Advances in Auctions

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Abstract

As a selling mechanism, auctions have acquired a central position in the free market economy all over the globe. This development has deepened, broadened, and expanded the theory of auctions in new directions. This chapter is intended as a selective update of some of the developments and applications of auction theory in the two decades since Wilson (1992) wrote the previous Handbook chapter on this topic.

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Contents

1 Introduction

Auction theory is a prominent and attractive topic of game theory and economic theory.[1] There are two main reasons for this: the first is that despite its simple rules, it is mathematically challenging and often leads to both surprising and elegant results. The second and probably the more important reason is that as a useful market mechanism, auctions have been widely practiced since ancient history.[2]

Both reasons have become even more relevant and convincing since the chapter "Strategic Analysis of Auctions" was written in the first volume of this Handbook (Wilson, 1992). The mathematical developments became more challenging as the theory branched in many directions, such as multi-unit auctions, dynamic auctions, combinatorial auctions, auctions with externalities, and more general incomplete information frameworks. The embracing of *free market* economic principles around the world made auctions the main vehicle for executing the huge volume of privatization that took place and is still occurring.

The theory and practice of auctions have stimulated one another. The massive use of spectrum auctions, online auctions and the privatization via auctions of big economic units (such as oil refineries and power plants) called for theoretical investigations, and the availability of new theoretical results and tools encouraged and facilitated the practical use of auctions, as reflected in the title of Milgrom's book, *Putting Auction Theory to Work* (Milgrom, 2004). The importance of mechanism design, of which auction theory is a central part, was recognized not only by politicians and regulators but also by the academic community, as expressed by awarding the 2007 Nobel Prize in Economics to Hurwicz, Maskin, and Myerson. Game theorists and economists became more and more involved in the practice of auctions both in helping regulators to design 'the right auction' and in consulting competing companies to choose the 'right bidding strategy' (for instance, Binmore, Cramton, Klemperer, McAfee, McMillan, Milgrom, Weber, Wilson, and Wolfstetter, to mention only a few, all acted in at least one of those roles). 'Econlit' lists 20 books[3] and 3,432 works with the word *Auction* or *Auctions* in the title that were written since 1992 (according to 'Google Scholar' the number is much higher – about 20,000).[4]

Given the volume of work done in the past two decades, we cannot hope to provide a fair and comprehensive presentation of all research and applications on auctions since Wilson's chapter in 1992, within the framework of one chapter. It is even clearer that our bibliography cannot be

[1] As an illustration of the importance of auction theory within game theory, see the recent textbook *Game Theory* (Maschler, Solan, and Zamir, 2013), where auction theory is presented and developed as one of its chapters.

[2] Herodotus describes Babylonian bridal auctions in 500 B.C. where revenue from the most attractive maidens was used to subsidize the less attractive ones (see Baye et al., 2012b, for an analysis). In 200 B.C., Ptolemy IV of Egypt ran an auction for the tax-farming rights for Palestine and Syria. When the highest bid was at 8000 talents of silver (232 tons), Joseph the Tobiad told the Pharaoh that the bidders were colluding and offered 16000 talents with the condition that he would be lent soldiers to help in the collection. The profits allowed him to be king of Israel (see Montefiore, 2011, and Adams, 1992).

[3] Among the relevant books are Cramton, Shoham, and Steinberg (2006), Klemperer (2004), Krishna (2002, 2009), Menezes and Monteiro (2005), Milgrom (2004), Paarsch and Hong (2006), Smith (1989), and Wolfstetter (1999).

[4] Econlit lists 1,993 academic journal articles, 977 working papers, 254 papers in collective volumes, 185 dissertations, and 20 books.

exhaustive. Instead, we made a selection of the theoretical themes of research and a sample of important applications of auction theory. As with any selection, it is prone to be biased by the taste of the selectors. We hope that, nonetheless, a non-expert reader will get a reasonably good picture, directly or indirectly through the references, of many of the important developments in auctions in the past few decades.

2 First-Price Auctions: Theoretical Advances

Theoretical and empirical studies of auctions are focused almost exclusively on the prominent solution concept of Nash equilibrium or more precisely Bayes-Nash equilibrium (BNE), as auctions are games of incomplete information. However, an explicit expression of the BNE is mathematically hard to obtain and so far it is available only for simple models, subject to highly restrictive assumptions. For example, Kaplan and Zamir (2012) provide explicit forms of the equilibrium bid functions in an asymmetric first-price auction with two buyers whose values v_1 and v_2 are uniformly distributed in $[\underline{v}_1, \overline{v}_1]$ and $[\underline{v}_2, \overline{v}_2]$, respectively. Furthermore, even the existence of a BNE is a difficult issue since auctions are games with discontinuous payoff functions and hence many of the standard results on the existence of Nash equilibrium cannot be applied in a straightforward way to auctions. Other tools, specifically tailored for this kind of model, are called for. In this section, we highlight the main results on existence of Nash equilibrium in auctions. We will mainly focus on auctions of a single indivisible object.[5]

Let us begin with the basic first-price auction model. Vickrey (1961) is the first to analyze auctions with independent, private values drawn from a uniform distribution. Riley and Samuelson (1981) extend Vickrey's analysis to n symmetric buyers, with values that are independent and identically distributed from a general distribution F that is strictly increasing and continuously differentiable. They find that there exists a unique equilibrium that is symmetric with bid function $b(v) = v - \int_{\underline{v}}^{v} F^{n-1}(x)dx/F^{n-1}(v)$ for all bidders; that is, each bidder bids the conditional expectation of the other bidders' highest value given that he wins the auction. Since this contribution, there have been various works extending it to auctions in which one or more of the key assumptions of Riley and Samuelson no longer hold. These assumptions are symmetry and independence of the value distributions and smoothness of the common distribution F.

2.1 Mixed-Strategy Equilibria

When one relaxes one of Riley and Samuelson's (1981) three basic assumptions about the distributions (symmetry, independence and smoothness), for a first-price auction, a pure-strategy Nash equilibrium may not exist. A simple example is the case of two buyers each with a value drawn from the discrete distribution of 0 and 1 each with probability 1/2. The equilibrium here is unique and in mixed strategies. In a classic example, Vickrey (1961) shows that when one buyer's value is commonly known there is a unique equilibrium where that buyer uses a mixed strategy.[6]

[5] For a recent survey of existence and characterization results, see de Castro and Karney (2012).
[6] See Martínez-Pardina (2006) for a generalization of this situation.

Jackson and Swinkels (2005) demonstrate that without independence even when both buyers have private values drawn from a continuous distribution, there may not exist a pure-strategy monotonic equilibrium in a first-price auction. As an example consider an auction with two buyers with values v_1 and v_2 uniformly distributed on the triangle $v_1 \geq 0$, $v_2 \geq 0$, and $v_1 + v_2 \leq 1$. The intuition is that a buyer with value 1 knows that the other buyer has value 0 and hence in equilibrium he should not bid above 0. By monotonicity, each buyer should bid 0 for all his values ,which cannot be an equilibrium.

Jackson and Swinkels then prove the existence of equilibrium in mixed strategies in a general model that covers in particular both first-price and all-pay auctions as special cases. More specifically, they consider an environment with multiple identical items of the same indivisible object. Each buyer has an additive, but not necessarily linear, utility for multiple items, he is endowed with a finite (possibly zero) number of items, and he may want to sell or buy some items. Formally, the type $\theta_i = (e_i, v_i)$ of buyer i consists of the number e_i of items that he is endowed with and his values $v_i = (v_{i1}, v_{i2}, \ldots, v_{i\ell})$, where v_{ik} is his marginal utility from the k-th item and ℓ is the total number of items in the economy. The key assumptions of the model are that the type space Θ is compact and the distribution F on Θ is continuous and absolutely continuous with respect to the product of its marginal distributions (which they call *imperfect correlation*). Finally, the utility functions over the payoff are assumed to be continuous.

2.2 Asymmetric Buyers: Existence of Mixed and Pure-Strategy Equilibria

The symmetry of buyers was central and crucial for the results of Milgrom and Weber (1982). Existence results for models with asymmetric buyers were obtained by several authors, primarily by Lebrun (1996, 2006) and by Maskin and Riley (2000b).

An early general existence result for equilibria in first-price asymmetric auctions is due to Lebrun (1996) who proved that an *equilibrium in mixed (distributional) strategies* exists in an n-buyer first-price auction for an indivisible single good, under the following assumptions:

- The value distributions $F_1, \ldots, F_n$ are independent, with compact supports contained in $[c, K]$ for some finite c and K ($c < K$).
- None of the distributions F_i has an atom at c.
- The allowed bidding range is $[c, K]$.

The condition of no atoms at c is needed to avoid payoffs discontinuities at c, with the standard tie-breaking rule (all the highest bidders have equal probability of winning). This difficulty can be avoided by 'augmenting' the game to include messages to be sent by the bidders and to be used in case of a tie. In this augmented first-price auction, the existence is then proved without the assumptions of no atoms at c. However, this is not very appealing: not only is this 'augmentation' rather artificial, but the resulting equilibrium may include buyers using weakly dominated strategies.

This variant of an 'augmented' first-price auction, along with a second variant in which in case of a tie, the object is given to the bidder with the highest value, was used in a subsequent paper (Lebrun, 2002) to prove (under the assumptions in Lebrun, 1996) the upper-hemicontinuity of the Nash-equilibrium correspondence with respect to the valuation distributions. This implies the continuity of the Nash-equilibrium under the conditions that guarantee its existence and uniqueness (that is, when the Nash-equilibrium correspondence is single valued). Lebrun views these results as a proof of robustness of the theoretical existence results and the usefulness of the numerical approximations of the Nash equilibrium.

Lebrun (2006) provides results on the existence and uniqueness of the equilibrium in asymmetric first-price auctions with n risk-neutral bidders, reserve price r, and the fair tie-breaking rule. Lebrun makes the following assumptions:

- The value distributions $F_1, \ldots, F_n$ are independent with supports $[c_i, d_i]; c_i < d_i,\ i = 1 \ldots n$.
- For each $i = 1, \ldots, n$, the cumulative distribution F_i is differentiable over $(c_i, d_i]$ with derivative (density) f_i locally bounded away from zero over this interval. Assume further that $F_i(c_i) = 0$ for all i.

Assume without loss of generality that $c_1 \geq c_2 \geq c_i$ for all $i \geq 2$; then the main result is:

Theorem 1 *If the above assumptions hold and no bidder bids above his value, then under any of the following conditions, the first-price auction has one and only one equilibrium: (i) $r > c_1$; (ii) $c_1 > c_2$; (iii) There exists $\delta > 0$ such that F_i is strictly log-concave over $(c_1, c_1 + \ \delta) \bigcap (c_i, d_i)$, for all $i \geq 1$, that is, f_i/F_i, the reverse hazard rate, is strictly decreasing over this interval.*[7]

2.3 Relaxation of Symmetry and Independence

Athey (2001) proves existence of a Bayes-Nash equilibrium in nondecreasing pure strategies for a large class of games of incomplete information. Her general result was widely used and in particular it was adapted and used for auctions by Athey herself and other authors (e.g., Reny and Zamir, 2004, to be discussed below). The novelty of Athey's work is that for studying the question of existence of pure-strategy Nash equilibrium (PSNE), which is basically a fixed-point problem, she introduces the *single-crossing condition* (SCC) which plays a central role in the proof of existence. Verifying this condition requires comparative statics analysis for a single-player problem (which is a simpler problem than showing the existence of a fixed point). The SCC of Athey is built on a related notion of *single-crossing property* due to Milgrom and Shannon (1994), who developed comparative statics analysis using only conditions that are ordinal (that is, invariant under order-preserving transformations of the utilities). Let X be a lattice, Θ a partially ordered set, and $h : X \times \Theta \to \mathbb{R}$ a real function on the product set.

Definition 2 *The function $h : \mathbb{R}^2 \to \mathbb{R}$ satisfies the (Milgrom-Shannon)* single-crossing property of incremental returns *(SCP-IR) in $(x; \theta)$ if, for all $x_H > x_L$ and all $\theta_H > \theta_L$, $h(x_H, \theta_L) - h(x_L, \theta_L) \geq (>) 0$ implies $h(x_H, \theta_H) - h(x_L, \theta_H) \geq (>) 0$. The function h satisfies weak SCP-IR if for all $x_H > x_L$ and all $\theta_H > \theta_L$, $h(x_H, \theta_L) - h(x_L, \theta_L) > 0$ implies $h(x_H, \theta_H) - h(x_L, \theta_H) \geq 0$.*

This condition requires that the incremental return to x, $h(x_H, \cdot) - h(x_L, \cdot)$, as a function of θ, crosses zero at most once, from below.

[7] For more about the role of log-concave probability in auction theory, see Bergstrom and Bagnoli (2005).

Athey considers a game of incomplete information with a finite set of players $I = \{1, \ldots, I\}$. The players' type sets are intervals in the real line, $T_i = [\underline{t}_i, \bar{t}_i]$, and the action sets are compact convex sets in the real line, $A_i \subset \mathbb{R}$. The prior probability on the types is assumed to have density (with respect to the Lebesgue measure). Finally a technical integrability condition is assumed on the utility functions (henceforth Assumption (A1)) to ensure that the objective function $U_i(a_i, t_i; \alpha_{-i})$ of player i of type t_i, when all other players are using nondecreasing strategies α_{-i}, is well defined and finite. For this model, she defines the following single-crossing condition.

Definition 3 *The Single-Crossing Condition for games of incomplete information (SCC) is satisfied if for each player $i \in I$, whenever every other player $j \neq i$ uses a strategy α_j that is nondecreasing (in his type), player i's objective function, $U_i(a_i, t_i; \alpha_{-i})$, satisfies single crossing of incremental returns (SCP-IR) in $(a_i; t_i)$.*

The method adopted by Athey to prove that the SCC is sufficient for the existence of PSNE is to prove it first for the case of finite action sets and then, for the case of a continuum of actions, to use a sequence of appropriately designed grid approximations to prove that a certain selection of a sequence of monotone PSNE for the finite approximations converges to a monotone PSNE in the original game. The result for finite action sets is:

Theorem 4 *If Assumption (A1) is satisfied, SCC holds, and the action set A_i is finite for all $i \in I$, then the game has a PSNE where each player's equilibrium strategy, $\beta_i(\cdot)$, is nondecreasing.*

The results for continuous actions sets are derived from Theorem 4 and the following theorem.

Theorem 5 *If Assumption (A1) is satisfied and the following hold: (i) for all $i \in I$, $A_i = [\underline{a}_i, \bar{a}_i]$, (ii) for all $i \in I$, $u_i(a, t)$ is continuous in a, and (iii) for any finite $A' \subset A = \bigtimes_{i \in I} A_i$, a PSNE exists in nondecreasing strategies, then a PSNE exists in nondecreasing strategies in the game where players choose actions from A.*

Athey applies these results to *supermodular* and *log-supermodular* utility functions and *affiliated types*. Then she shows that the results can be extended with additional assumptions to certain cases of discontinuous utility functions that include some auction games. However, a more general result on the existence of PSNE in asymmetric first-price auctions using Athey's result was given by Reny and Zamir (2004) (henceforth RZ).

RZ notice that the SCC can fail in two possible ways. First, if there are ties at winning bids and, second, if a buyer uses a strategy that yields a negative payoff. They define a single-crossing condition that states that SCC holds whenever we avoid these two possibilities.

Definition 6 *The Individually Rational, Tieless, Single-Crossing Condition (IRT-SCC) is satisfied if for each player $i \in I$, actions a_i, a_i', and nondecreasing strategies α_j such that $Pr(\max_{j \neq i} \alpha_j(\cdot) = a_i \text{ or } a_i') = 0$, $U_i(a_i', t_i; \alpha_{-i}) \geq 0$, then $U_i(a_i', t_i; \alpha_{-i}) \geq U_i(a_i, t_i; \alpha_{-i})$ is maintained when t_i rises if $a_i' > a_i$ and when t_i falls if $a_i' < a_i$.*

Using their definition IRT-SCC, RZ are able to prove the existence of PSNE for asymmetric buyers, interdependent values, and affiliated one-dimensional signals. They note, by way of a counterexample, that these results cannot extend to multidimensional signals.

A further significant generalization was obtained by Reny (2011), who proved the existence of monotone pure-strategy equilibria in Bayesian games with locally complete metric semilattices action spaces and type spaces that are partially ordered probability spaces. The novel idea that enabled this generalization was to use the Eilenberg and Montgomery (1946) fixed-point theorem rather than the Kakutani's fixed-point theorem used in Athey (2001). The point is that while Kakutani's theorem requires that the sets of monotone pure-strategy best replies be convex, which may be hard to establish, the Eilenberg and Montgomery fixed-point theorem requires that these sets be contractible. This condition turns out to be rather easy to verify for the general class of Bayesian games studied by Reny. Finally, we note that in a recent unpublished work Zheng (2013) proves the existence of a monotone PSNE in a first-price auction with (endogenous) resale.

2.4 Monotonicity and the Role of Tie-Breaking Rules

The study of Nash equilibrium in first-price auctions has concentrated primarily on *monotone pure-strategy equilibria* (MPSE). Whenever the existence of such MPSE is established, the natural questions to be asked are: Is there only one MPSE? Are there any *non-monotone* pure-strategy equilibria? Are there mixed-strategy equilibria (MSE)? These questions, which turned out to be related to each other, were partially or fully answered for various models (under various sets of assumptions), and some are not answered to date. The answers to these questions also often hinge on the *tie-breaking rule* used. For example, in the *general symmetric model* of Milgrom and Weber (1982) where they established a unique symmetric MPSE, it was only 24 years later that McAdams (2006) proved that there are no asymmetric MPSE and in a subsequent paper, McAdams (2007a), he ruled out non-monotone equilibria in this model (including mixed-strategy equilibria).[8] In these works, McAdams considered two tie-breaking rules:

- The standard *coin-flip rule* according to which the winner of the object is chosen by a uniform probability distribution over the set of buyers who submitted the highest bid.
- The *priority rule* according to which the winner is the buyer with the highest rank in a pre-specified (prior to the bidding) order (permutation) of the buyers.

The results of McAdams are for a general asymmetric model with any (finite) number of buyers with affiliated types and interdependent values. More precisely, he considers the general model of Reny and Zamir (2004) restricted by the additional assumption that the utility functions are strictly decreasing in the bids. In addition he considered two types of bidding sets: the *continuum price grid* where $b_i \in [0, \infty)$ and the *general price grid* where the bids are restricted to an arbitrary subset of $b_i \in [0, \infty)$. For this model, McAdams proves that:

- Given a continuum price grid and the coin-flip rule, every MSE of the first-price auction with no ties is outcome-equivalent to some MPSE.

[8] Earlier and weaker results in this direction were obtained in Rodriguez (2000).

- Given a general price grid and the priority rule, every MSE of the first-price auction is outcome-equivalent to some MPSE.
- Given the coin-flip rule, every MPSE of the first-price auction has no ties.

Thus, non-monotone equilibria can exist under the coin-flip rule but they are distinguishable: all non-monotone equilibria have positive probability of ties whereas all monotone equilibria have zero probability of ties. McAdams provided an example of a non-monotone pure-strategy equilibrium in a first-price auction (such an example requires three buyers with affiliated types and interdependent values).

In yet another paper, McAdams (2007b) addressed the issue of the uniqueness of the MPSE proved to exist in the RZ model. He proves this uniqueness under considerable restriction of the RZ model. In particular he assumes a symmetric model (roughly: the distribution and the utilities are invariant to any permutation of the buyers). For this he proves:

- There is a unique MPSE in the symmetric first-price auction, up to the bids made by a set of measure zero of types.

2.5 Revenue Comparisons

An important property of symmetric auctions with independent private values that is lost as soon as we relax the symmetry condition is *revenue equivalence* among a large class of auctions. Although symmetry is not a condition in the Revenue Equivalence Theorem of Myerson (1981), the asymmetry implies different equilibrium allocations for different types of auctions (e.g., first-price and second-price auctions), which violates the requirement that equilibrium allocations be the same. Now, the investigation of revenue comparisons between selling mechanisms in general and types of auctions in particular naturally follows. This comparison turns out to be rather difficult, and so far, no strong general results are available. Even the question of which of the two auctions, first price or second price, generates a higher revenue in an asymmetric setting is very partially answered. In this subsection, we review some of these results.

