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The Time Path of the Saving Rate: Hyperbolic Discounting and Short-Term Planning[1]

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Abstract. The standard neoclassical growth model with Cobb-Douglas production predicts a monotonically declining saving rate, when reasonably calibrated. Ample empirical evidence, however, shows that the transition paths of most countries' saving rates exhibit a statistically significant hump-shaped pattern. Prior literature shows that CES production may imply a hump-shaped pattern of the saving rate (Goméz, 2008). However, the implied magnitude of the hump falls short of what is seen in empirical data. We introduce two non-standard features of preferences into a neoclassical growth model with CES production: hyperbolic discounting and short planning horizons. We show that, in contrast to the commonly accepted argument, in general (except for the special case of logarithmic utility) a model with hyperbolic discounting is *not* observationally equivalent to one with exponential discounting. We also show that our framework implies a hump-shaped saving rate dynamics that is consistent with empirical evidence. Hyperbolic discounting turns out to be a major factor explaining the magnitude of the hump of the saving rate path. Numerical simulations employing a generalized class of hyperbolic discount functions, which we term *regular* discount functions, support the results.

Keywords and Phrases: Saving rate dynamics, non-monotonic transition path, hyperbolic discounting, regular discounting, short-term planning, neoclassical growth model

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1. Introduction

This paper analyzes the impact of hyperbolic discounting on transitional dynamics of the saving rate. It is well known that the standard neoclassical growth model with Cobb-Douglas technology, exponential discounting, and isoelastic preferences – a common framework in growth theory – exhibits a monotone transition path of the saving rate. For a reasonable calibration, it exhibits a monotonically declining transition path as an economy develops (Barro and Sala-i-Martin 2004, p.135 ff.). This property, however, is counterfactual. As is discussed in Section 2, ample empirical evidence suggests two regularities: an increase in the saving rate as an economy experiences growth of per capita income; and a non-monotone transition path, featuring a statistically significant hump.

The problem of the counterfactual prediction of transitional dynamics of the saving rate by the standard growth model (with Cobb-Douglas technology) has been addressed in the literature. Gómez (2008), among others, provides a solution by introducing a more flexible CES production technology. For important situations, the model with CES technology implies a humped transition path of the saving rate. This hump is confirmed by a number of numerical simulations in the present paper paper. However, the numerical simulations also suggest that for reasonable calibrations, the implied hump in the transitional dynamics of the saving rate has a significantly smaller amplitude than suggested by the empirical evidence.

In this paper, we modify the model with CES technology in two important respects. First, we allow preferences to exhibit hyperbolic discounting (in which case the pure rate of time preference declines over time). Empirically, there is abundant evidence for the pure rate of time preference to decline over time, i.e., for hyperbolic discounting (cf., e.g., Ainslie 1992, and Laibson 1997). Discount rates are time sensitive, exhibiting a "present bias": people tend to put especially high weight on a given gain or loss delayed in the near future as opposed to the same gain or loss delayed in the more distant future. Households do not foresee that their discount rate declines in delay. This introduces time-inconsistency. Second, households exhibit a short planning horizon (as opposed to an infinite planning horizon). Along the lines of Caliendo and Aadland (2007), Findley and Caliendo (2009), Caliendo and Findley (2014), and Findley and Caliendo (2014), naïve households, who are not aware of their future impatience, revise their initial intertemporal consumption plans at every instant in time.

Within this framework, we argue that hyperbolic discounting (and short-term planning) – by adding a *hyperbolic discounting effect* to the usual substitution- and income effects – magnifies the amplitude of the hump of the transitional path of the saving rate. Numerical simulations show that for reasonable calibrations, the implied hump in the transitional dynamics of the saving rate has amplitude corresponding to what is suggested by empirical evidence.

Specifically, we show the following three results. First, if the elasticity of marginal utility of consumption, θ, differs from unity, our framework is *not* observationally equivalent with a framework with exponential discounting. Intuitively, if $\theta \neq 1$, the propensity to consume out of wealth is affected by *both* the discount rate and the rate of interest. As long as the rate of interest changes over time, the propensity to consume under hyperbolic discounting develops differently from that under exponential discounting. Given the lack of observational equivalence, the transitional paths of the saving rate differ between our model and one with exponential discounting. Second, as long as per capita income rises over time, the hyperbolic discounting effect tends to raise the saving rate relative to a framework with exponential discounting, and thereby tends to heighten the saving hump. Intuitively, a declining discount rate over time affects the intertemporal substitution effect. At any per capita income level, future consumption becomes more attractive as it is less strongly discounted relative to exponential discounting. This hyperbolic discounting effect encourages a higher saving rate. Third, by presenting numerical simulations, we quantify the impact of hyperbolic discounting on the hump of the transitional path of the saving rate. For reasonable calibrations, it is shown that the implied hump has an amplitude of around 5 percentage points, which is well in accord with empirical evidence. With exponential discounting, in contrast, the amplitude of the hump amounts to about one percentage point. In course of the numerical simulations, we introduce the class of regular discount functions. This class captures cases in which the second order growth rate of the discount rate is a *constant* multiple of the first-order growth rate. Most discounting specifications employed in the prior literature are special cases of the regular discount function, notably exponential discounting (where the discount rate is constant), less-than-exponential discounting, classical hyperbolic discounting (Ainslie 1992), or zero discounting.

This paper is related to several previous studies on saving rate dynamics. Gómez (2008) and Smetters (2003) introduce a CES production technology with elasticities of substitution differing from one. They show that a CES between capital and labor below (above) unity might imply a hump shaped (inverse-hump shaped) transitional path of the saving rate. Litina and Palivos (2010) introduce endogenous technical progress. Both Gómez (2008) and Litina and Palivos (2010) identify conditions under which there is overshooting (undershooting) behavior of the transition paths of the saving rate. Antràs (2001) shows that the introduction of a minimum consumption level (Stone-Geary preferences) may also imply a hump shaped savings profile. In his model, the intertemporal elasticity of substitution rises over time, which first weakens the substitution effect and later on, the substitution effect dominates the income effect, thereby generating a hump shaped transitional path. He also provides econometric evidence in support of the humped transitional path of the saving rate both in OECD countries and in a larger cross-section of countries.

The following Section 2 provides empirical evidence supporting two observations: as per capita income grows, an economy's saving rate tends to rise, at least over some period; and the transitional path of a country's saving rate behaves non-monotonically over time and typically exhibits a hump in the order of about 5 percentage points. Section 3 discusses two aspects of hyperbolic discounting. First, it introduces *regular* discount functions and shows that most discount functions found in the literature are special cases of regular discount functions. Second, it argues that a sensible comparison of experiments with hyperbolic-versus- exponential discount functions requires the same overall level of impatience. Section 4 presents the model with hyperbolic discounting and short planning horizons. Based on the model, the main qualitative results are shown. Section 5, presents numerical simulations investigating the impact of hyperbolic discounting on the saving rate hump. The section shows that for reasonable calibrations, the saving rate hump amounts to an amplitude of roughly 5 percentage points under hyperbolic discounting (in accordance with empirical evidence), in contrast to roughly one percentage point under exponential discounting (contrary to empirical evidence). Section 6 concludes, and the Appendix contains a number of derivations and proofs of propositions.

2. The behavior of the saving rate: Empirical evidence

Data on gross national saving rates suggest two regularities: as a country develops, its saving rate tends to increase, at least over some range; and, over time, saving rate paths behave non-monotonically and typically exhibit a hump.

2.1 Rising saving rates along transitional paths

Maddison (1992) provides evidence for 11 countries whose savings account for about half of world savings. He finds that over the last hundred-twenty years, the saving rates of all but one country (U.S.A.) increased substantially over time. Table 1, which is based on Barro and Sala-i-Martin (2004), provides empirical evidence for national saving rates.

Table 1. Gross national saving rates (percent)

Period	Australia	Canada	France	India	Japan	Korea	U.K.	U.S.A.
1870-89	11.2	9.1	12.8	-	-	-	13.9	19.1
1890-09	12.2	11.5	14.9	-	12.0	-	13.1	18.4
1910-29	13.6	16.0	-	6.4	17.1	2.4	9.6	18.9
1930-49	13.0	15.6	-	7.7	19.8	-	4.8	14.1
1950-69	24.0	22.3	22.8	12.2	32.1	5.9	17.7	19.6
1970-89	22.9	22.1	23.4	19.4	33.7	26.2	19.4	18.5

Source: Barro, Sala-i-Martin (2004, p.15)

In all countries, except for the United States, present saving rates are significantly above their levels in late nineteenth century. Similar evidence is seen in East Asia for the last half century.

Table 2. Gross national saving rates in East Asian countries (percent)

Period	Hong Kong	Taipei	Singapore	Malaysia	Thailand	Indonesia	Philippines
1960’s	31	14	8	25	22	7	17
1970’s	32	27	35	29	26	19	21
1980’s	34	31	42	33	26	33	20
1993	37	28	50	41	35	34	14

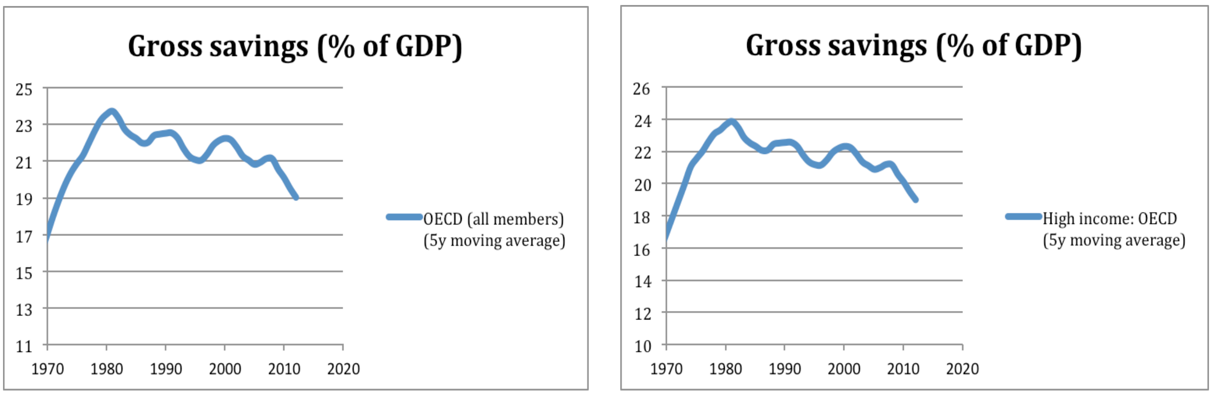
Source: Leipziger and Thomas (1997)

With the exception of the Philippines, gross national saving rates have increased in the Asian newly industrialized countries over the last fifty years, as shown in Table 2. Along the same lines, Loayza et al. (2000) show for 98 countries that private saving rates rise with the level of real per capita income.

