \max_{I^A, I^B, I^H} f\{C, V\} + V_H E(d\{a^A h\}) \\ & + V_A E(d\{a^A k^A\}) + V_B E(d\{a^B k^B\}) \\ & + \frac{1}{2} \left( V_{AA} (d\{a^A k^A\})^2 + V_{BB} (d\{a^B k^B\})^2 + 2V_{AB} d\{a^A h\} d\{a^A k^A\} \right) \end{aligned}$$

we then plug in the conjecture for the value function

$$(V\{a^A h, a^A k^A, a^B k^B, x^A, x^B\} = \frac{((a^A h + a^A k^A + a^B k^B)^F \{x^A, x^B\})^{1-\gamma}}{1-\gamma} \text{ with } x^i = \frac{a^i k^i}{a^A h + a^A k^A + a^B k^B}) \text{ and}$$

simplify

$$\begin{aligned}
0 &= \frac{\beta}{\rho} \left( \frac{c}{F} \right)^\rho + \phi \{x^A, x^B\} \\
&+ \ln \left( \frac{e^H + \theta}{\theta} \right) (x^A + x^B - 1) \left( \frac{\Gamma x^A F_A}{F} + \frac{\Gamma x^B F_B}{F} - \Gamma \right) \\
&+ \ln \left( \frac{e^B + \theta}{\theta} \right) x^B \left( -\frac{\Gamma x^A F_A}{F} - \frac{\Gamma (x^B - 1) F_B}{F} + \Gamma \right) \\
&+ \ln \left( \frac{e^A + \theta}{\theta} \right) x^A \left( -\frac{\Gamma (x^A - 1) F_A}{F} - \frac{\Gamma x^B F_B}{F} + \Gamma \right)
\end{aligned}$$

This is exactly what the current formulation of the model yields. To reduce the complexity of the notation and to maintain the notation from the recent literature on endogenous growth, I will use  $K^i = a^i k^i$  and  $H = a^A h$ .

Using Ito's Lemma we then have

$$\frac{d(a_t^i k_t)}{a_t^i k_t} = \frac{dK_t^i}{K_t^i} = \left( \Gamma \ln \left( 1 + \frac{I_t^i}{\theta K_t^i} \right) - \delta \right) dt + \sigma dZ_t^i$$

Similarly

$$\frac{d(a_t^A h_t)}{a_t^A h_t} = \frac{dH_t}{H_t} = \left( \Gamma \ln \left( 1 + \frac{I_t^H}{\theta H_t} \right) - \delta \right) dt + \sigma dZ_t^A$$

## 7.4 Processes for aggregate capital

$$\begin{aligned}
dK_t^i &= d \left( \int_J K_{j,t}^i dj \right) \\
&= \int_J (dK_{j,t}^i) dj \\
&= \int_J K_{j,t}^i \left( \Gamma \ln \left( 1 + \frac{I_{j,t}^i}{\theta K_{j,t}^i} \right) dt - \delta dt + \sigma dZ_t^i \right) dj \\
\Rightarrow dK_t^i &= \int_J K_{j,t}^i \left( \Gamma \ln \left( 1 + \frac{I_{j,t}^i}{\theta K_{j,t}^i} \right) dt \right) dj - K_t^i \delta dt + K_t^i \sigma dZ_t^i
\end{aligned}$$

We can conjecture and verify from the decentralized problem that

$$\frac{I_{j,t}^i}{K_{j,t}^i} \quad \text{is independent of } j \text{ since all firms in each sector are identical}$$

Therefore

$$\begin{aligned}
\frac{I_{j,t}^i}{K_{j,t}^i} &= \frac{I_t^i}{K_t^i} \\
\Rightarrow \frac{dK_t^i}{K_t^i} &= \Gamma \ln \left( 1 + \frac{I_t^i}{\theta K_t^i} \right) dt - \delta dt + \sigma dZ_t^i \quad \text{for } i \in \{A, B\}
\end{aligned}$$

Similarly

$$\frac{dH_t}{H_t} = \Gamma \ln \left( 1 + \frac{I_t^H}{\theta H_t} \right) dt - \delta dt + \sigma_h dZ_t^A$$

Despite the linear production processes, the adjustment cost implies that the planner will not simply invest all resources in one type of capital. For example, let us compare investing only in human capital or physical capital of Type  $A$  to investing in both. Given an initial positive stock of capital  $\{H_t, K_t^A\}$ , the expected growth rate of the stock if the planner invests in both is

$$\begin{aligned} E_t \left( \frac{dH_t + dK_t^A}{H_t + K_t^A} \right) &= \frac{H_t}{H_t + K_t^A} \Gamma \ln \left( 1 + \frac{I_t^H}{\theta H_t} \right) dt \\ &\quad + \frac{K_t^A}{H_t + K_t^A} \Gamma \ln \left( 1 + \frac{I_t^A}{\theta K_t^A} \right) dt - \delta dt \end{aligned} \quad (7.1)$$

If instead the planner were to invest entirely in  $K_t^A$  we would have

$$E_t \left( \widetilde{\frac{dH_t + dK_t^A}{H_t + K_t^A}} \right) = \frac{K_t^A}{H_t + K_t^A} \Gamma \ln \left( 1 + \frac{I_t^A + I_t^H}{\theta K_t^A} \right) dt - \delta dt \quad (7.2)$$

Now, with  $\sigma_h = \sigma$ , we have  $H = K^A$  and equations (7.1) and (7.2) become

$$\begin{aligned} E_t \left( \frac{dH_t + dK_t^A}{H_t + K_t^A} \right) &= \Gamma \ln \left( 1 + \frac{I_t^A}{\theta K_t^A} \right) dt - \delta dt \\ E_t \left( \widetilde{\frac{dH_t + dK_t^A}{H_t + K_t^A}} \right) &= \frac{1}{2} \Gamma \ln \left( 1 + 2 \frac{I_t^A}{\theta K_t^A} \right) dt - \delta dt \end{aligned}$$

Due to concavity

$$E_t \left( \frac{dH_t + dK_t^A}{H_t + K_t^A} \right) > E_t \left( \widetilde{\frac{dH_t + dK_t^A}{H_t + K_t^A}} \right)$$

Thus even if the agents were risk neutral, it would be preferable to invest in both types of capital despite having linear production. The simulation results will confirm that on average it is optimal to invest in all three types of capital. All the qualitative results would be unchanged with if the production function for were a Cobb-Douglas function in  $H$  and  $K^A$ . However, using a Cobb-Douglas (or other non-linear) production function makes the model significantly more difficult to solve.

If there were no adjustment cost the equations above would imply

$$E_t \left( \frac{dH_t + dK_t^A}{H_t + K_t^A} \right) = \frac{I_t^H}{H_t + K_t^A} dt + \frac{I_t^A}{H_t + K_t^A} dt - \delta dt = E_t \left( \frac{d\widetilde{H_t + K_t^A}}{H_t + K_t^A} \right)$$

Thus, aggregate production would be independent of the allocation of resources. The planner would be indifferent between putting all resources in any type of capital if  $\sigma_h = \sigma$  and would otherwise strictly prefer the less risky technology.

## 7.5 State variables

The state variables are  $x^i = \frac{K^i}{H + K^A + K^B} \in [0, 1]$ . With linear production, the boundaries  $\{0, 1\}$  are absorbing.

$$\begin{aligned} dx_t^i &= \mu_{x,i,t} dt + \sigma'_{x,i,t} dZ_t \\ \mu_{x,A,t} &= x^A \left( \begin{aligned} &\Gamma (x^A + x^B - 1) \ln \left( \frac{\Gamma p \{x^A, x^B\}}{\theta} \right) + (\Gamma - \Gamma x^A) \ln \left( \frac{\Gamma q^A \{x^A, x^B\}}{\theta} \right) \\ &- \Gamma x^B \ln \left( \frac{\Gamma q^B \{x^A, x^B\}}{\theta} \right) + \sigma_h^2 (x^A + x^B - 1)^2 \\ &- \sigma (2x^A - 1) \sigma_h (x^A + x^B - 1) + \sigma^2 ((x^A - 1)x^A + (x^B)^2) \end{aligned} \right) \\ \sigma_{x,A,t} &= \{x^A (\sigma_h (x^A + x^B - 1) + \sigma (1 - x^A)) \quad , \quad -\sigma x^A x^B\} \\ \mu_{x,B,t} &= x^B \left( \begin{aligned} &\Gamma (x^A + x^B - 1) \ln \left( \frac{\Gamma p \{x^A, x^B\}}{\theta} \right) - \Gamma x^A \ln \left( \frac{\Gamma q^A \{x^A, x^B\}}{\theta} \right) \\ &- \Gamma (x^B - 1) \ln \left( \frac{\Gamma q^B \{x^A, x^B\}}{\theta} \right) - 2\sigma x^A \sigma_h (x^A + x^B - 1) \\ &+ \sigma_h^2 (x^A + x^B - 1)^2 + \sigma^2 ((x^A)^2 + (x^B - 1)x^B) \end{aligned} \right) \\ \sigma_{x,B,t} &= \{x^B (\sigma_h (x^A + x^B - 1) - \sigma x^A) \quad , \quad -\sigma x^B (x^B - 1)\} \end{aligned}$$

## 7.6 The HJB

$$V \{H, K^A, K^B, x^A, x^B\} = \frac{((H + K^A + K^B) F \{x^A, x^B\})^{1-\gamma}}{1-\gamma}$$



$$\begin{aligned}
0 &= \max_{I^A, I^B, I^H} f\{C, V\} + V_H E(dH) \\
&+ V_A E(dK^A) + V_B E(dK^B) \\
&+ \frac{1}{2} \left( V_{AA} (dK^A)^2 + V_{BB} (dK^B)^2 + V_{HH} (dH)^2 + 2V_{AH} dH dK^A \right)
\end{aligned}$$

where  $E(dH)$ ,  $E(dK^A)$ , and  $E(dK^B)$  denote the drifts of  $H$ ,  $K^A$ , and  $K^B$

Plugging in the conjecture for  $V$  and simplifying a bit yields the desired result. The function  $\phi$  is

$$\begin{aligned}
\phi\{x^A, x^B\} &= F_B \left( \frac{\gamma x^A x^B F_A (2\sigma^2 x^A x^B + \sigma_h(\sigma - \sigma_h)(x^A + x^B - 1)^2 - \sigma(x^A + x^B - 1)(x^A(\sigma - \sigma_h) + x^B(\sigma + \sigma_h)))}{F^2} \right. \\
&\quad \left. + \frac{\gamma x^B (\sigma^2 ((x^A)^2 + (x^B - 1)x^B) - 2\sigma\sigma_h x^A (x^A + x^B - 1) + \sigma_h^2 (x^A + x^B - 1)^2)}{F} \right) \\
&+ \left( -\frac{\gamma (x^A)^2 F_A^2 (\sigma^2 ((x^A - 1)^2 + (x^B)^2) - 2\sigma\sigma_h (x^A - 1)(x^A + x^B - 1) + \sigma_h^2 (x^A + x^B - 1)^2)}{2F^2} \right. \\
&\quad \left. + \frac{\gamma x^A F_A (\sigma^2 ((x^A - 1)x^A + (x^B)^2) - \sigma\sigma_h (2x^A - 1)(x^A + x^B - 1) + \sigma_h^2 (x^A + x^B - 1)^2)}{F} \right) \\
&+ \left( \frac{(x^A)^2 F_{AA} (\sigma^2 ((x^A - 1)^2 + (x^B)^2) - 2\sigma\sigma_h (x^A - 1)(x^A + x^B - 1) + \sigma_h^2 (x^A + x^B - 1)^2)}{2F} \right. \\
&\quad \left. + \frac{x^A x^B F_{AB} (\sigma^2 ((x^A - 1)x^A + (x^B - 1)x^B) - \sigma\sigma_h (2x^A - 1)(x^A + x^B - 1) + \sigma_h^2 (x^A + x^B - 1)^2)}{F} \right) \\
&+ \left( -\frac{\gamma (x^B)^2 F_B^2 (\sigma^2 ((x^A)^2 + (x^B - 1)^2) - 2\sigma\sigma_h x^A (x^A + x^B - 1) + \sigma_h^2 (x^A + x^B - 1)^2)}{2F^2} \right. \\
&\quad + \frac{\rho (x^B)^2 F_{BB} (\sigma^2 ((x^A)^2 + (x^B - 1)^2) - 2\sigma\sigma_h x^A (x^A + x^B - 1) + \sigma_h^2 (x^A + x^B - 1)^2)}{2F\rho} \\
&\quad \left. + \frac{F(-2\beta - \rho(2\delta + \gamma(\sigma^2 ((x^A)^2 + (x^B)^2) - 2\sigma\sigma_h x^A (x^A + x^B - 1) + \sigma_h^2 (x^A + x^B - 1)^2))}{2F\rho} \right)
\end{aligned}$$

We find the boundary conditions by plugging  $\{x^A = 1, x^B = 0\}$ ,  $\{x^A = 0, x^B = 1\}$ , and  $\{x^A = 0, x^B = 0\}$  in equations (2.6)-(2.9) and solving the resulting non-linear equation for the constant  $F$ . When  $\sigma_h = \sigma$ , we have  $\bar{F} = \hat{F}$ .