Maskin and Riley (2000a) study an asymmetric environment with two buyers: a strong one and a weak one. This ranking of the buyers means that the distribution of the strong buyer's value, F_s, Conditionally Stochastic Dominates (CSD) the distribution of the weak buyer's value, F_w. Formally, CSD is defined as follows. For all $x < y$ in the common support of F_s and F_w,

$$Pr\{v_s < x | v_s < y\} = \frac{F_s(x)}{F_s(y)} < \frac{F_w(x)}{F_w(y)} = Pr\{v_w < x | v_w < y\}.$$

Under CSD,[9] in a first-price auction, the weak buyer bids more aggressively than a strong buyer. Furthermore, a strong buyer would prefer a second-price auction while a weak buyer would prefer a first-price auction. Revenue comparisons between the two formats are mixed. When there is a *shift* ($F_s(x + \alpha) = F_w(x)$, $\alpha > 0$) or *stretch* (e.g., for distributions that start at zero, $F_s(x) = \alpha \cdot F_w(x)$, $0 < \alpha < 1$, holds on the common support) a first-price auction generates higher

[9] Maskin and Riley (2000a) use a weaker definition.

revenue, while when there is a move of probability to the lower end of the support, a second-price auction generates higher revenue ($F_w(x) = \alpha \cdot F_s(x) + \delta$, $0 < \alpha < 1$, $\delta = 1 - \alpha$).[10]

Kirkegaard (2012) expands Maskin and Riley's (2000a) revenue comparison between first-price and second-price auctions. He finds that under CSD, no atoms, and the condition $\int_x^{F_s^{-1}(F_w(x))}(f_w(x) - f_s(y))dy \geq 0$ for all x, a first-price auction generates higher revenue than a second-price auction. This also holds in the presence of a reserve price. Kirkegaard (2014) provides additional sufficient conditions for revenue comparisons of first-price and second-price auctions. Outside the independent private value model, de Castro (2012) finds that with certain cases of dependence, revenue in a first-price auction may be higher than that in a second-price auction even under symmetry, in contrast to Milgrom and Weber (1982), who find the opposite holds in the case of affiliated values.

3 Multi-Unit Auctions

The literature on multi-unit auctions for environments in which buyers exhibit demand for multiple units of the same good is of great practical importance. The U.S. government treasury auctions three trillion dollars worth of securities annually (to finance public debts). The underpricing of one cent per \$100 would lead to a loss of \$300 million!

Treasury auctions are typically sealed-bid auctions in which each buyer submits a demand function (either as the total price $P(x)$ that he is willing to pay for x units, or equivalently his marginal bid $p(x)$ for the x-th unit). The submitted demands determine a clearing price and there are two commonly used pricing methods: uniform pricing, in which all units are sold at the highest price that clears the market, and discriminatory pricing, in which each buyer pays his own bid (for the units that he won). In practice, both pricing methods are used in treasury auctions of various countries and empirically it is not clear which one generates higher revenue (see, for example, Bartolini and Cottarelli, 1997, Binmore and Swierzbinski, 2000, and Brener et al., 2009). Theoretically, it is known that both types of auctions are inefficient (see, for example, Wilson, 1979, Back and Zender, 1993, and Ausubel et al., 2013) and both are vulnerable to manipulation and cooperative behavior among buyers (see Goswami et al., 1996, Kremer and Nyborg, 2003). In addition to sealed-bid auctions, ascending auctions are commonly used in wine auctions (see Février et al., 2005), timber auctions (see Athey and Levin, 2001, and Athey et al., 2011), spectrum auctions (see Section 10), and other instances where multiple units of the same object or similar objects are for sale. These can be run either simultaneously or sequentially.

3.1 Efficient Ascending-Bid Auctions

Effective auction design is guided by two principles; both are related to information. The first principle is that the price paid by the winner should be independent of his own bid, inducing the buyer to reveal his information (his true value of the good).[11] This was achieved brilliantly by

[10] For more precise (and general) definitions of shifts, stretches, and probability moves, see Maskin and Riley (2000a).

[11] The independence of the price of the bidder's own bid is not a sufficient condition for bidding the true value in equilibrium. As an example, in the equilibrium a third-price auction (of a single-unit indivisible object and symmetric

Vickrey in the second-price auction and was later generalized to the Vickrey-Clarke-Groves (VCG) mechanism. The second principle is to maximize the information of each bidder when placing his bid, as in the English ascending auction. This principle is important if the objective of the design is revenue maximization. When the buyers' values for the object are interdependent and signals are affiliated (roughly speaking, positively correlated), gathering more information about the other bidders' signals will typically induce buyers to bid more aggressively and hence increase seller revenue (increasing the expectation of the second-highest bid).[12]

Ausubel (2004) proposes an efficient auction design with these two features for multi-unit auctions. He calls it a "dynamic auction" but it is not dynamic in the same sense as in the dynamic auction literature where the dynamic component is either that of the buyers' population or that of the information available to a fixed population of buyers. The dynamic in Ausubel's auction is that of the auction mechanism: information is revealed (implicitly) during the process as the buyers change their demand with the change of prices. It is a dynamic auction in the same sense that the ascending English auction is dynamic. It could therefore be called an *open multi-unit auction*.[13]

The starting point of Ausubel is what he calls the 'static multi-unit Vickrey auction,' which is actually the VCG mechanism for selling homogeneous items. This is a sealed-bid auction that generalizes the second-price auction to multiple units: each buyer submits a sealed-bid demand function, which with discrete items is a list of prices that he is willing to pay for the first, second, third item, etc. A market clearing price is determined (where the number of bids above or equal to that price equals the number of units available). Each bid exceeding the clearing price is winning and for each unit won the buyer pays the opportunity cost of assigning this unit to him, that is, the bid of the buyer (other than himself) that was rejected as a result of him receiving the unit. For example, if a buyer receives two items, and the highest rejected prices other than his own were 5 and 3, he pays 5 for his first unit and 3 for his second. Note that these are neither the uniform nor the discriminatory prices and, just as in the second-price auction for a single unit, the prices paid by the buyer are independent of his bids for these two units (which are each of course at least 5). [14] The declared objective of Ausubel in this work was to design an open ascending auction that generalizes the single-unit English clock auction and is analogous to the multi-unit Vickrey auction in the same way that the English auction is analogous to the second-price auction for a single unit.

setting), every bidder bids more than his value (see Kagel and Levin, 1993). It is also not a necessary condition since in an environment having two buyers with values independently drawn from the uniform distribution, a first-price auction where the winning bidder pays half his bid will induce truthful bidding.

[12]However, it is not always true that the seller's revenue increases monotonically as more information is given to the buyers. A counterexample is provided by Landsberger et al. (2001), who show that when the (two) buyers (in addition to privately knowing their own values) commonly know (only) the ranking of their values, the revenue is higher than the case of complete information when the values are commonly known. This was also shown in Kaplan and Zamir (2000) in the context of the strategic use of seller information in a private-value first-price auction: revealing only part of his information may yield the seller higher revenue than revealing all of it.

[13]In the auction literature, it is also referred to as the Ausubel auction.

[14]For the pure independent private-value setting, the Vickrey multi-unit auction achieves an (ex-post) efficient outcome. This need not be the case when the values are interdependent. Perry and Reny (2002) modify the Vickrey auction to obtain efficiency also in the case of interdependent values (while maintaining the assumptions that the objects for sale are homogeneous and that each bidder's demand is downward sloping). Their mechanism consists of a collection of second-price auctions between each pair of bidders conducted over at most two rounds of bidding.

For a homogeneous multi-unit auction of M units for sale, the auction proceeds as follows: starting at a low per-unit price, the auctioneer raises the price continuously (or in discrete steps) while each of the buyers posts his demand (i.e., the number of units he is willing to buy) at the current price. The auction ends with the price at which the total demand equals the number of units offered for sale and each buyer receives the number of items that he demanded at this price. The payment made by each buyer is calculated as follows. As the process of increasing the price proceeds, at each price p_t for which the demand $d(p_t)$ at that price drops (by one unit or more), and $d(p_t) \geq M$, it is determined whether for buyer i the demand $d_{-i}(p_t)$ of all other buyers at that price is strictly less than M, that is, whether $d_{-i}(p_t) = M - k$ where $k > 0$. If so, then buyer i has *clinched* winning k units. If his last clinch at a previous stage was at price $p_{t'}$ when the demand of all other buyers was $d_{-i}(p_{t'}) = M - k'$, then buyer i pays a price p_t for each of the last $(k - k')$ units. In other words, the buyer buys "the newly clinched" units for the price p_t at which they were clinched.[15]

Illustrating the process is the following example of Ausubel (2004), loosely patterned after the first U.S. Nationwide Narrowband Auction in which there were five spectrum licenses and five bidders with the limitation that each can buy only up to three licenses. The bidders marginal values for the licenses (their inverse demand functions) are given as columns in Table 1 in millions of dollars.

Unit	A	B	C	D	E
1	123	75	125	85	45
2	113	5	125	65	25
3	103	3	49	7	5

Table 1: Marginal values of the buyers in Ausubel's (2004) example.

When the auctioneer starts from a low price, say 10, the demands will be $(3, 1, 3, 2, 2)$ for bidders (A,B,C,D,E), respectively. As the price goes up (say continuously), the demands become $(3, 1, 3, 2, 1)$ at price $p = 25$ (a bidder chooses not to buy when indifferent), then $(3, 1, 3, 2, 0)$ at price $p = 45$, then $(3, 1, 2, 2, 0)$ at price $p = 49$, and then $(3, 1, 2, 1, 0)$ at price $p = 65$. At this price, the total demand is 7, which still exceeds the supply of 5. Yet, the total demand of all bidders but A is $4 = 5 - 1$. In the language of Ausubel, 'bidder A has *clinched* winning one unit' at price 65 (expressing the fact that at that point it was guaranteed that he would end up with at least one unit). At price $p = 75$, the demand drops to $(3, 0, 2, 1, 0)$ and bidder A clinches winning two items, paying 75 for his second item, and bidder C clinches one item at price 75. The total demand is still higher than the supply and so the price goes on increasing until it reaches $p = 85$, where the demand becomes $(3, 0, 2, 0, 0)$ and hence supply equals demand. At this price 85, bidder A clinches

[15] If the demand drops from $d(p_{t-1}) > M$ to $d(p_t) < M$, then the total number of units clinched will exceed the supply M and the process ends at time t. (For example, the supply is 10, there are two symmetric buyers, there are 0 clinched units at $t - 1$, and the demand drops to $d(p_t) = 8$. Here, each buyer would clinch 10-(8/2)=6 units.) In this case, there may be many possible methods to assign a market clearing allocation of the objects, including random allocation.

his third unit ($3 = 5 - 2$) and bidder C clinches his second unit ($2 = 5 - 3$). The outcome of the auction is therefore: bidder A wins three licenses and pays $65 + 75 + 85 = 225$ million, and bidder C wins two licenses and pays $75 + 85 = 160$ million.

This mechanism yields an efficient allocation: the process allocates the items to the buyers who value them the most, and for this homogeneous discrete multi-item environment with independent values it yields the same allocation as the sealed-bid multi-unit Vickrey auction described previously.

For comparison, in a uniform-price ascending-clock auction, the closing price in this example would be 75 (assuming full information), buyer A would reduce his demand and ask for two items at price 75 and end the auction (preferring to get two items at price 75 each rather than three items at price 85 each) with the inefficient allocation $(2, 0, 2, 1, 0)$. Furthermore, the example can be slightly perturbed to make this inefficient outcome the unique outcome of iterated elimination of weakly dominated strategies.

The Ausubel model is for independent values but otherwise it is quite general: it allows for both discrete and divisible objects, both complete information when demand functions are commonly known, and incomplete information when demand functions are only privately known. Ausubel considers two versions of the model, one with a continuous-time clock and one with a discrete-time clock. For private values, the main result about this mechanism which he calls 'the alternative ascending-bid auction' is that sincere bidding by all bidders is an ex-post perfect equilibrium yielding the efficient outcome of the multi-unit Vickrey auction.

To provide a generalization of the Milgrom-Weber model of interdependent values, Ausubel needs to make some assumptions and limit his general model. He considers a continuous-time game with symmetric interdependent values. A seller offers M discrete and indivisible homogeneous goods. Each buyer i has constant marginal utility v_i for each unit up to a *capacity* of λ_i units and zero marginal utility for any additional unit beyond this capacity. The marginal values v_i are interdependent: they are derived from affiliated private signals. The main result for this model is that if all buyers have the same capacity λ, and M/λ is an integer, then under certain assumptions (analogous to those in the Milgrom-Weber model) both the multi-unit Vickrey auction and the alternative ascending-bid auction attain full efficiency. However, the alternative ascending-bid auction yields the same or higher expected revenue than the multi-unit Vickrey auction. This is analogue to the the Milgrom and Weber result for a single-unit auction: with affiliation, the ascending English auction generates higher revenue than its sealed-bid analogue, the second-price auction.

3.2 Multiple Heterogeneous Items

In a subsequent paper, Ausubel (2006) extends his multi-unit open ascending auction to K heterogeneous commodities with available supply $S = (S^1, \ldots, S^K)$ among n buyers. The starting point of Ausubel's mechanism is the Walrasian Tâtonnement process (also known as *dynamic clock auction*). One application of such auctions is to sell spectrum rights where many licenses are sold simultaneously and may vary slightly (see the subsequent section of this chapter for further discussion). In this selling mechanism, a 'fictitious' auctioneer announces a price vector for the K commodities and the buyers report their demand (vectors), the auctioneer then adjusts the prices by increasing the prices of commodities with positive excess demand and decreasing the prices of commodities with negative excess demand. The process continues until a price is reached in which demand equals supply for all commodities. The trade takes place only at the final (market clearing) price and the buyers' payments are linear in quantities; all buyers pay the same price for all units of the same commodity that was allocated to them.

This process is vulnerable to strategic manipulation as the linear payment method provides an incentive for the bidders to underreport their true demand at the announced price (see Ausubel, 2004, for the ascending auction, and Ausubel, 2002, for the sealed-bid auction.) Consequently the process typically does not result in a Walrasian outcome. An extreme empirical demonstration of this is the GSM spectrum auction in Germany (see Grimm et al., 2003, and Riedel and Wolfstetter, 2006). To overcome this drawback, Ausubel replaces the linear pricing by the non-linear pricing in 'clinches' introduced in his above-described one-commodity multi-unit ascending auction. Similarly to the process in the homogeneous case, the price vector $P(t)$ is changing (according to an adapted Walrasian Tâtonnement process), and units of each commodity ℓ are allocated (clinched) to buyer i at that price $P_\ell(t)$ when the demand of all other buyers for that commodity is less than the available supply S^ℓ. Thus, in Ausubel's dynamic mechanism, payments are made along the process and they are determined by the prices at various points in time. Hence, buyers pay different prices for units of the same commodity clinched to them at different stages in the process. As a result, there are technical issues to ensure that payments are well defined, i.e., do not depend on the trajectory of the price vector $p(t)$ provided it satisfies certain conditions.

When a bundle of heterogeneous objects is purchased by a bidder in an auction, there is the issue of *complementarity* and *substitutability* of different commodities that affects the efficiency of the auction. For his open ascending auction, Ausubel assumes a *substitutes condition* (or *gross substitutes*) which is needed for the existence of Walrasian equilibrium. This condition requires that if the prices of some commodities are increased while the prices of the remaining commodities are held constant, then a bidder's sincere demand weakly increases for each of the commodities whose prices were held constant.

The issue of complementarity was raised in environments where heterogeneous objects are sold in separate auctions. As it was pointed out in Milgrom (2000), this situation may lead to inefficiency due to what is called *exposure*: a bidder may purchase object A while paying more than his value for A alone or a bidder may purchase bundle AB while paying more than his value for the bundle. This situation may occur, for instance, if, while bidding for A, the price of the complementary object B is not yet determined. The price of B may eventually be too high for it to be worthwhile to purchase B, or the marginal benefit of purchasing B may be worthwhile but the total purchase price of bundle AB is higher than its value to the bidder.[16]

Zheng (2012) considers a sale of two objects A and B with two types of bidders: one type is interested in only one object and another type has an added value for the bundle AB. The author

[16]See Szentes and Rosenthal (2003a, 2003b) for the exposure problem in the chopsticks auction.

designs a mechanism consisting of two simultaneous ascending English auctions modified so as to allow for *jump bidding* (and other rules accompanying it). The author shows that this modified auction can avoid exposure and restore efficiency since the jump bids can serve as signals.

For divisible goods and strictly concave utility functions and mandatory participation, Ausubel proves that sincere bidding by every buyer is an ex-post perfect equilibrium of the auction game. With sincere bidding, the price vector converges to a Walrasian equilibrium price vector and the outcome is that of a modified VCG mechanism with the same initial price vector $P(0)$.

For the algorithmic aspects of multi-unit auctions see the chapter "Algorithmic Mechanism Design" in this Handbook.

4 Dynamic Auctions

The literature on dynamic auctions provides natural extensions to the well-established static auctions theory to common economic environments: situations where the populations of sellers and buyers, the amount of goods for sale, and the state of information of the various agents are changing dynamically. These changes present the agents and the market designers with dynamic decision problems. Examples are airlines managing their prices when customers enter the market at different times, government surplus auctions where the amount being sold is stochastically changing, owners of internet search engines managing their advertisements when the number of people searching for terms varies, and selling a start-up company when the actual value becomes clearer over time.

The framework of modeling and analyzing such economic environments is that of *dynamic mechanism design* within which *dynamic auctions* are a special case of such mechanisms. There is no clear line between the two bodies of literature and often a paper with the title 'Dynamic Auctions' is actually a paper on 'Dynamic Mechanism Design'. This is the case with the survey by Bergemann and Said (2011). This paper surveys the literature on dynamic mechanism design by grouping it in a two dimensional way. In the first dimension the distinction is between models in which the population of agents change over time while their private information is fixed and models in which the population of agents is fixed but the information of the agents is changing over time. In the second dimension, they group the works according to whether they aim to find mechanisms maximizing social welfare or maximizing revenue.

4.1 Dynamic Population

Due to the mathematical difficulties involved, each work studies a very special case resulting from restrictions and assumptions that have various degrees of plausibility. For this reason, it is hard to compare results obtained by different authors. The common approach to model a changing population of bidders is to have potential bidders enter (or possibly depart from) the market by some exogenous or endogenous process. Crémer et al. (2007) study a mechanism design problem in which a seller wishes to sell a single unit of an indivisible object to one of a finite set I of potential buyers. The problem becomes dynamic since the seller has to contact prospective bidders, at a cost, and bring the auction to their attention. He incurs a cost of c_i (which they call a search cost) to inform bidder i about the rules of the auction, the identity of the other bidders, and the distribution of their valuations. Bidder i then privately learns his type x_i and decides whether to participate or not. If he agrees to participate, he signs a binding contingency contract. While the types of the bidders are independent, there are interdependent valuations of the object: bidder i's value is $u_i(x_1, \ldots, x_I) = x_i + \sum_{j \neq i} e_{ij}(x_j)$. All bidders have the same discount factor $\delta \in (0, 1]$.

At the start of each period t the seller approaches a set of potential entrants. Those who decide to enter (after learning their type) and sign a contract send a message to the seller who decides whether to stop the mechanism and sell the object to one of the participants (entrants and incumbents) according to the contracts or to approach new bidders. There are variants of information disclosure policies, whether or not the seller reveals to the new entrants the messages received in previous stages. In principle, bidders may send a message at each stage they participate in but there is no loss of generality in limiting the mechanism to a single message at the entering stage. The authors apply a revelation principle argument to observe that any perfect Bayes equilibrium (PBE) outcome can be obtained by an *incentive feasible mechanism* in which the bidders communicate their types truthfully. The main result is that under some technical differentiability conditions, the optimal (seller's profit-maximizing) mechanism is an *optimal search procedure* relative to the *ex-post virtual utility* functions $V_i(x_1, \ldots, x_I) = x_i - \frac{1-F_i(x_i)}{f_i(x_i)} + \sum_{j \neq i} e_{ij}(x_j)$. An interesting part of the result is that the seller's optimal revenue does not depend on the information disclosure policy adopted.

For the special case of private values ($e_{ij}(\cdot) = 0$ for all i, j) and $\delta = 1$, the optimal mechanism can be implemented by a sequence of Myerson's optimal auctions where a new entrant joins at each period as long as the object is not sold. In the symmetric case where the bidders have the same distribution of values, the mechanism consists of a sequence of second-price auctions with a reserve price that declines over time. Crémer et al. show that in this special case the optimal design problem is similar to the Weitzman (1979) pandora problem of searching for the highest reward box when only one box can be opened (at a cost) at each period.

In a subsequent paper, Crémer et al. (2009) consider a variant of the model where the costly recruitment of bidders by the seller is replaced by a cost for a bidder to find out his valuation. The seller can force the bidder to pay an entry fee before finding out his type. They show that the seller can obtain the same profit as if he had full control over the bidders' acquisition of information and could directly observe their valuations once they are informed (intuitively because the bidders are ex-ante identical).

