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Abstract

In this paper, we reconsider the Foley model of Liquidity / Profit-Rate Cycles where such cycles are generated as bifurcations from initially attracting steady states if a parameter of the model crosses a critical value, for example the growth rate of money supply as in the Foley paper. We employ a slightly modified version of the Foley model and provide sufficient conditions for the local asymptotic stability of its balanced growth path. A second theorem then shows the existence of a Hopf-bifurcation derived from such a stable situation by decreasing the growth rate of liquidity to a sufficient degree. The generated cycles are studied from the numerical point of view in addition.

Keywords: Liquidity / Profit-Rate Cycle, Stability Conditions, Hopf-bifurcation.

JEL classifications: E32, E64.

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1 Introduction

Foley's analysis [Foley (1986)] is considered a benchmark in the study of the existence of liquidity-profit rate cycles in capitalist economies. He makes use of a model in the tradition of the nonlinear business cycle models developed by Goodwin (1982) and Hicks (1950), but he also takes into account social classes in a Kaleckian fashion [see Kalecki (1969)]. Besides, his model assigns to capital outlays a crucial role as one of the key variables of the aggregate demand and it focuses on the enterprise decisions concerning the outlay of capital to initiate production, and their borrowing to finance production. Arguably, this variable plays an important role in the analysis, since an attempt of the economic system to grow faster than the money supply would lead to higher interest rates, and consequently to a shortage of liquidity. Thus the growth rate of the economy adjusts to the growth rate of money through changes in the rate of capital turnover. [See Foley (1986, p. 372)].

In the present paper we further study the dynamics of these liquidity-profit cycles by departing from a slightly modified version of Foley's model, which assumes that only the interest rate is a function of the amount of money per unit of capital, whereas Foley assumes that profit/interest gap is a function of the amount of money per unit of capital. With this approach we emphasize the crucial role of two important variables, namely aggregate demand arising from the decisions of all the enterprises as to their capital outlays, and an interest rate, which must adjust to clear the market for loanable funds. Hence, the determinants of the market rate of interest are highlighted by the role of money in the economy. Like in the original model, a stable steady state growth path is shown to exist but such equilibrium may degenerate into a limit cycle if the growth rate of liquidity decreases to a sufficient degree. The generated cycles are studied from the numerical point of view in addition.

This paper is structured as follows: the next section presents a modified version of the Foley's liquidity /profit-rate cycle model. Section 3 studies the existence and stability of stationary points and shows the possibility of a Hopf bifurcation, which leads to the existence of a limit cycle in financial and real variable. Section 4 presents numerical simulations by considering plausible shapes of the assumed reaction functions. Section 5 concludes.

2 The Foley model reconsidered

Foley (1987) analyses the dynamics of a capitalist economy populated by profit-seeking firms in which labor is in surplus, and the productivity of labor makes the real wages consistent with given profit margins. The model departs from an explicitly disaggregated assessment of the economy and emphasizes money as a link amongst enterprises. The author then shows that when all firms are identical and the rate of growth of money is constant, the economy may be locally unstable and in fact giving rise to a limit cycle, i.e., economically to persistent business fluctuations due to a strong financial accelerator effect.

He considers two economic classes: capitalists and workers and follows the classical assumptions that all wages are consumed and all profits are invested, using a given wage share v, defined through $\frac{wL^d}{pY} = \frac{w}{px}$, where w is the nominal wage, where L^d stands for actual employment, p refers to prices, Y is actual production and x represents the output-labour ratio. It is assumed that prices, the wage level and the output-labour ratio are constant, which in the presence of a fixed proportions technology without technical change makes the wage share constant. By assuming in particular that $w = p = 1$ and that there is no capital stock depreciation, the variation in the stock of capital immediately reflects the fact that all profits are invested. Its law of motion is therefore given by:

$$\dot{K} = I = rK = (1 - v)\, Y \tag{1}$$

where r is the profit rate. The output is by assumption either consumed or invested in this Marxian supply side model. We therefore simply get:

$$Y = L^d + I, \quad \text{since} \quad w = p = 1. \tag{2}$$

An important new variable considered in this classical model is the aggregated debt of firms, denoted by D, which in Foley's micro-founded model must equal M, the monetary assets of firms, plus their aggregate holding of debt of other enterprises F, that is we have: $D = M + F$. Money supply is assumed to grow at the constant rate μ and is transferred as a subsidy to firms. Hence:

$$\dot{M} = \mu M \tag{3}$$

In what follows, let us consider real balances, loans and debt per unit of capital, namely $d = \frac{D}{K} = \frac{M}{K} + \frac{F}{K} = m + f$. Foley (1987) assumes that the variation of debt per unit of capital, $\frac{\dot{D}}{K}$, is a function of the difference between the profit- and interest-rate, $r - i$. He also assumes in his simplified model that the difference between the profit- and interest-rate depends only on the ratio $m = \frac{M}{K}$, namely $r - i = p\,(m)$ In his words, "[c]learly the higher is m, the lower will be the interest rate relative to the profit rate, since with a high m the enterprises are liquid and would have a high supply of loanable funds" [Foley (1987, p. 370)]. We thus have:

$$\frac{\dot{D}}{K} = B\,[r - i] = B[p\,(m)] \quad \text{with} \quad B^{'}\,[\cdot] > 0 \tag{4}$$

According to this rationale the higher is m, the lower will be the interest rate relative to the profit rate, since with a higher m the enterprises are liquid and would have a high supply of loanable funds. Here we introduce a modification to the model of Foley by instead assuming that only the interest rate is a function of the amount of money per unit of capital, namely $i = i\,(m)$, with $i^{'}\,(\cdot) < 0$. We consider this as a more meaningful assumption on the role of money in such an economy.