## 7.7 Martingale method

I will suppress the subscript  $j$  for clarity. Let  $\Lambda_t$  denote the state price density (SPD). Let

$$\begin{aligned}
\bar{\psi}_t &= p_t \left( -E_t \left[ \frac{d(\Lambda_t p_t)}{\Lambda_t p_t} \right] / dt + \sigma_h \tilde{\sigma}_{\Lambda, t}^{(A)} \right) \\
\psi_t^i &= q_t^i \left( -E_t \left[ \frac{d(\Lambda_t q_t^i)}{\Lambda_t q_t^i} \right] / dt + \sigma \hat{\sigma}_{\Lambda, i, t}^{(i)} \right)
\end{aligned}$$

$$\begin{aligned}\tilde{\sigma}_{\Lambda,t} &= \{\tilde{\sigma}_{\Lambda,t}^{(A)}, \tilde{\sigma}_{\Lambda,t}^{(B)}\}' = Diffusion \left[ \frac{d(\Lambda_t p_t)}{\Lambda_t p_t} \right] \\ \hat{\sigma}_{\Lambda,i,t} &= \{\hat{\sigma}_{\Lambda,i,t}^{(A)}, \hat{\sigma}_{\Lambda,i,t}^{(B)}\}' = Diffusion \left[ \frac{d(\Lambda_t q_t^i)}{\Lambda_t q_t^i} \right]\end{aligned}$$

Using the Martingale Method, I will derive some static constraints

$$\begin{aligned}\tilde{\Lambda}_t &\equiv \Lambda_t p_t \\ \frac{d\tilde{\Lambda}_t}{\tilde{\Lambda}_t} &\equiv -\tilde{r}_t dt - \tilde{\sigma}_{\Lambda,t} dZ_t \\ d(\tilde{\Lambda}_t H_t) &= H_t d\tilde{\Lambda}_t + \tilde{\Lambda}_t dH_t + d\tilde{\Lambda}_t dH_t \\ \Rightarrow \tilde{\Lambda}_T H_T - \tilde{\Lambda}_t H_t &= \int_t^T H_\tau d\tilde{\Lambda}_\tau + \int_t^T \tilde{\Lambda}_\tau dH_\tau + \int_t^T d\tilde{\Lambda}_\tau dH_\tau\end{aligned}$$

The transversality condition is  $\lim_{T \rightarrow \infty} E_t(\tilde{\Lambda}_T H_T) = 0$

$$\Rightarrow \Lambda_s H_s p_s = E_s \int_s^\infty \Lambda_t p_t H_t \left( \frac{\bar{\psi}_t}{p_t} - \Gamma \ln \left( 1 + \frac{\iota_t^H}{\theta} \right) + \delta \right) dt \quad (7.3)$$

Notice that

$$\tilde{r}_t = -Drift \left[ \frac{d(\Lambda_t q_t)}{\Lambda_t q_t} \right]$$

We can define

$$\begin{aligned}dp &\equiv \bar{\vartheta}_t dt + p_t \bar{\varphi}_t' dZ_t \\ \text{with} \\ \bar{\vartheta}_t &= -\bar{\psi}_t + p_t r_t + p_t \bar{\varphi}_t \sigma_{\Lambda,t}' + p_t \sigma_h \left( \sigma_{\Lambda,t}^{(A)} - \bar{\varphi}_t^{(A)} \right)\end{aligned} \quad (7.4)$$

Using the same steps we can derive

$$\Lambda_s K_s^i q_s^i = E_s \int_s^\infty \Lambda_t q_t^i K_t^i \left( \frac{\psi_t^i}{q_t^i} - \Gamma \ln \left( 1 + \frac{\iota_t^i}{\theta} \right) + \delta \right) dt \quad (7.5)$$

$$dq_t^i \equiv \vartheta_t^i dt + q_t^i \varphi_t^{i'} dZ_t$$

$$\vartheta_t^i = -\psi_t^i + q_t^i r_t + q_t^i \varphi_t^{i'} \sigma_{\Lambda,t} + q_t^i \mathbf{1}_i (\sigma_{\Lambda,t} - \varphi_t^i) \sigma$$

where  $\mathbf{1}_A = \{1, 0\}$ ,  $\mathbf{1}_B = \{0, 1\}$  and

$$\Lambda_s \tilde{W}_s = E_s \int_s^\infty \Lambda_t (C_t + I_t^H - \omega_t H_t) dt$$

We can write the agent and firm problems as

$$\begin{aligned} & \max_{\{C_{j,t}, I_{j,t}^H, H_{j,t}\}_{t=0}^\infty} E_0 \int_0^\infty f(C_{j,t}, V_{j,t}) dt \quad \text{s.t.:} \\ & \tilde{W}_0 + p_0 H_0 = E_0 \int_0^\infty \Lambda_t (C_{j,t} + I_{j,t}^H - \omega_t H_{j,t}) dt \\ & + E_0 \int_0^\infty \Lambda_t p_t H_{j,t} \left( \frac{\bar{\psi}_t}{p_t} - \Gamma \ln \left( 1 + \frac{I_{j,t}^H}{\theta H_{j,t}} \right) + \delta \right) dt \end{aligned}$$

Following [Duffie and Skiadas \(1994\)](#), the first order condition for consumption is

$$\begin{aligned} G f_C &= \Lambda_t \\ & \text{where} \\ G &= \exp\left(\int_0^t f_V ds\right) \end{aligned}$$

The first order condition for human capital implies

$$\begin{aligned} 0 &= -p_t \Gamma + \theta - \bar{\psi}_t + \omega_t + p_t \Gamma \ln \left( p_t \frac{\Gamma}{\theta} \right) \\ \Rightarrow \bar{\psi}_t &= \underbrace{\omega_t}_{\text{marginal benefit from human capital}} - \underbrace{\left( -p_t \left( \Gamma \ln \left( p_t \frac{\Gamma}{\theta} \right) - \delta \right) + p_t \Gamma - \theta \right)}_{\text{marginal cost to human capital}} \end{aligned}$$

The first order condition for investment in human capital implies

$$\frac{I_{j,t}^H}{H_{j,t}} = p_t \Gamma - \theta$$

Thus we have

$$\begin{aligned} \frac{I_{j,t}^H}{H_{j,t}} &= \frac{I_{k,t}^H}{H_{k,t}} \quad \forall \{j, k\} \in J \\ \Rightarrow \frac{I_{j,t}^H}{H_{j,t}} &= \frac{I_t^H}{H_t} = \iota_t^H \end{aligned}$$

The firms problems are

$$\begin{aligned} \max_{K_{j,t}^A, H_{j,t}, \epsilon_{j,t}^A} \int_0^\infty \Lambda_t (\mathcal{A}(K_{j,t}^A + H_{j,t}) - \omega_t H_{j,t} - I_{j,t}^A) dt \quad \text{s.t.:} \\ K_0^A q_0^A = E_0 \int_0^\infty \Lambda_t q_t^A K_{j,t}^A \left( \frac{\psi_t^A}{q_t^A} - \Gamma \ln \left( 1 + \frac{I_{j,t}^A}{\theta K_{j,t}^A} \right) + \delta \right) dt \end{aligned}$$

$$\begin{aligned} \max_{K_{j,t}^B, \epsilon_{j,t}^B} \int_0^\infty \Lambda_t (\mathcal{A}K_{j,t}^B - I_{j,t}^B) dt \quad \text{s.t.:} \\ K_0^B q_0^B = E_0 \int_0^\infty \Lambda_t q_t^B K_{j,t}^B \left( \frac{\psi_t^B}{q_t^B} - \Gamma \ln \left( 1 + \frac{I_{j,t}^B}{\theta K_{j,t}^B} \right) + \delta \right) dt \end{aligned}$$

The first order conditions for physical capital<sup>51</sup> imply

$$\psi_t^i = \underbrace{\mathcal{A}}_{\text{marginal product of physical capital}} - \underbrace{\left( -q_t^i \left( \Gamma \ln \left( q_t^i \frac{\Gamma}{\theta} \right) - \delta \right) + q_t^i \Gamma - \theta \right)}_{\text{marginal cost to physical capital}}$$

The first order conditions for investment in physical capital imply

$$\begin{aligned} \frac{I_{j,t}^i}{K_{j,t}^i} &= q_t^i \Gamma - \theta \\ \Rightarrow \frac{I_{j,t}^i}{K_{j,t}^i} &= \frac{I_t^i}{K_t^i} = \iota_t^i \end{aligned}$$

For firms of Type  $A$  the first order condition with respect to human capital implies

$$\omega_t = \mathcal{A}$$

By using the FOC's from the planner's problem we then have

$$\begin{aligned} p &= \frac{1}{\Gamma} (\iota^H + \theta) = \frac{1}{\beta} \left( \frac{c}{F} \right)^{1-\rho} (F - x^A F_A - x^B F_B) \\ q^A &= \frac{1}{\Gamma} (\iota^A + \theta) = \frac{1}{\beta} \left( \frac{c}{F} \right)^{1-\rho} (F - (x^A - 1) F_A - x^B F_B) \\ q^B &= \frac{1}{\Gamma} (\iota^B + \theta) = \frac{1}{\beta} \left( \frac{c}{F} \right)^{1-\rho} (F - x^A F_A - (x^B - 1) F_B) \end{aligned}$$

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<sup>51</sup>We can think of  $p$  and  $q^i$  as already including the Lagrange multipliers.

where  $\iota^H = I^H/H$  and  $\iota^i = I^i/K^i$ . Plugging in the first order conditions from the decentralized problem in the static constraints (7.3) and (7.5) yields

$$\Lambda_s S_s^H = \Lambda_s H_s p_s = E_s \int_s^\infty \Lambda_t p_t H_t (\mathcal{A} - e_t^H) dt = E_s \int_s^\infty \Lambda_t H_t p_t \underbrace{\left( \frac{\mathcal{A} - p_t \Gamma - \theta}{p_t} \right)}_{\text{Dividend Yield: } D^H/S^H} dt = E_s \int_s^\infty \Lambda_t D_t^H dt$$

$$\Lambda_s S_s^i = \Lambda_s K_s^i q_s^i = E_s \int_s^\infty \Lambda_t q_t^i K_t^i (\mathcal{A} - e_t^i) dt = E_s \int_s^\infty \Lambda_t K_t^i q_t^i \underbrace{\left( \frac{\mathcal{A} - q_t^i \Gamma - \theta}{q_t^i} \right)}_{\text{Dividend Yield: } D^i/S^i} dt = E_s \int_s^\infty \Lambda_t D_t^i dt$$

We can combine and re-write these last equations as

$$\begin{aligned} E_s \int_s^\infty \frac{\Lambda_t}{\Lambda_s} C_t dt &= p_s H_s + q_s^A K_s + q_s^B K_s^B \\ \Lambda_s S_s^H + \int_0^s \Lambda_t D_t^H dt &= E_s \int_0^\infty \Lambda_t D_t^H dt \end{aligned} \tag{7.6}$$

$$\Lambda_s S_s^i + \int_0^s \Lambda_t D_t^i dt = E_s \int_0^\infty \Lambda_t D_t^i dt \tag{7.7}$$

The resource constraint and market clearing conditions are

$$C_t + I_t^A + I_t^B + I_t^H = Y_t \quad \text{where } Y_t = \mathcal{A}(H_t + K_t^A + K_t^B)$$

$$\varpi_t^A \tilde{W}_t + \varpi_t^B \tilde{W}_t = S_t^A + S_t^B \quad \text{market for risky securities clears}$$

$$1 - \varpi_t^A - \varpi_t^B = 0 \quad \text{zero net bond holdings}$$

The last two conditions imply

$$\tilde{W}_t = S_t^A + S_t^B = q_t^A K_t + q_t^B K_t^B = E_t \int_t^\infty \Lambda_s (C_s + I_s^H - \omega_s H_s) ds$$

as expected. I will summarize some of the above results in the following proposition.

**Proposition 3. :** *We can re-write the agent and firms problems as follow*

- The agent solves

$$\begin{aligned} & \max_{\{C_{j,t}, I_{j,t}^H, H_{j,t}\}_{t=0}^{\infty}} E_0 \int_0^{\infty} f \{C_{j,t}, V_{j,t}\} dt \quad s.t.: \\ & \tilde{W}_0 + p_0 H_0 = E_0 \int_0^{\infty} \Lambda_t (C_{j,t} + I_{j,t}^H - \omega_t H_{j,t}) dt \\ & + E_0 \int_0^{\infty} \Lambda_t p_t H_{j,t} \left( \frac{\bar{\psi}_t}{p_t} - \Gamma \ln \left( 1 + \frac{I_{j,t}^H}{\theta H_{j,t}} \right) + \delta \right) dt \end{aligned}$$

- Firms A and B (respectively) solve

$$\begin{aligned} & \max_{K_{j,t}^A, H_{j,t}, I_{j,t}^A} \int_0^{\infty} \Lambda_t (\mathcal{A}(K_{j,t}^A + H_{j,t}) - \omega_t H_{j,t} - I_{j,t}^A) dt \quad s.t.: \\ & K_0^A q_0^A = E_0 \int_0^{\infty} \Lambda_t q_t^A K_{j,t}^A \left( \frac{\psi_t^A}{q_t^A} - \Gamma \ln \left( 1 + \frac{I_{j,t}^A}{\theta K_{j,t}^A} \right) + \delta \right) dt \\ & \max_{K_{j,t}^B, \epsilon_{j,t}^B} \int_0^{\infty} \Lambda_t (\mathcal{A}K_{j,t}^B - I_{j,t}^A) dt \quad s.t.: \\ & K_0^B q_0^B = E_0 \int_0^{\infty} \Lambda_t q_t^B K_{j,t}^B \left( \frac{\psi_t^B}{q_t^B} - \Gamma \ln \left( 1 + \frac{I_{j,t}^B}{\theta K_{j,t}^B} \right) + \delta \right) dt \end{aligned}$$

*Proof.* We can obtain these results by applying the Martingale Method from Karatzas and Shreve (1991).

## 7.8 Asset pricing formulas

Since markets are complete, there exists a unique state price density (SPD),  $\Lambda_t$ . I will conjecture and verify the following process for the SPD

$$\frac{d\Lambda_t}{\Lambda_t} = -r_t dt - \sigma_{\Lambda,t} dZ_t$$

From Duffie and Skiadas (1994) we have

$$\begin{aligned} \Lambda_t &= G f_C \\ &\text{where} \\ G &= \exp\left(\int_0^t f_V ds\right) \end{aligned}$$

Applying Ito's Lemma to  $\Lambda_t$  and collecting the drift and diffusion terms yields

$$r_t = (1 - \rho)\mu_{c,t} - \frac{1}{2}(\rho - 2)(\rho - 1)\sigma_{c,t}^2 - \frac{(\rho - 1)(\gamma + \rho - 1)\sigma_{c,t}\sigma_{v,t}}{\gamma - 1} - \frac{\rho(\gamma + \rho - 1)\sigma_{v,t}^2}{2(\gamma - 1)^2} + \beta$$

$$\sigma_{\Lambda,t} = -(\rho - 1)\sigma_{c,t} - \left(\frac{\rho}{\gamma - 1} + 1\right)\sigma_{v,t}$$

Thus the conjecture is verified.  $\mu_{c,t}$  and  $\sigma_{c,t}$  are the conditional mean and volatility of consumption growth

$$\frac{dC_t}{C_t} = \mu_{c,t}dt + \sigma_{c,t}dZ_t$$

Plugging the first order conditions for investment in the aggregate resource constraint yields

$$C_t = \mathcal{A}(H_t + K_t^A + K_t^B) - H_t(\Gamma p_t\{x_t^A, x_t^B\} - \theta) - K_t^A(\Gamma q_t^A\{x_t^A, x_t^B\} - \theta) - K_t^B(\Gamma q_t^B\{x_t^A, x_t^B\} - \theta)$$