As we move to multiple units for sale, Vulcano et al. (2002) and Pai and Vohra (2013) have similar models in which a seller with K identical items for sale faces a random arrival of buyers with a demand for one unit. The value for the item and the latest time that it has to be purchased by are the private information of the arriving buyer. Both papers address the revenue maximization problem. Board and Skrzypacz (2013) consider a seller with K identical indivisible items for sale in a discrete time span of $\{1, \ldots, T\}$. At time t, a random number N_t of buyers arrive to the market where $\{N_t\}_{t=1}^{\infty}$ are i.i.d. random variables. The number of buyers is observed by the seller but not by the buyers. Each buyer is interested in a single unit for which he has value v_i (his private information). The values are also i.i.d. There is a discount factor δ for the buyers' utilities.[17] Compared to the Crémer et al. model, in the Board and Skrzypacz model the seller has multiple units of the same object rather than one, the set of potential buyers is infinite, and the process of entering the market is random and exogenously given (while it is controlled strategically by the seller in the Crémer et al. model). The values are assumed to be i.i.d. while Crémer et al. allow for interdependent values.

The authors design a selling mechanism in which a buyer, when entering the market and observing his value v_i, declares his value to be $\tilde{v}_i$ and the mechanism determines (probabilistically) the allocation and the transfers. They prove that the optimal (profit-maximizing) allocation is obtained by the following mechanism: at time t with k units left to sell, the seller sells the next unit to the highest-valued agent with a value exceeding a certain *cutoff* x_t^k. Interestingly, the cutoff level x_t^k depends on the time remaining and the number of items for sale but not on the number of agents who have entered in the past and their values. The endogenously determined cutoffs x_t^k are decreasing both in t and k. The cutoff x_t^k is determined by an equation asserting that the seller is indifferent between selling to the agent with the cutoff value today and waiting one more period.

The implementation is achieved by setting, for $t < T$, prices p_t^k such that the 'cutoff agent' with value x_t^k will be indifferent between buying and not buying. At the last period $t = T$, the seller allocates the items to the k-highest-value buyers, subject to these values exceeding the static monopoly price. Note that it is sufficient to base these prices only on the items remaining. For example, the price for an item in period 3 would be the same if three items were allocated in period 1 and one item in period 2 or two items were allocated in each period.

For the case of a single item ($K = 1$), this allocation can be implemented by a sequence of second-price auctions with pre-determined reserve prices R_t. For the multiple-item case ($K > 1$), a sequence of second-price auctions cannot implement the optimal allocation since the optimal allocation may have more than one item allocated in the first period. The optimal allocation in period 2 depends upon how many items were allocated in period 1. Hence, if a second-price auctions with reserve prices were to be used to implement the optimal allocation, then the second-period reserve prices would need to be a function of how many items were sold in the first period (or equivalently the number of items remaining to be sold). Thus, as with prices in the optimal allocation, the mechanism needs to have reserve prices that depend upon the period and the number of remaining items, that is, R_t^k.

Board and Skrzypacz also consider the continuous time version of the model in which buyers enter the market according to a Poisson process at arrival rate λ, and instantaneous discount rate r. The optimal mechanism becomes simpler to implement in this case since at any time instant t at most one unit is sold. The cutoff values x_t^k are determined as before but now by the continuous form of the indifference condition for the seller. The implementation is made by setting a take-it-or-leave-it selling price p_t^k that makes the agent with value equal to the cutoff x_t^k indifferent between buying at time t and waiting.

[17] For a model of allocating heterogeneous objects to impatient agents arriving sequentially (with privately known characteristics) see Gershkov and Moldovanu (2009).

4.2 Repeated Ascending-Price Auctions

Said (2008) highlights the difference between static and dynamic auctions by demonstrating that even with a single object for sale and a random arrival of buyers (similar to the Board and Skrzypacz model) repeated sealed-bid second-price auctions may not lead to an efficient allocation.

In his leading example, two buyers are present in the market with values $v_1,\ v_2 \in [0,1]$, and w.l.o.g. assume $v_1 > v_2$. A third potential buyer with value $v_3 \sim F$, where F is the uniform distribution on $[0,1]$, may enter the market with probability $q \in [0,1]$. Assume that v_1 and v_2 are commonly known by the players but the value v_3 of the new entrant is his private information. Each buyer wishes to purchase exactly one unit of the object. There are three units that are sold via a sequence of three second-price auctions in which the buyers' bids are revealed after each round and subsequent rounds are discounted by δ per period. Note that at the first round, buyers 1 and 2 do not know whether buyer 3 has entered or not but, as the bids of the first round are announced, this becomes known to them in the second and third rounds. The efficient allocation should have the highest-valued buyer receiving the unit in the first round, the second-highest-valued buyer receiving the unit in round 2, and the third-highest-valued buyer (if there is one) receiving the last unit.

To see why this situation may lead to inefficiency, let us sketch the equilibrium of the specific case (adapted from Said, 2008) in which $v_1 = \frac{2}{3},\ v_2 = \frac{1}{3},\ \delta = \frac{9}{10}$, and $q > 0$. Buyer 1 knows that even if he loses the first round there is a probability of $1-q$ that he will be alone in the next round, will receive an item at price zero, and obtain a utility of δ times his value. Therefore, his bid in the first round will be at most $(1-(1-q)\delta)v_1 = (0.1+0.9q)v_1$, which is considerably lower than his value for small q. The same holds for buyer 2.

However, unlike buyers 1 and 2, buyer 3 knows whether or not he has entered the market in round 1. If entering in round 1, buyer 3 knows that he cannot pay zero for the item in round 2 since he would face another buyer. Since in any equilibrium, in round 1, buyer 1 outbids buyer 2, in round 2 buyer 3 faces buyer 2 who bids $(1-\delta)v_2$ (since the round 3 price is zero). From this, if buyer 3's value is larger than $\frac{1}{3}$, then his potential round 2 profit is $v_3 - \frac{1}{3}(1-\delta)$. Therefore, in round 1, buyer 3 will bid $v_3 - (v_3 - \frac{1}{3}(1-\delta))\delta = 0.1v_3 + 0.03$. Consequently, for small enough q, a buyer 3 entering the market with value slightly less than $v_1 = 2/3$ will win the first item, which is inefficient. This happens when $0.1v_3 + 0.03 > (0.1+0.9q)v_1$, for example, $v_3 = 0.6$ and $q = 0.03$.

The author suggests that this result is driven first by the fact that the future is discounted and hence the order in which objects are allocated matters. But more importantly, there is a fundamental information asymmetry: while the values of buyers 1 and 2 and their presence are commonly known, the presence of the new entrant and his value are his private information. With this insight, Said shows that efficiency is recovered if the sealed-bid second-price auction is replaced by its open-auction counterpart, namely, the open ascending price auction (while there is still one object per round). The intuition is that with this auction more information is revealed to make up for the initial asymmetry of information.

In a subsequent paper, Said (2012), building on this insight, proves similar results; namely, the repeated sealed-bid second-price auction is inefficient while an appropriate repeated ascending auction is efficient. However, this cannot be viewed as a 'more general' result since the model here differs from the one in Said's previous work. In Said's (2012) model, the author considers an infinite-horizon discrete-time process with a single seller. In each period $t \in \mathbb{N}$, the seller has K_t units of a homogeneous and indivisible good available for sale. The amounts K_t are independently distributed according to μ_t. Objects are perishable: any object that is not allocated or sold perishes at the end of each period. Each period t begins with the arrival of N_t buyers. The variables $\{N_t\}_{t=1}^{\infty}$ are independent with distributions $\lambda_t;\ t = 1, 2, \ldots$. Each buyer i present in the market wishes to obtain a single unit and is endowed with a privately-known value v_i for that single unit. The values v_i are independent and with distributions F_i. In addition, buyers may exogenously depart from the market (and never return) after each period, where the (common) probability of any buyer i surviving from period t to $t+1$ is $\gamma_t \in [0, 1]$. Otherwise, buyers remain present on the market until they obtain an object. Buyers are risk neutral, with quasilinear and time-separable preferences. All buyers, as well as the seller, discount the future with the common discount factor $\delta \in (0, 1)$. Finally, it is assumed that every buyer is aware of the presence of all other buyers in the market.

In this model, Said shows first that a sequence of second-price sealed-bid auctions with no bid disclosure does not admit an efficient equilibrium. Again, the main reason is that the dynamics creates asymmetry of beliefs between the incumbents and the entrants: the incumbents of a certain round are the losers of the previous round, and each knows his bid and the information disclosed at the end of the previous round (e.g., selling prices). Thus, an incumbent has information on the other incumbents that the new entrant at this period does not have. Also, even when private values are independent, market dynamics and repeated competition generate interdependence: the value of winning an object in a round compared to not winning it in that round depends upon the opportunity cost of not participating in subsequent rounds. This opportunity cost depends upon the competitors' values. Thus, the net values are interdependent.

Here again, efficiency can be restored, by replacing the sealed-bid second-price auction by its open-auction counterpart, namely, the ascending auction. In such an auction, a price clock rises continuously from zero and buyers drop out of the auction at various points. The auction ends as soon as the number of active buyers is K_t or less. Each one will then receive a unit and pay the closing price. If at the beginning of the round there are less than K_t buyers, the auction ends immediately at price zero. Here again, in principle, losers from the previous period seem to have an informational advantage over the new entrants of that period. However, it turns out that the information content of the dropping time in the previous round will be available again in the present period bidding strategy, and this time to all competitors: incumbents and entrants. In other words, in the ascending auction bidders are engaged in a revelation process and at each round a player reveals his information again. Technically, the game has an equilibrium in memory-free strategies. Indeed, the author provides such equilibrium-bidding strategies in the sequential ascending auction and proves that they generate the same allocations and transfers as in the truthful equilibrium of the *dynamic pivot mechanism* developed by Bergemann and Välimäki (2010) as the dynamic version of the VCG mechanism.

5 Externalities in Single-Object Auctions

Externalities in auctions are where a player cares not only about winning the object but also about who wins it in case of losing.[18] Two well-suited examples are given by Jehiel et al. (1996). After the breakup of the Soviet Union, Ukraine was left with 176 intercontinental nuclear missiles. Although neither the United States nor Russia had any interest in these old-fashioned missiles, they each paid Ukraine (in various ways) about one billion dollars to dismantle this arsenal. The second example is when China agreed in 1994 not to sell its M-9 and M-11 missiles to Arab countries; as a reward, the United States agreed to lift its one-year-old embargo on satellite exports to China. Although these are not proper auctions, they are cases in which a party is willing to pay, not to have an object, but to prevent a third party from having it. More generally, these are cases in which 'whoever wins' may affect the downstream interaction between the players. Typical economic examples are auctioning of a cost-reducing patent to oligopolists, auctioning spectrum licenses to incumbents, and a retailer competing in an auction for a neighboring plot of land against a competitor who, if he wins, intends to build a polluting factory on it.

5.1 A General Social Choice Model

Major contributions to this topic are by Jehiel and Moldovanu in numerous papers including a review (Jehiel and Moldovanu, 2006). In this review, they make the distinction between *allocative externality* due to the final allocation of the object and *informational externality* due to information held by the other competitors in the auction. They provide a general social choice mechanism in which multi-object auctions with both types of externalities, as well as combinations of the two and complementarities, are special cases. In this model there are $N+1$ agents, indexed by $i = 0, \dots, N$ (agent 0 is the seller) and K social alternatives, indexed by $k = 1, \dots, K$. Each agent gets a private ℓ-dimensional signal about the state of the world $\theta^i \in \Theta^i \subset \mathbb{R}^\ell$. The vector of all private signals is $\theta = (\theta^0, \dots, \theta^N) \in \Theta = \bigtimes_{i=0}^{N} \Theta^i$. Agents have quasi-linear utility functions $u^i(k; \theta; t^i) = v_k^i(\theta) + t^i$ that depends on the chosen alternative k, on the vector of private signals θ, and on monetary payments t^i.

For an auction allocating a set M of (heterogeneous) objects, a partition of M is $P = (P_1, \dots, P_N)$, where P_i is the bundle of objects allocated to agent i. In this model, the number of alternatives K is equal to the number of possible partitions. Thus, we can write utility as $v_P^i(\theta) + t^i$. In addition, agents receive a signal for each possible partition, and so $\ell = K$. Hence, we can also write, for each partition P, agent i's signal as θ_P^i.

The pure private value is obtained when both the signal θ_P^i and the value $v_P^i(\theta)$ depend only on the bundle P_i allocated to the agent; that is, for any two partitions P and P', $P_i = P_i'$ implies $\theta_P^i = \theta_{P'}^i \equiv \theta_{P_i}^i$ and $v_P^i(\theta) = v_{P'}^i(\theta) \equiv v_P^i(\theta_{P_i}^i)$. The pure allocative externalities case is obtained when $v_P^i(\theta_P) \equiv v_P^i(\theta_P^i)$; that is, while the value of an agent depends on the whole distribution of bundles (allocative externalities), it depends only on his own signal and not on the signals of

[18] Externalities can extend beyond the identity of the winner such as in the case where there is a positive spillover from the expenditure (see D'Aspremont and Jacquemin, 1988).

the other agents (no informational externalities). In the case of pure informational externalities and no allocative externalities, the agent cares only about his bundle and about the information of other agents about this bundle; that is, for any two partitions P and P', $P_i = P'_i$ implies $v^i_P(\theta) = v^i_{P'}(\theta) \equiv v^i_{P_i}(\theta)$ for all θ. In addition, since $v^i_{P_i}(\theta)$ depends on the signals of other agents only to the extent that they concern information 'about' the bundle P_i, the signals of any agent j can be partitioned into equivalence classes (with respect to their effect on $v^i_{P_i}(\theta)$) indexed by P_i; $\theta^j = (\theta^j_{P_i})_{P_i \subseteq M}$ and we can write $v^i_{P_i}(\theta)$ as $v^i_{P_i}(\theta^0_{P_i}, \ldots, \theta^N_{P_i})$. Finally, the dependence of the value v^i_P on the partition P accommodates complementarities and substitutabilities in the usual way.

5.2 Complete Information

Considering first the issue of allocative externalities, it turns out that traditional auction formats need not be efficient, and they may give rise to multiple equilibria and strategic non-participation. The first observation is that even in the simplest case of a single object and complete information, the very notion of value becomes *endogenous*: if we denote by v^i_i the value of agent i when i wins the object (e.g., a license of a cost reducing patent) and by v^i_j (typically negative) the externality exerted on agent i if agent j wins the object, then the net value for agent i for winning the object compared to losing the auction is $v^i_i - v^i_j$ if he expects agent j to win or $v^i_i - v^i_k$ if he expects agent k to win in the case where he loses the auction. Jehiel and Moldovanu (1996) demonstrate this point and show that, even in a simple second-price auction, not only is there no dominant strategy but there can be multiple equilibria with different allocations of the good. They illustrate this possibility in the following example: $N = 3$ and $v^i_i = v$ for all i, the externality terms are $v^2_1 = v^1_2 = -\alpha$, $v^3_1 = v^3_2 = -\gamma$, and $v^1_3 = v^2_3 = -\beta$, where $\alpha > \gamma > \beta > 0$. In one equilibrium, agents 1 and 2 compete (being 'afraid' of each other winning the object) and one of them wins and pays $v + \alpha$. In a second equilibrium, agent 3 competes with agent 1 (or with 2) and wins the object paying $v + \beta$ (since $\gamma > \beta$, agent 3 is more afraid from 1 than 1 is afraid of 3).

Another phenomenon pointed out by Jehiel and Moldovanu (1996) is that with the presence of allocative externalities there is *strategic non-participation*. By staying out, an agent may induce an outcome that turns out to be more favorable to him than the outcome that would have arisen if he had participated. An example where firms are the agents (taken from Jehiel and Moldovanu, 2006) has $N = 3$, where firms 1 and 2 are incumbents, while firm 3 is a potential entrant. The incumbents do not value the object per se (such as an innovation that is irrelevant for their production technologies): $v^1_1 = v^2_2 = 0$. Moreover, $v^2_1 = v^1_2 = 0$. The entrant has value $v^3_3 = v$, and it creates a negative externality of α on each of the incumbents, which is greater than his value; $v^1_3 = v^2_3 = -\alpha$ and $\alpha > v$. Consider a second-price auction. Clearly it is the interest of the incumbents to avoid entry by bidding slightly above v. However, there is a free-rider problem between the two incumbents: each one would prefer to let the other deter the entrant. Indeed, in this example, in any equilibrium (there are three) at least one incumbent firm chooses not to participate with positive probability.

5.3 Incomplete Information

Models with allocative externalities and incomplete information give rise to various situations that depend on the nature of the private information of the agents. When the private information of agent i is the allocative externality v_j^i caused to him by other agents, the resulting situation has no informational externalities. When the private information of agent i is the allocative externality v_i^j that he causes to others, the environment becomes that of private interdependent values. (Of course, having only partial information about these two types of externalities is also conceivable.)

As we just said, models of incomplete information and allocative externalities are closely related to the research on auctions with interdependent values. For example, Jehiel and Moldovanu (2000) consider a two-bidder second-price auction in which each of the private signals is the corresponding private value, that is, $\theta^i = v_i^i$, and the externalities are functions of both signals, that is, $v_j^i = v_j^i(\theta^i, \theta^j)$. With no reserve prices, this is analoguous to the Milgrom and Weber (1982) model with interdependent values $v^i = \theta^i - v_j^i(\theta^i, \theta^j)$, $i = 1, 2,\ j = 3 - i$. However, with the presence of a reserve price r, there is the possibility the seller keeping the object and hence the 'net value' of bidder i for winning the object is either θ^i if the alternative is that the seller keeps it, or $\theta^i - v_j^i(\theta^i, \theta^j)$ if the alternative is that bidder j wins it. Jehiel and Moldovanu (2000) find that in this case there is discontinuity in the bidding range in equilibrium. This can happen when the reserve price is binding, that is, when the value including the externalities may be less than the reserve price. For example, take the symmetric case of two buyers with values in $[\underline{v}, \overline{v}]$ with constant (negative) externality $-e$. In a second-price auction with a reserve price r satisfying $\underline{v} + e \leq r \leq \overline{v} + e$, a (basically unique) symmetric equilibrium strategy is bidding $v + e$ when $v \geq r$ and bidding 0 when $v < r$. In other words, there is no relevant bid in the interval $(r, r + e)$. In such a case the optimal reserve price may well be below the seller's value for the object. For positive externalities they show that entry fees and reserve prices need not lead to the same revenue, in contrast to the case with no externalities. Variants of this model were studied by Moldovanu and Sela (2003), Goeree (2000), Das Varma (2003), and Molnar and Virag (2004).

Fan et al. (2013, 2014) have a model where a patent is auctioned between two firms engaged in a Cournot duopoly where the winner of the auction receives the patent for a fixed fee and then collects royalties from the loser. Since the fixed fee and the royalties depend upon information obtained by bidding in the auction, the bids in this auction have a signaling content that affects the downstream competition. Hence, the externality is bid-dependent.

On the issue of revenue-maximizing auctions, Jehiel et al. (1999) consider a model in which the private information of bidder i is his value v_i^i and all externalities v_j^i caused to him (he receives a multidimensional signal). They find that a second-price auction with an appropriately determined entry fee is the revenue-maximizing mechanism in a class of mechanisms where the object is always sold. Similar models are studied by Das Varma (2002) and Figueroa and Skreta (2004). Caillaud and Jehiel (1998) study the possibility of collusion in the presence of negative externalities.

Informational externalities were studied at early stages by Wilson (1967) and Milgrom and Weber (1982) in single-object auctions with symmetric bidders. The focus was later shifted to multi-object auctions with asymmetric bidders. The main approach of most studies is that of mechanism design, investigating the issues of incentive compatibility, revenue equivalence, and revenue and welfare maximization in *direct-revelation mechanisms*. The breadth, depth, and technical aspects of this literature make a systematic detailed presentation of the main results not feasible within this section. For further reference, we suggest the works of Maskin (1992), Krishna and Maenner (2001), Jehiel and Moldovanu (2001), Jehiel and Moldovanu (2006), and the numerous references listed there.

6 Auctions with Resale

In the standard private-value auction model, if a buyer wins the auction it is for him worth his private value. This is even the case if another buyer has a higher private value. In this section, we look at what happens when there is the possibility of resale by the winner of the auction to another buyer with a higher value. For this analysis, there must be two stages in the game. In the first stage, the object is sold by the seller via an auction. In the second stage, the winning buyer has the option of selling the object to one of the other buyers. Here we need to specify both what information is revealed after the first stage and the selling procedure of the second stage. For example, are bids revealed after the first stage and does the winning buyer have full bargaining power in the second stage? The main issue to address is the effect of the introduction of resale on the outcome of the auction, specifically, on the allocation of the object and on the revenue.