2.2 Hump in the saving rate along transitional paths

That the gross saving rates are lower in the nineteen eighties than earlier is a well documented regularity (cf. Shafer et al. 1992). Schmidt-Hebbel and Servén (1999) as well as Antràs (2001) demonstrate that for most of 24 OECD countries, as well as for the OECD as a whole, the transitional paths of the saving rates exhibit a statistically significant hump when considering the last half century. Maddison (1992) shows that in many countries, after World War II, the saving rate exhibits overshooting. Similar trends are reported by Bosworth et al. (1991), Christiano (1989), Chari et al. (1996), and Tease et al. (1991). Specifically, Antràs (2001, p.1) finds that "there is clear evidence of a hump shape in the series. This is confirmed … where the series is detrended, using the Hodrick-Prescott filter, to remove business cycle fluctuations. Different tests on the series corroborated the statistical significance of the hump".

More recent data from the World Bank national accounts data, and OECD National Accounts data files confirm these findings, as shown in Figure 1. The figure shows gross savings as a share of the GDP for OECD member countries (all members, and a subgroup of high-income countries) as well as for the world as a whole. The hump of the saving rate is obvious in the two diagrams showing the savings rates for OECD countries over time.



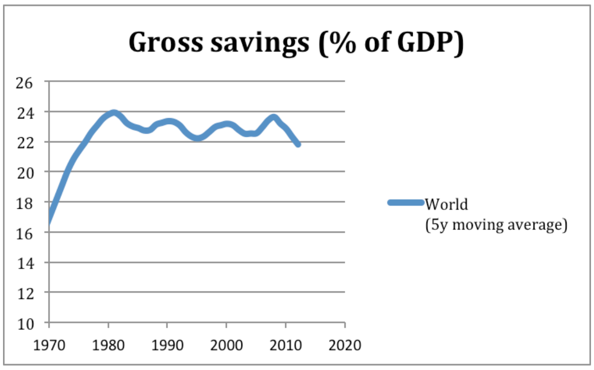


Figure 1. Gross savings as percentage of GDP (1965 – 2012)

Source: World Bank indicators: World Bank national accounts data, and OECD National Accounts data files.

Notes. Gross savings are calculated as gross national income less total consumption, plus net transfers; High-income economies are those in which 2012 GNI per capita was $12,616 or more.

The hump of the saving rate over time is less pronounced in the diagram showing the saving rates for the world as a whole. In line with the argument pursued in this paper, many countries experience a saving rate hump in course of developing from a low- to a medium-/high income country (see Antràs 2001). In the diagram exhibiting world data, the transitional paths of countries with a high GDP (and a declining saving rate) are aggregated with those of countries with a low GDP (and an increasing saving rate). As a consequence, the saving rate hump for the period of the last half century is less pronounced for the world as a whole than for a group of similar countries (OECD) or for individual countries.

Remark. (Magnitude of the "saving rate hump") Considering aggregate data for developed countries in the last 50 years, Figure 1 suggests a saving rate hump of around 5 percentage points. In the following, we argue that – in contrast with a reasonably calibrated model with exponential discounting – our framework with hyperbolic discounting implies a saving rate hump of the right magnitude.

3. Hyperbolic discounting

Psychologists and behavioral economists have established the fact that a household's discount rate declines over time (cf., e.g., Ainslie 1992, and Laibson 1997, Thaler 1981). In his seminal paper, Thaler (1981) reports that when individuals are given a choice between one apple today and two tomorrow, most choose one apple today. However, when individuals are given a choice between one apple in 50 days versus two apples in 51 days, the same group of decision-makers choose two apples in violation of the stationarity of the discount rate. As a conclusion, the pure rate of time preference is considered to decline in delay (time).

Accordingly, we allow the discount rate (pure rate of time preference), ρ_τ, to depend on delay. In the special case of exponential discounting, $\rho_\tau = \rho$ is stationary. In the more general case of hyperbolic discounting, ρ_τ declines in *delay*, τ. We use the notation $\rho_{t|t_0}$ (or simply ρ_τ) for an individual's discount rate of date t as seen from date t_0, that is, for a delay of $\tau = t - t_0$ periods.

We define a household's discount function by $D(t-t_0) \equiv e^{-\int_0^{t-t_0} \rho_s \, ds}$, or equivalently, by $D(\tau) \equiv e^{-\int_0^{\tau} \rho_s \, ds}$. For $\tau = 0$, $D(0) = 1$, regardless of whether an individual is an exponential or a hyperbolic discounter. The discount rate, then, is the rate at which the discount function declines in delay: $\rho_\tau = -\frac{\dot{D}_\tau}{D_\tau}$. For an exponential discounter, $D(\tau) \equiv e^{-\rho\tau}$ implying $-\frac{\dot{D}_\tau}{D_\tau} = \rho$.

3.1 Regular hyperbolic discounting

In the following, we specify a rather general class of discount functions that encompasses most special cases employed in the prior literature. Following the concepts employed by Groth et al. (2010), we call this class the class of *regular* discount functions. This class is defined by the property that the second-order growth rate of the discount function is proportional to the first-order growth rate.

The *first-order* growth rate of the discount factor is given by $g_D = \dot{D}_\tau / D_\tau = -\rho_\tau < 0$. The second-order growth rate of the discount factor is given by $g_{2,D} = \dot{g}_D / g_D = \dot{\rho}_\tau / \rho_\tau$. Following Groth et al. (2010), we call discount functions regular, if

$$g_{2,D} = \beta g_D, \; \beta \geq 0, \tag{1}$$

where the *constant* β is called the *hyperbolic discounting coefficient*. Given $D_0 = 1$, the second order differential equation (1) has the unique solution

$$D_\tau = (1+\rho_0\beta\tau)^{-1/\beta}, \ \rho_\tau = \frac{\rho_0}{1+\rho_0\beta\tau}. \tag{2}$$

The regular discount functions (2) encompass a number of special cases, depending on the specific value of the hyperbolic discounting coefficient. First, if $\beta = 0$, $\rho_\tau = \rho_0$. This is the case of conventional *exponential* discounting. Second, if $\beta > 0$, the discount rate declines in τ. This is the case of hyperbolic discounting. If $\beta = 1$, $D_\tau = (1+\rho_0\tau)^{-1}$. This is the case of *classical hyperbolic* discounting.[2] As the hyperbolic discounting coefficient rises, the rate of decline of the discount rate becomes larger, and as the hyperbolic discounting coefficient approaches infinity, the discount rate declines to zero instantly. Table 3 summarizes regular discount functions.

Table 3. Regular discount functions

	β	ρ_τ	D_τ
Regular discounting (general)	$\beta \geq 0$	$\rho_0 / (1+\rho_0\beta\tau)$	$(1+\rho_0\beta\tau)^{-1/\beta}$
Exponential discounting	$\beta = 0$	ρ_0	$e^{-\rho_0\tau}$
Classical hyperbolic discounting	$\beta = 1$	$\rho_0 / (1+\rho_0\tau)$	$1/(1+\rho_0\tau)$
No-discounting	$\beta \to \infty$	0	1

Figure 2 shows regular time paths of the discount rate for various values of the hyperbolic discounting coefficient. The figure illustrates that these paths capture *the whole spectrum* of discount rate paths between exponential discounting, less-than-exponential (that is, hyperbolic) discounting, and no discounting at all.

[2] In the original, classical psychological literature, hyperbolic discount functions like $1/\tau$ or $(1+\rho_0\tau)^{-1}$ were used (Ainslie, 1992).

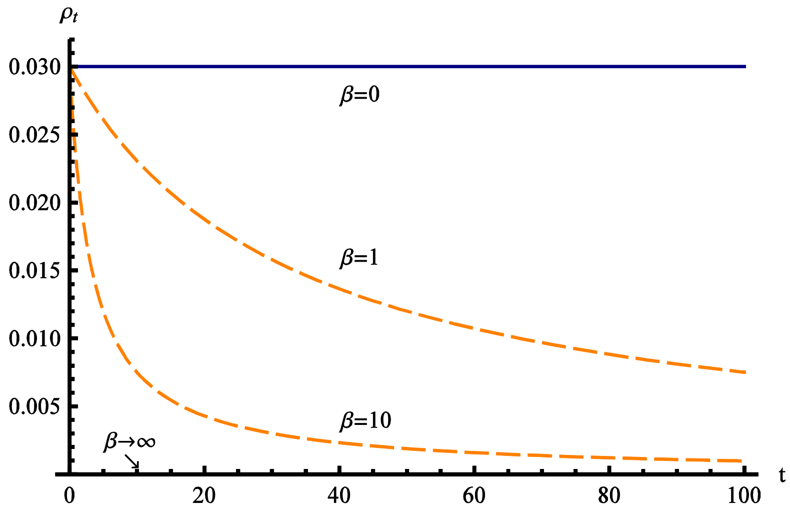


Figure 2. Time paths of the discount rate with $\rho_0 = 0.03$.