Applying Ito's Lemma to  $C_t$  yields

$$\mu_{c,t} = \frac{1}{C_t} \left( \begin{aligned} &\frac{\partial C_t}{\partial H_t} \times H_t \left( \Gamma \ln \left( 1 + \frac{I_t^H}{\theta H_t} \right) - \delta \right) + \frac{1}{2} \frac{\partial^2 C_t}{\partial H_t^2} \times H_t \sigma_h^2 + \frac{\partial^2 C_t}{\partial H_t \partial K_t^A} \times H_t K_t^A \sigma \sigma_h \\ &+ \frac{\partial C_t}{\partial K_t^A} \times K_t^A \left( \Gamma \ln \left( 1 + \frac{I_t^A}{\theta K_t^A} \right) - \delta \right) + \frac{1}{2} \frac{\partial^2 C_t}{\partial (K_t^A)^2} \times K_t^A \sigma^2 \\ &+ \frac{\partial C_t}{\partial K_t^B} \times K_t^B \left( \Gamma \ln \left( 1 + \frac{I_t^B}{\theta K_t^B} \right) - \delta \right) + \frac{1}{2} \frac{\partial^2 C_t}{\partial (K_t^B)^2} \times K_t^B \sigma^2 \end{aligned} \right)$$

$$\sigma_{c,t} = \frac{1}{C_t} \left\{ \frac{\partial C_t}{\partial H_t} \times H_t \sigma_h + \frac{\partial C_t}{\partial K_t^A} \times K_t^A \sigma, \quad \frac{\partial C_t}{\partial K_t^B} \times K_t^B \sigma \right\}$$

The value function can be written as

$$V\{H + K^A + K^B, x^A, x^B\} = \frac{1}{1 - \gamma} \left( (H + K^A + K^B) F\{x^A, x^B\} \right)^{1 - \gamma}$$

$$\sigma_{v,t} = \left\{ \begin{aligned} &\frac{(\gamma - 1)\sigma x_t^A (F_A(x_t^A - 1) + F_B x_t^B - F)}{F} \\ &+ \frac{(\gamma - 1)\sigma_h (x_t^A + x_t^B - 1) (-F_A x_t^A - F_B x_t^B + F)}{F} \\ &+ \frac{(\gamma - 1)\sigma x_t^B (F_A x_t^A + F_B (x_t^B - 1) - F)}{F} \end{aligned} \right\}$$

with

$$dV_t = -f\{C_t, V_t\}dt + V_t \sigma_{v,t} dZ_t$$

I define the risky security prices as

$$dS_t = \left( \mu_t' \text{diag}(S_t) - D_t \right) dt + \begin{pmatrix} S_t^A \varsigma_t^{A'} \\ S_t^B \varsigma_t^{B'} \end{pmatrix} dZ_t$$

where  $S_t^i = K_t^i q_t^i$ . Using the static constraints from [Section 7.7](#) of the Appendix we can write

$$\begin{aligned} S_t^i &= E_t \int_t^\infty \frac{\Lambda_\tau}{\Lambda_t} D_\tau^i d\tau \\ D_t^i &= K_t^i q_t^i \left( \frac{\mathcal{A} - q_t^i \Gamma - \theta}{q_t^i} \right) \end{aligned}$$

Similarly for human capital we have

$$\begin{aligned} S_t^H = H_t p_t &= E_t \int_t^\infty \frac{\Lambda_\tau}{\Lambda_t} D_\tau^H d\tau \\ D_t^H &= H_t p_t \left( \frac{\mathcal{A} - p_t \Gamma - \theta}{p_t} \right) \\ dS_t^H &= (\mu_t^H S_t^H - D_t) dt + S_t^H \varsigma_t^{H'} dZ_t \end{aligned}$$

The static constraints in [Section 7.7](#) of the Appendix imply that the gains processes adjusted by the SPD is Martingales, therefore

$$\begin{aligned} E_t \left( \Lambda_T S_T^i + \int_0^T \Lambda_t D_t^i dt \right) &= \Lambda_t S_t^i + \int_0^t \Lambda_t D_t^i dt \\ \Rightarrow 0 &= E_t \left( d \left( \Lambda_t S_t^i + \int_0^t \Lambda_t D_t^i dt \right) \right) \\ \Rightarrow \mu_t^i &= r_t + \sigma_{\Lambda,t} \cdot \varsigma_t^i + \frac{D_t^i}{S_t^i} - \frac{D_t^i}{S_t^i} = r_t + \sigma_{\Lambda,t} \cdot \varsigma_t^i \end{aligned}$$

where  $\varsigma_t^A = \frac{1}{q_t^A} \frac{\partial q_t^A}{\partial x_t^A} \sigma_{x,A,t} + \frac{1}{q_t^A} \frac{\partial q_t^A}{\partial x_t^B} \sigma_{x,B,t} + \{\sigma, 0\}'$  and  $\varsigma_t^B = \frac{1}{q_t^B} \frac{\partial q_t^B}{\partial x_t^A} \sigma_{x,A,t} + \frac{1}{q_t^B} \frac{\partial q_t^B}{\partial x_t^B} \sigma_{x,B,t} + \{0, \sigma\}'$ . The returns on the risky securities are

$$dR_t^i = \frac{D_t^i}{S_t^i} dt + \frac{dS_t^i}{S_t^i} = \mu_t^i dt + \varsigma_t^{i'} dZ_t = (r_t + \sigma_{\Lambda,t} \cdot \varsigma_t^i) dt + \varsigma_t^{i'} dZ_t$$



Equivalently, using Ito's Lemma we can write

$$dR_t^i = \frac{D_t^i}{S_t^i} dt + \frac{dS_t^i}{S_t^i} = \underbrace{\frac{\mathcal{A} - q_t^i \Gamma + \theta}{q_t^i}}_{\text{dividend yield} = D_t^i/S_t^i} dt + \underbrace{\frac{dq_t^i}{q_t^i} + \frac{dK_t^i}{K_t^i} + \frac{dK_t^i}{K_t^i} \times \frac{dq_t^i}{q_t^i}}_{\text{capital gains} = dS_t^i/S_t^i}$$

The total wealth is

$$S_t^w = p_t H_t + q_t^A K_t^A + q_t^B K_t^B$$

The marketable wealth is

$$S_t^m = q_t^A K_t^A + q_t^B K_t^B$$

It will be convenient to define

$$\omega_i = \frac{S_t^i}{S_t^w}$$

Let us derive the return on the total wealth portfolio. The above results imply that the gains process for the total wealth portfolio is Martingale when adjusted by the SPD

$$0 = E_t \left( d \left( \Lambda_t S_t^w + \int_0^t \Lambda_t (D_t^H + D^A + D^B) dt \right) \right)$$

$$\begin{aligned} 0 &= \frac{D_t^H}{S_t^H} (1 - \omega_A - \omega_B) + \frac{D_t^A}{S_t^A} \omega_A + \frac{D_t^B}{S_t^B} \omega_B - r_t \\ &+ \mu_t^w - \frac{D_t^H}{S_t^H} (1 - \omega_A - \omega_B) - \frac{D_t^A}{S_t^A} \omega_A - \frac{D_t^B}{S_t^B} \omega_B \\ &- \sigma_{\Lambda, t} \cdot (\varsigma^H (1 - \omega_A - \omega_B) + \varsigma^A \omega_A + \varsigma^B \omega_B) \end{aligned}$$

$$\Rightarrow \mu_t^w = r_t + \sigma_{\Lambda, t} \cdot \varsigma_t^w$$

where

$$\mu_t^w = \mu^H (1 - \omega_A - \omega_B) + \mu^A \omega_A + \mu^B \omega_B$$

and

$$\varsigma_t^w = \varsigma^H (1 - \omega_A - \omega_B) + \varsigma^A \omega_A + \varsigma^B \omega_B$$

Let  $D_t^w$  denote the dividend on the total wealth portfolio

$$\frac{D_t^w}{S_t^w} = \frac{D_t^H + D^A + D^B}{p_t H_t + q_t^A K_t^A + q_t^B K_t^B} = \frac{D_t^H}{S_t^H} (1 - \omega_A - \omega_B) + \frac{D_t^A}{S_t^A} \omega_A + \frac{D_t^B}{S_t^B} \omega_B = \frac{C_t}{S_t^w}$$

Thus the return on the total wealth portfolio is

$$dR_t^w = \frac{D_t^w}{S_t^w} dt + \frac{dS_t^w}{S_t^w} dt = (r_t + \sigma_{\Lambda,t} \cdot \varsigma^w) dt + \varsigma_t^{w'} dZ_t = \mu_t^w + \varsigma_t^{w'} dZ_t$$

Similarly the return on the market portfolio is

$$dR_t^m = \frac{D_t^m}{S_t^m} dt + \frac{dS_t^m}{S_t^m} dt = (r_t + \sigma_{\Lambda,t} \cdot \varsigma^m) dt + \varsigma_t^{m'} dZ_t = \mu_t^m + \varsigma_t^{m'} dZ_t$$

where

$$\begin{aligned}\varsigma^m &= \varsigma^A \frac{S^A}{S^m} + \varsigma^B \frac{S^B}{S^m} = \frac{S^w}{S^m} (\varsigma^A \omega_A + \varsigma^B \omega_B) \\ \mu_t^m &= \mu^A \frac{S^A}{S^m} + \mu^B \frac{S^B}{S^m} = \frac{S^w}{S^m} (\mu^A \omega_A + \mu^B \omega_B)\end{aligned}$$

We can write the return on the market portfolio as a value-weighted sum of the two risky securities return

$$\begin{aligned}dR_t^m &= dR_t^A \frac{S^A}{S^m} + dR_t^B \frac{S^B}{S^m} \\ dR_t^m &= \frac{S^w}{S^m} (dR_t^A \omega_A + dR_t^B \omega_B)\end{aligned}$$

Notice that when  $\rho = 0$

$$\begin{aligned}\frac{D_t^w}{S_t^w} &= \frac{D_t^H + D^A + D^B}{p_t H_t + q_t^A K_t^A + q_t^B K_t^B} = \frac{A(H + K^A + K^B) - (\epsilon^H - \epsilon^A - \epsilon^B)}{p_t H_t + q_t^A K_t^A + q_t^B K_t^B} \\ \Rightarrow \frac{D_t^w}{S_t^w} &= \frac{C}{p_t H_t + q_t^A K_t^A + q_t^B K_t^B} = \beta\end{aligned}$$

Back to the more general case we have, the beta with the wealth portfolio

$$\beta_{i,w} = \frac{\text{cov}(dR_t^i, dR_t^w)}{\varsigma_t^{m'} \varsigma_t^m} = \frac{\varsigma_t^{i'} \varsigma_t^m}{\varsigma_t^{w'} \varsigma_t^w}$$

Beta with the market portfolio

$$\beta_{i,m} = \frac{\text{cov}(dR_t^i, dR_t^m)}{\varsigma_t^{m'} \varsigma_t^m} = \frac{\varsigma_t^{i'} \varsigma_t^m}{\varsigma_t^{m'} \varsigma_t^m}$$

The expected excess return of stock  $A$  relative to stock  $B$  is

$$\begin{aligned} E_t(dR_t^A) - E_t(dR_t^B) &= \sigma_{\Lambda,t} \cdot (\varsigma_t^A - \varsigma_t^B) dt \\ &= Cov_t\left(-\frac{d\Lambda_t}{\Lambda_t}, \frac{dq_t^A}{q_t^A}\right) - Cov_t\left(-\frac{d\Lambda_t}{\Lambda_t}, \frac{dq_t^B}{q_t^B}\right) + \sigma_{\Lambda,t} \cdot \{1, -1\}' \sigma dt \end{aligned}$$

After simulating the model, I find that the instantaneous covariance between the SPD and the price of physical capital of Type  $A$  is on average smaller than that between the SPD and the price of physical capital of Type  $B$ . In particular, I find

$$E\left(\sigma_{\Lambda,t} \cdot \{1, -1\}' \sigma\right) dt > E\left(Cov_t\left(-\frac{d\Lambda_t}{\Lambda_t}, \frac{dq_t^A}{q_t^A}\right) - Cov_t\left(-\frac{d\Lambda_t}{\Lambda_t}, \frac{dq_t^B}{q_t^B}\right)\right) > 0$$

Thus most of the action comes from the fact that the price of risk associated with  $dZ^A$  is on average larger than that associated with  $dZ^B$ . This is because negative shocks to firms of Type  $A$  are costlier to the agents since they coincide with negative shocks to the human capital.

Let us define the  $Q$  measure as the measure under which the gains processes are Martingale. Let  $\xi_t$  denote the corresponding Radon-Nikodym derivative (with  $\xi_0 = 1$ ).  $\Lambda_t = \xi_t e^{-\int_0^t r_s ds}$ . Thus,  $\xi_0 = 1 \Rightarrow \Lambda_0 = 1$ .

## 7.9 Market Completeness

With  $Corr_t\left(\frac{dH_t}{H_t}, \frac{dK_t^A}{K_t^A}\right)/dt = Corr_t(dZ_t^A, dZ_t^H)/dt = \varrho = 1$  markets are dynamically complete as long as the two risky securities (claims on the profits of firms of types  $A$  and  $B$ ) are traded. Thus with  $\varrho = 1$  the agents can implicitly issue claims on the value of their human capital. Now, let  $1 > \varrho > 0$ . In that case, markets are dynamically complete only if we explicitly introduce a third risky security by allowing agents to issue claims on the value of their human capital. With complete markets, we can use the solution to the planner's problem. The asset pricing formulas would be slightly different and the Value premium would be lower (compared to when  $\varrho = 1$ )

$$\begin{aligned} E_t(dR_t^A) - E_t(dR_t^B) &= Cov_t\left(-\frac{d\Lambda_t}{\Lambda_t}, \frac{dq_t^A}{q_t^A}\right) - Cov_t\left(-\frac{d\Lambda_t}{\Lambda_t}, \frac{dq_t^B}{q_t^B}\right) \\ &\quad + \left\{\sigma_{\Lambda,t}^{(1)} + \sigma_{\Lambda,t}^{(3)}\varrho, \sigma_{\Lambda,t}^{(2)}\right\} \cdot \{1, -1\}' \sigma dt \end{aligned}$$

Notice that with  $\varrho = 1$  we have  $Cov_t\left(\frac{dH_t}{H_t}, \frac{dK_t^A}{K_t^A}\right) = \sigma_h \sigma dt$  and with  $1 > \varrho > 0$  it is  $Cov_t\left(\frac{dH_t}{H_t}, \frac{dK_t^A}{K_t^A}\right) = \tilde{\sigma}_h \sigma \varrho dt$ . Thus, we can always set  $\sigma_h = \tilde{\sigma}_h \varrho$  so this covariance is the same in both models. Furthermore, the expected excess return of Security  $A$  with  $\varrho = 1$  is  $Cov_t\left(-\frac{d\Lambda_t}{\Lambda_t}, \frac{dq_t^A}{q_t^A}\right) - Cov_t\left(-\frac{d\Lambda_t}{\Lambda_t}, \frac{dq_t^B}{q_t^B}\right) +$

$\sigma_{\Lambda,t} \cdot \{1, -1\}' \sigma dt$ . Thus we again recover a simple mapping between the two models. Hence, I conjecture that the quantitative results should not be very different. Furthermore, I should be able to calibrate the parameters of the model with  $1 > \varrho > 0$  (and complete markets) so that the key moments are exactly the same as those in the model with  $\varrho = 1$ . The time paths will nonetheless not be identical because of non-linearities, state dependence, and because the introduction of the additional shock  $dZ^H$  changes the time paths of the state variables.