6.1 First-Price and Second-Price Auctions

Under symmetry, adding this possibility of resale does not change the equilibrium in a first-price auction or the symmetric equilibrium (bid your value) in a second-price auction. This is shown when the bids are revealed (Haile, 2003), when the values are revealed (Gupta and Lebrun, 1999), and when the losing bids are not revealed (Hafalir and Krishna, 2008). Under asymmetry, this is no longer the case and revenue equivalence between first-price and second-price auctions breaks down. Hence, it is important to study resale in asymmetric auctions.

We begin with an example adapted from Hafalir and Krishna (2008). A weak buyer has a value drawn from the uniform distribution on $[0, 1]$ and a strong buyer has a value drawn from the uniform distribution on $[0, 4]$. Consider the first-price auction with resale where the winner in the auction in the first stage, without seeing the losing bid, can make a take-it-or-leave-it offer to the loser. The equilibrium bid functions are $\frac{5}{4}v_w$ for the weak buyer and $\frac{5}{16}v_s$ for the strong buyer. If the weak buyer wins in the first stage, he would make a take-it-or-leave-it offer to the strong buyer of double what he paid, $\frac{5}{2}v_w$. Notice that for both buyers the distribution of bids is uniform on $[0, \frac{5}{4}]$. Revenue is thus $\frac{2}{3} \cdot \frac{5}{4} = \frac{5}{6}$. In a second-price auction with resale there is an equilibrium where buyers bid their values (and, hence, there is no resale), revenue is $\frac{3}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{3} = \frac{3}{8} + \frac{1}{12} = \frac{11}{24}$, which is less than in the first-price auction with resale. By comparison, in a first-price auction without resale, the equilibrium bid functions are:[19]

$$b_w(v_w) = \frac{16 - 4\sqrt{16 - 15v_w^2}}{15v_w}, \quad b_s(v_s) = \frac{4\sqrt{16 + 15v_s^2} - 16}{15v_s}.$$

Revenue in this equilibrium is approximately 0.59, which is lower than the revenue in the equilibrium of a first-price auction with resale.

Hafalir and Krishna show that these results generalize to any two distributions of values. Namely, in the first-price auction with resale the bid distributions are identical for the two buyers and the revenue of the first-price auction with resale is always higher than that of the second-price auction where the buyers bid their values. Thus, with the introduction of resale, there is a ranking between a first-price auction and a second-price auction that doesn't exist without resale (provided we only consider the bid-one's-value equilibrium in the second-price auction).[20]

The model of Hafalir and Krishna was studied earlier by Garrat and Troger (2006) who consider the special case in which the value distribution of one of the buyers is degenerate at 0. In other words, one of the buyers is a speculator who has no value for the object per se and this is known to the other buyer. Their main finding is that speculators can affect the equilibrium in auctions with resale. In the first-price auction with resale they find that the speculative bidder uses a mixed-strategy equilibrium and makes zero profit but influences the equilibrium, compared to a first-price auction without resale, by increasing the selling price (and hence the revenue) and changing the allocation to a possibly inefficient one. In a second-price auction there are multiple equilibria where sometimes the speculator makes a profit. Again, the speculator changes the equilibrium by increasing revenue and decreasing efficiency. The focus of Garrat and Troger is on the role of speculation in auctions with resale while the focus of Hafalir and Krishna is on the ranking of the seller's revenue, but as we see in our above example of Hafalir and Krishna, the weak buyer bids $\frac{5}{4}v_w$, which is above his value, and so speculation is present in first-price auctions even in the non-degenerate case.

6.2 Seller's Optimal Mechanism

Zheng (2002) and Lebrun (2012) search for the optimal mechanism when resale is permitted. Myerson (1981) finds the optimal mechanism (i.e., maximizing seller revenue) when resale can be prevented. For example, assume that buyer 1 has a value uniform on $[0, 1]$ and buyer 2 has a value that is drawn uniformly from $[0, 2]$. The optimal mechanism should allocate the object to the buyer with the highest virtual surplus,[21] which means that buyer 1 should get the object if $v_1 \geq v_2 - \frac{1}{2}$. This can be implemented by running a second-price auction with a minimum bid of 1 but where buyer 1 receives a coupon that will reimburse him for his payment up to $\frac{1}{2}$. For instance, if buyer 1 bids $\frac{3}{2}$ and buyer 2 bids $\frac{5}{4}$, then buyer 1 will win the auction at a price of $\frac{5}{4}$, yet pay $\frac{5}{4} - \frac{1}{2} = \frac{3}{4}$.

[19] These were from Griesmer et al. (1967). Also, see Kaplan and Zamir (2012) for the general solution to the uniform case.

[20] See Section 2.5 for examples by Maskin and Riley (2000a) where the revenue between a first-price auction and a second-price auction can be ordered in either way.

[21] Virtual surplus for a buyer with value v drawn from the distribution $F(v)$ is $v - \frac{1-F(v)}{f(v)}$.

For such a mechanism, buyer 2 will bid his value (or drop out if his value is below 1) and buyer 1 will bid his value plus $\frac{1}{2}$ (or drop out if his value is below $\frac{1}{2}$).

Once resale is introduced, this mechanism will no longer work since if buyer 1 wins, he will then try to sell the object to buyer 2, increasing his surplus. Zheng (2002) proposes a mechanism that will work under resale. The buyers bid for the object and the winner pays some function of his bid (a form of a first-price auction). This function should be set such that buyer 1 bids $v_1 + 1$ and buyer 2 bids v_2. If buyer 1 wins, then he will believe that buyer 2's value is uniform on $[0, v_1 + 1]$ and will make a take-it-or-leave-it offer at $v_1 + \frac{1}{2}$ which would result in the buyer with the highest virtual surplus getting the object as in the optimal mechanism without resale (since the allocation is the same as in the optimal mechanism, the revenue is equivalent). The key insight into this mechanism is that the buyer winning the auction sets a resale auction such that the object is resold whenever the current owner does not have the highest virtual surplus. The winner of the resale auction will do likewise (as will any subsequent winners). While the necessary conditions for an optimal mechanism in Zheng (2002) are restrictive, Mylovanov and Tröger (2009) are able to relax them such that even for three or more bidders there are several examples of distributions that satisfy them (besides the uniform distribution).

Lebrun (2012) shows that optimal allocation for revenue is an equilibrium of a standard second-price auction with resale (albeit with bidder-specific entry fees). This seemingly contradicts Hafalir and Krishna (2008), who prove that a second-price auction is inferior to a first-price auction with resale; however, rather than focusing on the bid-your-value equilibrium, Lebrun takes one of the inefficient equilibria present in Garratt and Troger (2006). In order for this to be optimal, the beliefs of the winner about the value of the loser must be very specific. In order for them to be so, the buyers must use particular mixed-strategies in the first period.

6.3 Further Results

Gupta and Lebrun (1999) consider a different model of two-buyer first-price auctions with resale in which they make the assumption that the values are revealed after the first stage.[22] Under this rather strong assumption, the authors provide an equilibrium for any second-stage sale price function $\pi(v_1, v_2)$, for example, $max\{v_1, v_2\}$. If the asymmetric value distributions are F_1 and F_2, then the first-price auction with resale will have the same bid distribution as the symmetric first-price auction (without resale) with value distribution G given by $G^{-1}(q) = \pi(F_1^{-1}(q), F_2^{-1}(q)),\ q \in [0, 1]$. As a consequence, under symmetry the equilibrium doesn't change under resale.

Bikhchandani and Huang's (1989) early contribution to auctions with resale is one of the few that deal with common values. They theoretically investigate treasury bill auctions where the securities can be resold, and find that revenue is higher using a discriminatory auction (similar to first price) rather than a uniform-price auction (similar to second price).

Garratt et al. (2009) show that resale can help reach a collusive equilibria in second-price auctions. With two bidders with values drawn independently from the uniform distribution on

[22] Gupta and Lebrun also assume that the support of the value distribution of both buyers is the same.

$[0, 1]$, bidders can flip a coin (or use a public sunspot) where, depending upon the outcome, one of the bidders bids 1 and the other bids 0. A bidder with a value of 1 receives on average 1/2. However, with a different distribution on $[0, 1]$, the expectation can be lower than 1/2, meaning that a bidder with value 1 would prefer the bid-one's-value equilibrium. With resale, the bidder with a value of 1 will receive more than 1/2 on average since when he bids 0, he will still receive the object by buying it on the resale market. The extra profit is enough to make this collusive equilibrium worthwhile.

In this section, we mentioned just some of the recent results on auctions with resale. In Haile (2003), the reason motivating a resale is not the inefficiency in the primary auction but rather information gained afterwards. Additional contributions of note are Bose and Deltas (2007), Pagnozzi (2007, 2009), Harfalir and Krishna (2009), Cheng and Tan (2010), Lebrun (2010a, 2010b), Cheng (2011), Che et al. (2013), Virág (2013), and Xu et al. (2013).

Finally, we note new work on an environment in which the seller cannot commit to not reattempt to sell the object if he fails to do so in the auction (see Vartiainen, 2013, and Skreta, 2013). This topic is somewhat related to auctions with resale since in both cases, the seller cannot prevent post-auction trade.

7 All-Pay Auctions

A contest is a situation in which players exert effort in an attempt to win a prize. Greater effort increases the chances of winning. All the efforts are sunk while only the winner gets the prize (hence the name *all-pay* as all participants pay a cost). In the literature, contests have been used to describe environments including patent races, sport competitions, court cases, lobbying, political campaigns, promotions, military, and rent-seeking. Explicitly, contests have been used as a tool to spur innovation since the 1700s. The Longitude Prize of £20,000 established by the British Parliament in 1714 induced John Harrison to invent the marine chronometer, for which he won the prize in 1765 (see Sobel, 1996). Motivated by the Orteig Prize, Charles Lindbergh became the first person to fly non-stop solo across the Atlantic in 1927 (Berg, 1998). The self-financed Feymnann prize inspired nanotechnology. Such contests have gained more traction in recent years. For example, the X-Prize Foundation created a number of high-profile prizes (the *Ansari Space Prize* to privately launch a reusable manned spacecraft, the *Lunar Prize* to privately land an unmanned probe on the moon, and the *Tricorder Prize* for a palm-sized instant medical evaluation tool), a number of companies offer (or offered) design prizes (*Topcoder* uses computer programming contests to generate needed code, and *Netflix* held a contest to help design a better movie recommendations system), and the US military holds the DARPA challenges (races) to improve robotic designs. These real-world contests have sparked increased academic interest in the field.

An important ingredient in describing a contest with the set of participants $I = \{1, \ldots, n\}$ is the contest success function, which takes the efforts $\{x_i\}_{i=1}^{n}$ of the agents and converts them into each agent's probability of winning: $P_i : \mathbb{R}^n \to [0, 1]$. The (expected) utility of a risk-neutral player

i with a value of winning the prize v_i and a cost of effort $c_i(x_i)$ is

$$P_i(x_1, \ldots, x_n)v_i - c_i(x_i).$$

There are several popular contest success functions in the literature. The Generalized Tullock (1980) success function is

$$P_i(x_1, \ldots, x_n) = \frac{x_i^r}{\sum_j x_j^r} \tag{1}$$

where $r > 0$ is a parameter.

The Lazear-Rosen (1981) success function is

$$P_i(x_1, \ldots, x_n) = P(x_i + \epsilon_i \geq \max_{j \neq i}\{x_j + \epsilon_j\})$$

where ϵ_i are random noise variables with the same distribution.

Perhaps the most "natural" is the auction success function where the prize is awarded (with probability 1) to the competitor that exhibited the greatest effort, and in case of tie, the prize is awarded randomly to one of the maximizers with equal probability to each. A contest with this success function is equivalent to an all-pay (first-price) auction and the auction success function is mathematically the limit of the Generalized Tullock success function (1) when $r \to \infty$:

$$P_i(x_1, \ldots, x_n) = \begin{cases} \frac{1}{\#\{j | x_j = \max_j\{x_j\}\ \}} & \text{if } x_i = \max_j\{x_j\}, \\ 0 & \text{otherwise.} \end{cases}$$

In this section, we will focus on the theoretical literature on all-pay auctions, which is where the contest and auction theory literature overlap. This literature uses two main environments: complete information and incomplete information about the bidders' values of the prize or the bidders' costs. We will begin by describing core developments in these two environments before delving into further advances. Unlike first-price auctions, many research questions have non-trivial results under complete information and hence many topics are covered in the literature using both environments' tools. Intuition gained from the complete-information environment usually carries over to the incomplete-information environment. Therefore, in many cases we cover the complete-information case in more detail. When using the auction terminology we will refer to competitors also as bidders or players and to the effort x_i as the bid of player i.

7.1 Complete Information

Baye et al. (1996) build upon the work of Hilman and Samet (1987), Hillman (1989), and Hillman and Riley (1989) to characterize the all-pay auction with complete information with two or more players. They allow the values for the prize to be asymmetric (it is possible that $v_i \neq v_j$). In the complete-information models, all values $v_i,\ i \in I$, are common knowledge among all participants. For each player i, $c_i(x_i) = x_i$.[23]

[23] For the case of $c_i(x_i) = c_i \cdot x_i$, the same analysis can be applied by redefining values such that $\tilde{v}_i = v_i / c_i$.

Before describing the equilibria, it is useful to describe several properties that hold in equilibrium, and to see why. First, the equilibrium involves mixed strategies. In a pure-strategy equilibrium each player i chooses an effort level x_i with probability 1. If in such an equilibrium there are no ties with the winning bid, then the winning bidder i can profitably deviate to $(x_i + x_j)/2$ where x_j is the second-highest bid. If there is a tie at the winning bid, then a winning bidder i can choose $x_i + \epsilon$ and gain at least $\frac{v_i}{n} - \epsilon$; hence, there is no pure-strategy equilibrium.

Second, in equilibrium any player that chooses zero with positive probability earns zero profits. This is because there must be at least one player who chooses a strictly positive bid with probability one. If not, a player bidding zero would be able to have a discrete jump in profits by increasing his bid to some $\epsilon > 0$.

Third, under symmetry (when $v_i = v$ for all i) all players make zero profits in equilibrium (which is not necessarily symmetric). If one player makes a positive profit, then all players must make a positive profit since any other player can bid at the top of the support of the player making the positive profit (which must be less than v for him since he is making a positive profit). All players cannot be making a positive profit since at least one player bidding at the bottom of the union of supports of all strategies must be making zero profit (because if all players choose this point with positive probability a player will be able to discretely increase profit by increasing his bid by an arbitrarily small amount).

The following describes the equilibria in more detail.

If $v_1 = \ldots = v_m > v_{m+1} \geq \ldots \geq v_n$ with $m \geq 2$, then there is an equilibrium where the first m bidders bid symmetrically according to $F(x) = \left(\frac{x}{v_1}\right)^{\frac{1}{m-1}}$ and the remaining bidders bid zero and all bidders make zero profit. For $m \geq 3$, there is also a continuum of equilibria that are revenue equivalent in which a subset of at least two of the first m bidders randomize continuously on $[0, v_1]$. The remaining of the first m bidders randomize continuously on $[b_i, v_1]$ and bid 0 with positive probability if $b_i > 0$. If two or more players randomize continuously over a common interval, their CDFs are identical over that interval.

If $n = 2$ and $v_1 > v_2$ or if $n > 2$ and $v_1 > v_2 > v_3 \geq \ldots \geq v_n$, then there is a unique equilibrium. In that equilibrium, player i chooses his bid randomly according to the cumulative distribution F_i where

$$F_1(x) = \frac{x}{v_2},\ 0 \leq x \leq v_2,\ \text{and}\ F_2(x) = \frac{x + v_1 - v_2}{v_1},\ 0 \leq x \leq v_2, \tag{2}$$

$$F_i(x) = 1 \text{ for } x \geq 0,\ i \geq 3.$$

Notice that only the two players with the highest two values are active, and all players except player 1 are making zero profit, since all of them bid zero with positive probability (including player 2, since $F_2(0) > 0$). Since each player must be indifferent among all bids in his support and all players $i > 1$ bid zero with positive probability while player 1 bids strictly above zero with certainty, player i (> 1) bidding zero always loses and thus has no positive profit anywhere in his support.

If $v_1 > v_2 = \ldots = v_m > v_{m+1} \geq \ldots \geq v_n$ with $m \geq 3$, then there exists a continuum of equilibria that are not necessarily revenue equivalent to each other and these are parameterized by $\{b_i\}_{i=2}^m \in [0, v_2]^{m-1}$ where $b_i = 0$ for a least one i. Player 1 randomizes continuously on the interval $[0, v_2]$ and players $i \in \{m+1, \ldots, n\}$ bid zero. The remaining players {2, ..., m} choose 0 with probability α_i and randomize continuously on $[b_i, v_2]$ where $\Pi_{i=2}^m \alpha_i = (v_1 - v_2)/v_1$. Each of these equilibria has the property that for any $i,\ j \in \{2, \ldots, m\}$ and $x \geq \max\{b_i,\ b_j\}$, we have $F_i(x) = F_j(x)$; that is, the CDFs of i and j are the same in the intersection of their supports.

For an example where revenue is not equivalent between two equilibria, take the case of $m = n = 3$ with $v_1 = 2$ and $v_2 = 1$. One equilibrium (corresponding to $b_2 = 0$ and $b_3 = 1$) has

$$F_1(x) = x,\ F_2(x) = \frac{x+1}{2},\ 0 \leq x \leq 1,\ F_3(0) = 1.$$

While another equilibrium (corresponding to $b_2 = b_3 = 0$) has

$$F_1(x) = x\left(\frac{2}{x+1}\right)^{\frac{1}{2}},\ F_2(x) = F_3(x) = \left(\frac{x+1}{2}\right)^{\frac{1}{2}},\ 0 \leq x \leq 1.$$

The first generates revenue of 0.75 while the second generates revenue of $\frac{5-2\sqrt{2}}{3} \approx 0.724$.

7.2 Incomplete Information

In the incomplete information models of all-pay auctions, values are drawn from commonly known distributions, while a player's value is his private information. Thus, this is a game of incomplete information (a Bayesian game) in which the type of a player i is his value v_i.

The symmetric all-pay auction with i.i.d. values drawn from distribution F has a unique equilibrium in which each player bids

$$b(v) = F(v)^{n-1}v - \int_0^v F(\tilde{v})^{n-1} d\tilde{v}.$$

The asymmetric two-bidder case with common support was shown by Amann and Leininger (1996) also to have a unique equilibrium. They also showed uniqueness and existence for more general winning payments of the form $(1-\lambda)x_i + \lambda x_{-i}$ for $\lambda \in [0, 1)$ (while the losing payment is still x_i).

Parreiras and Rubinchik (2010) examine the incomplete information case where there are three or more heterogeneous bidders. They find that (unlike the two-bidder case) there can be cases where an individual bidder will either not bid the entire range of equilibrium bids (including having gaps in his support) or even completely drop out of the contest and bid 0.

7.3 Multiple Prizes

In this section, we consider contests where the prize is a divisible amount of money and address the issue of how to divide the prize among the contestants according to their place. Galton (1902) proposed that one should divide prize money between first and second places in a ratio of 3 to 1. His objective (presumably out of fairness) was to make the ratio of the prizes equal to the ratio of the difference between the first-place and third-place competitors' efforts to the difference between the second-place and third-place competitors' efforts, notably ignoring any incentive effect of the prizes. This is a ratio of the difference between the first-order and the third-order statistic to the difference between the second-order and the third-order statistic, and when efforts are i.i.d. from a normal distribution this ratio is equal to 3. (Lazear and Rosen, 1981, note the loose connection to the concept of marginal product.)