3.2 Hyperbolic- versus exponential discounting: a controlled experiment

In the subsequent sections, we argue that hyperbolic discounting is the key ingredient for explaining the magnitude of empirically observed humps of transitional paths of the saving rate. With exponential discounting alone, the magnitude of the humps cannot be explained. This raises one question. How can a model with exponential discounting be sensibly compared with one with hyperbolic discounting? Findley and Caliendo (2014) and Caliendo and Findley (2014) argue that psychologists have always used the "overall level of impatience" as a key measure. They argue that for a comparison to be sensible (a *controlled* experiment) the overall level of impatience must be the same for exponential and hyperbolic consumers. Experiments that do not control for the overall level of impatience may implicate spurious results that are entirely due to different overall levels of impatience – not to hyperbolic, as opposed to exponential discounting.[3]

Let $h \in \mathbb{R}_{++}$ be the number of periods characterizing an individual's planning horizon. For $t \in [0, h]$, the overall level of impatience, $I(h)$, is given by the area above the discounting curve $D(\tau)$ and below one (see Figure 3, below). That is,

[3] As shown below, generally, there is no observational equivalence between exponential- and hyperbolic discounting.

$$I(h) \equiv \int_0^h 1 - D(\tau)\,d\tau \ . \qquad (3)$$

Consider an exponential and a hyperbolic discount function: $D_E(\tau)$, and $D_H(\tau)$. Then, the overall level of impatience is the same if and only if

$$\int_0^h 1 - D_E(\tau)\,d\tau = \int_0^h 1 - D_H(\tau)\,d\tau \ \Leftrightarrow \ \int_0^h D_E(\tau)\,d\tau = \int_0^h D_H(\tau)\,d\tau \ . \qquad (4)$$

Considering (2), for any given $\rho_0 \equiv \rho_{H0}$, (4) allows for calculating that specific value of the exponential discount rate, ρ_E, for which the overall level of impatience for the planning period $t \in \left[0, h\right]$ is the same. Figure 3 shows one example for $\beta = 50,\ h = 20$ and $\rho_{H0} = 0.06$. According to (4), the "controlled" value of the exponential discount rate corresponds to $\rho_E = 0.0064$ (see Appendix).

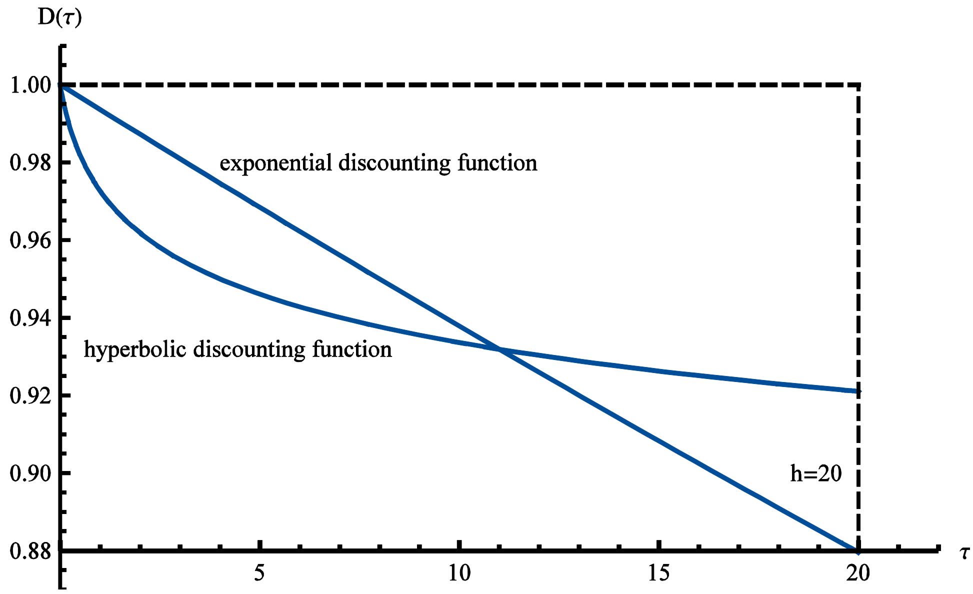


Figure 3. Exponential- and hyperbolic discount functions with the same overall level of impatience: $\beta = 50,\ h = 20,\ \rho_{H0} = 0.06,\ \rho_E = 0.0064$.

In the following analysis, we distinguish between *uncontrolled* experiments not satisfying (4) and *controlled* experiments satisfying (4).

4. The neoclassical growth model with hyperbolic discounting and short-term planning

We modify the standard neoclassical growth model in that we allow preferences to exhibit two non-standard features. First, naïve individuals may discount hyperbolically rather than exponentially. They consider their future discount rates to be lower than the present one. However, when future arrives, they realize that their discount rate, then, is higher than planned. That is, hyperbolic discounting introduces time-inconsistency. Second, individuals exhibit a short planning horizon. That is, they are not planning from t_0 to infinity, they rather plan from t_0 to some finite period $t_0 + h$. In accordance with the individuals' naïveté, they not only consider their discount rate to be constant, they also consider the time t_0- wage and interest rates to remain constant during their short planning horizon *at* their t_0 levels. As shown below, due to time-inconsistency, individuals re-optimize and form new plans at every instant t. Following the procedure introduced by Caliendo and Aadland (2007), and Findley and Caliendo (2009), we provide an exact analytical solution to this growth model with hyperbolic discounting and short-term planning, in which consumers form time-inconsistent saving plans. In the subsequent subsection, we consider the impact of hyperbolic discounting and short-term planning on the transitional dynamics of the saving rate.

4.1 The model

4.1.1 Households

The economy is populated with a large number, L_t, of identical, naïve, infinitely lived households. Each household inelastically supplies one unit of labor per unit of time. A household's preferences are described by an instantaneous CRRA utility function with absolute elasticity of marginal utility of consumption equal to $\theta \geq 0$. At every date t, a household is planning ahead for $h \in \mathbb{R}_{++}$ periods (short-term planning) and plans a sequence of consumption $\{c_t\}_{t_0}^{t_0+h}$ so as to maximize the present value of intertemporal utility

$$U(t_0;h) = \int_{t_0}^{t_0+h} \frac{c_t^{1-\theta} - 1}{1-\theta} L_t D(t - t_0)\, dt, \; t \in [t_0, t_0 + h] \quad (5)$$

subject to the flow budget constraint

$$\dot{k}_t = (r_t - n)k_t + w_t - c_t, \quad k_{t_0} > 0 \text{ given}, \quad t \in [t_0, t_0 + h] \tag{6}$$

– where k, w, r respectively denote capital per capita, wage rate and the interest rate – and the terminal condition

$$k_{t_0+h} = 0 \ . \tag{7}$$

Two remarks are in order. First, in (5), $L_t = e^{nt}$ represents the population size with $n \geq 0$ representing the exogenous growth rate of the population. Furthermore, the discount function $D(t - t_0)$ is considered to be a regular discount function as given by (2). This discount function encompasses the special case of exponential discounting as the limit when the hyperbolic discounting coefficient β approaches zero.

Second, at every point t, a household solves a short-horizon (fixed-endpoint) control problem. The solution to this optimal control problem is *planned* consumption from the perspective of t_0. The (fixed-endpoint) terminal condition $k_{t_0+h} = 0$ indicates that the household is concerned only with the "next" *h* periods. It does *not* imply that wealth (capital) is *actually* equal to zero at $t_0 + h$, as the household's planning horizon is continuously sliding forward. As the planning horizon is sliding forward, previous consumption plans are invalidated, and the household re-optimizes and updates its consumption plan at every *t*. That is, although a household plans to exhaust its resources within *h* periods, it never actually exhausts its resources in finite time, as it keeps re-planning its consumption plans.

At $t = t_0$, the Hamiltonian of the control problem becomes:

$$H(c_t, k_t, \mu_t; t_0) = \frac{c_t^{1-\theta} - 1}{1-\theta} e^{nt} D(t - t_0) + \mu_t[(r_t - n)k_t + w_t - c_t], \ \ t \in \left[t_0, t_0 + h\right] \ . \tag{8}$$

As demonstrated in the Appendix, the optimal consumption plan – as seen from date t_0 – is given by:

$$\tilde{c}(t\,|\,t_0) = \frac{\tilde{k}_{t_0} + \int_{t_0}^{t_0+h} \tilde{w}_\tau \, e^{-\hat{R}(\tau,t_0)+\gamma(\tau-t_0)} \, d\tau}{\int_{t_0}^{t_0+h} D(\tau-t_0)^{1/\theta} e^{-\frac{\theta-1}{\theta}\hat{R}(\tau,t_0)-\gamma t_0+n/\theta\tau} \, d\tau} e^{\hat{R}(t,t_0)/\theta} L_t^{1/\theta} e^{-\gamma t} D(t-t_0)^{1/\theta} \ , \tag{9}$$

where $\hat{R}(t,t_0) \equiv \int_{t_0}^{t} (r_{t_0} - n)\, dt = (r_{t_0} - n)(t - t_0)$, as households consider the rate of interest – as seen from t_0 – fixed. A tilde denotes units of effective labor. Let the exogenous rate of technical progress be given by γ, then: $\tilde{x}_t \equiv x_t e^{-\gamma t}$, $x \in \{k, w, c\}$ (see also below). Due to time-inconsistency, the household follows this consumption plan *only* at $t = t_0$. So, we consider the envelope, by setting all $t = t_0$. In the resulting expression, we then replace t_0 by t, which directly yields the exact analytical solution to this optimal control problem in which naïve consumers form time-inconsistent saving- and consumption plans.

$$\tilde{c}_t = \frac{\tilde{k}_t + \int_t^{t+h} \tilde{w}_t \, e^{-(r_t-\gamma-n)(\tau-t)} \, d\tau}{\int_t^{t+h} D(\tau-t)^{1/\theta} e^{-\left(\frac{\theta-1}{\theta}r_t - n\right)(\tau-t)} \, d\tau} \tag{10}$$

Equation (10) presents optimal consumption of a short-sighted household with hyperbolic discounting. Consumption is derived as the envelope of infinitely many initial values from a continuum of planned time paths. The numerator of (10) represents total (physical and human) wealth, and the denominator represents the inverse of the propensity to consume out of total wealth. Specifically, let

$$\Delta \equiv \int_t^{t+h} D(\tau-t)^{1/\theta} e^{-\left(\frac{\theta-1}{\theta}r_t - n\right)(\tau-t)} d\tau \ , \tag{11}$$

then, the propensity to consume out of total wealth is given by Δ^{-1}, and equation (10) reads $\tilde{c}_t = \Delta^{-1}\left[\tilde{k}_t + \int_t^{t+h} \tilde{w}_t\, e^{-(r_t-\gamma-n)(\tau-t)}\, d\tau\right]$.

The important insight from (10) consists in the fact that – as the propensity to consume generally depends on calendar time t via r_t – there exists no observational equivalence between exponential- and hyperbolic discounting. By *observational equivalence*, we mean that for every hyperbolic discount function $D_H(\tau)$ there exists an exponential discount function $D_E(\tau)$ so that the observed consumption paths are the same under both discount functions.