With  $1 > \varrho > 0$ , if we do not allow agents to issue claims on the value of their human capital, markets are incomplete. As a result the price of risk is greater since the agents can no longer smooth consumption as much as before due to the restriction on security holdings. In particular,  $\sigma_{\Lambda,t}^{(3)}$  (and hence  $\{\sigma_{\Lambda,t}^{(1)} + \sigma_{\Lambda,t}^{(3)}\varrho, \sigma_{\Lambda,t}^{(2)}\} \cdot \{1, -1\}' \sigma$ ) is greater (relatively to the case with complete markets and  $1 > \varrho > 0$ ). Thus, market incompleteness can undo the effect<sup>52</sup> of a decrease in  $\varrho$ . When markets are dynamically complete, the Martingale Method is simple. If we introduced markets incompleteness, we would need to directly solve for the competitive equilibrium. With incomplete markets the Martingale Method would be more complex as it would require solving a min-max problem. We could instead use dynamic programming to solve for the competitive equilibrium.

## 7.10 Allowing for firm liabilities

Let us assume that at time  $t$  the agents hold an amount  $d_t^i S_t^i$  of debt from firms of Type  $i$ . As before,  $\varpi_t^i$  is the fraction of financial wealth the agent invests in the equity of Type  $i$ . The securities market clearing conditions would be

$$\varpi_t^A \tilde{W}_t + \varpi_t^B \tilde{W}_t = ((1 - d_t^A)S_t^A + (1 - d_t^B)S_t^B) \quad \text{equity market clears}$$

$$(1 - \varpi_t^A - \varpi_t^B) \tilde{W}_t = d_t^A (\tilde{W}_t - S_t^A) + d_t^B (\tilde{W}_t - S_t^B) \quad \text{debt market clears}$$

where financial wealth is

$$\tilde{W}_s = S_t^A + S_t^B$$

By Walras Law one of these equations is redundant. Indeed, we can re-write these equations as

$$(\varpi_t^A + \varpi_t^B - 1) \tilde{W}_t = -d_t^A S_t^A - d_t^B S_t^B \quad \text{equity market clears}$$

$$(1 - \varpi_t^A - \varpi_t^B) \tilde{W}_t = d_t^A S_t^A + d_t^B S_t^B \quad \text{debt market clears}$$

---

<sup>52</sup>Of course market incompleteness would also affect all other endogenous variables of the model. Nonetheless, I expect that the mechanism highlighted above will be the dominant one.

Notice that when  $d_t^A = d_t^B = 0$  we recover the outcome of the baseline model. When we allow for firm debt the return on equity becomes

$$\begin{aligned} d\tilde{R}_t^i &= dR_t^i - r_t d_t^i dt \quad \text{for } i \in \{A, B\} \\ \Rightarrow d\tilde{R}_t^i &= (r_t(1 - d_t^i) + \sigma_{\Lambda,t} \cdot \varsigma_t^i) dt + \varsigma_t^i \cdot dZ_t \end{aligned}$$

The model's BE/ME ratio is unchanged because of the Modigliani-Miller Theorem (the firm value is unaffected by how it is financed)

$$BE/ME = \frac{Assets - Liabilities}{Equity} = \frac{K_t^i - d_t^i S_t^i / q_t^i}{S_t^i - d_t^i S_t^i} = \frac{K_t^i - d_t^i K_t^i}{S_t^i - d_t^i S_t^i} = \frac{1 - d_t^i}{(1 - d_t^i) q_t^i} = \frac{1}{q_t^i}$$

Let us assume that  $d_t^i$  is constant and that firm debt pays the risk-free rate (as in He and Krishnamurthy; 2012). We can obtain the parameters  $\{d^A, d^B\}$  by using data on the Value (firms of Type A) and Growth firms (of Type B)

$$Leverage = \frac{Liabilities}{Assets} = \frac{d^i S_t^i / q_t^i}{S_t^i / q_t^i} = d^i$$

Using data from Compustat and the Fama and French breakpoint for Value and Growth, I find the ratios of total liabilities to total assets

$$d^A = 0.62$$

$$d^B = 0.58$$

The Type A security excess return becomes

$$d\tilde{R}_t^A - d\tilde{R}_t^B = (r_t(d^B - d^A) + \sigma_{\Lambda,t} \cdot (\varsigma_t^A - \varsigma_t^B)) dt + (\varsigma_t^A - \varsigma_t^B) \cdot dZ_t$$

With an average risk-free rate of 1% we would have

$$E(d\tilde{R}_t^A - d\tilde{R}_t^B) = \underbrace{E(r_t(d^B - d^A))}_{\approx -0.0005} dt + E(dR_t^A - dR_t^B)$$

### 7.11 Cox, Ingersoll and Ross (1985a)

To better understand the implications of the adjustment costs, I will now consider the case without adjustment costs as in Cox, Ingersoll and Ross (1985).

$$\begin{aligned}\frac{dH_{j,t}}{H_{j,t}} &= \frac{I_{j,t}^H}{H_{j,t}}dt - \delta dt + \sigma_h dZ_t^A \\ \frac{dK_{j,t}^i}{K_{j,t}^i} &= \frac{I_{j,t}^i}{K_{j,t}^i}dt - \delta dt + \sigma dZ_t^i \quad \text{for } i \in \{A, B\}\end{aligned}$$

From the Martingale Method in Section 7.7 we have

$$\begin{aligned}\max_{\{C_{j,t}, I_{j,t}^H, H_{j,t}\}_{t=0}^\infty} E_0 \int_0^\infty f\{C_{j,t}, V_{j,t}\} dt \quad \text{s.t.:} \\ \tilde{W}_0 + p_0 H_0 = E_0 \int_0^\infty \Lambda_t (C_{j,t} + I_{j,t}^H - \omega_t H_{j,t}) dt \\ + E_0 \int_0^\infty \Lambda_t p_t H_{j,t} \left( \frac{\bar{\psi}_t}{p_t} - \frac{I_{j,t}^H}{H_{j,t}} + \delta \right) dt\end{aligned}$$

The FOC for investment in human capital implies

$$p_t = 1$$

Similarly from the firms problems we have

$$q_t^i = 1 \quad \text{for } i \in \{A, B\}$$

This is the same result reported in Section 5.3 of Eberly and Wang (2011) and Appendix A.4.3 of Kozak (2012). Therefore, without adjustment cost the BE/ME is the same for both types of securities.

In particular, the FOC's from planner's problem become

$$\begin{aligned}1 &= \frac{1}{\beta} \left( \frac{c}{F} \right)^{1-\rho} (F - x^A F_A - x^B F_B) \\ 1 &= \frac{1}{\beta} \left( \frac{c}{F} \right)^{1-\rho} (F - (x^A - 1)F_A - x^B F_B) \\ 1 &= \frac{1}{\beta} \left( \frac{c}{F} \right)^{1-\rho} (F - x^A F_A - (x^B - 1)F_B)\end{aligned}$$

They imply

$$c = F \left( \frac{\beta}{F} \right)^{\frac{1}{1-\rho}}$$

$$F_A = F_B = 0$$

This is exactly as shown in Appendix A.4.3 of Kozak (2012). The HJB becomes

$$0 = \frac{\beta}{\rho} \left( \frac{c}{F} \right)^\rho + \phi \{x^A, x^B\} + \iota^H(1 - x^A - x^B) + \iota^B x^B + \iota^A x^A$$

where

$$\iota^H(1 - x^A - x^B) + \iota^B x^B + \iota^A x^A = \mathcal{A} - c$$

$$c = F \left( \frac{\beta}{F} \right)^{\frac{1}{1-\rho}}$$

$$\phi \{x^A, x^B\} = \frac{-2\beta - \rho \left( 2\delta + \gamma \left( \sigma^2 \left( (x^A)^2 + (x^B)^2 \right) - 2\sigma\sigma_h x^A(x^A + x^B - 1) \right) + \sigma_h^2 (x^A + x^B - 1)^2 \right)}{2\rho}$$

We can thus find  $F$

$$F = \left( \frac{\rho(A + \phi \{x^A, x^B\})\beta^{\frac{1}{\rho-1}}}{\rho - 1} \right)^{-\frac{1-\rho}{\rho}}$$

As in Eberly and Wang (2011) and Kozak (2012),  $x^A$  and  $x^B$  are no longer state variables but (constant) choice variables which maximize the value function (at date zero).  $\{x^A, x^B\}$  solve

$$0 = \max_{x^A, x^B} \left( \frac{\rho(A + \phi \{x^A, x^B\})\beta^{\frac{1}{\rho-1}}}{\rho - 1} \right)^{-\frac{1-\rho}{\rho}}$$

The solution is

$$x^B = \bar{x}^B = 1/2$$

$$x^A = \bar{x}^A \in [0, 1/2]$$

To summarize, we have

$$F = \bar{F} = \left( \frac{\rho(A + \phi\{\bar{x}^A, \bar{x}^B\})\beta^{\frac{1}{\rho-1}}}{\rho - 1} \right)^{-\frac{1-\rho}{\rho}}$$

and the optimal consumption to capital ratio and total investment to capital ratio

$$c = \bar{c} = \bar{F} \left( \frac{\beta}{\bar{F}} \right)^{\frac{1}{1-\rho}}$$

$$x^A_{\iota^A} + x^B_{\iota^B} + (1 - x^A - x^B)_{\iota^H} = \bar{\iota} = \mathcal{A} - \bar{c}$$

The asset pricing implications are as follows

$$\bar{\psi}_t = r_t + \sigma_h \sigma_{\Lambda,t}^{(A)}$$

$$\psi_t^i = r_t + \sigma \sigma_{\Lambda,t}^{(i)}$$

The first order condition for human capital implies

$$0 = -\bar{\psi}_t + \omega_t$$

$$\Rightarrow \bar{\psi}_t = \mathcal{A}$$

Similarly, the first order condition for physical capital implies

$$\psi_t^i = \mathcal{A}$$

Thus we must have

$$\mathcal{A} = r_t + \sigma_h \sigma_{\Lambda,t}^{(A)}$$

$$\mathcal{A} = r_t + \sigma \sigma_{\Lambda,t}^{(A)}$$

$$\mathcal{A} = r_t + \sigma \sigma_{\Lambda,t}^{(B)}$$

which implies

$$\sigma_h = \sigma$$



$$\sigma_{\Lambda,t}^{(A)} = \sigma_{\Lambda,t}^{(B)}$$

$$\mathcal{A} = r_t + \sigma \sigma_{\Lambda,t}^{(A)}$$

The diffusions are

$$\sigma_{v,t} = \left\{ -(\gamma - 1)\sigma x^A + (\gamma - 1)\sigma_h (x^A + x^B - 1) \quad , \quad -(\gamma - 1)\sigma x^B \right\}$$

$$\sigma_{c,t} = \left\{ (1 - x^A - x^B)\sigma_h + x^A\sigma \quad , \quad x^B\sigma \right\} \quad (\text{since } c \text{ is constant})$$

$$\sigma_{\Lambda,t} = -(\rho - 1)\sigma_{c,t} - \left( \frac{\rho}{\gamma - 1} + 1 \right) \sigma_{v,t}$$

$$\sigma_{\Lambda,t}^{(B)} = \gamma \sigma x^B$$

$$\sigma_{\Lambda,t}^{(A)} = \gamma \sigma (1 - x^B)$$

Thus,

$$r_t = \mathcal{A} - \frac{1}{2}\gamma\sigma^2$$

$$\sigma_{\Lambda,t}^{(A)} = \sigma_{\Lambda,t}^{(B)} = \frac{1}{2}\gamma\sigma$$

The volatility of returns are

$$\varsigma_t^A = \{\sigma, 0\}'$$

$$\varsigma_t^B = \{0, \sigma\}'$$

and

$$dR_t^i = \left( r_t + \frac{1}{2}\gamma\sigma^2 \right) dt + \varsigma_t^{i'} dZ_t$$

$$\Rightarrow dR_t^i = \mathcal{A}dt + \varsigma_t^{i'} dZ_t$$

Thus the expected excess return  $(E_t(dR_t^A) - E_t(dR_t^B))$  is zero.

## 7.12 Alternative calibration

I search for a combination of  $\{\mathcal{A}, c^*, F^*, \gamma, \theta, \rho, \sigma\}$  over the corresponding seven-dimensional space to satisfy the following seven conditions: an expected output growth of 2% in a one-capital economy, a risk-free rate of 0.90%, a volatility of output of 4% , a consumption-to-output ratio of 90% in a one-capital economy, the resource constraint in a one-capital economy, the first-order condition for investment in a one-capital economy, and the HJB in a one-capital economy.

Variable	Name/Calculation	Value
$\mathcal{A}$	Marginal Product of Capital	20.63%
$\beta$	Subjective Discount Factor	7.00%
$c^*$	$\frac{\text{Consumption}}{\text{Capital}}$ in one-capital economy	18.63%
$\delta$	Depreciation	0.00%
$F^*$	Normalized Value Function, $F$ , in one-capital economy	0.0791
$\gamma$	Risk Aversion Parameter	59.35
$\Gamma = \theta$	Adjustment Cost Parameters	2.73%
$\rho$	Implied IES Parameter = 2.0025	0.5006
$\sigma = \sigma_h$	Standard deviation of Capital Growth	5.50%

Table 17: Parameters of the model

Moments	Data/Targets (%)	Model (%)
Mean Consumption Growth	2.00	2.02
Mean Output Growth*	2.00	2.02
Mean Risk-free Return*	0.90	0.90
Standard Deviation of Risk-free Return	2.00	0.14
Mean Return of Value Stocks	10.32	10.97
Standard Deviation of Value Stocks	16.73	4.66
Sharpe Ratio of Value Stocks	56.31	216
Mean Return of Growth Stocks	6.24	9.34
Standard Deviation of Growth Stocks	16.62	4.15
Sharpe Ratio of Growth Stocks	32.13	204
Mean Value premium	4.08	[1.64, 1.65]
Sharpe Ratio of Value premium	38.50	41.42
Mean Market Return	7.16	10.05
Standard Deviation of Market Return	15.45	3.92

Table 18: The parameters are set to to match some annual moments. The moments with \* are some of the moments that I target in the calibration. To obtain the model's moments, I simulate 25 years of observations 10,000 times with a monthly frequency. I then take the sample mean (or standard deviation) over time and then its average across simulations. All values are in annual units.