Glazer and Hassin (1988) is the first paper to analyze this problem of the allocation of prize money in all-pay auctions with complete information and find that with symmetry and risk aversion, the expected sum of the efforts is maximized by having homogeneous prizes: the prize money is divided evenly among all but the last-place contestant, who should not get a prize. Under the complete-information environment, Clark and Riis (1998) study homogeneous prizes when the prizes are given either simultaneously or sequentially with the restriction that each participant can win at most one prize. They find that both formats generate the same expected effort. Also under complete information, Barut and Kovenock (1998) characterize the equilibria for heterogeneous prizes and give the expected revenue.[24]

Moldovanu and Sela (2001) readdress Galton's question of how to divide prize money for an all-pay auction with incomplete information about costs when a designer wishes to maximize total effort. More specifically, they use a cost function of the form $c(x) \cdot \theta_i$ where $1/\theta_i$ is the ability of player i and is private information. If $c(x)$ is linear or concave then a designer should use one prize. If $c(x)$ is convex, then using multiple prizes may be optimal. The general intuition is that when $c(x)$ is linear, then the incentive problem is equivalent to one obtained if we divide by θ_i and transform the problem to one with incomplete information about the value of the prize, but with complete information about ability (which is then the same for all players). But revenue equivalence holds in this new formulation. Moreover, we can think of this as auctioning an object (the prize money) where the probability of winning the prize for the n-th highest bid is p_n rather than giving a fraction f_n of the prize to the n-th place bidder, where $p_n = f_n$. We know from standard optimal auction results that it is revenue-maximizing for the object to be allocated (if it is allocated at all) to the highest bidder (it is optimal to set $p_1 = 1$.) This also holds when the object must be allocated. Hence, we should only have one prize in an all-pay auction (since then $f_1 = p_1 = 1$). Having a concave cost function in an all-pay auction further favors having just one prize. With convex costs, the marginal cost of effort increases in effort. Thus, revenue-maximizing design would provide incentives that increase effort of the lower types at the expense of the effort of higher types. One way to do this is to have multiple prizes such that a lower type has a higher chance of winning at least a part of the prize. This is also the intuition for having bid caps under convex costs of bidding (see Gavious et al., 2002, and Section 7.7 of this chapter).

Glazer and Hassin (1988) is also the first paper to analyze the multiple prize problem under incomplete information. As with Moldovanu and Sela (2001), they use the cost function $c(x) \cdot \theta_i$ but they have the additional assumption that $1/\theta_i$ is uniformly distributed on $(0, 1]$. They find that when the players are risk-neutral, then a revenue-maximizing designer would want one prize (which

[24] While Clark and Riis (1998) assume all prizes are the same, they allow the prize value to differ among players. In contrast, Barut and Kovenock (1998) assume that prizes may be different, but each particular prize has the same value for all players.

Moldovanu and Sela proved more generally). Under risk-aversion, they find that multiple prizes are indeed optimal (with three or more players). Here, the intuition is that multiple prizes are superior for incentives since they reduce the chance that an agent exerting a high effort will receive any reward. Overall, combining these results one sees that in general with either risk-aversion and/or convex costs it is possible that multiple prizes will enhance revenue. For a longer review of the multiple-prize literature see Sisak (2009).

7.4 Bid-Dependent Rewards

In many of the examples of contests, the reward for winning depends upon the winning bid. This may occur naturally such as in patent races where an earlier patent (obtained from a higher level of effort) is worth more in present-value terms. It may also be set by the designer of the contest. A race director may set a higher prize for breaking a record time. The Longitude prize, described above, had higher prizes set for more accurate methods. Kaplan et al. (2003) analyze a complete-information all-pay auction with a reward for winning of $v(x)$ and with the same cost of effort, $c(x)$, for all firms. We have a symmetric equilibrium where each firm chooses x randomly according to the distribution F given by

$$F(x) = \max_{x' \leq x} \left(\frac{c(x')}{v(x')} \right)^{\frac{1}{n-1}}.$$

When $\frac{c(x)}{v(x)}$ is nondecreasing in x, this simplifies to $F(x) = \left(\frac{c(x)}{v(x)}\right)^{\frac{1}{n-1}}$.

Kaplan et al. (2003) also analyze this environment with two asymmetric contestants. Here, the equilibrium is more complicated. Siegel (2009) defines a player i's reach as the highest bid at which his utility of winning is 0, that is, the largest x such that $v_i(x) = c_i(x)$. Designate player 1 as the player with the largest reach. Denote by π_1 the profit player 1 makes in equilibrium, which is equal to $\max_{x \geq x^*} v_1(x) - c_1(x)$, where x^* is the reach of player 2. We can then define $q_1(x) = \frac{c_1(x)+\pi_1}{v_1(x)}$ and $q_2(x) = \frac{c_2(x)}{v_2(x)}$. Subsequently, we denote $g_1(x) = \min_{x' \geq x} q_1(x')$ and $g_2(x) = \min_{x' \geq x} q_2(x')$ and G_1 and G_2 denote the sets where g_1 and g_2, respectively, are strictly increasing. We then have the equilibrium distribution functions as $F_i(x) = Max\{1 - g_j(\sup_{x'<x} G_i), 0\}$.

This analysis is extended to three or more heterogeneous players in Siegel (2009, 2010, 2014b, 2014c). In addition to the analysis being an important and difficult technical achievement, Siegel finds that the equilibrium with three or more heterogeneous players can have qualitatively different behavior than in the two-heterogeneous-players model.

Kaplan et al. (2002) analyze a variant of this environment with incomplete information about the value of the reward when it is bid-dependent. When types θ_i are i.i.d. from the uniform distribution and the reward is multiplicatively separable in effort and type θ_i, that is, total reward is $\theta_i \cdot R(x)$, where $R(x)$ is a function of effort, the equilibrium effort is given by $b(\theta) = u^{-1}(\theta^n)$ where $u(x) = R(x)^{-\frac{n}{n-1}} \int_0^x \frac{n}{n-1} c'(t) R(t)^{\frac{1}{n-1}} dt$. Interestingly, they find that an increase in the costs or a reduction in rewards may increase the expected sum of the efforts and/or the maximum effort. This is because the information rents (the equilibrium expected profit given one's private information) may be lowered by such an increase.

To see this, by the envelope theorem, the information rent of a type θ is given by

$$\int_0^\theta F(\hat{\theta})^{n-1} R(x(\hat{\theta})) d\hat{\theta}.$$

From this equation we can see why a decrease in the rewards may increase bids. If there is a decrease only for low values of the reward function R, keeping bids constant, this will decrease the rents for all types, not just low types. This requires that the equilibrium profit decrease for all types, even those whose reward function for their equilibrium bid is unaffected by the change. Thus, those types must see an increase in their bid in order for the profits to decrease. Thus, the overall effect can be an increase in bids.

An optimal design approach has been applied to such rewards. Cohen et al. (2008) determine the optimal bid-dependent reward in an all-pay auction under incomplete information when a designer cares about the sum of the efforts or the maximum effort. Kaplan and Wettstein (2013) find the optimal bid-dependent reward under complete information when the designer cares about the maximum effort.

7.5 Contests versus Lotteries

A contest may also be used as a method to allocate good. For instance, colleges routinely use waiting in line as a means to distribute the right to buy basketball tickets. In other cases, instead of a contest, tickets are distributed by means of a lottery such as with tickets to Michael Jackson's funeral. In both of these examples, the objective was to maximize welfare rather than revenue. Chakravarty and Kaplan (2013) find the optimal mechanism when an agent of type θ, exerting effort x, has a utility $v(\theta) - c(x)g(\theta)$ where $g(\theta)$ represents the agent's waiting cost and $v(\theta)$ is the agent's value and the type θ drawn i.i.d. from the uniform distribution on $[0, 1]$.[25] The agent's willingness to pay in terms of cost of effort is then $\frac{v(\theta)}{g(\theta)}$, that is, how high an agent would be willing to set cost $c(x)$ in order to obtain the object. They find that if $\left(\frac{v(\theta)}{g(\theta)}\right)'' \leq 0$, then a lottery is optimal. On the other hand, the condition $\left(\frac{v(\theta)}{g(\theta)}\right)'' \geq 0$ is necessary for a contest to be optimal. Still, the optimal mechanism does not only depend upon the willingness to pay. For instance, when $v(\theta) = \theta$ and $g(\theta) = (1-\theta)^2$, a contest is optimal. When $v(\theta) = \theta(1-\theta)$ and $g(\theta) = (1-\theta)^3$, a lottery is optimal while $\frac{v(\theta)}{g(\theta)}$ is the same in both cases.

Hartline and Roughgarden (2008), Condorelli (2012), and Yoon (2011) also made contributions in the analysis of maximizing the bidder's surplus as opposed to the revenue. This objective is closely related to the objective of the work of McAfee and McMillian (1992) in which a bidding ring wants to maximize expected surplus of its members. In earlier literature, Taylor et al. (2003) and Koh et al. (2006) compare lotteries to contests rather than looking at when either of these is an optimal mechanism. Boyce (1994) made an early contribution comparing lotteries to auctions and to queues in both the case where (after the lottery) post-allocation selling is permitted and in the case where it is not permitted.

[25]Following Milgrom (2004, page 111), there is no loss of generality.

7.6 All-Pay Auctions with Spillovers

Baye et al. (2012a) generalize the complete information all-pay auctions allowing for spillovers. If player 1 bids x and player 2 bids y, then player 1's payoff for winning is $v_1 - W(x, y)$ and for losing is $-L(x, y)$ (with ties broken by a coin toss). The degree to which y affects these two payoffs is the spillover. They assume that $W(x, y)$ and $L(x, y)$ are linear functions of x and y. Even with the restriction of linearity, this model covers many economic environments: the all-pay auction that corresponds to $W(x, y) = L(x, y) = x$, the first-price auction that corresponds to $W(x, y) = x$ and $L(x, y) = 0$, and the second-price auction that corresponds to $W(x, y) = y$ and $L(x, y) = 0$. A case with a negative spillover is when $W(x, y) = L(x, y) = x + \frac{y}{2}$. Other cases include: R&D with spillovers (D'Aspremont and Jacquemin, 1988), Varian's model of sales (Varian, 1980), and competition with price-matching policies (Baye and Kovenock, 1994). Baye et al. characterize all the possible equilibria, both pure and mixed.

Baye et al. (2005) analyze an incomplete information version of this framework in application to litigation systems where the value from winning is private information. Here, they consider the following spillover functions:

$$W(x, y) = \beta x + (1 - \alpha)y,$$

$$L(x, y) = \alpha x + (1 - \beta)y.$$

They classify the legal systems according to the values of α and β.

In the American legal system each side pays his own cost; thus $\beta = \alpha = 1$. In the British system $\alpha = 1$, $\beta = 0$, the loser pays both costs. In the (Dan) Quayle system (a system suggested by the former vice president of the United States, 1989-1993), we have $\alpha = 2$, $\beta = 1$; that is, the loser pays his own costs and reimburses the winner up to his (the loser's) costs. In the Matthew system (from the Gospel of Matthew 5:39-41) the winner should pay not only his own costs but reimburse the loser for a fraction of them (the winner's costs) too: $\alpha = 1$, $\beta > 1$. Of these systems, the British yields the highest legal costs and the lowest litigant utility while the Matthew system yields the lowest legal costs and the highest litigant utility.

7.7 Bid Caps

In the two-player all-pay auction under complete information, Che and Gale (1998) analyze the results of the imposition of a bid cap. In many situations that are modelled as all-pay auctions such caps are in place. For instance, in sports many leagues including four major U.S. sports leagues impose a salary cap.[26] In many political contests, there are spending caps in place on campaign spending. The application in their paper was lobbying. In their model, the value of the prizes are v_1, v_2 such that $v_1 > v_2$ and there is a bid cap of m imposed where $m < v_2$. Without a bid cap, the equilibrium is given by (2). The expected revenue in this case is $\frac{v_2(v_1+v_2)}{2v_1}$, which is less than v_2.

[26] In professional sports, teams can be thought of as competing with each other in their spending for players. Higher spending increases the team's chances of winning the league championship.

When $m \leq \frac{v_2}{2}$, both players bidding at the cap of m is an equilibrium. (When $m = \frac{v_2}{2}$, there are also equilibria where player 1 bids m with probability of $\frac{2m}{v_1}$ or more.) Thus, for m such that $\frac{v_2(v_1+v_2)}{4v_1} < m < \frac{v_2}{2}$, in equilibrium there is higher revenue (equal to $2m$) with a cap than without a cap.

Kaplan and Wettstein (2006) argue that in many contests a designer lacks the ability to impose a hard cap. They show that in the Che and Gale (1998) model imposing a flexible cap decreases revenues. While Che and Gale (2006) show that adding asymmetry of costs to their model (in addition to asymmetry of values), imposing a flexible cap (resulting in raised costs) may also increase revenue. Gavious et al. (2002) show that with incomplete information about values and a convex bid function, imposing a rigid cap may increase revenues. Pastine and Pastine (2010) study bid caps when the seller prefers to sell to one of the bidders (as can happen in the context of political lobbying). Pastine and Pastine (2013) extend their analysis to include soft (flexible) bid caps. In general, there is also a strong connection between bid caps and financially contained bidders, which have been studied by Che and Gale (1996) and Pai and Vohra (2014) in all-pay auctions.

7.8 Research Contests

A significant motivation provided by many authors for the study of contests (as we do in this chapter) is the use of contests to spur innovation. There have been a number of papers that have specifically tailored contests to this application. Dasgupta (1986) and Kaplan et al. (2003) specifically use an all-pay auction with complete information, while Pérez-Castrillo and Wettstein (2012) use an all-pay auction with incomplete information. Taylor (1995) considers an innovation contest where a firm attempting an innovation takes a draw from a distribution $F(q)$ for a quality q at cost c. For cost $x \cdot c$, the firm can take x independent draws and use the highest quality. (This is equivalent to one draw from the distribution $F^x(q)$.) The buyer gives a prize to the firm with the highest quality. Interestingly, the success function is mathematically equivalent to a Tullock success function with $r = 1$ (see equation (1) on page 28).[27] Fullerton and McAfee (1999) solve this for asymmetric costs of draws. Fullerton et al. (2002) replace the prize in a Taylor (1995) style tournament with a first-price auction using a 'scoring rule' that takes both price and quality into account.[28] Schöttner (2008) uses the Lazear-Rosen success function to model the innovation draw. Ding and Woffstetter (2011) have each firm only take one draw (for a fixed cost of entry) but model the case where firms that have too high a quality will not enter the contest and try to bargain separately.

In an interesting variant of an all-pay auction, Che and Gale (2003) have a scoring rule similar to Fullerton et al. (2002). In their paper, a buyer wishes to purchase an innovation from one of several potential firms. Each firm i expends effort $c(x_i)$ to create innovation with a quality

[27] To see this for discrete xs: if firm 1 chooses x_1 draws and firm 2 chooses x_2 draws, then there are $x_1 + x_2$ draws in all. Each draw has an equal chance of having the highest quality. Thus, the chance of the highest quality being among the x_1 draws is $x_1/(x_1 + x_2)$.

[28] See Che (1993) and Asker and Cantillon (2008, 2010) for design competitions with a 'scoring rule' but without an all-pay component.

x_i (measured in monetary units). Afterwards, each firm offers its design to the buyer for a price p_i (chosen by the firm). The buyer chooses the firm offering highest surplus s_i which equals $x_i - p_i$ (the quality minus the price). Denote by $G_i(s_i)$ the probability that firm i with surplus s_i will be offering the highest surplus, that is, $G_i(s_i) = P(s_i > \max\{s_{-i}\})$. Then, firm i chooses effort x_i, surplus s_i, and price p_i to solve

$$\max_{x_i, s_i, p_i} G_i(s_i)p_i - c(x_i) \text{ s.t. } x_i - p_i = s_i.$$

Substituting the constraint into the maximand and the first-order conditions w.r.t. x_i and p_i yields $G_i'(s_i)p_i = c'(x_i)$ and $G_i'(s_i)p_i = G_i(s_i)$, respectively. Combining the two yields: $G_i(s_i) = c'(x_i)$. For arguments similar to the all-pay auction with complete information, firms also make zero profits. Hence $G_i(s_i)p_i = c(x_i)$. Together, we find that firms set prices $p_i = c(x_i)/c'(x_i)$ and choose surpluses by the cumulative distribution set by $G_i(s_i) = c'(x_i)$.

For example, if $c(x) = x^2$ and $n = 2$, then $p_i = \frac{x_i}{2}$. This implies $s_i = \frac{x_i}{2}$ and $G_i(s_i) = F_i(s_i) = 2x_i = 4s_i$. Hence, in the equilibrium each firm chooses a surplus s_i according to the uniform distribution on $[0, 4]$, and sets $x_i = 2s_i$ and $p_i = s_i$. Note that an all-pay auction with a bid-dependent reward of $\frac{x}{2}$ with a cost function of x^2 has a symmetric equilibrium of each firm choosing x according to $F(x) = 2x$. Thus, it has an equivalent equilibrium. Kaplan and Wettstein (2013) prove that the profit from the Che-Gale innovation contest is also the highest expected profit the buyer can earn in a bid-dependent all-pay auction.[29]

7.9 Blotto Games

Introduced by Borel (1921), a Colonel Blotto game has two players (Colonels) each deciding how to divide their respective army across $n > 2$ battles; that is, they each must choose the size of the force x_i to place in battle i such that the sum of their forces in each battle is less than or equal to the size of their total army. The player with the largest force in a particular battle wins that battle. Winning each battle is worth the same and a tie in a battle is worth half as much as a win. Note that the winner of each battle can also be determined by different success functions (such as Tullock) and overall payoffs can be determined by a winner-take-all for the player who wins most of the battles.

The Blotto game is in essence n all-pay auctions run simultaneously when the cost of using part of one's army in one battle is the opportunity cost of using the troops in another battle. For now assume that both armies are the same size (and equal to one). For similar reasons to those in the all-pay auction with complete information, there is no pure-strategy Nash equilibrium and there cannot be a specific size of force in a battle strictly greater than zero chosen with positive probability (also, at most one player can place an atom at zero for each particular battle). Denote F_i as the distribution played in battle i. Given the other player's choice of F_is, we can now write the Lagrangian of a player's problem where the Lagrange multiplier λ is the shadow price that represents the opportunity cost of using the troops. Hence, a player will choose his strategy to

[29] This is not the case when there is asymmetry among firms.

maximize:

$$\max_{x_1,x_2,\ldots,x_n,\lambda\geq 0} F_1(x_1) + F_2(x_2) + \ldots + F_n(x_n) + \lambda(1 - \sum_i x_i). \quad (3)$$

The first-order condition is $F_i'(x_i) = \lambda$ and must hold for all x_i in the support of F_i. Integrating, yields $F_i(x_i) = \lambda x_i + c_i$. However, for the player to be indifferent to shifting forces between two different battles, we must have $c_i = c_j$ for all i, j. Thus, expectation of this uniform distribution is $(1-c)\lambda/2$. Hence for $\sum_i x_i = 1$ to hold, we must have $\lambda = \frac{2}{n(1-c)}$. However, since the support of each distribution must be the same for both players and they both cannot have an atom at the bottom of the support, we must have $c = 0$ and each player chooses, for each battle, the x_i according to a uniform distribution on $[0, \frac{2}{n}]$.

While each player deploys an army in each battle according to a uniform distribution, there must be a connection between each battle since the total size of the army equals one for each realization and not just in expectation. This joint distribution is what is called a copula in mathematics, but it requires the additional property that the sum of the armies in each battle always be equal. Borel and Ville (1938) provide a solution for $n = 3$ by choosing $x_1 \sim U[0, \frac{2}{3}]$ and half the time setting $x_2 = \frac{2}{3} - \frac{x_1}{2}$ and $x_3 = \frac{1}{3} - \frac{x_1}{2}$, while the other half of the time swapping how one sets x_2 and x_3. Gross and Wagner (1950) extend Borel and Ville's result to n armies and provide an additional method for finding a joint distribution.[30]

Roberson (2006) gives the complete set of equilibrium marginal distributions for general n and asymmetric army sizes. Roberson shows that when the ratio r of army sizes is between $\frac{2}{n}$ and $\frac{n}{2}$, the equilibrium is similar to asymmetric complete-information all-pay auctions given by (2).[31] For intermediate asymmetries in army sizes where $\frac{1}{n-1} \leq r < \frac{2}{n}$ or $\frac{n}{2} < r \leq n-1$, the equilibrium is similar to Che and Gale (1998) but with a bid cap placed only on the weaker firm. He also provides a specific method for finding a joint distribution (making use of n-copulas).

Notice that having battles worth different amounts is just a matter of adding different weights in front of the F_is. This yields a solution where the upper bound of each uniform distribution of each battle is proportional to its weight. This case was studied for symmetric army sizes and general n by Gross (1950), Laslier (2002), and Thomas (2013). Also, Gross and Wagner (1950) solved the case of $n = 2$ with heterogeneous values of battles and asymmetric army sizes. The case for $n > 2$ with heterogeneous values of battles and asymmetric army sizes has not been generally solved. Another version of the game has all the utility going to the player winning the majority of the battles. This version for the $n = 3$ case with symmetric army sizes was solved in Borel and Ville (1938).

Adamo and Matros (2009) solve the Blotto game when the army size is incomplete information. Namely, if the distribution of each army size G is concave, then equally dividing one's army is an equilibrium. Hart (2008) solves the Blotto game when the armies can be divided only discretely.

[30] Laslier and Picard (2002) measure the inequality of the equilibria in Gross and Wagner (1950) using the Gini coefficient. Weinstein (2012) provides another method of finding a joint distribution and examines Laslier and Picard's analysis.