Proposition 1. *Consider a naïve household with a short planning horizon,* $h \in \mathbb{R}_{++}$*, and* $\theta \neq 1$*, and a time-dependent rate of interest* r_t*. Then, the model with hyperbolic discounting is* <u>*not*</u> *observationally equivalent to a corresponding model with exponential discounting.*

Proof. Suppose, contrary to the proposition, observational equivalence holds. Then, for some date t_0 it must be true that

$$\int_{t_0}^{t_0+h} D_H(\tau-t_0)^{1/\theta} e^{-\left(\frac{\theta-1}{\theta}r_{t_0}-n\right)(\tau-t_0)} d\tau = \int_{t_0}^{t_0+h} D_E(\tau-t_0)^{1/\theta} e^{-\left(\frac{\theta-1}{\theta}r_{t_0}-n\right)(\tau-t_0)} d\tau \qquad (12)$$

Note that the discount functions do not depend on calendar time t_0 (by definition, they only depend on delay). Also, observe that the weight $e^{-\left(\frac{\theta-1}{\theta}r_{t_0}-n\right)(\tau-t_0)}$ changes in calendar time as long as the rate of interest is not stationary. Therefore, if (12) holds at some date t_0, since generally $D_H(\tau-t_0) \neq D_E(\tau-t_0)$, it is not possible that (12) also holds at $t \neq t_0$, a contradiction. ||

Proposition 1 contrasts sharply with the commonly accepted argument that a model with hyperbolic discounting is observationally equivalent to one with exponential discounting (cf. Barro 1999). Based on the important result established by Proposition 1, we will investigate the impact of hyperbolic discounting on the "savings hump," below.

Corollary 1. (Observational equivalence)

Consider a naïve household with a short planning horizon, $h \in \mathbb{R}_{++}$, and logarithmic utility $\theta = 1$. Then, the model with hyperbolic discounting is observationally equivalent to a corresponding model with exponential discounting.

Corollary 1 can immediately be seen when setting $\theta = 1$ in (12). With *logarithmic* utility, for every hyperbolic discount function there exists an exponential discount function so that the observed consumption paths are the same under both discount functions. Below, we will therefore not consider the special case of logarithmic utility. For hyperbolic discounting and short-term planning to have an impact on the "savings hump," we will restrict attention to $\theta \neq 1$ (generally to $\theta > 1$, see below).

It is worth noting that observational equivalence does not imply that a controlled experiment is pursued. The mere fact that there exists an exponential discount function that gives rise to the same observed consumption path as the hyperbolic discount function does *not* imply that the overall level of discounting is the same under both discount functions (cf. (4)).

Corollary 2. (Observational equivalence and a controlled experiment)

(i) If and only if $\theta = 1$, observational equivalence meets the requirement of a controlled experiment; that is, the overall level of discounting is the same under both discount functions.

(ii) If the rate of interest is stationary, $r_t = r$, observational equivalence may occur. But as long as $\theta \neq 1$, the overall level of discounting between a hyperbolic- and an exponential discount function differs.

4.1.2 Production and the market economy

Output, Y, is produced via the CES technology

$$Y_t = A\left[\alpha K_t^{(\sigma-1)/\sigma} + (1-\alpha)(T_t L_t)^{(\sigma-1)/\sigma}\right]^{\sigma/(\sigma-1)},\ A > 0,\ 0 < \alpha < 1,\ \sigma > 0, \tag{13}$$

where L is labor input, K is capital input, and T is an index of labor-augmenting productivity that evolves through exogenous disembodied technical change:

$$T_t = e^{\gamma t},\ \gamma \geq 0. \tag{14}$$

Parameter α represents the weight of capital in production, and parameter σ denotes the elasticity of substitution between capital and labor.

We consider a closed economy so that national income accounting implies

$$Y_t = C_t + I_t, \tag{15}$$

where C_t is aggregate consumption. Then, the capital stock develops according to

$$\dot{K}_t = Y_t - C_t - \delta K_t,\ \delta > 0, \tag{16}$$

where δ is the rate of depreciation of capital.

Let a tilde denote variables in units of effective labor (TL). Then, production function (13) becomes

$$\tilde{y}_t = A\left[\alpha \tilde{k}_t^{(\sigma-1)/\sigma} + (1-\alpha)\right]^{\sigma/(\sigma-1)}, \tag{17}$$

and resource constraint (16) reads: $\dot{\tilde{k}}_t = \tilde{y}_t - \tilde{c}_t - (n+\gamma+\delta)\tilde{k}_t$.

We now embed the described technology into a market economy with perfect competition. The representative firm chooses inputs so as to maximize the profit for a given real wage, $w_t = \partial Y_t / \partial L_t$, and capital rental rate, $q_t = \partial Y_t / \partial K_t$. That is, the rate of return on holding capital is given by $r_t = q_t - \delta$.

4.2 Behavior of the saving rate

From (10) and (17), the saving rate becomes

$$s_t = 1 - \frac{\tilde{c}_t}{\tilde{y}_t} = 1 - \Delta^{-1}\left[\tilde{k}_t / \tilde{y}_t + \int_t^{t+h} \tilde{w}_t / \tilde{y}_t\, e^{-(r_t-\gamma-n)(\tau-t)}\, d\tau\right],$$

$$\text{where } r_t = \alpha A^{(\sigma-1)/\sigma} \tilde{k}_t^{-1/\sigma} \tilde{y}_t^{1/\sigma} - \delta,\ \tilde{w}_t = \frac{1-\alpha}{\alpha}(r_t+\delta)\tilde{k}_t^{1/\sigma},\ \tilde{y}_t = A\left[\alpha \tilde{k}_t^{(\sigma-1)/\sigma} + (1-\alpha)\right]^{\sigma/(\sigma-1)}. \tag{18}$$

As an economy develops (when $\tilde{k}_t$ increases over time), whether the saving rate increases or decreases (possibly non-monotonically) along the transition path depends on whether $\tilde{c}_t$ increases by more or by less than $\tilde{y}_t$. In general, the behavior of the saving rate is complicated along the transition path as a substitution effect opposes an income effect. As the return on saving declines, ceteris paribus households tend to lower the saving rate over time (substitution effect). At the same time, as $\tilde{k}_t$ rises, the difference between current and permanent income decreases. The desire for consumption smoothing requires a household in an economy distant from the steady state to consume more relative to actual income than a household in an economy close to the steady state. As the economy develops, then, consumption relative to income declines. This income effect tends to raise the saving rate over time.

For the special case of Cobb-Douglas production, it has been demonstrated by Barro and Sala-i-Martin (2004, p.135 ff) that the dynamics of the saving rate is *always* monotonic – a *counterfactual* prediction, as shown in Section 2. For the more general case of CES production, however, Smetters (2003) and Gómez (2008) demonstrate that for important cases the transitional path of the saving rate exhibits a hump.[4] Specifically, they analyze the dynamics of $z_t \equiv \tilde{c}_t / \tilde{y}_t$ in a framework with an infinite planning horizon, perfect foresight, and exponential discounting. In such a framework, if $\sigma < 1$, the $\dot{z}(r,z) = 0$ locus typically exhibits a U-shaped pattern in (r,z) space. Let the minimum of the $\dot{z}(r,z) = 0$ locus occur at (r_0, z_0). If parameters are such that $z_0 > 0$ and $r^* < r_0$ (with r^* representing the steady state rate of interest), then the saddle path is U-shaped in (r,z) space. That is, as $\tilde{k}_t$ increases over time, $s_t = 1 - z_t$ increases first and decreases later on. The transitional path of s_t exhibits a saving rate hump (Gómez 2008, pp. 203f).

Numerical simulations employing the framework with CES production and exponential discounting, however, show that for reasonable parameterizations the saving rate humps have very small amplitudes. These amplitudes are not consistent with the much larger amplitudes shown by empirical evidence (Section 2).

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[4] We refer to the hump in the transitional path of the saving rate as the *saving rate hump*.

The consideration of hyperbolic discounting in the framework with short-term planning and CES production adds a third effect that we term *hyperbolic discounting effect*. This additional effect tends to heighten the saving hump. As shown in the following section, the hyperbolic discounting effect allows for amplitudes of the saving rate hump that are consistent with the empirical evidence.

In the following, we compare the propensities to consume between a hyperbolic- and an exponential discounter. The hyperbolic discounting effect is caused by the fact that the change in the propensity to consume over time differs between exponential and hyperbolic discount functions, irrespective of whether (4) is satisfied or not.[5] Specifically, consider the integrals of the exponentiated discount functions:

$$\int_0^h D_H(\tau)^{1/\theta} d\tau = \frac{\theta\left(-1+\left(1+h\beta\rho_0\right)^{1-\frac{1}{\beta\theta}}\right)}{\left(-1+\beta\theta\right)\rho_0}, \quad \int_0^h D_E(\tau)^{1/\theta} d\tau = \frac{\theta\left(1-\mathrm{e}^{-\frac{h\rho_E}{\theta}}\right)}{\rho_E} \quad . \tag{19}$$

Assumption 1. $\beta \geq 1,\ \theta > 1$ (A.1)

Assumption 1 mildly restricts regular discount functions to those consistent with classical hyperbolic discounting or those for which the discount rate declines more strongly as compared to classical hyperbolic discounting. The restriction on the elasticity of marginal utility of consumption implies an intertemporal elasticity of substitution of less than unity.[6] Under (A.1) – for a *given* wealth (in (18), the term in square brackets) – the income effect exceeds the substitution effect. That is, holding the wealth constant, as $\tilde{k}_t$ increases, the saving rate rises. However, as $\tilde{k}_t$ increases, so does wealth, as seen by the term in square

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[5] Considering (18), the hyperbolic discounting effect is not related to the wealth to income ratio.

[6] That $\theta > 1$ is overwhelmingly suggested by the literature. Hall (1988, p. 350) favors a value of (at least) $\theta = 5$.

brackets in (18).[7] This wealth effect raises the consumption share in income and thereby lowers the saving rate.

The following three lemmas are useful for characterizing the hyperbolic discounting effect (see Proposition 2).