This alternative calibration yields a Value premium that is very close to that from the main calibration. The Sharpe ratios are much larger because the return volatilities are small. All other results of the model are qualitatively identical and quantitatively similar to those from the main

calibration.

The volatility of equity returns is much lower than what is observed in data. We can nonetheless reconcile the two by adjusting for leverage. Indeed, the model does not include debt financing. In data, firms are levered and the observed volatility of equity is the volatility of levered equity, which is much higher than that of equity with zero leverage.

### 7.13 More on the conditional CAPM

In the table below I reproduce the regressions from Table 4 while introducing some error in  $\beta_t^{i,m}$ ,  $\beta_t^{i,w}$ ,  $\varsigma_t^m$ , and  $\varsigma_t^w$ . In particular, for each observation of  $\{\varsigma_t^A, \varsigma_t^B, \varsigma_t^m, \varsigma_t^w\}$  construct  $\{\hat{\varsigma}_t^A, \hat{\varsigma}_t^B, \hat{\varsigma}_t^m, \hat{\varsigma}_t^w\}$  where  $\hat{\varsigma}_t^A = \varsigma_t^A + \epsilon_t^A$ ,  $\hat{\varsigma}_t^B = \varsigma_t^B + \epsilon_t^B$ ,  $\hat{\varsigma}_t^m = \varsigma_t^m + \epsilon_t^m$ ,  $\hat{\varsigma}_t^w = \varsigma_t^w + \epsilon_t^w$  and the  $\epsilon$ 's are Normal random variables with mean zero and standard deviation equal to that of the corresponding  $\varsigma$ . Lastly, I define  $\hat{\beta}^{i,w} = \frac{\hat{\varsigma}_t^i \cdot \hat{\varsigma}_t^w}{\hat{\varsigma}_t^m \cdot \hat{\varsigma}_t^w}$  and  $\hat{\beta}^{i,m} = \frac{\hat{\varsigma}_t^i \cdot \hat{\varsigma}_t^m}{\hat{\varsigma}_t^m \cdot \hat{\varsigma}_t^m}$ .

Conditional CAPM Regressions									
(1) Wealth Portfolio: $E_t(dR_t^A) - E_t(dR_t^B) = \underbrace{\alpha_0}_{\text{Pricing Error}} + \underbrace{\alpha_1}_{\text{Slope}} \times \hat{\varsigma}_t^w \cdot \hat{\varsigma}_t^w (\hat{\beta}_t^{A,w} - \hat{\beta}_t^{B,w}) dt$									
(2) Market Portfolio: $E_t(dR_t^A) - E_t(dR_t^B) = \underbrace{\alpha_0}_{\text{Pricing Error}} + \underbrace{\alpha_1}_{\text{Slope}} \times \hat{\varsigma}_t^m \cdot \hat{\varsigma}_t^m (\hat{\beta}_t^{A,m} - \hat{\beta}_t^{B,m}) dt$									
	$\gamma = (1 - \rho) = 1$			$\gamma \neq (1 - \rho) = 1$			$\gamma > (1 - \rho) \neq 1$		
	(1)	(2)	Diff.	(1)	(2)	Diff.	(1)	(2)	Diff.
$\alpha_0$ (%)	0.34	1.1	-0.71	0.98	1.8	-0.78	0.9	2.	-1.1
t-stat	6.3	32.	-26.	7.7	20.	-13.	7.1	24.	-17.
$\alpha_1$	0.66	0.63	0.03	1.5	1.5	0.01	1.9	0.05	1.9
t-stat	24.	24.	0.22	18.	17.	0.41	20.	17.	3.
$R^2$	0.66	0.65	0.00	0.51	0.5	0.01	0.56	0.48	0.08
t-stat	23.	23.	0.11	8.3	6.8	0.27	11.	5.9	1.2

Table 19: Each panel shows the regression results for three different specifications for preferences. To obtain the regression coefficients I simulate the model 10,000 times and run a regression for each simulation. The coefficients displayed are the average coefficients across simulations. I then calculate the standard errors as the standard deviation of the statistics across simulations. The t-stats are the ratio of the average statistic and their standard error. In the columns labelled (1), I include the results from the CAPM regressions of excess returns against excess betas on the wealth portfolio. In the columns labelled (2), I include the results from the CAPM regressions of excess returns against excess betas on the market portfolio. The column labelled *Diff* shows the difference between the statistics from regressions (1) and (2) as well as the corresponding t-stats.

## 7.14 Deterministic model

I now turn off all the shocks in the model by setting  $\sigma = \sigma_h = 0$ . The processes for the state variables become

$$\begin{aligned} dx_t^i &= \mu_{x,i,t} dt \quad \text{for } i \in \{A, B\} \\ \mu_{x,A,t} &= x_t^A \left( \Gamma (x_t^A + x_t^B - 1) \ln \left( \frac{\Gamma p_t}{\theta} \right) + (\Gamma - \Gamma x_t^A) \ln \left( \frac{\Gamma q_t^A}{\theta} \right) - \Gamma x_t^B \ln \left( \frac{\Gamma q_t^B}{\theta} \right) \right) \\ \mu_{x,B,t} &= x_t^B \left( \Gamma (x_t^A + x_t^B - 1) \ln \left( \frac{\Gamma p_t}{\theta} \right) - \Gamma x_t^A \ln \left( \frac{\Gamma q_t^A}{\theta} \right) + (\Gamma - \Gamma x_t^B) \ln \left( \frac{\Gamma q_t^B}{\theta} \right) \right) \end{aligned}$$

At the deterministic steady state we have

$$\begin{aligned} 0 &= \Gamma (x^A + x^B - 1) \ln \left( \frac{\Gamma p}{\theta} \right) + (\Gamma - \Gamma x^A) \ln \left( \frac{\Gamma q^A}{\theta} \right) - \Gamma x^B \ln \left( \frac{\Gamma q^B}{\theta} \right) \\ 0 &= \Gamma (x^A + x^B - 1) \ln \left( \frac{\Gamma p}{\theta} \right) - \Gamma x^A \ln \left( \frac{\Gamma q^A}{\theta} \right) + (\Gamma - \Gamma x^B) \ln \left( \frac{\Gamma q^B}{\theta} \right) \end{aligned}$$

which implies that

$$\begin{aligned} p &= q^A = q^B \\ dR^A &= dR^B = r dt = ((1 - \rho)\mu_c + \beta) dt \\ \mu_c &= \Gamma \ln \left( p \frac{\Gamma}{\theta} \right) - \delta \end{aligned}$$

In steady state,  $x^A$  and  $x^B$  are indeterminate because  $H$ ,  $K^A$  and  $K^B$  are risk-less and have the same productivity,  $\mathcal{A}$ . Using **Corollary 2** and the HJB (2.10), we then have that  $p$  and  $F$  jointly solve

$$\begin{aligned} p &= \frac{1}{\beta} \left( \frac{\mathcal{A} - \Gamma p + \theta}{F} \right)^{1-\rho} F \\ 0 &= \frac{\beta}{\rho} \left( \frac{\mathcal{A} - \Gamma p + \theta}{F} \right)^\rho - \frac{\beta}{\rho} + \Gamma \ln \left( p \frac{\Gamma}{\theta} \right) - \delta \end{aligned}$$

Lastly, in steady state we also have  $\iota^A = \iota^B = \iota^H = \Gamma p - \theta$ .

## 7.15 Joint distribution of the state variables

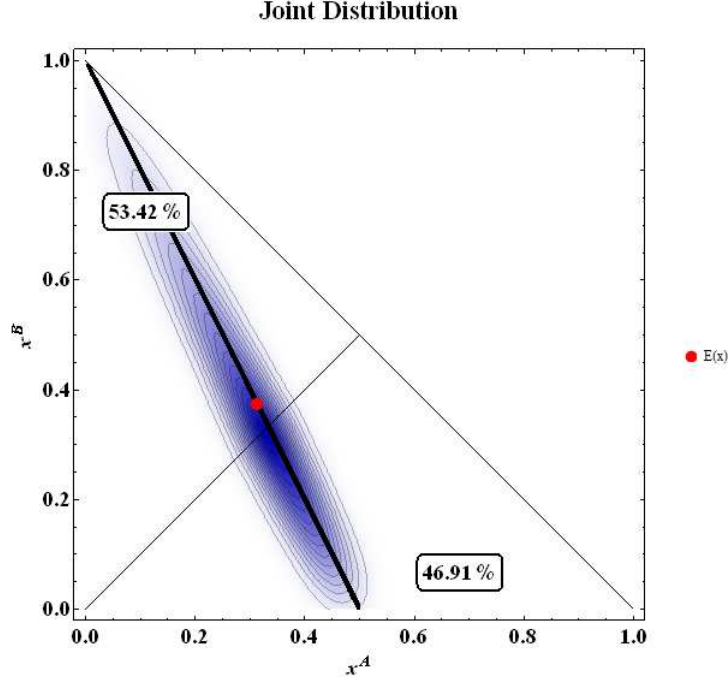


Figure 11: I pool all observations (across time and simulations) from the model for  $\{x^A, x^B\}$  to produce the joint distribution.

The red dot shows the mean of the state variables. Because  $x^A \in [0, 1]$  and  $x^B \in [0, 1]$ , the state space is triangular (and is delineated with the thin black line). However because of the strong symmetry of the model, there is a strong relationship between  $x_t^A$  and  $x_t^B$  and the realizations of  $\{x_t^A, x_t^B\}$  are in a tight band. In particular, the thick black curve is actually a scatter plot from 10,000 simulation of the model. The dark shading shows where the observations are relatively more concentrated. I also plot the 45-degree line and the percentage of observations in the two halves of the state space. Furthermore, from the 10,000 simulations of the model I find that the boundaries are never visited:  $x_t^A \in [0.0038, 0.4976]$  and  $x_t^B \in [0.0034, 0.9924]$ .

## 7.16 Long run risk

Following Borovicka et al. (2011) and Hansen (2011), I construct risk-price elasticities for securities  $A$  and  $B$ . Let  $x$  denote the vector of state variables. We have

$$dx_t = \mu_{x,t}dt + \sigma_{x,t}dZ_t$$

$$\mu_{x,t} = \begin{pmatrix} \mu_{x,A,t} \\ \mu_{x,B,t} \end{pmatrix} \quad \text{and} \quad \sigma_{x,t} = \begin{pmatrix} \sigma'_{x,A,t} \\ \sigma'_{x,B,t} \end{pmatrix}$$

I introduce the perturbation with  $\zeta_t \{\epsilon\}$  where

$$\ln(\zeta_t \{\epsilon\}) = \int_0^t -\frac{1}{2} \epsilon^2 \alpha \cdot \alpha ds + \int_0^t \epsilon \alpha \cdot dZ_s \quad \text{and} \quad \alpha = \left\{ \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\}'$$

The risk-price elasticities are

$$\pi^i \{x, t\} = \underbrace{\frac{1}{t} \frac{d}{d\epsilon} \ln \{E(S_t^i \zeta_t \{\epsilon\} | x_0 = x)\}}_{\text{risk-exposure elasticity}} \Big|_{\epsilon=0} - \underbrace{\frac{1}{t} \frac{d}{d\epsilon} \ln \{E(\Lambda_t S_t^i \zeta_t \{\epsilon\} | x_0 = x)\}}_{\text{risk-value elasticity}} \Big|_{\epsilon=0}$$

for  $i \in \{A, B\}$

The first step is to produce the dynamic valuation decomposition for  $S_t^i$  and  $\Lambda_t S_t^i$ . Since the approach is the same in both cases, I will produce the decomposition for a generic multiplicative functional  $M_t$  and later specialize to  $M_t = S_t^i$  and  $M_t = \Lambda_t S_t^i$ .

$$d \ln(M_t) = \mu_{M,t} dt + \sigma_{M,t} \cdot dZ_t$$

$$\Rightarrow M_t = M_0 \exp \left( \int_0^t (\mu_{M,s}) ds + \int_0^t \sigma_{M,s} \cdot dZ_s \right)$$

I look for an eigenfunction  $e \{x\} = \exp \{g \{x\}\}$  and an eigenvalue  $\nu$  which solve<sup>53</sup>

$$\frac{E(M_t e \{x_t\} | x_0 = x)}{M \{x\}} = \exp(\nu t) e \{x\}$$

Since this equation holds for any  $t$ , it can be localized as

$$\begin{aligned} \lim_{t \downarrow 0} \frac{E(M_t e \{x_t\} | x_0 = x) - \exp(\nu t) e \{x\} M \{x\}}{t} &= 0 \\ \Rightarrow \frac{\partial e}{\partial x'} (\mu_x + \sigma_x \sigma_M) + \frac{1}{2} \text{trace} \left\{ \left( \frac{\partial}{\partial x} \left( \frac{\partial e}{\partial x'} \right) \right) \sigma_x \sigma_x \right\} + e \left( \mu_M + \frac{1}{2} \sigma_M \cdot \sigma_M \right) &= \nu e \\ \Rightarrow \frac{\partial g}{\partial x'} (\mu_x + \sigma_x \sigma_M) + \frac{1}{2} \frac{\partial g}{\partial x'} \sigma_x \sigma_x \frac{\partial g}{\partial x} + \frac{1}{2} \text{trace} \left\{ \left( \frac{\partial}{\partial x} \left( \frac{\partial g}{\partial x'} \right) \right) \sigma_x \sigma_x \right\} &+ \left( \mu_M + \frac{1}{2} \sigma_M \cdot \sigma_M \right) = \nu \end{aligned} \quad (7.8)$$

---

<sup>53</sup>See Section 6.2 of Hansen (2011).