[31] The difference is that v_i is replaced by the respective army size times $\frac{n}{2}$.

Roberson and Kvasov (2012) extend the analysis of Blotto games to an environment where a player saves an expense proportional to the portion of his army that is not used in a battle.

There is a reason for the strong connection to the all-pay auction with complete information (Roberson, 2006). While the size of the army is fixed and it does not cost more to field the whole army than to field half the army, there is an opportunity cost in that using part of the army in one battle means that it cannot be used in a different battle. This is represented by the shadow price equal to λ in (3). Hence, it is no surprise that both the all-pay auction and the Colonel Blotto game have similar equilibria since they both consist of strategies that are uniform distributions.

7.10 Other Topics

There are many topics on all-pay auctions not covered in detail in this chapter. For instance, several papers use the basic contest building block in a more complex environment: Moldovanu and Sela (2006) analyze if it is optimal to run one contest or break it down into playoff contests. Konrad and Kovenock (2009) study a sequence of contests where there is a prize for the winner of each individual contest and another for the majority winner of the sequence.[32] There are also studies where the prize is not exactly a monetary prize but rather the utility for winning depends upon the number of prizes given out, as with As in a course (see Modovanu et al., 2007, and Dubey and Geanakoplos, 2010), or losing the contest can come at an additional loss of utility (beyond wasted effort) at a cost to the designer of such a punishment (see Moldovanu et al., 2012, and Thomas and Wang, 2013). Under incomplete information, there is also progress beyond the IPV case (see Lizzeri and Persico, 2000, and Siegel, 2014a). There is also a large body of literature on experiments in all-pay auctions (see Dechenaux et al., 2012, for a review). For a more detailed theoretical review on contests, see Konrad (2009).

8 Incorporating Behavioral Economics

In recent years both the literature and the public have paid increased attention to behavioral economics. Since in many auction experiments, subjects do not bid according to standard theory, auction theory has begun to follow the trend. Probably the most natural behavioral factor to incorporate is a utility for winning the auction. Mathematically this is equivalent to shifting the distribution of values (see Cox et al., 1988). Other more complex ways in which behavioral economics has been incorporated into auction theory are explored in this section.

8.1 Regret

In an auction, there are two types of regret: winner's regret and loser's regret. With winner's regret, the winning bidder regrets if he overpaid and could have won with a lower bid. With loser's regret, a losing bidder regrets if he could have profitably won the auction by submitting a higher bid.

[32]For other papers on sequential contests where the values of winning each contest are not independent, see Sela (2011, 2012).

For a single-object first-price auction, Engelbrecht-Wiggans (1989) (henceforth EW) expresses the utility of the buyer in an auction as the expected profit from winning minus α_1 times the expectation of an increasing function of the overpayment (winner's regret) minus α_2 times the expectation of an increasing function of the missed surplus (loser's regret):

$$P(b > B) \cdot (v - b) - \alpha_1 \cdot E[h(b - B)] - \alpha_2 \cdot E[g(v - B)] \tag{4}$$

where B is the maximum bid of the other buyers and $g(x)$ and $h(x)$ are two functions that are both positive (and typically monotone increasing) for $x > 0$ and 0 for $x \leq 0$. For the special case $g(x) = h(x) = x^+$ and $\alpha_1 = \alpha_2 = \alpha$,[33] EW proved that the equilibrium bid function is the same with and without regret.

To see this, let F be the cumulative distribution of B and let $f = F'$ be its density. Now, in the special case we can rewrite (4) as:

$$F(b)(v - b) - \alpha \left(\int_0^b (b - B) f(B) dB + \int_b^v (v - B) f(B) dB \right).$$

The first-order condition for the maximization of this function of b is:

$$f(b)(v - b) - F(b) + \alpha \left(\int_0^b f(B) dB - (v - b) f(b) \right) = 0$$

or

$$f(b)(v - b) - F(b) + \alpha \left(f(b)(v - b) - F(b) \right) = 0. \tag{5}$$

Equation (5) is the same whether $\alpha = 0$ (without regret) or $\alpha > 0$ (with regret).

EW also shows that when $\alpha_1 > \alpha_2$ (and $g(x) = h(x) = x^+$), the equilibrium bid function is lower than without regret, and vice versa when $\alpha_1 < \alpha_2$. Logically, when the costs of increasing his bid go up relative to the benefits, a buyer will decrease his bid. Engelbrecht-Wiggans and Katok (2008) (henceforth EWK) show that with n buyers having values drawn from the uniform distribution starting at 0, the equilibrium bid function will be

$$b(v) = \frac{n - 1}{n + \frac{\alpha_1 - \alpha_2}{1 + \alpha_2}} \cdot v.$$

With the restrictions that $h(x) = g(x) = 0$ for $x \leq 0$ and $h(x)$, $g(x)$, $h'(x)$, and $g'(x)$ positive for $x > 0$, Filiz-Ozbay and Ozbay (2007) (henceforth FOO) prove the rather intuitive result that when there is only loser's regret, $\alpha_1 = 0$ and $\alpha_2 > 0$, the equilibrium bid functions are higher than in the no-regret equilibrium and when there is only winner's regret, $\alpha_1 > 0$ and $\alpha_2 = 0$, the equilibrium bid functions are lower than in the no-regret equilibrium. Consequently, revenue is higher with only loser's regret and lower with only winner's regret than in the no-regret equilibrium.

FOO run a one-shot first-price auction experiment with these three regret conditions as the three treatments by varying the feedback. For the no-regret treatment, there is no feedback

[33] The notation x^+ denotes x if $x > 0$ and 0 otherwise.

except that one has won or lost. For the winner's-regret treatment, in addition the winner sees the second-highest bid. For the loser's-regret treatment, the winning bid is announced. Using groups of four buyers having values drawn from the uniform distribution, they find that indeed the relative ordering of bidding among treatments matches the theoretical predictions.

EWK also run an experiment but with subjects playing repeatedly against computerized strategies. They go further in developing the feedback treatments by telling the winner explicitly the difference between his bid and the second-highest for winner's regret (rather than leaving it up to them to calculate), or telling the losers the exact amount of the money left on the table (the difference between their value and the winning bid for loser's regret). They also add a treatment with both types of regret. As with FOO, they find the bidding among the treatments rank according to theory and that bids with no regret are similar to those with both types of regret, which is consistent with the theory under the assumptions that $\alpha_1 = \alpha_2$ and $g(x) = h(x) = x^+$.

These results suggest that a seller would have higher revenue by announcing the winning bid and keeping the losing bids secret as opposed to only announcing the winner's identity or publicizing all bids. Regret need not be purely psychological. A situation with regret can also exist when an agent is bidding on behalf of a client. The agent wishes to avoid the displeasure of his client. This possibility also holds true for many of the other topics in this section.

8.2 Impulse Balance

Impulse Balance Theory was introduced by Ockenfels and Selten (2005) (henceforth OS). It is similar to regret except that rather than a utility function being explicitly specified, bidders have an impulse to adjust their bid functions upwards or downwards whenever ex post it is rational to do so. This impulse is proportional to the potential gains (if measurable). For instance, if in a first-price auction one bids 10 and the second-highest bid is 7, then the downwards impulse of the winner is proportional to 3. If instead one loses to a bid of 12 when one's value is 14, then his upwards impulse is proportional to 2. These impulses are in balance if the upward impulse is equal to λ times the downward impulse, where λ is a weight constant that is specific to the agent. If half the time one faces a bid of 7 and half the time one faces a bid of 12 (always with a value of 14), then bidding 10 would be impulse-balanced if $\lambda = 2/3$.

OS analyze impulse balance in an n-bidder first-price auction when values are drawn from the uniform distribution on $[0, 1]$. The approach is not that of best reply and equilibrium. Rather, they define the notion of a bidder's strategy to be impulse-balanced given the strategy of the other bidders. More specifically, they assume that all bidders use a linear bid function $b(v) = av$, and determine for which value of the parameter a each bidder's strategy is impulse-balanced (for a given impulse weight λ).

As a theory attached to an experimental work, OS consider two repeated auction treatments: the full-feedback treatment (denoted F) in which all bids (in particular, the highest and second-highest) are announced at the end of the auction, and the no-feedback treatment (denoted NF) in which only the winning bid is announced. The F treatment is similar to the both-regrets treatments of EWK and the NF treatment is similar to the loser's-regret treatments of OO and EWK. The main differences in design is that OS is repeated with random matching (unlike the one-shot OO) and against human bidders (unlike EWK). (Note also that OS have two bidders rather than four bidders in OO.)

In the F treatment, the downward and upward impulses are defined as the expected forgone profits in case of winning and losing respectively and they are given by:

$$I_-^F = E[(b-B)^+] = \frac{a}{n(n+1)}, \tag{6}$$

and

$$I_+^F = E[(v-B)^+ \cdot \mathbf{1}_{b<B}] = \frac{(1-a)^2(n-1)}{2(n+1)}. \tag{7}$$

In the NF treatment, as the foregone profit is not observable for the winner (he does not know how much further he could have lowered his bid and still have won), the impulses are driven just by the fact of winning or losing the auction. Therefore, OS define the impulses to be the probability of these events respectively; that is, the downwards impulse is the probability of winning and the upwards impulse is the probability that a bidder loses when it would be profitable to increase his bid so as to win the auction. For this model, these impulses are given by:

$$I_-^{NF} = P(b > B) = \frac{1}{n}, \tag{8}$$

$$I_+^{NF} = P(b < B \ \& \ v > B) = \frac{(1-a)(n-1)}{n}. \tag{9}$$

The condition that the strategy $b(v) = av$ be impulse balanced is $I_+^F = \lambda I_-^F$ in the F treatment and $I_+^{NF} = \lambda I_-^{NF}$ in the NF treatment. Solving these equalities shows that if $0 < \lambda < \frac{1}{2}$, then $a^{NF} > a^F > \frac{n-1}{n}$. Recalling that $b(v) = \frac{n-1}{n}v$ is the equilibrium bid of this auction without impulse biases, this means that there is overbidding in both conditions, and there is more overbidding without feedback. This is in agreement with both regret theory and OS's experimental results (which are consistent with the experimental results of OO and EWK).

8.3 Reference Points

Rosenkranz and Schmitz (2007) (henceforth RS) analyze a model in which the utility of the buyer may depend upon a reference point r; any price paid above this number decreases the buyer's utility and any price below it increases his utility. Formally, a buyer with value v purchasing the object at price p has utility

$$u(v, p, r) = v - p + z(r - p) \tag{10}$$

where z is a function satisfying $z' \geq 0$ and $z(0) = 0$.

While RS analyze this model with a linear z, their solution can be generalized, using the envelope theorem, to any non-decreasing function z. In a second-price auction, for the usual reasons, a buyer will set a bid $b_s(v, r)$ such that $u(v, b_s(v, r), r) = 0$. If $z(r - v) > 0$ then $b_s(v, r) > v$. Likewise, if $z(r - v) < 0$, then $b_s(v, r) < v$.

In a first-price auction, if we define the strictly increasing function $w(b,r) := b - z(r-b)$ and its inverse w.r.t. b by $w_r^{-1}(\cdot)$, then the equilibrium bid function is

$$b(v,r) = w_r^{-1}\left(v - \frac{\int_0^v F^{n-1}(\tilde{v})d\tilde{v}}{F^{n-1}(v)}\right). \tag{11}$$

RS show that if the reference point r is affected by the minimum bid, then unlike the standard case without reference points, the optimal minimum bid would depend upon the number of bidders. RS also show that in some cases revenue increases if the minimum bid is kept secret and, in which case the buyers have only an exogenous reference point. Finally, RS prove revenue equivalence for the case of linear z (which does not hold for non-linear z). This equivalence does not hold if the reserve price is kept secret.

8.4 Buy-It-Now Options

A buy-it-now option allows a bidder to buy an object for a specified price before the auction finishes. This option is typically available only for a limited time (for instance, until someone bids on the item or until the reserve price is reached), after which the option vanishes and the auction proceeds normally. The buy-it-now option is available in auctions on eBay (since 1999) and accounts for a large portion of their sales.[34]

Reynolds and Wooders (2009) show that theoretically under risk-neutrality a buy-it-now option cannot improve revenue, but there can be gains if the buyers are risk averse. Shunda (2009) shows that if the buy-it-now option can influence the reference point (in the utility function), then it can improve revenue even under risk-neutrality. The intuition is that a buy-it-now option can push up the reference point and thereby increase revenue.

8.5 Level-K Bidding

The level-k model was introduced by Stahl and Wilson (1994, 1995) and Nagel (1995). In Nagel's (1995) experimental study of the guessing game, subjects choose a number from 0 to 100 and the number that was closest to 2/3 the average won a prize (ties were broken randomly). Many guesses were bunched around 33 and 22. According to Nagel, a subject who chooses 33 believes that other subjects are choosing a number randomly is classified as L_0 level. Thus, the subjects who choose 33 think one level ahead of L_0 subjects and are classified as level L_1. Those who choose 22 are level L_2 since they think one level ahead of level L_1; that is, if all other subjects are level L_1 players and thus choose 33, it is best to choose 22. The limit of this iterative process goes to the equilibrium of 0.

Crawford and Iriberri (2007) use this type of analysis to derive behavior in auctions. A level L_0 player will bid randomly by choosing a bid uniformly over the whole range of possible bids (from 0 to the highest possible value).[35] Like in the guessing game, a level L_k player ($k > 0$) will

[34] Before being discontinued, Yahoo! Auctions also had a buy-it-now option that allowed the seller to change the price during the auction as opposed to eBay that only permits the price to be set at the beginning.

[35] Also considered by Crawford and Iriberri (2007) is a "truthful" L_0 that bids his value.

best-react under the assumption that all other players are level L_{k-1} players. As an example, in a first-price auction with two buyers and a minimum bid of m, a level L_1 player with a value above m will choose a bid $b = \max\{\arg\max_{\tilde{b}} \tilde{b}(v - \tilde{b}), m\} = \max\{\frac{v}{2}, m\}$. Note that this is independent of the distribution of the players' values since level L_0 is assumed to bid uniformly. In particular, if the values are drawn independently from a distribution on $[0, 1]$ and $m = 0$, level L_1 bidding coincides with equilibrium bidding under the uniform distribution and hence bidding is the same for all levels L_k for $k \geq 1$. On the other hand, if the values of the two players are affiliated, as the level L_1 bidding will still be $\frac{v}{2}$ if both players are level L_1 players, they will not take into account the winner's curse and potentially overbid in the auction (and in expectation lose money).

Crawford et al. (2009) explore the possibility of a different choice of the level L_0 player, letting such a player bid uniformly on the range from the minimum bid m to the highest possible value. Hence, now a level L_1 player with a value above m will choose a bid $b = \arg\max_{\tilde{b}} (\tilde{b} - m) \cdot (v - \tilde{b}) = \frac{v+m}{2}$. (And a level L_2 player with $v > m$ will choose $b = \arg\max_{\tilde{b}} F(2\tilde{b} - m)(v - \tilde{b})$). Their assumption is somewhat unpalatable in that such an L_0 player will always bid more than m (even if $v < m$). We suggest that perhaps a more reasonable L_0 player for a first-price auction will be one that bids only above the minimum bid if $v > r$ and never bids above his value. In this case for $m = 0$ and uniform values on $[0, 1]$, an L_1 player will choose $b = \arg\max_{\tilde{b}}(\tilde{b} + \int_{\tilde{b}}^{1} \frac{\tilde{b}}{\tilde{v}} d\tilde{v})(v - \tilde{b})$. An L_1 player's inverse bid function would be $v(b) = \frac{-b+2b\ln(b)}{\ln(b)}$. However, an L_2 player will then in turn bid the highest possible bid from an L_1 player (approximately 0.34) for a range of possible values.

For experimental studies on level-k models see the chapter "Behavior Game Theory Experiments and Modeling" in this Handbook.

8.6 Spite

The behavioral considerations thus far have looked only at individual biases, traits, and reasoning. Yet bidders may also have preferences about the surpluses of the other bidders. In particular, they may have spiteful preferences; namely, they are willing to sacrifice some of their own surplus in order to lower the surplus of the other bidders.

Morgan et al. (2003) present a two-bidder model in which a bidder's utility when losing is inversely related to the surplus of the winner. More specifically, for some $\alpha > 0$, in a first-price auction the utility equals

$$P(b > B) \cdot (v_i - b) - \alpha \cdot E[v_{-i} - B | b < B] \tag{12}$$

and in a second-price auction, the utility equals:

$$P(b > B) \cdot (v_i - E[b_{-i} | b > B) - \alpha P(b < B) \cdot E[v_{-i} - b | b < B]. \tag{13}$$

The equilibrium bid function in the first-price auction is:

$$b(v) = v - \frac{\int_0^v F(\tilde{v})^{1+\alpha} d\tilde{v}}{F(v)^{1+\alpha}}.$$

By taking the derivative w.r.t. α, it is straightforward to show that the second term is decreasing in α and thus, for $\alpha > 0$, the bid function is higher than the bid function without spite ($\alpha = 0$) and bidding is increasing in spite. The equilibrium bid function in the second-price auction is

$$b(v) = v + \frac{\int_v^1 (1 - F(\tilde{v}))^{\frac{1+\alpha}{\alpha}} d\tilde{v}}{(1 - F(v))^{\frac{1+\alpha}{\alpha}}}.$$

Since this is clearly greater than v and is increasing in α (by straightforward verification), there is overbidding in the second-price auction. Again, the bid function is higher than the bid function without spite and it is increasing in the spite α. It is thus possible that the winner in fact loses in the sense that he has negative utility and would be better off not having won. Morgan et al. (2003) also demonstrates a similar result for English auctions.

Let us now look at the case where there is asymmetry in values. Take the same preferences but where one bidder has a value equal to 9 and the other bidder has a value equal to 0 and there is complete information about values. Assume that $\alpha = 1/2$ and consider a second-price auction in which when both bidders submit the same bid, the high-value bidder wins. In this case, we claim that both bidders submitting the same bid b^*, where $3 < b^* < 6$, is an equilibrium. The high-value buyer will win at price b^* and have utility $9 - b^*$. The low-value bidder will have utility $-(9 - b^*)/2$. To see that this is an equilibrium, the only candidate for a profitable deviation of the high-value bidder is to bid just below b^* and lose the auction in which case his utility will be $b^*/2$. For the low-value bidder, the only candidate for a profitable deviation is to bid just above b^*, win the auction, and have a utility of $-b^*$. For $3 < b^* < 6$ we have $9 - b^* \geq b^*/2$ and $-(9 - b^*)/2 \geq -b^*$; hence, both deviations are not profitable.

In this simple example, we have the low-value bidder overbidding ($b^* > 0$) because of spite and the high-value underbidding ($b^* < 9$) to essentially counterbalance this spite. Nishimura et al. (2011) show that such spite and counter-spite hold more generally in both English and second-price auctions.

8.7 Ambiguity

In analyzing auction models, we usually assume that bidders know the probability of winning given a certain bid. In many cases, the bidder may be uncertain about this probability. This can be due to not knowing the distribution of values or strategies of the other bidders. Uncertainty about the relevant probabilities is known as *ambiguity* and its effects on behavior have been demonstrated and studied in a large number of papers starting from the seminal work of Ellsberg (1961). In general, decision makers are, as pointed out by Ellsberg, ambiguity-averse. Theoretical models for studying ambiguity and explaining its effects were laid out by Gilboa (1987), Gilboa and Schmeidler (1989), Schmeidler (1989), Sarin and Wakker (1992), and others.

One main method for incorporating ambiguity into theory is to replace the additive probability of the agent with a non-additive probability measure known as *capacity*. This is a real function $c(\cdot)$ defined on the subsets of the probability space Ω and satisfying $c(\emptyset) = 0, \quad c(\Omega) = 1$, and $A \subseteq B \Rightarrow c(A) \leq c(B)$ for any $A \subseteq \Omega$ and $B \subseteq \Omega$. The additivity condition of a probability measure: $P(A) + P(B) = p(A \cup B) + P(A \cap B)$ is replaced by the *convexity* condition: $c(A) + c(B) \leq c(A \cup B) + c(A \cap B)$ for any $A \subseteq \Omega$ and $B \subseteq \Omega$. The degree of the convexity (that is, the difference between the two sides of the inequality) reflects the degree of ambiguity aversion.