Lemma 1. *Suppose Assumption 1 holds. Then,*

$$\lim_{\beta\to 0}\int_0^h D_H(\tau)^{1/\theta}d\tau = \int_0^h D_E(\tau)^{1/\theta}d\tau \,.$$

Proof. Consider (19). Taking the limit of $\int_0^h D_H(\tau)^{1/\theta}d\tau$ as β approaches zero directly yields the result. ||

The lemma confirms that the known limit result $\lim_{\beta\to 0} D_H(\tau) = D_E(\tau)$ extends to the integral of the exponentiated discount functions. Exponential discounting, in the version presented in Lemma 1, remains the special case of hyperbolic discounting in which $\beta = 0$.

Lemma 2. *The integral* $\int_0^h D_H(\tau)^{1/\theta}d\tau$ *rises in beta.*

Proof. See Appendix. ||

Lemma 2 states that the overall level of impatience – that is, $1-\int_0^h D_H(\tau)^{1/\theta}d\tau$ – is the lower the higher is the rate of decline of the hyperbolic discount rate (the higher is β).

Lemma 3. *Let the weight* $E(r_t,\tau) \equiv e^{-\left(\frac{\theta-1}{\theta}r_t - n\right)(\tau - t)}$. *Suppose Assumption 1 holds and* $\tau > t$. *Then:*

(i) $\dfrac{\partial E(r_t,\tau)}{\partial \tilde{k}_t} = \dfrac{\partial E(r_t,\tau)}{\partial r_t}\dfrac{\partial r_t}{\partial \tilde{k}_t} > 0$;

(ii) $\lim_{\tilde{k}_t\to\infty} E(r_t,\tau) \geq 1$;

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[7] It can easily be veryfied from (18) that, given the CES production technology, both $\tilde{k}_t/\tilde{y}_t$ and $\tilde{w}_t/\tilde{y}_t$ are increasing functions of $\tilde{k}_t$.

(iii) $\lim_{\tilde{k}_t \to 0} E(r_t, \tau) = \begin{cases} 0 & \text{for } \sigma \geq 1 \\ <1 & \text{for } \sigma < 1 \end{cases}$.

Proof. (i) Follows from the assumption that $\theta > 1$. (ii) $\lim_{\tilde{k}_t \to \infty} \tilde{y}'(\tilde{k}) = 0$. If $n + \delta = 0$, $\lim_{\tilde{k}_t \to \infty} E(r_t, \tau) = 1$. If $n + \delta > 0$, $\lim_{\tilde{k}_t \to \infty} E(r_t, \tau) > 1$. (iii) If $\sigma \geq 1$, $\lim_{\tilde{k}_t \to 0} r_t = \infty$. If $\sigma < 1$, $0 < \lim_{\tilde{k}_t \to 0} r_t < \infty$. The result directly follows. ||

Lemma 3 shows that as an economy develops (as $\tilde{k}_t$ increases) the weight $E(r_t, \tau)$ strictly monotonically increases. For a very high rate of interest $E(r_t, \tau) \approx 0$, and for a very low rate of interest $E(r_t, \tau) \approx 1$.

Proposition 2. (Hyperbolic discounting effect) *Suppose Assumption 1 holds. Then*

$$\Delta_H \equiv \int_t^{t+h} D_H(\tau - t)^{1/\theta} e^{-\left(\frac{\theta - 1}{\theta} r_t - n\right)(\tau - t)} d\tau \geq \int_t^{t+h} D_E(\tau - t)^{1/\theta} e^{-\left(\frac{\theta - 1}{\theta} r_t - n\right)(\tau - t)} d\tau \equiv \Delta_E .$$

Specifically, if $E(r_t, \tau) > 0$*, the inequality is strict:* $\Delta_H > \Delta_E$*. Moreover, as an economy develops (as* $\tilde{k}_t$ *increases over time), the strictly positive difference* $\Delta_H - \Delta_E > 0$ *becomes larger.*

Proof. See Appendix. ||

Proposition 2 defines the hyperbolic discounting effect as the *increasing difference over time* in the change of the propensity to consume between exponential and hyperbolic discount functions. As shown in the proof of the proposition, the *difference* is due to the fact that compensation rule (4) for the exponential discount rate $\rho_E(\beta)$ refers to the integrals $\int_t^{t+h} D(\tau - t) d\tau$, *not* to the exponentiated integrals $\int_t^{t+h} D(\tau - t)^{1/\theta} d\tau$. That is, $\rho_E(\beta)$ equalizes $\int_t^{t+h} D_H(\tau - t) d\tau$ and $\int_t^{t+h} D_E(\tau - t) d\tau$, but *not* the integrals of the exponentiated discount functions. Let $\rho_E |_{\theta > 1}$ be the exponential discount rate that equalizes the integrals of the exponentiated discount functions. Generally $\rho_E |_{\theta = 1} \neq \rho_E |_{\theta > 1}$. As shown in the proof, $\rho_E |_{\theta = 1} > \rho_E |_{\theta > 1}$. As the propensities to consume depend on the exponentiated discount functions, generally $\Delta_H \neq \Delta_E$. With $\theta > 1$, the adjusted – according to (4) – exponential discount rate is larger than the one required to equalize the integrals of the exponentiated discount functions. Therefore, $\Delta_H > \Delta_E$.

The *increase* in the difference $(\Delta_H - \Delta_E) > 0$ *over time* is due to the weight $E(r_t, \tau)$. With $\theta > 1$, the income effect dominates the substitution effect, and as r_t declines over time $E(r_t, \tau)$ rises over time (thereby lowering the propensity to consume). The rise in $E(r_t, \tau)$ increases the difference $\Delta_H - \Delta_E$ over time.

The propensity to consume is given by Δ^{-1}. As $\tilde{k}_t$ increases over time, the hyperbolic discounting effect lowers the propensity to consume in a hyperbolic discounting framework *relative to an exponential discounting framework.* That is, as $\tilde{k}_t$ increases over time, the hyperbolic discounting effect tends to raise the saving rate relative to a framework with exponential discounting, and thereby tends to heighten the saving hump.

Intuitively, a declining discount rate over time affects the intertemporal substitution effect. At any per capita income level, future consumption becomes more attractive as it is less strongly discounted relative to exponential discounting. This hyperbolic discounting effect encourages a higher saving rate, and hence investment in capital accumulation. At the same time, the induced higher saving rate and resulting faster capital accumulation yields higher per capita income level at any future date (again relative to that under the exponential case), which in turn raises the above-mentioned wealth effect, tending to lower the saving rate, according to (18).

The critical consideration in explaining possible dynamics of the saving rate over time is to know how these opposing effects of a declining discount rate themselves change over time as, on the one hand, the discount rate decreases with time and the per capita income rises on the other hand. Obviously, the extent of the hump and its timing depend critically on the particular hyperbolic discount function assumed (the value of the hyperbolic discounting coefficient β), on the utility function assumed (value of the elasticity of intertemporal substitution), and on the production function used (implying how fast output per head grows as capital is accumulated). Thus, for example, depending on the assumed values of parameters, it may be the case that the hump in the savings rate occurs relatively soon but is not very pronounced, or alternatively it may occur after a longer period of time with a higher hump, or various other possible combinations of these, each reflecting the experience of different countries as they have developed. It may also be the case that for a set of parameter values, the hump does not occur, implying that either the intertemporal substitution or wealth effect dominates for a very long time period.

Taking the hyperbolic discounting effect into account, in addition to the substitution and income (wealth) effects, may give rise to a humped transitional path of the saving rate – even with Cobb-Douglas production technology. The numerical simulations presented in the next section address the question of the quantitative impact of hyperbolic discounting on the saving rate hump.

5. Results from numerical simulations

Proposition 2 implies the qualitative result that the hyperbolic discounting effect tends to heighten the saving rate hump. Considering reasonable calibrations, three main *quantitative* questions suggest themselves. First, by how much does the introduction of hyperbolic discounting (a rise in β) heighten the saving rate hump? Second, how does the length of the planning horizon affect the saving rate hump? Third, how different is the saving rate hump between exponential and hyperbolic discounting in a controlled experiment? The numerical simulations that follow address these questions.

To that end, we consider an adverse shock on the predetermined state variable $\tilde{k}$. At time zero, starting from an initial steady state, we reduce the value of the predetermined variable. The resulting time paths show the *non-linearized* transitions of the saving rate (and other variables of interest) from far away from the steady state to the steady state equilibrium. These transition paths are interpreted as showing the development of the saving rate (and other variables) as a country develops, i.e., as its stock of capital $\tilde{k}$ increases.[8]

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[8] We employ the Mathematica implementation of the Relaxation Algorithm (Trimborn et al., 2008) to produce the numerical results documented in this paper. The code is available from the authors upon request. Notice that

Table 4. Baseline values of background parameters

Preference parameters	$\rho_0 = 0.06$, $\theta = 5$
Production parameters	$A = 2$, $\alpha = 0.7$, $\gamma = 0.02$, $\delta = 0.06$, $\sigma = 0.8$
Population growth rate	$n = 0.01$

Note. The hyperbolic discounting coefficient, β, as well as the planning horizon h vary across simulations.

What we call baseline values of the background parameters are listed in Table 4. The graphs below are based on these parameter values, which may be considered standard and noncontroversial. Notice that ρ_0 represents the *hyperbolic* discount rate at $\tau = 0$. The value for the elasticity of marginal utility is in line with the estimate of Hall (1988, p.350).

The parameters of primary interest are β and h. The empirical literature does not provide firm conclusions as to possible magnitudes of these parameters. To clarify the potential quantitative roles of β and h for the saving rate hump, we consider different values of these parameters in the numerical simulations.

For $\beta = 50$ and $h = 50$, together with the baseline values of the background parameters, important stylized facts for a modern industrialized economy are reproduced by the model. Per capita consumption grows at a rate of 2% per year, around 25% of output is devoted to investment, and the output-capital ratio is 0.37 in steady state.

Figure 3, presents transition paths of the saving rates for the baseline values of background parameters and for various values of β. The calculations of the transition paths are based on the Relaxation Algorithm (Trimborn et al., 2008).

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the shock is introduced on the state variable, not on a specific parameter. All parameters take on the same values before and after the shock.