$\nu$  is also the long run growth (or decay) rate,  $\nu = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \left( E \left( M_t \middle| x_0 = x \right) \right)$ . To obtain  $\nu$  we can collect the constant terms from the above eigenfunction equation (7.8) by evaluating this equation at the  $x^A = x^B = 0$  boundary. As expected, I find that  $\nu$  is larger for  $M_t = S_t^A$  (Value stocks) than  $M_t = S_t^B$  since Value stocks have greater returns on average. In particular,

$$\begin{aligned}\nu^A &= \beta - \frac{A - \Gamma q^A \{0, 0\} + \theta}{q^A \{0, 0\}} + \frac{1}{2} \gamma \sigma_h (2\sigma + (-2 + \rho) \sigma_h) + (1 - \rho) \left( \Gamma \ln \left( \frac{\Gamma p^A \{0, 0\}}{\theta} \right) - \delta \right) \\ \nu^B &= \beta - \frac{A - \Gamma q^B \{0, 0\} + \theta}{q^B \{0, 0\}} + \frac{1}{2} \gamma (-2 + \rho) \sigma_h^2 + (1 - \rho) \left( \Gamma \ln \left( \frac{\Gamma p^A \{0, 0\}}{\theta} \right) - \delta \right)\end{aligned}$$

where  $q^B \{0, 0\} > q^A \{0, 0\} > p^A \{0, 0\}$ . The difference in the long-run growth of stock prices between Value and Growth is

$$\nu^A - \nu^B = \frac{(q^A \{0, 0\} - q^B \{0, 0\}) (A + \theta)}{q^A \{0, 0\} q^B \{0, 0\}} + \gamma \sigma \sigma_h > 0$$

Similarly, for  $M_t = \Lambda_t S_t^i$ , I find

$$\begin{aligned}\bar{\nu}^A &= - \frac{A - \Gamma q^A \{0, 0\} + \theta}{q^A \{0, 0\}} \\ \bar{\nu}^B &= - \frac{A - \Gamma q^B \{0, 0\} + \theta}{q^B \{0, 0\}}\end{aligned}$$

Thus, following Section 6.4 of Borovicka et al. (2011), the limiting Value premium is

$$\lim_{t \rightarrow \infty} \left\{ \begin{array}{l} \frac{1}{t} \left[ \ln \left( E \left( S_t^A \middle| x_0 = x \right) \right) - \ln \left( E \left( \Lambda_t S_t^A \middle| x_0 = x \right) \right) \right] \\ - \frac{1}{t} \left[ \ln \left( E \left( S_t^B \middle| x_0 = x \right) \right) - \ln \left( E \left( \Lambda_t S_t^B \middle| x_0 = x \right) \right) \right] \end{array} \right\} = (\nu^A - \bar{\nu}^A) - (\nu^B - \bar{\nu}^B) = \gamma \sigma \sigma_h$$

The  $\sigma \sigma_h$  term comes from the relative covariance of human capital growth with the asset growth of Type A firms; that is  $Cov \left( \frac{dH}{H}, \frac{dK^A}{K^A} \right) - Cov \left( \frac{dH}{H}, \frac{dK^B}{K^B} \right) = \sigma \sigma_h dt$ . This is by design. Nonetheless, it is interesting that we are able to characterize the limiting Value premium by the product of the risk aversion governing parameter and the relative covariance of human capital growth with the asset growth of Value firms.

Continuing with the decomposition, I can now write

$$M_t = \exp(\nu t) \tilde{M}_t \frac{\hat{e} \{x_t\}}{\hat{e} \{x\}}$$

where  $\hat{e} \{x\} = 1/e \{x\}$  and  $\tilde{M}_t$  is Martingale. Notice that at the boundaries we have

$$M_t = \exp(\nu t) \tilde{M}_t \frac{\hat{e} \{x_t\}}{\hat{e} \{x\}} = \exp(\nu t) \tilde{M}_t$$

Therefore, we can set

$$e\{1, 0\} = e\{1, 0\} = e\{0, 0\} = 1$$

$$\Rightarrow g\{1, 0\} = g\{0, 1\} = g\{0, 0\} = 0$$

The above boundary conditions are necessary for solving the partial differential equation in (7.8).

$$\ln(\tilde{M}_t) = \ln(M_t) + \ln(e\{x_t\}) - \ln(e\{x\}) - \nu t$$

$$d\ln(\tilde{M}_t) = -\frac{1}{2}\tilde{\sigma}_{M,t} \cdot \tilde{\sigma}_{M,t}dt + \tilde{\sigma}_{M,t} \cdot dZ_t$$

$$\Rightarrow \tilde{\sigma}_{M,t} = \sigma_{M,t} + \sigma_{x,t} \frac{\partial}{\partial x} g\{x_t\}$$

I now define an alternative measure such that for any function  $n\{x_t\}$  we have

$$E\left(\tilde{M}_t \exp(n\{x_t\}) \middle| x_0 = x\right) = \tilde{E}\left(\exp(n\{x_t\}) \middle| x_0 = x\right)$$

$d\tilde{Z}_t$  is a standard Brownian motion increment under the alternative measure and

$$dZ_t = \Theta_t dt + d\tilde{Z}_t$$

$$dx_t = \mu_{x,t} dt + \sigma_{x,t} (\Theta_t dt + d\tilde{Z}_t)$$

$$dn = \left(\frac{\partial}{\partial x} n\{x\}\right) dx + O(dx^2) = \mu_n dt + \left(\frac{\partial}{\partial x} n\{x\}\right) \sigma_x dZ$$

Therefore,  $E\left(\tilde{M}_t \exp(n\{x_t\}) \middle| x_0 = x\right) = \tilde{E}\left(\exp(n\{x_t\}) \middle| x_0 = x\right)$  implies  $\Theta_t = \tilde{\sigma}_{M,t}$ .

Now, we have

$$\frac{1}{t} \frac{d}{d\epsilon} \ln\{E(M_t \zeta_t\{\epsilon\} | x_0 = x)\} \Big|_{\epsilon=0} = \frac{1}{t} \frac{d}{d\epsilon} \ln\left\{\exp(\nu t) \tilde{E}\left(\frac{\hat{e}\{x_t\}}{\hat{e}\{x\}} \zeta_t\{\epsilon\} | x_0 = x\right)\right\} \Big|_{\epsilon=0}$$

$$\begin{aligned} \frac{1}{t} \frac{d}{d\epsilon} \ln\{E(M_t \zeta_t\{\epsilon\} | x_0 = x)\} \Big|_{\epsilon=0} &= \frac{1}{t} \frac{\tilde{E}\left(\hat{e}\{x_t\} \left(\int_0^t \alpha \cdot d\tilde{Z}_s\right) \middle| x_0 = x\right)}{\tilde{E}(\hat{e}\{x_t\} | x_0 = x)} \\ &+ \frac{1}{t} \frac{\tilde{E}\left(\hat{e}\{x_t\} \left(\int_0^t \alpha \cdot \tilde{\sigma}_{M,s} ds\right) \middle| x_0 = x\right)}{\tilde{E}(\hat{e}\{x_t\} | x_0 = x)} \end{aligned}$$



Using the innovation representation

$$\begin{aligned}
\hat{e}\{x_t\} &= \int_0^t \chi\{x_s, t-s\} d\tilde{Z}_s + \tilde{E}(\hat{e}\{x_t\} | x_0 = x) \\
\chi\{x, t-s\} &= \sigma_x \frac{\partial}{\partial x} \tilde{E}(\hat{e}\{x_t\} | x_s = x) \\
\chi\{x, t-s\} &= \tilde{E}(\hat{e}\{x_t\} | x_s = x) \underbrace{\sigma_x \frac{\partial}{\partial x} \ln(\tilde{E}(\hat{e}\{x_t\} | x_s = x))}_{\phi\{x_s, t-s\}} \\
\Rightarrow \frac{1}{t} \frac{d}{d\epsilon} \ln\{E(M_t \zeta_t\{\epsilon\} | x_0 = x)\} \Big|_{\epsilon=0} &= \frac{1}{t} \left( \frac{\tilde{E}(\hat{e}\{x_t\} \int_0^t \epsilon\{x_s, t-s\} ds | x_0 = x)}{\tilde{E}(\hat{e}\{x_t\} | x_0 = x)} \right) \\
\epsilon\{x, t\} &= \alpha \cdot \left( \phi\{x, t\} + \sigma_M\{x\} + \sigma_x\{x\} \frac{\partial}{\partial x} g\{x\} \right)
\end{aligned}$$

Notice that

$$\begin{aligned}
\phi\{x, t\} &= \sigma_x \frac{\partial}{\partial x} \ln(\tilde{E}(\exp(-g\{x_t\}) | x_0 = x)) \\
\phi\{x, 0\} &= -\sigma_x \frac{\partial}{\partial x} g\{x\}
\end{aligned}$$

I will assume that process for the state vector,  $x$ , is stochastically stable under the alternative measure. Thus  $\lim_{t \rightarrow \infty} \tilde{E}(\exp(-g\{x_t\}) | x_s = x) = \tilde{E}(\exp(-g\{x_t\}))$  and

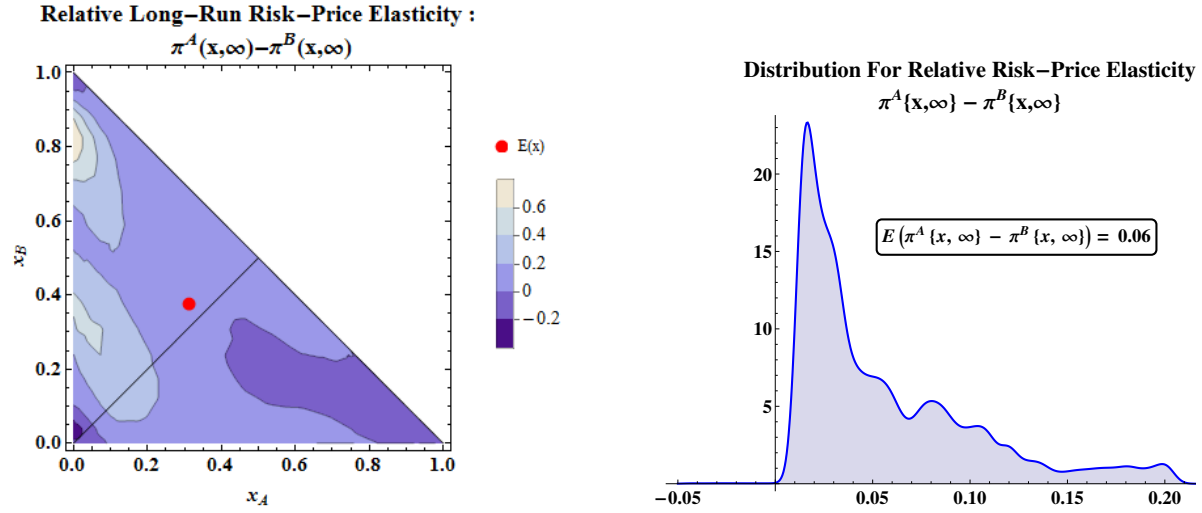
$$\lim_{t \rightarrow \infty} \phi\{x, t\} = 0$$

### 7.16.1 Limiting behavior

Following the approach from Section 3.4 of Borovicka et al. (2011) I derive the limiting behavior. The long horizon risk-price elasticities are

$$\pi^i\{x, \infty\} = \alpha \cdot \left( \sigma_\Lambda\{x\} + \sigma_x \frac{\partial}{\partial x} g_g^i\{x\} - \sigma_x \frac{\partial}{\partial x} g_v^i\{x\} \right) \quad i \in \{A, B\}$$

where  $g_g^i\{x\}$  and  $g_v^i\{x\}$  solve equation (7.8) for  $M_t = S_t^i$  and  $M_t = \Lambda_t S_t^i$  respectively. When we condition on the mean value of the state vector ( $\bar{x}$ ) we find  $\pi^A\{\bar{x}, \infty\} > \pi^B\{\bar{x}, \infty\}$ . Thus, in the model the Type  $A$  firms are indeed more exposed to long-run risk than the Type  $B$  firms.



(a) Contour plots for the relative long-run risk-price elasticity. The red dot shows the location of the mean of the state variables. The 45-degree line separates the two halves of the domain of the state variables.

(b) Distribution of the relative risk-price elasticities from 10,000 simulations of the model.

Figure 12: Figure (a) shows the relative long-horizon risk-price elasticity as a function of the state variables. Figure (b) shows the distribution of the relative risk-price elasticities from simulations of the model.

From Figure 12 we can see that  $\pi^A\{\bar{x}, \infty\} > \pi^B\{\bar{x}, \infty\}$  (where  $\bar{x}$  is the mean of the state variables, that is  $\bar{x} = E(x)$ ) and  $E(\pi^A\{x, \infty\}) > E(\pi^B\{x, \infty\})$ . Thus the Type A firms are indeed more exposed to long-run risk.

## 7.16.2 Transitional dynamics

Let

$$f\{x, t-s\} = \tilde{E}\left(\exp(-g\{x_t\}) \middle| x_s = x\right)$$

$$\Rightarrow f\{x, 0\} = \exp(-g\{x\})$$

Stochastic stability implies

$$\begin{aligned} \lim_{t-s \rightarrow \infty} \phi\{x, t-s\} &= \lim_{t-s \rightarrow \infty} \sigma_x \frac{\partial}{\partial x} \ln \left( \tilde{E}\left(\exp(-g\{x_t\}) \middle| x_s = x\right) \right) \\ &= \lim_{t-s \rightarrow \infty} \sigma_x \frac{\partial}{\partial x} \ln(f\{x, t-s\}) \end{aligned}$$

$$\Rightarrow \lim_{t-s \rightarrow \infty} \frac{\partial}{\partial x} \ln(f\{x, t-s\}) = \{0, 0\}'$$

A conditional expectation is Martingale, therefore

$$\Rightarrow 0 = \frac{\partial}{\partial s} f + \left( \frac{\partial}{\partial x'} f \right) (\mu_x + \sigma_x \sigma_M + \sigma_x \sigma_x g_x) + \frac{1}{2} \text{trace} \left\{ \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x'} f \right) \sigma_x \sigma_x \right\}$$

Let  $u = t - s$ , now we can find  $f\{x, u\}$  by solving

$$\begin{aligned} 0 = & -\frac{\partial}{\partial u} f + \left( \frac{\partial}{\partial x'} f \right) (\mu_x + \sigma_x \sigma_M + \sigma_x \sigma_x g_x) \\ & + \frac{1}{2} \text{trace} \left\{ \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x'} f \right) \sigma_x \sigma_x \right\} \end{aligned} \quad (7.9)$$

$$f\{x, 0\} = \exp(-g\{x\})$$

$$\lim_{u \rightarrow \infty} \frac{\partial}{\partial x} \ln(f\{x, u\}) = \{0, 0\}'$$

The risk-price elasticities are

$$\begin{aligned} \pi^i\{x, t-s\} = & \frac{1}{t} \left( \frac{\tilde{E}(\hat{e}_g^i\{x_t\} \int_0^t \epsilon_g^i\{x_s, t-s\} ds | x_0 = x)}{\tilde{E}(\hat{e}_g^i\{x_t\} | x_0 = x)} \right) \\ & - \frac{1}{t} \left( \frac{\tilde{E}(\hat{e}_v^i\{x_t\} \int_0^t \epsilon_v^i\{x_s, t-s\} ds | x_0 = x)}{\tilde{E}(\hat{e}_v^i\{x_t\} | x_0 = x)} \right) \quad \text{for } i \in \{A, B\} \end{aligned}$$

where

$$\hat{e}_g^i\{x_t\} = 1/\exp(g_g^i\{x_t\})$$

$$\hat{e}_v^i\{x_t\} = 1/\exp(g_v^i\{x_t\})$$

$$\epsilon_g^i\{x, t-s\} = \alpha \cdot \left( \phi_g^i\{x, t-s\} + \varsigma^i\{x\} - \sigma_\Lambda\{x\} + \sigma_x\{x\} \frac{\partial}{\partial x} g_g^i\{x\} \right)$$

$$\epsilon_v^i\{x, t-s\} = \alpha \cdot \left( \phi_v^i\{x, t-s\} + \varsigma^i\{x\} - \sigma_\Lambda\{x\} + \sigma_x\{x\} \frac{\partial}{\partial x} g_v^i\{x\} \right)$$