To see this, let A and B be two complementary events, that is, $A \cup B = \Omega$ and $A \cap B = \emptyset$. Now the convexity condition is $c(A) + c(B) \leq 1$. If the decision maker has symmetric information about the occurrence of the two events, then he assigns to them equal probabilities, i.e., $c(A) = c(B)$. But, as Schmeidler says, this equal probability need not be $\frac{1}{2}$ each if the information about the occurrence of these events is meager. This is, for example, the case for the events of drawing a red (R) or a black (B) ball from the Ellsberg urn containing 100 balls with an unknown composition of red and black balls. If $c(R) = c(B) = \frac{3}{7}$ then $c(A) + c(B) < c(A \cup B) + c(A \cap B) = 1$ and the difference between the two sides ($\frac{1}{7}$ in this case) is a measure of the uncertainty, or the *ambiguity*, of the decision maker about the probability of the events.

A convex capacity can be conveniently represented as a composition $c = \phi \circ P$ where P is an additive probability measure and the *probability transformation* ϕ is a function $\phi : [0, 1] \rightarrow [0, 1]$, where the degree of ambiguity aversion is reflected in the convexity of ϕ. In the absence of ambiguity, an agent with utility function $u : \Omega \rightarrow \mathbb{R}$ defined on the state space Ω on which he has subjective (additive) probability P considers the *expected utility*, which can be written as $\int_\tau P(\{\omega \in \Omega | u(\omega) \geq \tau\}) d\tau$. In the presence of ambiguity, the agent considers what is called the Choquet Expected Utility (CEU), which is the expectation according to the transformed probability $c = \phi \circ P$, which can be written as:

$$\int_\tau \phi(P(\{\omega \in \Omega | u(\omega) \geq \tau\})) d\tau. \tag{14}$$

Instead of maximizing expected utility, an ambiguity-averse agent will maximize this Choquet Expected Utility. Salo and Weber (1995) study the effects of ambiguity in first-price auctions. In their model, an ambiguity-neutral bidder assumes that the other bidders' values are independent and uniformly distributed on $[0, \overline{v}]$. The bidder's CEU profit $\pi_i(v)$ can thus be written as:

$$\pi_i(v) = \phi \left[\left(\frac{B^{-1}(b_i)}{\overline{v}} \right)^{N-1} \right] (v_i - b_i) \tag{15}$$

where the random variable B is the highest bid of the other bidders. Using the envelope theorem to find $\pi_i'(v)$ and then integrating and setting $\pi_i(0) = 0$, we find the equilibrium bid function is

$$B(v_i) = v_i - \frac{\int_0^{v_i} \phi([\frac{\tau}{\overline{v}}]^{N-1}) d\tau}{\phi([\frac{v_i}{\overline{v}}]^{N-1})}. \tag{16}$$

If one restricts the ratio $\frac{\phi(\alpha x)}{\phi(x)}$ to only depend upon α, then it would have the form $\phi(x) = x^{\frac{1}{K}}$. Here, $K < 1$ expresses ambiguity aversion ($\phi(x) < x$) and the smaller K implies more ambiguity aversion. $K > 1$ expresses ambiguity-loving ($\phi(x) > x$). The bid function now reduces to:

$$B(v_i) = \frac{N-1}{N-1+K}\, v_i. \tag{17}$$

We see here the somewhat intuitive result that increasing ambiguity aversion increases the bid function in a first-price auction (and vice versa for ambiguity-loving). Other models of ambiguity have been applied to auctions with similar theoretical results (see Lo, 1998, Ozdenoren, 2002, Chen et al., 2007). However, experimentally, Chen et al. (2007) find that bidding behavior is consistent with ambiguity-loving behavior (lower bids when faced with ambiguity).[36]

Ambiguity could exist about the number of bidders in the auction. Salo and Weber (1995) also examine this ambiguity with CEU and find that the bids are higher (indicating the seller should not reveal the number). Levin and Ozdenoren (2004) revisit this question with maxmin expected utility and also find that bidding should be higher in a first-price auction than in a second-price auction. They also find that if the buyers are more pessimistic than the seller, the seller will want to maintain the ambiguity of the bidders and to refrain from revealing his information about the number of bidders.

When looking at ambiguity aversion of a seller, Turocy (2008) finds that an ambiguity-averse seller will prefer a first-price auction to a second-price auction if bidders make small payoff errors, that is, errors in choosing strategies that result in lowering payoffs by ϵ or smaller.

For detailed discussion of non-expected utility theory and its implications on auctions when the object itself is a risky prospect (such as a lottery ticket), see the chapter "Ambiguity and Nonexpected Utility" in this Handbook.

9 Position Auctions in Internet Search

Here we discuss research on a fairly recent auction mechanism that has been the main revenue-driving force behind several large internet companies: the auctioning off of advertisements based upon the keywords used in an internet search. In such a mechanism, several advertisement placements per search are sold. The placements intrinsically vary in quality with the better positions going to the higher bidders.

In 1998, Goto.com introduced the first successful position auction for search results (see Davis et al. U.S. Patent 6,269,361). When a particular term was searched, Goto.com listed the results in the order of the advertisers' willingness to pay per click (the advertiser's bid was listed along with the results). This company founded by Bill Gross's Idealab was eventually rebranded as Overture and sold to Yahoo! in 2003 for $1.6 billion. In 2002, Google used the advertisers' quality along with their bids to determine the position of advertisements in their AdWords auction (while listing search results independent of bids). This generated $42 billion in revenues in 2012. In 2004, Google agreed to pay Yahoo! 2.7 million shares (worth $300 million at the time) for patent infringement.

The main driving force behind these auctions is that not only is having an ad posted or link listed valuable, but placement matters too. Being listed first makes it more likely to be chosen. In politics, Orr (2002) argues that in Australia voters were more likely to choose the candidates the higher they were on the ballot.

[36] See Dickhaut et al., 2011, for a first-price auction experiment with ambiguity and a common value.

9.1 First-Price Pay-Per-Click Auctions

In its typical form, a position auction is somewhat of a misnomer: the advertisers do not bid on positions but rather on clicks; that is, the bid represents an amount to be paid when a user is directed to their site.

Assume there are n advertisers. Each advertiser a submits a bid b_a for a click whose value for that advertiser is v_a. There are also a limited number of positions where each position i gets x_i clicks (note that at this stage we assume that the number of clicks per ad depends only upon ad position and not also on ad quality). The lower i indicates the better location, that is, $x_i > x_{i+1}$. Thus, the value for position i for advertiser a is $v_a x_i$. Assume that there is complete information about the bidders' values.

The initial Goto.com position auction was run using the natural extension of first-price auctions. The advertisers are listed in decreasing order of their bids. The advertiser with the highest bid gets the most-preferred position of 1. The advertiser with the second-highest bid gets the second-highest position of 2. This is continued until we run out of advertisers or positions. Ties are broken randomly. The price paid for each click is equal to the bid of the advertiser receiving that click. Advertiser i (that is, the one in position i) will receive $(v_i - b_i)x_i$.

Since many users searched using the same keywords, the same keywords were auctioned off repeatedly. Bids were solicited for a number of searches in advance with an option of changing one's bid at a later time. This created a problem with first-price position auctions. To see this, assume that there are two advertisers and two positions. When first place has an $x > 0$ chance of a click and second place has a $y > 0$ chance of a click (and $x > y$), then under complete information, there is no pure-strategy Nash equilibrium. The advertiser who is listed second will either wish to lower his bid to zero (if it is not already at zero) and stay in second place or raise his bid just enough to come in first. If the second-place advertiser submitted a bid of zero, then the first-place advertiser would wish to lower his bid to \$0.01. This cannot be an equilibrium since then the second-place advertiser would like to bid \$0.02. Through similar logic, ties can be ruled out. Although this example resembles a first-price auction for the right to be listed first (which would have a pure-strategy Nash equilibrium), it differs in that the loser pays also for the second-place listing.

In position auctions, mixed-strategy equilibria are problematic in that the equilibrium is not efficient (sometimes the low-valued advertiser will be listed higher than an advertiser with a higher value), and effort is then wasted in altering bids and strategizing. Edelman and Ostrovsky (2007) show a sawtooth pattern of cycles in prices in data from June 2002 to June 2003 on Overture.[37] For this reason, Yahoo! switched to using a second-price position auction, which is referred to by Edelman et al. (2007) (henceforth, EOS) as a Generalized Second-Price Auction.

[37] This sawtooth pattern occurs for a similar reason in Edgeworth prices (see Maskin and Tirole, 1988 and Noel, 2008) and empirically is found elsewhere in gasoline markets (see Noel, 2007a, 2007b).

9.2 Second-Price Pay-Per-Click Auctions

In second-price position auctions, advertisers are ordered according to their bids and assigned positions in the same manner as in the first-price position auction. For all positions strictly less than n, the price paid per click for the advertiser assigned to position i is now $p_i = b_{i+1}$. The price per click of the advertiser ordered last is zero.

Varian (2007) (henceforth, VAR) shows that the Nash equilibria of the second-price position auctions (under complete information) must satisfy constraints consisting of an advertiser in position i not wanting to move up to a higher position j, $j < i$ (recall that a higher position corresponds to a lower index) and not wanting to move down to a lower position j, $j > i$. To move higher, an advertiser would need to bid slightly above the bid of the advertiser in position j and pay that bid, b_j, which is equal to p_{j-1} (we define $p_0 = b_1$). To move to a lower position, an advertiser needs to pay the price of the advertiser in position j, p_j (by bidding slightly under his bid). Hence, the incentive constraints are:

$$(v_i - p_i)x_i \geq (v_i - p_{j-1})x_j \quad \text{for } j < i, \tag{18}$$

$$(v_i - p_i)x_i \geq (v_i - p_j)x_j \quad \text{for } j > i. \tag{19}$$

VAR also shows that in the case of complete information (about the values) there are multiple equilibria, as with the standard second-price auction under complete information. However, EOS show that, unlike in the standard second-price auction, truthtelling is no longer a dominant strategy nor even always an equilibrium. To see this let us look at an example where the highest position has a 60% chance of a click, second place has a 50% chance of a click, and third place has a 20% chance of a click. There are three advertisers: one with a value of 3 per click, another with a value of 2, and the last with a value of 1. If everyone bid truthfully, then the advertiser with a value of 3 would be in first place with utility $(v_1 - p_1)x_1 = (3 - 2)0.6 = 0.6$. However, by bidding 1.9, he could be in second place with utility $(v_1 - p_2)x_2 = (3 - 1)0.5 = 1$. Hence, there is an incentive for this advertiser to deviate.

If we define envy as desiring to swap positions and prices paid with another advertiser, then we can define the envy-free equilibrium as an outcome in which there is no envy between any two advertisers,[38] that is,

$$(v_i - p_i)x_i \geq (v_i - p_j)x_j \text{ for all } i, j. \tag{20}$$

Note that, as the name suggests, this concept is indeed a refinement of the notion of equilibrium, since (20) is equivalent to (19) for $j > i$, and implies (18) for $j < i$ since $p_j < p_{j-1}$.

An envy-free equilibrium is in particular locally envy-free; that is, an advertiser does not envy the advertiser located one position above or below his position, that is,

$$(v_i - p_i)x_i \geq (v_i - p_j)x_j \text{ for all } i, j = i \pm 1. \tag{21}$$

[38]This is called Symmetric Nash Equilibrium by Varian (2007). Note that there are other equilibria. As Borges et al. (2013) mention, if $x_1 = 3, x_2 = 2, x_3 = 1$, and $v_1 = 16, v_2 = 15, v_3 = 14$, then under complete information one advertiser bidding 11, another 9, and another 7 will be a Nash equilibrium. Thus, a non-symmetric equilibrium could be inefficient since in this example the one valuing a click at 14 could be in the first position.

This concept defined by EOS was proved by VAR to be equivalent to the (global) envy-free equilibrium defined by (20). Indeed, (20) implies (21), while (21) implies (20) using the 'directional' transitivity of the envy-free conditions (that is, the transitivity of 'not envying someone above you' and 'not envying someone below you').[39]

VAR shows that the equilibrium with the highest revenue is the highest revenue envy-free equilibrium given by $b_{n+1} = 0$ and $b_{i+1} = (b_{i+2}x_{i+1} + v_i(x_i - x_{i+1}))/x_i$ (in this case, (21) holds with equality for $j = i + 1$). In our example, we would thus have $b_3 = 1.2$, $b_2 = 1.5$, and $b_1 > 1.5$ with prices $p_1 = 1.5$, $p_2 = 1.2$, and $p_3 = 0$. Notice that while this is envy-free, it does require advertiser 3 to bid above his value (of 1).

The Vickrey-Groves-Clark (VCG) mechanism with position auctions will induce truthtelling as dominant strategies. With this mechanism, each advertiser i has to pay the externality that he imposes on the other advertisers. Without advertiser i, the advertisers above him would not change their position while each advertiser j positioned below him ($j > i$) would move up one position gaining $v_j(x_{j-1} - x_j)$, which is the advertiser j's value times the difference in position clicks. Hence, the entire gain would be $\sum_{j=i+1}^{n} v_j(x_{j-1} - x_j)$. Thus, the VCG mechanism would ask advertisers their true valuations $\tilde{v}$ and sort them into positions according to the reported valuations. It would then require the advertiser in position i to pay a total of $P_i = \sum_{j=i+1}^{n} \tilde{v}_j(x_{j-1} - x_j)$. That is, $P_{i-1} = P_i + v_i(x_{i-1} - x_i)$ where $P_n = 0$. Notice that these payments would cause (21) to hold with equality (where $p_i x_i = P_i$ and $j = i - 1$). As shown by both VAR and EOS, this corresponds to the envy-free equilibrium and Nash equilibrium with the lowest revenue to the seller. In our example, this is where $p_1 = 0.833$, $p_2 = 0.6$, and $p_3 = 0$.

9.3 Other Formats

An English position auction (called a Generalized English Auction by EOS) is one in which the auctioneer increases the price per click. Advertisers decide when to drop out. The first advertiser to drop out when the remaining advertisers are fewer than the number of positions pays zero and receives the last position. The advertisers dropping out after that pay the price at which the previous advertiser dropped out and get the position one higher than that advertiser. EOS shows that under incomplete information this has a unique perfect Bayesian equilibrium that is equivalent to the VCG mechanism.

Another way to obtain the VCG outcome is for the advertisers to use a mediator. Ashlagi et al. (2009) give conditions under which the VCG outcome will be implemented as an equilibrium of a mediated second-price position auction, that is, a mechanism in which a mediator bids on behalf of the advertisers based upon messages received from them. The advertisers also have the option of ignoring mediation and bidding independently but they have incentives to agree to mediation. Because of this option, the mediator cannot simply utilize the English position auction (which EOS

[39] If i prefers position i to position j (i.e., i does not envy j) and j prefers position j to position k (i.e., j does not envy k) where either $i > j > k$ or $i < j < k$, then $(v_i - p_i)x_i \geq (v_i - p_j)x_j = (v_j - p_j)x_j + (v_i - v_j)x_j \geq (v_j - p_k)x_k + (v_i - v_j)x_j = (v_i - p_k)x_k + (v_i - v_j)(x_j - x_k) \geq (v_i - p_k)x_k$. Thus, i also prefers position i to position k (i.e., i does not envy k).

showed leads to the VCG outcome) and use the results to bid in the second-price position auction resulting in the VCG outcome. Reaching the VCG outcome through a mediator is more difficult, since the mediator must be able to induce the advertisers bidding in the auction to use his service.

So far we have discussed the current pay per click. There are two other methods for collecting payments: pay per action, meaning pay based upon the business resulting from the click, and pay for impression (position). For the advertiser, pay per action is ideal as it eliminates several problems that arise in pay per click: the expected value of a click is still uncertain to the advertiser, learning the distribution is costly, and even if the expectation is known, many small advertisers prefer to pass the risk on to the search engine. Moreover, pay per click is subject to click fraud (where owners of websites click fictitiously). However, Agarwal et al. (2009) demonstrate a weakness in pay per action for the seller. Namely, the advertisers know better the potential actions of the searcher and exploit this. For instance, they can use a bait-and-switch technique where the searcher clicks on an ad for a 50-inch TV and winds up buying a 60-inch TV since the other is out of stock. The advertiser could offer less in return for the purchase of the 60-inch TV than the 50-inch TV while the seller would think that the action of purchasing the 50-inch TV is likely.

Some websites, rather than pay per click, use pay for impression. With paying for impression, an advertiser in position i with bid B_i would pay B_{i+1} independently of the number of clicks obtained in that position. Note the effective bid per click would be B_i/x_i and the effective payment per click would be B_{i+1}/x_i, which is different from the effective bid per click of the advertiser in position $i+1$. However, any envy-free equilibrium under pay for click is equivalent to one in pay per impression with $B_i = b_i x_{i-1}$. If the $\{b_i\}$ are an envy-free equilibrium under pay per click, by (20) we have:

$$(v_i - b_{i+1})x_i \geq (v_i - b_{j+1})x_j \text{ for all } i, j. \tag{22}$$

By substitution, this implies that

$$v_i x_i - B_{i+1} \geq v_i x_j - B_{j+1} \text{ for all } i, j. \tag{23}$$

These are sufficient conditions for an envy-free equilibrium in the pay-for-impression auction.

The method Google actually uses for position auctions is to take into account the ad quality of the advertiser in addition to his bid per click. Two different advertisers in the same position may receive a different number of clicks. Varian (2007) models this such that advertiser a in position j will have a click-through rate of $x_j e_a$ where e_i is the advertiser's quality. Rather than being ordered by his pay-per-click bid of b_i, an advertiser is ordered by his quality-adjusted bid-per-click of $b_i e_i$. Thus, whereas before an advertiser wanting to move up from position i to position $j < i$ would need to raise his bid to slightly above b_{i-i}, the advertiser now needs to raise his quality-adjusted bid from $b_i e_i$ to slightly above $b_{i-1}e_{i-1}$. This entails raising his bid to slightly above $b_{i-1}e_{i-1}/e_i$ and paying $b_{i-1}e_{i-1}/e_i$ per click as compared to $b_{i+1}e_{i+1}/e_i$ per click in his current position i. Likewise, whereas before an advertiser wanting to move down from position i to position $j > i$ needed to lower his bid from b_i to slightly below b_{i+1}, the advertiser now needs to lower his quality-adjusted bid from $b_i e_i$ to slightly below $b_{i+1}e_{i+1}$. This entails lowering his bid to slightly below $b_{i+1}e_{i+1}/e_i$

and paying $b_{i+2}e_{i+2}/e_i$ per click as compared to $b_{i+1}e_{i+1}/e_i$ per click. The incentive constraints for not wanting to move up or down are then:

$$x_i e_i(v_i - b_{i+1}e_{i+1}/e_i) \geq x_j e_i(v_i - b_j e_j/e_i) \qquad \text{for } j < i, \tag{24}$$

$$x_i e_i(v_i - b_{i+1}e_{i+1}/e_i) \geq x_j e_i(v_i - b_{j+1}e_{j+1}/e_i) \quad \text{for } j > i. \tag{25}$$

Bringing the e_i inside the parentheses yields:

$$x_i(v_i e_i - b_{i+1}e_{i+1}) \geq x_j(v_i e_i - b_j e_j) \qquad \text{for } j < i, \tag{26}$$

$$x_i(v_i e_i - b_{i+1}e_{i+1}) \geq x_j(v_i e_i - b_{j+1}e_{j+1}) \quad \text{for } j > i. \tag{27}$$

This is the same structure as before in (18) and (19) except that, in the pay-per-click auction, the value v_i is replaced by $v_i e_i$ and the bid b_i is replaced by $b_i e_i$. Hence, all the properties stated for the pay per click hold.

Borges et al. (2013) generalize the second-price position auction to include values that are position-dependent and click-through-rates that are advertiser-dependent. Advertiser a in position k will have a value of v_a^k and receive c_a^k clicks. Being in a particular position may also enhance advertiser utility independent of clicks received. This enhancement, called impression value, depends upon advertiser a and position k and is denoted by w_a^k. Hence, the advertiser paying price p would have utility

$$(v_a^k - p)c_a^k + w_a^k. \tag{28}$$

Borges et al. find that as long as $v_a^k c_a^k > v_a^{k+1} c_a^{k+1}$, a symmetric Nash equilibrium exists. They also collected data from 2004 Yahoo! auctions that used this method and found empirical support for the hypothesis that $v_a^1 > v_a^2 > \ldots > v_a^n$.

Unlike auctioning off inanimate objects such as cars, the consumer viewing these advertisements is a player in the game and his surplus and strategy are of concern. An early business model of Goto.com assumed that consumers would prefer to have search results displayed in the order of those willing to pay the most for a click. While Goto.com's business model did not survive (against Google's page rank method), the question of which design benefits the consumer the most is important. Furthermore, once the consumer is considered to be a player, the value of each position and the number of ads that each player clicks should be endogenously determined. Athey and Ellison (2009) do just that (for ads rather than search results) and find that click-weighted auctions are socially optimal as search costs go to zero, whereas this need not be the case with strictly positive search costs. They also find that advertisers hide information about themselves in the click-weighted auction but not in the pay-per-click auction.