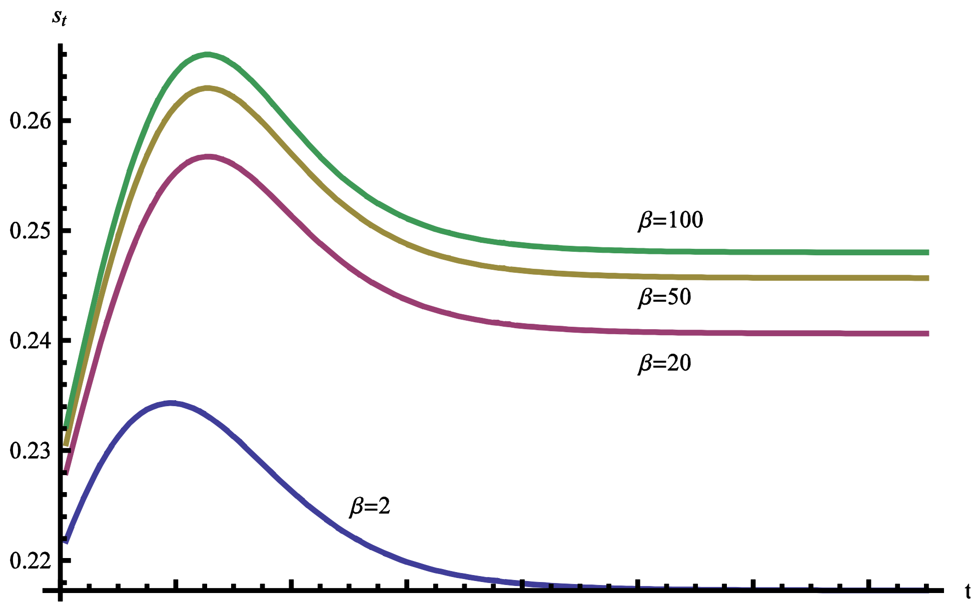

Figure 4. The saving rate hump as the hyperbolic discounting coefficient β rises.

Notes. $h = 50$.

Result 1. *The amplitude of the saving rate hump significantly increases in the hyperbolic discounting coefficient* β. *A value of* $\beta = 50$ *implies an amplitude of the saving hump that accords with empirical evidence.*

The figure shows the saving rate hump for various values of the hyperbolic discounting coefficient. The amplitude of the saving rate hump increases in β. This effect of hyperbolic discounting was already identified, qualitatively, in Proposition 2. Quantitatively, the figure suggests a very small amplitude of the saving rate hump for small values of the hyperbolic discounting coefficient. For example, $\beta = 2$ ($\beta < 2$) implies an amplitude of roughly one (less than one) percentage point. Higher values of β imply a much larger amplitude. $\beta = 50$ ($\beta > 50$) implies an amplitude of about four (more than four) percentage points.[9] The latter amplitudes correspond well with empirical evidence.

[9] A lower initial value of the hyperbolic discount rate further accentuates the saving rate hump.

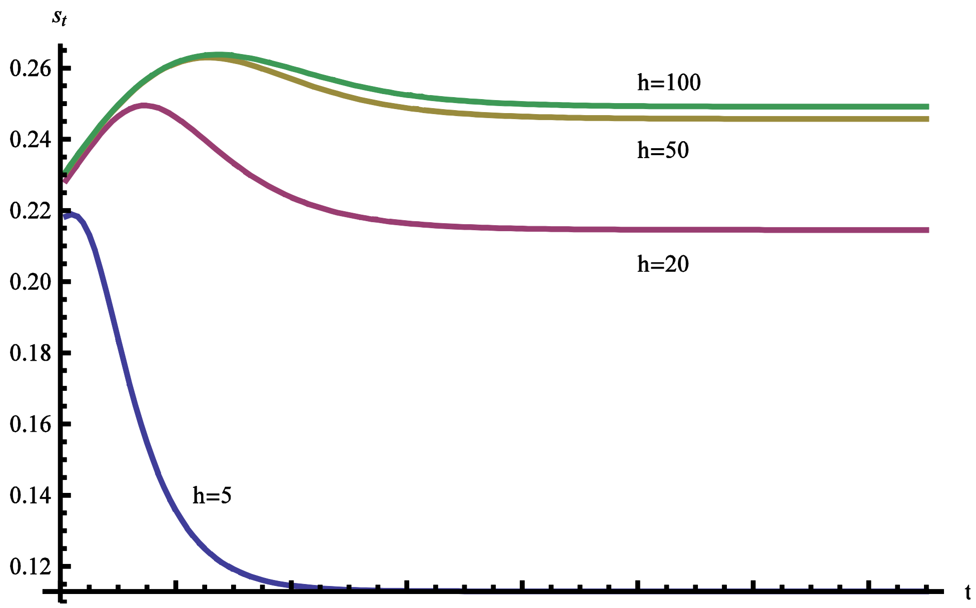

Figure 5. The saving rate hump as the planning horizon parameter h rises.
Notes. $\beta = 50$.

It is worth emphasizing that even if β is close to zero, the model does imply a saving rate hump. The existence of such a hump under a CES production technology was already demonstrated by Smetters (2003) and Gómez (2008). However, as indicated by Figure 4, the implied saving rate hump is significantly smaller as compared with empirical evidence.

The second quantitative question refers to the sensitivity of the above result with respect to the length of the planning horizon, h.

Result 2. *The planning horizon parameter h strongly affects both the amplitude of the saving rate hump as well as the steady state level of the saving rate. A rise in h raises the steady state saving rate.*

Figure 5 shows the saving rate hump for various values of the short-term planning horizon parameter h. For given β, a rise in h raises

$$\int_t^{t+h} D_H(\tau - t)^{1/\theta} d\tau = \frac{\theta\left[-1 + (1+\beta h \rho_0)^{(\beta\theta-1)/\beta\theta}\right]}{(\beta\theta - 1)\rho_0} ,$$

thereby increasing Δ_H and lowering the propensity to consume Δ_H^{-1}. According to (18), the lower propensity to consume raises the saving rate both along the (hump-shaped) transitional path and in steady state.

The figure suggests that for $\beta = 50$ together with the baseline values of the background parameters, a value of the short-term planning horizon parameter of $h = 50$ implies a saving rate hump that is consistent with empirical evidence. Too low a value of h results in a steady state saving rate too low as to be empirically supported (cf. $h = 5$ in Figure 5). Too high a value of h results in a very large steady state saving rate. As a consequence, the amplitude of the saving rate – viewed as difference between the peak of the saving rate hump and the steady state level of the saving rate – becomes lower as suggested by empirical evidence.

The final quantitative question refers to the difference between hyperbolic- and exponential discounting in a *controlled* experiment, that is when the exponential discount rate is adjusted such that the overall level of impatience is the same under both discount functions. The numerical simulations are based on the baseline values of the background parameters. For a fixed initial value of the hyperbolic discount rate, ρ_0, and a given β, the corresponding adjusted exponential discount rate is calculated. The following table shows the parameter specifications for β, ρ_0, ρ_E employed in the simulations.

Table 5. Adjusted exponential discount rates ρ_E

	$\beta = 2$	$\beta = 20$	$\beta = 50$
$\rho_0 = 0.06$	0.0270	0.0065	0.0033
$\rho_0 = 0.20$	0.0516	0.0089	0.0043

Note. The overall level of impatience is the same under both discount functions; $h = 50$.

Table 5 shows that for reasonable values of the hyperbolic discounting coefficient β, the adjusted exponential discount rate becomes very small, independently of whether the initial hyperbolic discount rate is relatively "small" ($\rho_0 = 0.06$) or "large" ($\rho_0 = 0.2$). Specifically, for our preferred value of $\beta = 50$, the adjusted exponential discount rate is close to zero (below 0.005) for both initial hyperbolic discount rates.

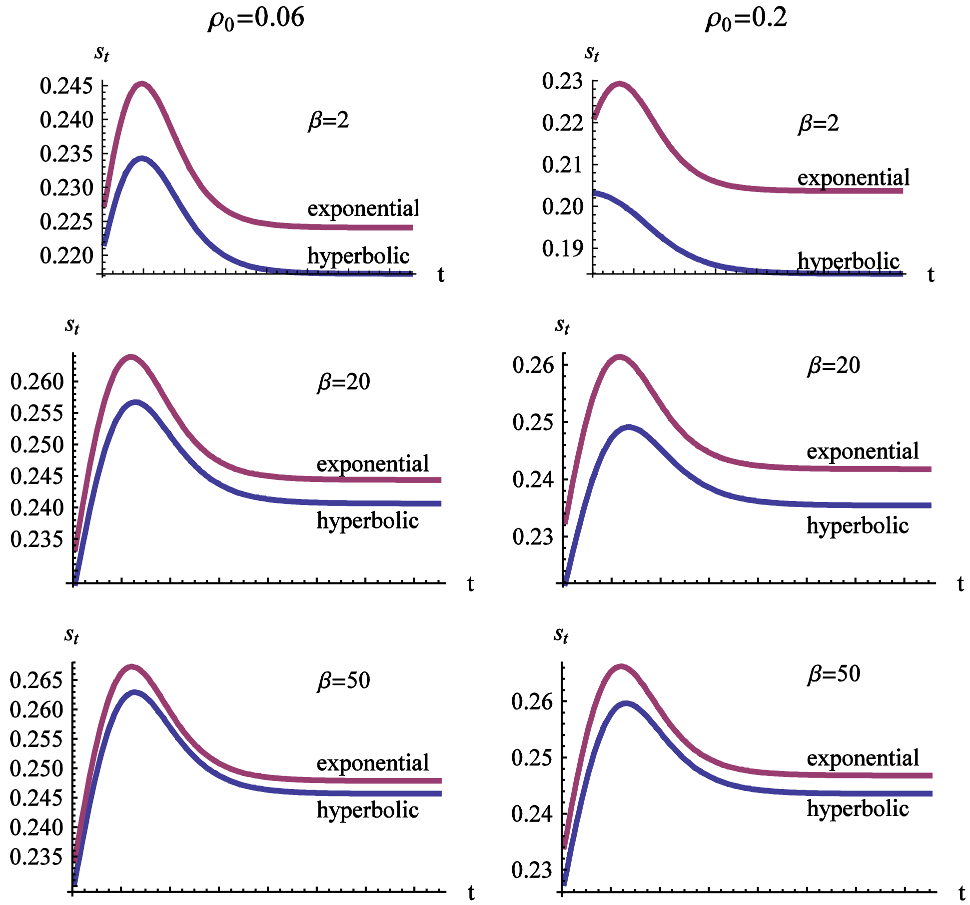

Figure 6. The saving rate hump under exponential- and hyperbolic discounting.