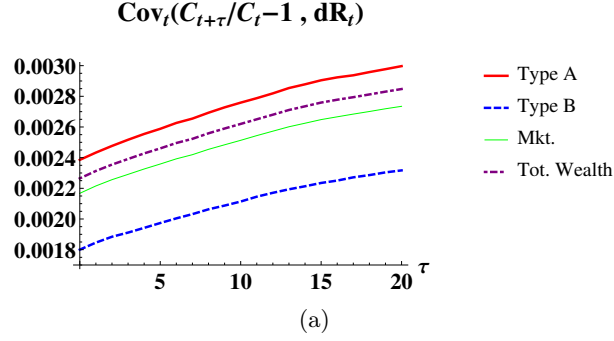
$$\phi_g^i\{x, t-s\} = \sigma_x \frac{\partial}{\partial x} f_g^i\{x, t-s\}$$

$$\phi_v^i\{x, t-s\} = \sigma_x \frac{\partial}{\partial x} f_v^i\{x, t-s\}$$

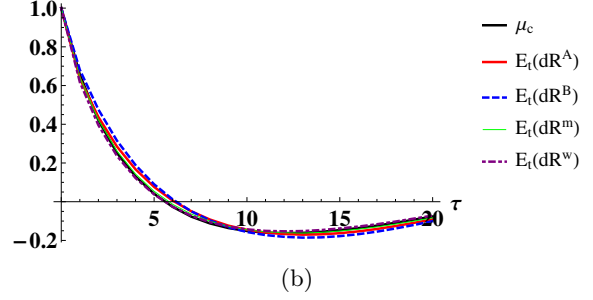
and  $f_g^i\{x, u\}$  and  $f_v^i\{x, u\}$  solve equation (7.9) for  $M_t = S_t^i$  and  $M_t = \Lambda_t S_t^i$  respectively.

### 7.16.3 Autocorrelations and covariances with consumption growth

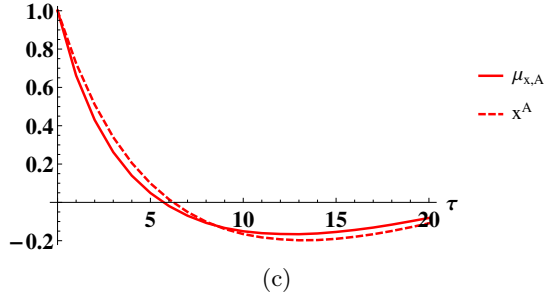
#### Covariances with consumption growth



#### Autocorrelations with lag of τ years



#### Autocorrelations with lag of τ years



#### Autocorrelations with lag of τ years

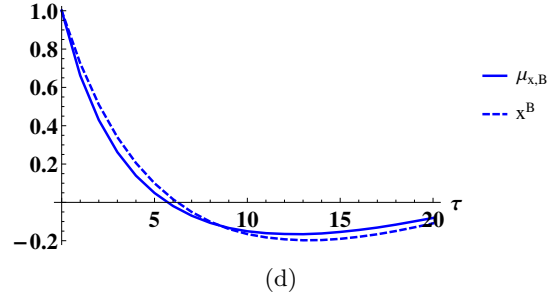


Figure 13: These figures show covariances of security returns with consumption growth over  $\tau$  years and autocorrelations with lags of  $\tau$  years.

First, I write the process for consumption as

$$d\ln C_t = \mu_{c,t}dt - \frac{1}{2}\sigma_{c,t} \cdot \sigma_{c,t}dt + \sigma_{c,t} \cdot dZ_t$$

Then, I calculate the covariances between equity returns and consumption growth as follows.

$$\begin{aligned} \text{Cov}_t\left(\frac{C_{t+\tau}}{C_t} - 1, dR_t^i\right) &= \text{Cov}_t\left(\exp\left(\int_t^{t+\tau} \left(\mu_{c,s} - \frac{1}{2}\sigma_{c,s} \cdot \sigma_{c,s}\right)ds + \int_t^{t+\tau} \sigma_{c,s} \cdot dZ_s\right), dR_t^i\right) \\ \text{Cov}_t\left(\frac{C_{t+\tau}}{C_t} - 1, dR_t^i\right) &= E_t\left(\exp\left(\int_t^{t+\tau} \left(\mu_{c,s} - \frac{1}{2}\sigma_{c,s} \cdot \sigma_{c,s}\right)ds + \int_t^{t+\tau} \sigma_{c,s} \cdot dZ_s\right) dR_t^i\right) \\ &\quad - E_t\left(\exp\left(\int_t^{t+\tau} \left(\mu_{c,s} - \frac{1}{2}\sigma_{c,s} \cdot \sigma_{c,s}\right)ds + \int_t^{t+\tau} \sigma_{c,s} \cdot dZ_s\right)\right) E_t(dR_t^i) \end{aligned}$$

$$Cov_t \left( \frac{C_{t+\tau}}{C_t} - 1, dR_t^i \right) = E_t \left( \exp \left( \int_t^{t+\tau} (\mu_{c,s} - \frac{1}{2} \sigma_{c,s} \cdot \sigma_{c,s}) ds + \int_t^{t+\tau} \sigma_{c,s} \cdot dZ_s \right) \varsigma_t^i \cdot dZ_t \right)$$

$$Cov_t \left( \frac{C_{t+\tau}}{C_t} - 1, dR_t^i \right) \approx E_t \left( \varsigma_t^i \cdot dZ_t \times \exp \left( \sum_{j=0}^{\tau} (\mu_{c,t+j} - \frac{1}{2} \|\sigma_{c,t+j}\|^2) dt + \sum_{j=0}^{\tau} (\sigma_{c,t+j} \cdot dZ_{t+j}) \right) \right)$$

As expected the returns on the Type *A* security covary much more with consumption growth over horizons  $\tau > 0$  compared to the Type *B* security. This result is reminiscent of that from [Hansen, Heaton and Li \(2008\)](#) and shows that the Type *A* security is relatively more exposed to long horizon risk than the Type *B* security. However, in my model it is also true at shorter horizons. The contemporaneous correlations between consumption growth and equity returns are in [Table 20](#). They are larger than what is observed in data because the model is not tailored to match these moments.

	$dR_t^A$	$dR_t^B$	$dR_t^m$	$dR_t^w$
$Corr_t \left( \frac{dC_t}{C_t}, dR_t \right)$	0.90	0.74	0.98	1.0

Table 20: Contemporaneous correlations between consumption growth and equity returns

The autocorrelations are somewhat large for expected consumption growth ( $\mu_{c,t}$ ), expected returns, as well as the levels and expected growth rate of the state variables ( $x_t^i$  and  $\mu_{x,i,t}$  for  $i \in \{A, B\}$  respectively). That is because the adjustment costs prevent instantaneous reallocation of capital. As a result the capital shares,  $\{x_t^A, x_t^B\}$ , have large autocorrelations for  $\tau < 3$ . Since the expected consumption growth, the expected returns and the expected growth rate of the capital shares are functions of state variables, these drifts also display strong autocorrelations for  $\tau < 3$ .

## 7.17 Controlling for the number of employees

Since Growth firms tend to hire more employees, in the regressions below I control for the level (or growth rate) of employment.

			$R^2$	#Obs.	
(1)	$\ln (Asset)_{i,t} = a_1 \ln (Wage)_{i,t} + a_2 \ln (Wage)_{i,t} \times ValueDummy_{i,t}$ $+ a_3 \ln (Emp)_{i,t} + \varepsilon_{i,t}$		0.96	15,051	
(2)	$\Delta \ln (Asset)_{i,t} = a_1 \Delta \ln (Wage)_{i,t} + a_2 \Delta \ln (Wage)_{i,t} \times ValueDummy_{i,t}$ $+ a_3 \Delta \ln (Emp)_{i,t} + \varepsilon_{i,t}$		0.41	13,089	
		Coef.	Std.Err.	T-stat	P-value
	$\ln (Wage)$	1.68	0.00	391.53	0.00
(1)	$\ln (Wage) \times ValueDummy$	0.11	0.01	21.69	0.00
	$\ln (Emp)$	−0.92	0.01	−134.67	0.00
	$\Delta \ln (Wage)$	0.30	0.01	53.83	0.00
(2)	$\Delta \ln (Wage) \times ValueDummy$	0.07	0.01	7.75	0.00
	$\Delta \ln (Emp)$	0.47	0.01	50.34	0.00

Table 21: Panel (1) shows the results from regressing the natural logarithm of firm assets against the natural logarithm of the wage bill, the wage bill times an indicator variable for Value firms, as well as the natural logarithm of the number of employees. Panel (2) shows the results from regressing the annual change in the natural logarithm of assets against the change in the natural logarithm of the wage bill, the interaction of the latter with the Value firm indicator, as well as the change in the natural logarithm of the number of employees. The wage bill is the sum of salaries, wages, pension costs, profit sharing and incentive compensation, payroll taxes and other employee benefits (As measured by the *XLR* variable from the firm income statement in Compustat).

## 7.18 Univariate betas of monthly equity returns with monthly returns on human Capital

	Aggregate Human Capital		~Obs
	$\beta_{i,h}$	P-val	
Value Portfolio	0.91	0.07	600
Growth Portfolio	0.04	0.93	600
	Aggregate Human Capital		~Obs
	$\beta_{i,h}$	P-val	
Value Firms	0.71	0.00	500,000
Growth Firms	0.52	0.00	500,000

Firm level returns on human capital data is only available annually from Compustat. Thus, I am unable run the above regressions at a monthly frequency for portfolio or firm level human capital returns.

## 7.19 Annually Updated $\beta_h$

$\beta_h$ Quintiles	BE/ME Quintiles				
	Low	2	3	4	High
Low	-0.18	3.72	5.09	4.54	5.19
2	1.44	3.42	5.20	6.84	6.94
3	1.65	2.26	3.77	4.83	8.04
4	4.58	3.30	4.23	6.69	9.19
High	4.60	3.48	6.39	7.19	8.71

(a) This table shows the annualized average real monthly return of 25 dual-sort portfolios. The dual-sort portfolios are from groupings of securities by BE/ME quintiles and  $\beta_h$  quintiles.  $\beta_h$  is the slope from a rolling 12 month univariate regression of equity returns against human capital returns. The shaded cells show where the monotone increasing relationship between BE/ME and average equity returns breaks down.

$\beta_h$ Quintiles				
Low	2	3	4	High
Expected Return				
$E(dR_t)$				
1.73	2.84	3.70	3.88	5.35

(b) Mean annualized real monthly returns for  $\beta_h$ -quintile portfolios.

Table 22: These tables use data from CRSP and Compustat.

## 7.20 Extension of the model

The model yields a Sharpe ratio for the Value premium ( $\frac{E_t(dR_t^A) - E_t(dR_t^B)}{\sqrt{\varsigma_t^A \cdot \varsigma_t^A - 2\varsigma_t^A \cdot \varsigma_t^B + \varsigma_t^B \cdot \varsigma_t^B}}$ ) which is fairly small compared to what is observed in data. This is because the local covariance of the two risky securities ( $\varsigma_t^A \cdot \varsigma_t^B dt$ ) is fairly small. I can correct this without affecting the average Value premium much by introducing some common variation in the two types of firms which is locally orthogonal to (or has the same local covariance with) both  $dZ_t^A$  and  $dZ_t^B$ . I can do so by making the parameter which governs the risk aversion ( $\gamma_t$ ) or the adjustment cost parameter ( $\Gamma_t = \theta_t$ ) stochastic (and mean-reverting for stationarity). In the current model, the Tobin Q's have a rather low volatility and most of the volatility of the capital gains comes from the volatility of capital. With  $\theta_t$  or  $\gamma_t$  stochastic, the Tobin Q's would be more volatile and we could have highly volatile capital gains while keeping the volatility of capital low; which would be more factual. This model would be more difficult to solve because there would be an additional state variable,  $\theta_t$  or  $\gamma_t$ . I go through some of the details of one of the aforementioned extensions of the model below.

### 7.20.1 Stochastic Risk Aversion

Let  $\gamma = \gamma_t$  where

$$d\gamma_t = \vartheta_0 (\bar{\gamma} - \gamma_t) dt + \vartheta_1 \gamma_t dZ_t^\gamma$$

We can obtain the solution for  $\gamma_t$  using the steps below.

$$\begin{aligned}
(d\gamma_t + \vartheta_0\gamma_t dt - \vartheta_1\gamma_t dZ_t^\gamma) \exp \left\{ \left( \vartheta_0 + \frac{1}{2}\vartheta_1^2 \right) t - \int_0^t \vartheta_1 dZ_s^\gamma \right\} &= \vartheta_0 \bar{\gamma} \exp \left\{ \begin{array}{c} (\vartheta_0 + \frac{1}{2}\vartheta_1^2) t \\ - \int_0^t \vartheta_1 dZ_s^\gamma \end{array} \right\} dt \\
\Rightarrow d \left( \gamma_t \exp \left\{ \left( \vartheta_0 + \frac{1}{2}\vartheta_1^2 \right) t - \vartheta_1 Z_t^\gamma \right\} \right) &= \vartheta_0 \bar{\gamma} \exp \left\{ \begin{array}{c} (\vartheta_0 + \frac{1}{2}\vartheta_1^2) t \\ - \vartheta_1 Z_t^\gamma \end{array} \right\} dt \\
\Rightarrow \gamma_t &= \frac{\vartheta_0 \bar{\gamma} \int_0^t \exp \left\{ (\vartheta_0 + \frac{1}{2}\vartheta_1^2) n - \vartheta_1 Z_n^\gamma \right\} dn + \gamma_0}{\exp \left\{ (\vartheta_0 + \frac{1}{2}\vartheta_1^2) t - \vartheta_1 Z_t^\gamma \right\}}
\end{aligned}$$

Notice that with  $\gamma_0 = \bar{\gamma}, \vartheta_0 > 0, \vartheta_1 > 0$  we have  $\gamma_t \geq 0$ . Furthermore,

$$E_0(\gamma_t) = \bar{\gamma} (1 - \exp\{-\vartheta_0 t\}) + \gamma_0 \exp\{-\vartheta_0 t\}$$

$$\lim_{t \rightarrow \infty} E_0(\gamma_t) = \bar{\gamma}$$

To maintain market completeness, I will set  $dZ_t^\gamma = dZ_t^A + dZ_t^B$ . For homogeneity, I need to modify the value function slightly. In particular, I will use an aggregator,  $\langle \tilde{f}, \tilde{A}_v \rangle$ , where

$$\begin{aligned}
\tilde{f}\{C_t, \tilde{V}_t\} &= \frac{\beta}{\rho} \left( \frac{C_t^\rho}{\tilde{V}_t^{\rho-1}} - \tilde{V}_t \right) \\
\tilde{A}_v\{\tilde{V}_t\} &= -\frac{\gamma_t}{\tilde{V}_t}
\end{aligned}$$

the new value function is  $\tilde{V}_t \equiv ((1-\gamma)V_t)^{\frac{1}{1-\gamma}}$  and  $\tilde{f}\{C_t, \tilde{V}_t\} \equiv \tilde{V}_t^{\gamma_t} f\{C_t, V_t\}$ . The process for the value function is

$$d\tilde{V}_t = - \left( \tilde{f}\{C_t, \tilde{V}_t\} + \frac{1}{2} \tilde{V}_t^2 \tilde{A}_v\{\tilde{V}_t\} \sigma_{v,t} \cdot \sigma_{v,t} \right) dt + \tilde{V}_t \sigma_{v,t} dZ_t$$

The variance multiplier,  $\tilde{A}_v\{\tilde{V}_t\}$ , introduces a penalty for volatility in the value function.