Jerath et al. (2009) show that once consumers' actions are taken into account, the position of the advertisers may no longer be sorted by the one with the highest value first. A clear example is two brands of a product: a well-known brand and a lesser-known brand. A well-known brand may get the same number of clicks per position in all positions while the number of clicks of the lesser-known brand may vary significantly more based upon position. In this case, the lesser-known brand may be willing to pay more for the improvement of position and thus be listed in a higher position. For example, when typing "JFK Car rentals" the advertised list on the RHS listed a site called Jfkcarrentals.net first and Avis second. Likewise, when searching for "shoes" in the UK, it listed Schuh first and Clarks second.

10 Spectrum Auctions

One of the most celebrated achievements of auction theory is the widespread use of spectrum auctions.[40] Many auction theorists are hired either to advise governments on design or help companies on bidding strategies (see McMillan, 1994, for those hired just in the earlier auctions).

In the early spectrum auctions (pre-2000) several issues arose: the exposure problem (analogous to the danger of buying a left shoe without buying a right shoe), strategic demand reduction, and tacit collusion (see Section 3 of this chapter for a detailed description of the former two). The exposure problem was particularly problematic in the U.S. where the spectrum was divided up into more than 200 geographical regions and there were complementarities in owning spectrum in neighboring regions. This pushed economists to develop solutions that were eventually implemented in the 4G auctions.

Strategic demand reduction was a major concern in European spectrum auctions, in particular, where no entry was allowed or occurred. For example, in the 1999 German GSM auction, there were four incumbents competing essentially for ten identical blocks of spectrum.[41] The auction was over after just three rounds of bidding with jump-bidding occurring in the first round followed by strategic reduction. While this may be considered a case of collusion, Grimm et al. (2003) (and more generally Riedel and Wolfstetter, 2006) demonstrate that such an auction has a unique subgame-perfect equilibrium where a drastic demand reduction occurs immediately. (Essentially, bidders immediately reduce demand to the number of blocks they would acquire if everyone bid truthfully.) Goeree et al. (2013) confirmed this behavior in an experiment.

10.1 3G Auctions

In April 2000, the 3G rights were sold off in the UK for 39 billion Euros (£22 billion) (630 Euros per capita). This was coined at the time "the biggest auction ever" (Binmore and Klemperer, 2002). This record lasted all of four months until in Germany the rights were sold off for 51 billion Euros (615 euros per capita).

Besides the magnitude of the revenue, the European 3G auctions are a particularly interesting case study to examine since all the countries had roughly the same spectrum to auction off at roughly the same time. They varied only slightly in the number of incumbents and in GDP per capita.[42] Also, there were only three different designs used. Furthermore, there were no combina-

[40]The idea of selling spectrum rights by auction dates back to Coase (1959). There is a large literature on the design of spectrum auctions including Banks et al. (2003), Cramton (1997), Klemperer (2004), Plott (1997), Plott and Salmon (2004), Porter and Smith (2006), and Weber (1997). See Hoffman (2011) and Wolf (2012) for a recent summaries.

[41]One was slightly larger and ownership of that block was determined in a secondary auction.

[42]The incumbents were firms that were already in the telecom industry that owned 2G licenses.

toric considerations and networks could successfully operate solely in one country. Each country had roughly 120MHz of spectrum to auction off. This can be divided into six blocks of 2 x 10MHz, four blocks of 2 x 15MHz, or three small blocks of 2 x 10MHz and two large blocks of 2 x 15MHz.

The three basic designs used were: UK-Ascending, Anglo-Dutch, and German-Endogenous. In the UK design, licenses were specified ahead of time and grouped into categories. Initially, the exact spectrum was not specified within a category. An ascending auction was used. Rounds continued as long as there were more active bidders than licenses. Bidders could remain active and continue bidding if they held or beat the previous high bid for a license. Once license ownership was determined for licenses within the same category, there would be an additional sealed-bid auction to determine who gets which specific frequencies.[43]

In the Anglo-Dutch design, the auction begins the same way as the UK design, but switches once the number of active bidders dropped to one plus the number of licenses. At this point, a sealed-bid auction occurs. If K was the number of licenses, then the K top bidders each would get one license for the same price equal to the lowest winning bid. For instance, if there were four licenses, the price would be set to the fourth-highest bid. For determining the specific blocks, the last stage again resembles the UK design.

The German design had an endogenous number of licenses in that the number of owners could range from four to six. The spectrum was divided into 12 lots. Firms were permitted to bid on either two or three lots. If a firm did not remain active on at least two lots, they were considered dropped from the auction. If the auction stopped with a firm owning a single lot, then that firm is dropped and there is an auction for the single lot. Afterwards, a specific spectrum is determined in a way similar to the UK design.

Results from six countries are displayed in Table 2.[44] Of these, the two successful auctions were Germany and the UK. Between them, Germany appears more successful since it not only generated higher revenue overall, but did so adjusting for GDP.[45] Furthermore, it granted six licenses rather than five, which means a more competitive market afterwards. While it had the same number of incumbents as the UK, it had fewer entrants, making the outcome more impressive in its competitiveness given that overall there were fewer bidders (although, as Klemperer, 2004, points out, this smaller number of entrants could have been a function of the auction design).

Other 3G auctions did not fare so well. Using the UK design, Switzerland with the highest GDP per capita and the lowest number of incumbents was poised for a high per capita sales price (Wolfstetter, 2003). There were also ten firms vying for the four licenses. However, in the end, there were only four entrants competing for the four licenses that went for the reserve price which was set too low (despite a governmental attempt to change the rules at the last minute). This drop in numbers was in part due to the government allowing joint bidding (Klemperer, 2004).

In Austria, the use of the German design failed to push bids much beyond the reserve price.

[43] This stage is also useful in ameliorating the exposure problem where there is an additional advantage of holding the same specific frequencies in neighboring countries.

[44] The table is from Wolfstetter (2003) with added data from Klemperer (2004).

[45] In the UK, it was 630/25415=0.0248 Euros per Euro of GDP, while in Germany it was 615/23020=0.0267 Euros per Euro of GDP. See Table 2.

The six bidders each received a license. Apparently, the stronger bidders preferred to have a smaller license with a market size of six to trying to get a larger license (and with fewer competitors) by bidding on a second smaller license since that would run the risk of increasing all the prices without changing the outcome.

The main weakness of the UK ascending auction is that if there is little private information, even if the number of potential bidders is higher than the number of licenses, then the weaker bidders will realize when they cannot win and will choose not to enter. We then have the same number of bidders as licenses and this leads to the licenses going at the reserve prices. This is particularly problematic if the number of licenses equals the number of incumbents since who is strong is clear. Determining who is strongest among the entrants can take more time. Thus, the fact that the UK ran first may have helped it attract more entrants, while in Switzerland this was not the case.

The Anglo-Dutch design gives a weaker bidder a chance of winning and hence can attract more entrants (at the cost of efficiency). After the failures of the UK ascending design, with four incumbents and four licenses, Denmark used the Anglo-Dutch design (albeit only at the second stage).[46] The auction not only attracted an entrant, but the newcomer replaced one of the incumbents. While the price was 95 Euros per capita, given the drop in the stock market due to the 2000 dot.com crash, this result was successful.

Using an ascending auction for at least part of the auction is problematic in that bidders can communicate indirectly through the auction. In the German auction, bidding at the end of day 11, after 138 rounds, had two top bids by T-Mobile and one by Mannesmann Mobilfunk equal to DM6.666 billion:[47] potentially implying that they should split six ways. Although it was expensive to communicate in such a manner, it is conceivable that it was an attempt to collude. It is also possible to communicate one's strength by entering a jump bid and thereby increasing the bid significantly.

The flexibility of the German design garnered criticism in that it left more combinations for which the bidders could collude. There were also complaints about the complexity of its rules. Also, it was not clear whether the government had the knowledge to optimally set the number of different companies owning licenses at the end of the auction. On the other hand, allowing the auction to endogenously determine this number, runs the risk of resulting in too small a number of companies, and hence hurting the competitiveness of the industry. Limiting the number of licenses that one firm can own ensures a minimum level of after-auction firm participation. The advantage of the German design is that when the auction would normally end in a typical ascending auction where each bidder was permitted only one unit, there is potential for the stronger bidders to push the price up further by bidding for an additional unit (and keeping entrants out). As we saw with the German auction, this can end in higher prices without reducing the after-auction firm participation.[48]

[46] In the end, it did not matter since there were only five bidders.

[47] While bidding was in Deutsch Marks, which was the currency in circulation at the time, the exchange rate was fixed to the Euro.

[48] Ironically, the two entrants returned their licenses within one year (although they could not recover the money spent) in order to avoid the obligation of building infrastructure to provide nationwide coverage. Telefonica was one

Country	Date	Bidders	Licenses	Incum.	Euro/capita	GDP per capita
UK	Mar.-Apr. 2000	13	5	4	630	25,415
Netherlands	July 2000	9/6	5	5	170	24,250
*Germany	July-Aug. 2000	12/7	4-6	4	615	23,020
Italy	Oct. 2000	8/6	5	4	210 R	19,451
*Austria	Oct. 2000	6	4-6	4	103 R	24,045
Switzerland	Nov.-Dec. 2000	10/4	4	3	19 R	35,739
**Denmark	Sept. 2001	-/5	4	4	95	30,034

Table 2: The European 3G auctions in 2000. A * indicates used the German Design. A ** indicates using the second phase of the Anglo-Dutch auction. All others used the UK design (Italy had the additional rule that they would reduce the number of licenses if there was not an excess of serious bidders). An R indicates sold at or close to reserve price. For bidders x/y, represents that x bidders were thinking of entering bids but in the end only y bidders entered. Both the German and Austrian auction ended with six licenses awarded. GDP per capita is in 2000 dollars (from the IMF) using then current exchange rates.

See Illing and Klüh (2004) for a collection of papers that interpret and debate the 3G auctions in Europe.

10.2 4G Auctions

After the 3G auctions, there was the start of a new wave of spectrum auctions, the 4G auctions. These were different from the 3G auctions in many ways. For one, the market had matured. While smart phones were in widespread use, the expectations had been reduced from the dot.com boom. The competition was more Bertrand like resulting in lower profit for the firms. For these reasons, revenue was markedly smaller. Also, the number and type of licenses up for sale was much more varied across countries. In such a mix, there were many complementarities in the blocks. Finally, the number of firms considering entry and incumbents in the market were better known in advance. The need to attract new entrants seemed less of an issue than the need to ensure efficiency in fitting the new licenses to existing ownerships.

The two main designs used are the Simultaneous Multi-Round Auction (SMRA) and the Combinatorial Clock Auction (CCA).[49] These designs combine elements from both the Anglo-Dutch design and the German design. The CCA is significantly more complex than even the German 3G design[50] and allows for combinatoric bids. Like the German design, the CCA has a built-in flexibility

of those firms and even more ironically took over an incumbent and in 2010 bought back its returned license.

[49] Among the other designs, Rothkopf et al. (1998) propose using Hierarchical package bidding. This combinatorial auction limits the combinations that can be bid upon to hierarchies of blocks. For instance, one can bid for blocks A, B, C, and D separately, A and B together, C and D together, or A, B, C, and D all together, but one cannot bid on B and C together. This restriction makes computing the maximum revenue significantly easier and allows subjects to better understand the pricing tradeoffs. It was used in practice by the FCC (see Goeree and Holt, 2010). One drawback is that the packages must be predetermined by the auctioneer. The FCC also tried using Modified Package Bidding (also known as Resource Allocation Design) designed by Kwasnica et al. (2005). This design sets shadow prices for the blocks such that the revenue-maximizing bidders are willing to pay the prices (given their bids) and the other bidders are not. This is helpful in price discovery.

[50] Indeed, in the recent auctions in Switzerland and Austria, a bidder could submit more than 1500 bids in the last

about the amount of spectrum each winning firm can obtain but many implementations have caps imposed on how much spectrum a single bidder can win.

The SMRA is, in essence, an enhanced version of the UK-ascending auction (although a version of it proposed by McAfee, Milgrom, and Wilson has been used by the FCC since 1994). It has been used recently in Germany, Hong Kong, Belgium, Spain, Norway, Sweden, Finland, and other countries. In this format, each specific block of spectrum is bid on separately. In each round, bidders are allowed to bid in each block in which they are active. Bids can be increased in only specified increments (to avoid information transmitted in jump bids). One remains active in round t by either having the leading bid in round $t-2$ or submitting a higher bid than the $t-1$ leading bid in round $t-1$. In some versions, in earlier rounds of the auction one can also remain active for x blocks by satisfying those conditions in only a fraction of the blocks (such as half) for which one was active in round $t-1$. Depending upon the specific rules chosen, either the highest bid or all the bids are shown after each round. The overall auction ends when there is no improvement on the bids in the previous round. The main differences from the UK auction are that the activity rules are block-specific, the bidders are allowed to win more than one block, and the bid increases are restricted. As mentioned, there is a possibility of a cap on the number of blocks one can be active on.

The CCA was introduced by Cramton (2009, 2013), which incorporated many elements of the Proxy-Clock auction proposed by Ausubel et al. (2006). Maldoom (2007) describes an implementation by a private company, dotEcon, that ran many of the European 4G auctions using this format (UK, Ireland, Holland, Denmark, Switzerland). Here, bidders are allowed to bid on lots: combinations of spectrum blocks (with a potential cap on the total spectrum size a bidder can own). The CCA has two main stages: the clock stage, during which bidders can bid on just one package in each round, followed by the sealed-bid stage, in which bidders can bid on all possible combinations, restricted by their activity during the clock stage.[51]

Rather than asking the bidders to choose the bids, the clock increases the price and the bidders decide which package of blocks (if any) to bid on. Each block is worth a number of eligibility points and buyers can choose only licenses whose total points are within their eligibility point budget (and spectrum cap). The initial budgets are set by the financial considerations of the bidders and the budgets are reduced to the number of points used in the previous round. The price increase for various lots is determined by the interest in that lot. The clock stops when the demand is less than or equal to the supply. This can lead to overshooting where the demand is strictly less than the supply or it can lead to the highest bid on a single lot below the set threshold. To fix this, there is sealed-bid auction where bids must be consistent with the initial pattern of bids. (The sealed-bid phase also serves to fight strategic demand reduction and collusion.) The auctioneer then chooses the combination of bids (subject to feasibility) that maximizes revenue

stage.

[51]In order to reduce the number of combinations, as with the 3G auctions some blocks are classified as equivalent and bid upon as abstract blocks. After the two main stages, there is a subsequent stage in which (as similar to the UK 3G auction) specific blocks are bid upon among winners of the auction in each equivalent class of blocks.

(which is also efficient if bids are truthful) from both the sealed-bid stage and all rounds of the clock stage. The prices paid are determined by what is called Vickrey-nearest-core pricing. While in Vickrey pricing each individual must pay the externality that they impose on other bidders, in Vickrey-nearest-core pricing each collection of winning bidders must pay the joint externality that is imposed on the other non-winning bidders.

To demonstrate Vickrey-nearest-core pricing, we use an example from Cramton (2013). There are two lots: A and B. Bidder 1 values A at 28. Bidder 2 values B at 20. Bidder 3 values the package of A and B together at 32. Bidder 4 values A at 14. Bidder 5 values B at 12. The efficient allocation gives A to bidder 1 and B to bidder 2. In Vickrey-Clark-Groves (VCG) pricing, the price of A is 14 and B is 12.[52] However, this is below what the value of A and B together for bidder 3. In order to get a set of prices in the core, the sum of the prices for A and B must be at least 32 (their combined value for bidder 3). This constraint is satisfied by the set of prices that has the minimum revenue within the core (MRC). The closest set of prices (geometrically) from the MRC to the Vickrey prices is 17 for bidder 1 and 15 for bidder 2. This also has the advantage of higher revenue than the Vickrey pricing.

Using Vickrey-nearest-core pricing is motivated by the fact that using Vickrey pricing may lead not only lead to outcomes that are not in the core, but to lower revenue, as shown by Day and Milgrom (2008). Day and Milgrom also show that pure VCG pricing may give the bidders an advantage of using a shill bidder (entering as two bidders), and may be non-monotonic revenue-wise in the number of bidders. They provide an example with two licenses: A and B. Bidder 1 values them together at 10 and each separately at 0. Bidder 2 values the first license at 10 and the second at 0. Bidder 3 values the first license at 0 and the second at 10. With all 3 bidders, the VCG price for either item is 0. With just bidders 1 and 3, the price of item A is 0 but that of item B is 10. Hence in this case, bidder 3 can gain by entering a bidder to bid for item A, thereby, reducing B's price. (Remember that in VCG pricing, a bidder pays the externality he causes by receiving his allocation. With all three bidders, bidders 2 and 3 receive A and B, respectively, each receiving 10 in value. Without bidder 3, either bidder 2 gets A or bidder 1 gets A and B. In either case, the value is 10. Hence, bidder 3 does not impose an externality on the other bidders.)

In support of using Vickrey-nearest pricing, Day and Raghavan (2007) show that the Vickrey-nearest-core prices minimize the sum of the maximum gains of deviating from truthful bidding. Day and Milgrom (2008) suggest that the Vickrey-nearest-core prices maximize the incentive for truth telling among the prices in the core and Day and Cramton (2013) show that the Vickrey-nearest-core prices are unique. However, Goeree and Lien (2012) show that it still distorts incentives for truthful bidding and Erdil and Klemperer (2010) claim that incentives for truthful bidding are superior with a reference point other than VCG, one that does not depend upon the bids of the bidders involved. This would be minimizing incentives locally for small deviations from

[52] With VCG, the efficient allocation is chosen and a bidder pays the externality he imposes on others. If bidder 1 does not receive the allocation of A, the next-best allocation is bidder 4 gets A and bidder 2 gets B. This has a total value of 34. This is 14 above the value that the other bidders get when bidder 1 gets A (and bidder 2 gets B). Hence the VCG price for package A is 14.

truthful bidding.

The CCA consistency rules (see Bichler et al., 2013, for a description) are essentially that the final-round bids must be consistent by revealed preferences from the earlier rounds. For instance, if lot A and lot B are the same price in a preliminary round and a bidder chooses to bid on A and not on B, then in the final, sealed-bid round, that bidder cannot bid more for lot B than for lot A. (Similar rules apply for quantity reductions.)

While neither design has emerged as dominant in recent auctions, there are some concerns. For the SMRA auction, when there are complementaries, larger bidders may worry about being left exposed by purchasing only part of a complementary lot of blocks. Because of this the SMRA can result in smaller bidders winning at low prices and thus excluding bidders may be beneficial for seller revenue (see Goeree and Lien, 2014). There are other potential problems as well, as in the case in which a large bidder that has complementarities over a large number of lots is held up by a small bidder. For instance, if bidder A values ten specific licenses together at 100 and each individually only at 1, then a bidder B can speculate by buying one of the ten and then demand a large amount from bidder A. In the Advanced Wireless Services (AWS) auction run by the FCC (see Cramton, 2013), blocks of spectrum were divided up differently geographically. Because of this holdup problem, a block of spectrum sold for 12 times the price of the combined price of a similar block that could be created by buying several geographically divided blocks. Despite these problems, Bichler et al. (2013) experimentally find that the CCA does worse than the SMRA in both efficiency and revenue. Due to its complexity, one may question the feasibility of subjects comprehending the CCA design,[53] but experiments done on teams of bidders recruited from a class on auction theory, who also had two weeks to prepare, yielded similar results. Cramton (2013) claims that part of the problem is that bidders were not provided with the right bidding tools that would have enabled them to easily place bids on the relevant packages.

11 Concluding Remarks

As mentioned in the Introduction, we could not hope to provide a fair representation of all the work going on in auctions in the past two decades. There are many important topics that we did not cover, among them, collusion and corruption, budget constraints, and experiments.[54]

Even in the topics covered in the chapter, we were not exhaustive, but we hope to have given the reader a place to start. Some topics are covered more deeply and formally in other chapters of this handbook: Combinatorial Auctions and Algorithmic Mechanism Design.

[53]See Guala (2001) for a discussion on how experiments are a useful part of spectrum auction design.

[54]We refer the reader to the book of Marshall and Marx (2012) and papers by Skrzypacz and Hopenhayn (2004), Che and Kim (2006, 2009), Pavlov (2008), Garratt et al. (2009), and Rachmilevitch (2013a, 2013b, 2014) for collusion in auctions, Che and Gale (1996, 1998), Zheng (2001), Pitchik (2009), and Kotowski (2013) for budget constraints, and Kagel (1995) and Kagel and Levin (2014) for surveys of auction experiments.

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