Notes. The exponential discount rate is adjusted such that the overall level of impatience is the same under both discount functions.

Figure 6 suggests two important insights. First, the saving rate paths under hyperbolic and adjusted exponential discounting are similar in all cases. In fact, under $\theta = 1$, the paths would coincide, according to Corollary 2(i). However, with $\theta \neq 1$, observational equivalence is ruled out by Proposition 1. That is, the difference between the saving rate paths implied by adjusted exponential versus hyperbolic discounting is entirely due to the fact that $\theta > 1$ in the simulations.

Second, as seen for $\beta = 50$, adjusted exponential discounting implies a saving rate hump of the same magnitude as hyperbolic discounting. That is, not only hyperbolic discounting tends to raise the amplitude of the saving rate but so does also an exceedingly low exponential discount rate. The latter effect is due to the fact that a low(ering of the) discount rate weakens the substitution effect by which the saving rate tends to decline as $\tilde{k}_t$ increases (as the rate of interest declines). However, as shown in Table 5, the adjusted exponential discount rate required to produce a saving rate hump of the right magnitude (with $\beta = 50$) must be extremely small. For $\rho_0 = 0.06$, the adjusted exponential discount rate equals $\rho_E = 0.003$, which corresponds to only about *one tenth* of what is considered a standard estimate of the exponential discount rate (in a standard Ramsey model).

As a consequence, for the baseline values of the background parameters, a saving rate hump of the right magnitude is implied by two frameworks: (i) a model with hyperbolic discounting with $\rho_0 = 0.06$ and $\beta = 50$; (ii) a model with exponential discounting with $\rho_E = 0.003$. It must be emphasized though, that in the model with exponential discounting, a constant discount rate of $\rho_E = 0.003$ is *not* considered a sensible estimate.

All of these results hold irrespective of whether ρ_0 is "small" or "large". We summarize these results in

Result 3. (Controlled experiments) *If the exponential discount rate is adjusted so that the overall level of impatience is the same under both discount functions:*
(i) the implied transitional paths of the saving rate is similar in both models (exponential and hyperbolic discounting);
(ii) the value of the adjusted exponential discount rate that is needed to imply a saving rate hump of the right magnitude is only about one tenth of what is considered a standard estimate of the exponential discount rate.

6. Conclusions

Our paper considers the impact of hyperbolic discounting on the humped transitional dynamics of the saving rate. The standard neoclassical growth model with Cobb-Douglas technology exhibits – for a reasonable calibration – a monotonously declining transition path of the saving rate, as per capita incomes increase. This property is counterfactual and therefore unappealing for the analysis of policy shocks on transitional dynamics of an economy. In response, the prior literature shows that the saving rate exhibits a humped transitional dynamics in a neoclassical growth model with CES production technology. For a reasonable calibration, however, the amplitude of the hump implied by a framework with exponential discounting – roughly one percentage point – falls significantly short of an amplitude of roughly five percentage points, as suggested by empirical evidence. In our paper, we show that the introduction of hyperbolic discounting (as opposed to exponential discounting) amplifies the magnitude of the implied hump in the saving rate dynamics. Specifically, according to the numerical simulations, our framework with hyperbolic discounting implies an amplitude of the saving rate hump well fitting the empirical evidence.

The main mechanism at work is a hyperbolic discounting effect that is not present in a framework with exponential discounting. The hyperbolic discounting effect works via the propensity to consume out of wealth. If the elasticity of marginal utility, θ, differs from unity, the propensity to consume is not constant over time. Specifically, if $\theta > 1$, the propensity to consume decreases, over time, by more in a framework with hyperbolic discounting than in the same framework with exponential discounting. Other things equal, this hyperbolic discounting effect tends to raise the saving rate over time. Eventually, the hyperbolic discounting effect is dominated by the intertemporal substitution effect that tends to lower the saving rate (as the rate of interest declines) over time.

For the analysis of the effects of hyperbolic discounting we introduce the concept of a regular discount function. The class of regular discount functions captures all cases in which the second order growth rate of the discount rate is a constant multiple of the first-order growth rate. Most discounting specifications employed in the prior literature are special cases of the class of regular discount functions.

Several questions are open for future research. First, considering the general class of regular discount functions, empirical estimates of the key parameter, β, are scarcely available, if any. A robust estimate of β would greatly help to apply a framework with hyperbolic discounting to economic analyses of all kinds. Second, in the present paper, the saving rate was defined to be the difference between income and consumption relative to income. In a more sophisticated framework, at least one addition is warranted: a public sector. Clearly, a public sector interacts with a household's saving rate. Specifically, the introduction of social security systems over the last 50 years, in many countries, probably has affected the development of saving rates. Notwithstanding these open questions, we still hope to have shed some light on the impact of hyperbolic discounting on saving rate dynamics in a neoclassical growth model.

7. Appendix

A.1 Exponential discount rate in a "controlled" experiment

Considering (2) and (4), for any given hyperbolic discount rate at $\tau = 0$, ρ_0, the exponential discount rate satisfying (4) is given by

$$\rho_E(\beta) = \frac{\rho_0(\beta-1)(1+h\beta\rho_0)^{\frac{1}{\beta}-1}}{1-(1+h\beta\rho_0)^{\frac{1}{\beta}-1}} + \frac{1}{h}\text{ProductLog}\left[\frac{e^{\frac{h\rho_0(\beta-1)}{1-(1+h\beta\rho_0)^{1-\frac{1}{\beta}}}} h\rho_0(\beta-1)}{1-(1+h\beta\rho_0)^{1-\frac{1}{\beta}}}\right]. \tag{20}$$

As $\beta \to 0$, it follows that $\rho_E \to \rho_0$.

A.2 The optimal (time-inconsistent) consumption plan as seen from t_0

At $t = t_0$, the Hamiltonian of the control problem becomes:

$$H(c_t, k_t, \mu_t; t_0) = \frac{c_t^{1-\theta}-1}{1-\theta} e^{nt} D(t-t_0) + \mu_t[(r_t - n)k_t + w_t - c_t], \quad t \in [t_0, t_0 + h].$$

The Maximum principle implies: $c_t = \mu_t^{-1/\theta} L_t^{1/\theta} D(t-t_0)^{1/\theta}$. As $\dot{\mu}_t / \mu_t = -(r_t - n)$,

$$c_t = \mu_{t_0}^{-1/\theta} e^{\hat{R}(t,t_0)/\theta} L_t^{1/\theta} D(t-t_0)^{1/\theta}, \tag{21}$$

with $\hat{R}(t,t_0) \equiv \int_{t_0}^{t} (r_{t_0} - n)\,dt = (r_{t_0} - n)(t - t_0)$, as households consider the rate of interest – as seen from t_0 – fixed. Considering (21) in the flow budget constraint (6) yields

$$\dot{k}_t - (r_t - n)k_t = w_t - \mu_{t_0}^{-1/\theta} e^{\hat{R}(t,t_0)/\theta} L_t^{1/\theta} D(t-t_0)^{1/\theta},$$

the solution of which reads:

$$k_t = k_{t_0} e^{\hat{R}(t,t_0)} + \int_{t_0}^{t} \left[w_\tau - \mu_{t_0}^{-1/\theta} e^{\hat{R}(\tau,t_0)/\theta} L_\tau^{1/\theta} D(\tau - t_0)^{1/\theta} \right] e^{-\hat{R}(\tau,t)} d\tau \quad . \tag{22}$$

Next, we transform variables into units of effective labor, denoted by a tilde. Let the exogenous rate of technical progress be given by γ, then: $\tilde{x}_t \equiv x_t e^{-\gamma t}$, $x \in \{k, w, c\}$. Then, (22) becomes:

$$\tilde{k}_t = \tilde{k}_{t_0} e^{\hat{R}(t,t_0)-\gamma(t-t_0)} + \int_{t_0}^{t} \left[\tilde{w}_\tau - \mu_{t_0}^{-1/\theta} e^{\hat{R}(\tau,t_0)/\theta} L_\tau^{1/\theta} e^{-\gamma\tau} D(\tau - t_0)^{1/\theta} \right] e^{-\hat{R}(\tau,t)+\gamma(\tau-t)} d\tau \quad . \tag{23}$$

Considering (23) together with the terminal condition $k_{t_0+h} = 0$, and solving for the costate variable yields:

$$\mu_{t_0}^{-1/\theta} = \frac{\tilde{k}_{t_0} + \int_{t_0}^{t_0+h} \tilde{w}_\tau \, e^{-\hat{R}(\tau,t_0)+\gamma(\tau-t_0)} \, d\tau}{\int_{t_0}^{t_0+h} D(\tau - t_0)^{1/\theta} e^{-\frac{\theta-1}{\theta}\hat{R}(\tau,t_0)-\gamma t_0 + n/\theta\tau} d\tau} \quad . \tag{24}$$

Considering (24) in (21) yields an expression for planned consumption as seen from t_0:

$$\tilde{c}(t \mid t_0) = \frac{\tilde{k}_{t_0} + \int_{t_0}^{t_0+h} \tilde{w}_\tau \, e^{-\hat{R}(\tau,t_0)+\gamma(\tau-t_0)} \, d\tau}{\int_{t_0}^{t_0+h} D(\tau - t_0)^{1/\theta} e^{-\frac{\theta-1}{\theta}\hat{R}(\tau,t_0)-\gamma t_0 + n/\theta\tau} d\tau} e^{\hat{R}(t,t_0)/\theta} L_t^{1/\theta} D(t - t_0)^{1/\theta} \quad . \qquad ||$$