In the case where  $\gamma$  is constant, this new aggregator is ordinaly equivalent to the original aggregator. Indeed, following [Kozak \(2012\)](#), let us define the change of variables  $\chi\{y\} = \frac{y^{1-\gamma}}{1-\gamma}$ .



This implies that  $V_t = \chi \left\{ \tilde{V}_t \right\}$  and

$$\begin{aligned} \tilde{f} \left\{ C_t, \tilde{V}_t \right\} &= \frac{f \left\{ C_t, V_t \right\}}{\chi' \left\{ \tilde{V}_t \right\}} \\ \tilde{A}_v \left\{ \tilde{V}_t \right\} &= \frac{\chi'' \left\{ \tilde{V}_t \right\}}{\chi' \left\{ \tilde{V}_t \right\}} \end{aligned} \quad (7.10)$$

Recall that the original aggregator was a normalized aggregator  $\langle f, A_v \rangle$  with

$$A_v \left\{ V_t \right\} = A_v \left\{ \chi \left\{ \tilde{V}_t \right\} \right\} = 0$$

Thus we can re-write  $\tilde{A}_v \left\{ \tilde{V}_t \right\}$  as

$$\tilde{A}_v \left\{ \tilde{V}_t \right\} = \chi' \left\{ \tilde{V}_t \right\} A_v \left\{ \chi \left\{ \tilde{V}_t \right\} \right\} + \frac{\chi'' \left\{ \tilde{V}_t \right\}}{\chi' \left\{ \tilde{V}_t \right\}} \quad (7.11)$$

Equations (7.10) and (7.11) imply that the aggregator  $\langle \tilde{f}, \tilde{A}_v \rangle$  and  $f$  are ordinally equivalent (see Section 1.4 of [Duffie and Epstein; 1992b](#)). Thus the corresponding SPD's are the same.

$$G_t = \exp \left( \int_0^t f_v \left\{ C_s, V_s \right\} ds \right)$$

$$\Lambda_t = G_t f_c \left\{ C_t, V_t \right\}$$

$$\frac{d\Lambda_t}{\Lambda_t} = -r_t dt - \sigma_{\Lambda,t} \cdot dZ_t$$

$$r_t = -f_v \left\{ C_t, V_t \right\} - E_t \left[ \frac{df_c \left\{ C_t, V_t \right\}}{f_c \left\{ C_t, V_t \right\}} \right]$$

$$\sigma_{\Lambda,t} = -\mathcal{L} \left[ \frac{df_c \left\{ C_t, V_t \right\}}{f_c \left\{ C_t, V_t \right\}} \right]$$

where  $\mathcal{L}[\cdot]$  denotes the loading on  $dZ_t$

$$f \left\{ C_t, V_t \right\} = \tilde{V}_t^{-\gamma} \tilde{f} \left\{ C_t, \tilde{V}_t \right\}$$

$$\tilde{V}_t = ((1 - \gamma)V_t)^{\frac{1}{1-\gamma}}$$

$$f_v = \tilde{f}_v - \gamma \frac{\tilde{f}}{\tilde{V}_t}$$

$$f_c = \tilde{V}_t^{-\gamma} \tilde{f}_c$$

$$\frac{df_c}{f_c} = \frac{d\tilde{f}_c}{\tilde{f}_c} - \gamma \frac{d\tilde{V}_t}{\tilde{V}_t} - \gamma \left( \frac{d\tilde{f}_c}{\tilde{f}_c} \right) \left( \frac{d\tilde{V}_t}{\tilde{V}_t} \right)$$

$$\Rightarrow r_t = - \left( \tilde{f}_v - \gamma \frac{\tilde{f}}{\tilde{V}_t} \right) - E_t \left[ \frac{d\tilde{f}_c}{\tilde{f}_c} - \gamma \frac{d\tilde{V}_t}{\tilde{V}_t} - \gamma \left( \frac{d\tilde{f}_c}{\tilde{f}_c} \right) \left( \frac{d\tilde{V}_t}{\tilde{V}_t} \right) \right]$$

$$\Rightarrow \sigma_{\Lambda,t} = -\mathcal{L} \left[ \frac{d\tilde{f}_c \{C_t, \tilde{V}_t\}}{\tilde{f}_c \{C_t, \tilde{V}_t\}} - \gamma \frac{d\tilde{V}_t}{\tilde{V}_t} \right]$$

Now, when the parameter which governs the risk aversion is stochastic ( $\gamma = \gamma_t$ ) we can guess and verify that the process for the SPD is

$$\frac{d\Lambda_t}{\Lambda_t} = -r_t dt - \sigma_{\Lambda,t} \cdot dZ_t$$

$$r_t = - \left( \tilde{f}_v - \gamma_t \frac{\tilde{f}}{\tilde{V}_t} \right) - E_t \left[ \frac{d\tilde{f}_c}{\tilde{f}_c} - \gamma_t \frac{d\tilde{V}_t}{\tilde{V}_t} - \gamma_t \left( \frac{d\tilde{f}_c}{\tilde{f}_c} \right) \left( \frac{d\tilde{V}_t}{\tilde{V}_t} \right) \right]$$

$$\sigma_{\Lambda,t} = -\mathcal{L} \left[ \frac{d\tilde{f}_c \{C_t, \tilde{V}_t\}}{\tilde{f}_c \{C_t, \tilde{V}_t\}} - \gamma_t \frac{d\tilde{V}_t}{\tilde{V}_t} \right]$$

The following verification proof below is adapted from [Kozak \(2012\)](#). Recall that the agent's is

$$\begin{aligned} & \max_{\{C_{j,t}, I_{j,t}^H, H_{j,t}, \varpi_{j,t}\}_{t=0}^{\infty}} E_0 \int_0^{\infty} \tilde{f} \{C_{j,t}, \tilde{V}_{j,t}\} dt \quad s.t.: \\ d\tilde{W}_{j,t} &= \left( \tilde{W}_{j,t} r_t + \varpi_{j,t} \cdot \tilde{W}_{j,t} (\mu_t - \mathbf{1} r_t) - C_{j,t} - I_{j,t}^H + \omega_t H_{j,t} \right) dt + \tilde{W}_{j,t} \varpi'_{j,t} dZ_t \\ \frac{dH_{j,t}}{H_{j,t}} &= \Gamma \ln \left( 1 + \frac{I_{j,t}^H}{\theta H_{j,t}} \right) dt - \delta dt + \sigma_h dZ_t^A \end{aligned}$$

where  $\varsigma_t = \begin{pmatrix} \varsigma_t^{A'} \\ \varsigma_t^{B'} \end{pmatrix}$  is a  $2 \times 2$  matrix. The corresponding HJB is

$$\begin{aligned} 0 = & \max_{C_{j,t}, I_{j,t}^H, H_{j,t}, \varpi_{j,t}} \left( \tilde{f} + \frac{1}{2} \tilde{V}^2 \tilde{A}_v \sigma_{v,t} \cdot \sigma_{v,t} \right) dt \\ & + \tilde{V}_X \cdot dX + \frac{1}{2} dX (V_{XX}) dX + \tilde{V}_w dW + \frac{1}{2} \tilde{V}_{ww} dW^2 + \left( \tilde{V}_{Xw} \cdot dX \right) dW \end{aligned}$$

where

$$X = \{x^A, x^B, \gamma\} \quad \text{and} \quad dX = \mu_X dt + \sigma_X \cdot dZ$$

$$\left( \tilde{V}_{Xw} \cdot dX \right) \equiv \tilde{V}_{x^A w} dx^A + \tilde{V}_{x^B w} dx^B + \tilde{V}_{\gamma w} d\gamma$$

$$\sigma_{v,t} = \mathcal{L} \left[ \frac{d\tilde{V}}{\tilde{V}} \right] = \tilde{W}_{j,t} \frac{\tilde{V}_w}{\tilde{V}} \varpi'_{j,t} \varsigma_t + \frac{\tilde{V}_x \cdot \sigma_X}{\tilde{V}}$$

The FOCs for  $C_{j,t}$  and  $\varpi_{j,t}$  are

$$\tilde{f}_c = \tilde{V}_w$$

$$\tilde{V}^2 \tilde{A}_v \tilde{W}_{j,t} \frac{\tilde{V}_w}{\tilde{V}} \sigma'_{v,t} \varsigma_t + \tilde{W}_{j,t} \tilde{V}_w (\mu_t - \mathbf{1} r_t) + \tilde{W}_{j,t}^2 \tilde{V}_{ww} \varpi'_{j,t} \varsigma_t^2 + \tilde{W}_{j,t} \left( \tilde{V}_{Xw} \cdot \sigma_X \right)' \varsigma_t = 0$$

This implies

$$\begin{aligned} \sigma_{\Lambda,t} &= -\mathcal{L} \left[ \frac{d\tilde{V}_w}{\tilde{V}_w} \right] + \gamma_t \sigma_{v,t} \\ \sigma'_{\Lambda,t} &= -\tilde{W}_{j,t} \frac{\tilde{V}_{ww}}{\tilde{V}_w} \varpi'_{j,t} \varsigma_t - \frac{\left( \tilde{V}_{Xw} \cdot \sigma_X \right)'}{\tilde{V}_w} + \gamma_t \sigma'_{v,t} \\ \tilde{W}_{j,t} \sigma'_{\Lambda,t} \varsigma_t &= -\tilde{W}_{j,t}^2 \frac{\tilde{V}_{ww}}{\tilde{V}_w} \varpi'_{j,t} \varsigma_t^2 - \tilde{W}_{j,t} \frac{\left( \tilde{V}_{Xw} \cdot \sigma_X \right)'}{\tilde{V}_w} \varsigma_t + \tilde{W}_{j,t} \gamma_t \sigma'_{v,t} \varsigma_t \\ \Rightarrow \sigma'_{\Lambda,t} \varsigma_t &= (\mu_t - \mathbf{1} r_t) \end{aligned} \tag{7.12}$$

Now, the gains processes adjusted by the SPD is Martingales, therefore

$$E_t \left( \Lambda_T S_T + \int_0^T \Lambda_t D_t dt \right) = \Lambda_t S_t + \int_0^t \Lambda_t D_t dt$$

$$\begin{aligned}
0 &= E_t \left( d \left( \Lambda_t S_t + \int_0^t \Lambda_t D_t dt \right) \right) \\
\mu_t &= \mathbf{1}r_t + \sigma'_{\Lambda,t} \varsigma_t + \frac{D_t}{S_t} - \frac{D_t}{S_t} \\
\Rightarrow \sigma'_{\Lambda,t} \varsigma_t &= (\mu_t - \mathbf{1}r_t)
\end{aligned} \tag{7.13}$$

Since equations (7.12) and (7.13) are the same, we have confirmed that the guess for  $\frac{d\Lambda_t}{\Lambda_t}$  when  $\gamma_t$  is stochastic is indeed correct.

### 7.20.2 Boundary conditions

Following Kozak (2012), we can guess and verify that the value function takes the form

$$\tilde{V} \{H + K^A + K^B, x^A, x^B, \gamma\} = (H + K^A + K^B) F \{x^A, x^B, \gamma\}$$

Solving the model requires solving a system of five PDE's (three FOC's for investment, the HJB and the resource constraint) with three boundary conditions. I obtain the boundary conditions by solving the model with  $x^A = 1$ ,  $x^B = 1$ , and  $1 - x^A - x^B = 1$ . These boundary problems reduce to solving second order ODE's for  $F \{1, 0, \gamma_t\} = F \{0, 1, \gamma_t\} = F \{0, 0, \gamma_t\} = F \{\gamma_t\}$ . I solve these second order ODE's with the following conditions:  $F \{\gamma_t = 0\} = 1$  (since with risk-neutral preferences the value function is linear in total capital) and  $F \{\gamma_t = \infty\} = 0$  (since the value function is zero for an infinitely risk averse agent). Solving this extension of the model requires a lot of computing power.

### 7.21 Solving PDEs

First, following Chap 6 of Judd (1998) I re-write the state variables as  $\{\tilde{x}^A, \tilde{x}^B\}$ , functions of Chebyshev nodes. I then approximate  $F\{x^A, x^B\}$  with a complete<sup>54</sup> Chebyshev polynomial,  $\tilde{F}\{\tilde{x}^A, \tilde{x}^B\}$ . I define the residual function,  $R$ , as the PDE where I plug in the approximation  $\tilde{F}$  and the as well as the nodes  $\{\tilde{x}^A, \tilde{x}^B\}$ . Using the collocation approach, the vector of polynomial coefficients  $\alpha$ , is chosen to solve  $R\{\alpha\} = 0$  on the grid  $\{\tilde{x}^A, \tilde{x}^B\}$ . I first choose the size of the  $n \times n$  grid. Then I start with low order polynomials and solve for  $\alpha$ . I then use an interpolation method to obtain the solution for  $F\{x^A, x^B\}$  over the continuous state space  $\{x^A \in [0, 1], x^B \in [0, 1 - x^A]\}$ . I plug this function,  $F\{x^A, x^B\}$ , back into the PDE and examine the size of the PDE errors over the continuous state space  $\{x^A \in [0, 1], x^B \in [0, 1 - x^A]\}$ . I steadily increase the degree of the polynomial and

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<sup>54</sup>The approximation function will be composed of complete orthogonal basis functions.

repeat the procedure until the PDE errors are minimized. I use the [AMPL](#) modeling language to write the problem and I solve it using [SNOPT](#) on the [NEOS](#) server.