A.3 Proof of Lemma 2

Let $h \in \mathbb{R}_{++}$. The exponentiated discount function $D_H(\tau)^{1/\theta} : [0,h] \to [0,1]$ is strictly monotonically decreasing. $D_H(0) = 1$, and as long as $\beta, \theta < \infty$, $D_H(\mathrm{h})^{1/\theta} > 0$. Consider $\beta_1 > \beta_0$. For every $\tau > 0$, a rise in β from β_0 to β_1 *lowers* the hyperbolic discount rate ρ_τ. As a consequence, graphically speaking, the discount curve $D_H(\tau;\beta_1)^{1/\theta}$ is located weakly above $D_H(\tau;\beta_0)^{1/\theta}$, and it is located strictly above $D_H(\tau;\beta_0)^{1/\theta}$ for all $\tau > 0$. The area below the curves, then, must be larger under $D_H(\tau;\beta_1)^{1/\theta}$ than under $D_H(\tau;\beta_0)^{1/\theta}$, that is,

$$\int_0^h D_H(\tau;\beta_1)^{1/\theta} d\tau > \int_0^h D_H(\tau;\beta_0)^{1/\theta} d\tau \, . \quad ||$$

A.4 Proof of Proposition 2

(i) Suppose $E(r_t,\tau)=0$. Then, the weak inequality in Proposition 2 becomes

$$\int_t^{t+h} D_H(\tau-t)^{1/\theta} E(r_t,\tau)\,d\tau = \int_t^{t+h} D_E(\tau-t)^{1/\theta} E(r_t,\tau)\,d\tau$$
$$= \int_t^{t+h} \left[D_H(\tau-t)^{1/\theta} - D_E(\tau-t)^{1/\theta} \right] E(r_t,\tau)\,d\tau = 0.$$

(ii) Suppose $E(r_t,\tau)>0$. We want to show that $\Delta_H > \Delta_E$.

$$\Delta_H > \Delta_E \;\Leftrightarrow\; \int_t^{t+h} \left[D_H(\tau-t)^{1/\theta} - D_E(\tau-t)^{1/\theta} \right] E(r_t,\tau)\,d\tau > 0. \tag{25}$$

A sufficient condition for the strict inequality to hold is:

$$\int_t^{t+h} \left[D_H(\tau-t)^{1/\theta} - D_E(\tau-t)^{1/\theta} \right] d\tau > 0 \;\Leftrightarrow\; \int_0^h D_H(\tau)^{1/\theta} d\tau > \int_0^h D_E(\tau)^{1/\theta} d\tau . \tag{26}$$

The right hand side of the equivalence follows from the fact that the discount functions are independent of calendar time. Two cases have to be distinguished.

Case (ii.a) $\rho_E = \rho_0$, where ρ_0 denotes the hyperbolic discount rate at $\tau = 0$. In this case, the exponential discount rate is not adjusted so that the overall level of impatience is the same for exponential and hyperbolic discounters. By Lemma 1, $\lim_{\beta\to 0} \int_0^h D_H(\tau)^{1/\theta} d\tau = \int_0^h D_E(\tau)^{1/\theta} d\tau$. By Lemma 2, $\frac{\partial}{\partial\beta}\left[\int_0^h D_H(\tau)^{1/\theta} d\tau\right] > 0$. Hyperbolic discounting occurs when $\beta > 0$. Consequently, $\int_0^h D_H(\tau)^{1/\theta} d\tau > \int_0^h D_E(\tau)^{1/\theta} d\tau$, and the sufficient condition (26) holds.

Case (ii.b) $\rho_E(\beta)$ is adjusted so that the overall level of impatience is the same for exponential and hyperbolic discounters, and (4) is satisfied. Notice that $\int_0^h D_E(\tau)^{1/\theta}$ necessarily declines in ρ_E. The higher the discount rate the lower the discount factor D_E for all $\tau > 0$, and the lower the area under the curve $D_E(\tau)$. A sufficient condition for (26) to hold is:

$$\rho_E(\beta)\,|_{\theta>1} < \rho_E(\beta)\,|_{\theta=1} . \tag{27}$$

Adjustment rule (4) requires the exponential discount rate to be adjusted according to $\rho_E(\beta)\,|_{\theta=1}$. Let $\rho_E(\beta)\,|_{\theta>1}$ be the exponential discount rate for which $\int_0^h D_H(\tau)^{1/\theta} d\tau = \int_0^h D_E(\tau)^{1/\theta} d\tau$. Under condition (27), if instead the larger $\rho_E(\beta)\,|_{\theta=1}$ is applied

– as required by adjustment rule (4) – it follows that $\int_0^h D_H(\tau)^{1/\theta} d\tau > \int_0^h D_E(\tau)^{1/\theta} d\tau$. In order to validate (27), we calculate

$$\rho_E\,|_{\theta>1} = \frac{\rho_0\left(\beta\theta-1\right)\left(1+h\beta\rho_0\right)^{\frac{1}{\beta\theta}-1}}{1-\left(1+h\beta\rho_0\right)^{\frac{1}{\beta\theta}-1}} + \frac{\theta}{h}\text{ProductLog}\left[\frac{e^{\frac{h\rho_0(\beta\theta-1)}{\theta\left(1-(1+h\beta\rho_0)^{1-\frac{1}{\beta\theta}}\right)}}\,h\rho_0\left(\beta\theta-1\right)}{\theta\left(1-\left(1+h\beta\rho_0\right)^{1-\frac{1}{\beta\theta}}\right)}\right]. \qquad (28)$$

Consider $\rho_E\,|_{\theta>1}$ as a function of θ, $\rho_E\,|_{\theta>1}(\theta)$. Then, the following two properties are easily verified: (a) $\lim_{\theta\to 1}\rho_E\,|_{\theta>1}(\theta) = \rho_E\,|_{\theta=1}$; (b) $\frac{\partial}{\partial\theta}\rho_E\,|_{\theta>1}(\theta) < 0$. Thus, inequality (27) holds. As a consequence, considering (26) and (25), $\Delta_H > \Delta_E$.

(iii) Steps (i) and (ii) establish that $\int_0^h D_H(\tau)^{1/\theta} d\tau > \int_0^h D_E(\tau)^{1/\theta} d\tau$. That is, for $E(r_t,\tau) > 0$, (25) holds: $\Delta_H > \Delta_E$. As $\tilde{k}_t$ increases (as r_t decreases) the weight $E(r_t,\tau)$ increases, according to Lemma 3. That is, the positive difference $\int_0^h \left[D_H(\tau)^{1/\theta} - D_E(\tau)^{1/\theta}\right] d\tau$ is multiplied with an increasing strict positive weight $E(r_t,\tau)$ as $\tilde{k}_t$ increases. Consequently, as $\tilde{k}_t$ rises, the strictly positive difference $(\Delta_H - \Delta_E)$ increases. ||

References

Ainslie, G. (1992), Picoeconomics, Cambridge: Cambridge University Press.

Antràs, P. (2001), Transitional Dynamics of the Savings Rate in the Neoclassical Growth Model, Mimeo, Department of Economics, Harvard University.

Barro, R.J. (1999), Laibson Meets Neoclassical in the Neoclassical Growth Model, Quarterly Journal of Economics 114, 1125—1152.

Barro, R.J., X. Sala-i-Martin (2004), Economic Growth, Cambridge MA, MIT Press.

Bosworth, B., G. Burtless, J. Sabelhaus (1991), The Decline in Saving: Evidence from Household Surveys, Brookings Papers on Economic Activity 1, 183—256.

Caliendo, F.N., D. Aadland (2007), Short-Term Planning and the Life-Cycle Consumption Puzzle, Journal of Economic Dynamics and Control 31, 1392—1415.

Caliendo, F.N., T.S. Findley (2014), Discount Functions and Self-Control Problems, Economics Letters, forthcoming.

Chari, V.V., P.J. Kehoe, E.R. McGrattan (1996), The Poverty of Nations: A Quantitative Exploration, NBER Working Paper 5414.

Christiano, L.J. (1989), Understanding Japan's Saving Rate: The Reconstruction Hypothesis, Federal Reserve Bank of Minneapolis Quarterly Review 13, 10—25.

Findley, T.S., F.N. Caliendo (2009), Short Horizons, Time Inconsistency, and Optimal Social Security, International Tax and Public Finance 16, 487–513.

Findley, T.S., F.N. Caliendo (2014), Interacting Mechanisms of Dynamic Inconsistency, Journal of Economic Psychology, forthcoming.

Gómez, M.A. (2008), Dynamics of the Saving Rate in the Neoclassical Growth Model with CES Production, Macroeconomic Dynamics 12, 195—210.

Groth C., K.J. Koch, T.M. Steger (2010), When Growth is Less Than Exponential, Economic Theory 44, 213—242.

Hall, R.E. (1988), Intertemporal Substitution in Consumption, Journal of Political Economy 96, 339—357.

Laibson, D. (1997), Golden Eggs and Hyperbolic Discounting, Quarterly Journal of Economics 62, 443—477.

Leipziger, D., V. Thomas (1997), An Overview of East Asian Experience, in: D. Leipziger (ed.), Lessons from East Asia, Ann Arbor: University of Michigan Press.

Litina, A., T. Palivos (2010), The Behavior of the Saving Rate in the Neoclassical Optimal Growth Model, Macroeconomic Dynamics 14, 482—500.

Loayza, N., K. Schmidt-Hebbel, L. Servén (2000), What Drives Private Saving around the World?, Policy Research Working Paper 2309, Washington D.C.: The World Bank.

Maddison, A. (1992), A Long-Run Perspective on Saving, Scandinavian Journal of Economics 94, 181—196.

Schmidt-Hebbel, K., L. Servén (1999), The Economics of Saving and Growth: Theory, Evidence, and Implications for Policy, Cambridge: Cambridge University Press.

Shafer, J.R., J. Elmeskov, W. Tease (1992), Saving Trends and Measurement Issues, Scandinavian Journal of Economics 94, 155—175.

Smetters, K. (2003), The (Interesting) Dynamic Properties of the Neoclassical Growth Model with CES Production, Review of Economic Dynamics 6, 697—707.

Tease W., A. Dean, J. Elmeskov, P. Hoeller (1991), Real Interest Rate Trends: The Influence of Saving, Investment and other Factors, OECD Economic Studies 17, 107—144.

Thaler, R. (1981), Some Empirical Evidence on Dynamic Inconsistency, Economic Letters 8, 201-207.

Trimborn, T., K.J. Koch, T.M. Steger (2008), Multidimensional Transitional Dynamics: A Simple Numerical Procedure, Macroeconomic Dynamics 12, 301—319.