In this case, $B\,[p\,(r, m)] = B[r - i\,(m)]$ and expression (4) may be rewritten as:

$$\frac{\dot{D}}{K} = B[r - i(m)] \quad \text{with} \quad B'[\cdot] > 0,\, i'(\cdot) < 0 \tag{4'}$$

According to this specification, firms increase their debt with an increasing profitability spread by adding further loans to their liquidity position. In what follows we intend to study if this different specification may change the dynamics of the macroeconomic Foley model (1987). One of the most important outcomes of the Foley's analysis is the existence of a limit cycle. Here we also intend to show that this modified version exhibits such a limit cycle. In order to show this result we carry on the analysis both in terms of general functions as well as in terms of a particular example. In each case it is possible to find so-called Hopf-bifurcation parameters where in general there happens the death of a stable corridor around the steady state or the birth of a limit cycle around it.

To close the model a so-called output expansion function is added which is given by:

$$\frac{\dot{Y}}{Y} = A(r, d) \quad \text{with} \quad A_d > 0,\, A_r > 0 \tag{5}$$

and which therefore states that the growth rate of real output increases with the rate of profit and also increases when the debt to capital ratio is increased. Since we have assumed Say's Law to hold, this is indeed a supply side phenomenon and in the latter case based on the higher degree of liquidity relative to the capital stock.

In order to study the dynamics of this economy let us reduce the above model to three autonomous nonlinear differential equations. The first may be obtained from the fact that $r = \frac{(1-v)Y}{K}$ holds. Taking logs and differentiating this expression with respect to time gives: $\hat{r} = \hat{Y} - \hat{K}$, since v is constant. From expressions (1) and (5) we conclude that:

$$\dot{r} = rA(r, d) - r^2 \tag{6}$$

From expression (4)' it is possible to obtain after some algebraic manipulation:

$$\dot{d} = B[r - i(m)] - rd \tag{7}$$

and from expression (3) we conclude that:

$$\dot{m} = m(\mu - r) \tag{8}$$

We thus get a system of three non-linear differential equations in the three state variables (r,m,d) which we will consider in the following in this order of its state variables. In the next section we will study the existence and stability of stationary points for system (6) – (8).

3 Existence and Stability of Stationary Points and Hopf bifurcation

Let us assume that all functions are sufficiently smooth so that all solutions to initial value problems exist uniquely. Evaluating system (6) – (8) in steady state yields the following system of equations:

$$0 = r[A(r,d) - r] \tag{9}$$
$$0 = m(\mu - r) \tag{10}$$
$$0 = B[r - i(m)] - rd \tag{11}$$

From expression (9), $r = 0$ or $A(r,d) = r$. If $r = 0$ then from expression (10), we obtain $m = 0$. Then the first point to be considered is $P_1(0,0,z_1^*) \quad <=> B(p(0,0)) = 0$. From the economic viewpoint this may be considered an uninteresting case since it entails the non-existence of money and no economic growth. If $r \neq 0$ then, from (9), $A(r,d) = r$. From expression (10), $m = 0$ or $r = \mu$. Then the second point to be considered is: $P_2 = (r_2^*, 0, d_2^*) <=> A(r_2^*, d_2^*) = r_2^*; B(p(r_2^*, 0)) = r_2^* d_2^*$ But since $m = 0$ it is also uninteresting one from an economic viewpoint.

Now if $m \neq 0$, $r = \mu$ and substituting into expression (10) we obtain a single equation for $d^* \quad A(\mu, d^*) = \mu$ Now, we consider the function $F(r,d) = A(r,d) - r$, where $F(\mu, d^*) = 0$ and $\frac{\partial F}{\partial d}(\mu, d^*) = \frac{\partial A}{\partial d}(\mu, d^*) \neq 0$. Thus, by using the Implicit Function Theorem, there exist neighborhoods $U \subset R^+$ of $r^* = \mu > 0$ and $V \subset R^+$ of $d^* > 0$ such that there is a unique $d = h(r)$ defined for $r \in U$ and $d \in V$ satisfying $F(r, d(r)) = 0$, or equivalently, $A(r, d(r)) = r$. By substituting $r^* = \mu > 0$ into expression (11) we obtain the following equation for $m^* : B(p(\mu, m^*)) = \mu d^*$.

Let $G(m) = B(p(\mu, m)) - \mu d^*$. Hence, because $B'(p(\mu, m)) > 0$ and $\frac{\partial p}{\partial m}(\mu, m^*) > 0$, we obtain $G'(m) = B'(p(\mu, m)) \frac{\partial p}{\partial m}(\mu, m^*) > 0$; we suppose that there is a unique solution $m^* > 0$ to expression (11) that is, $G(m^*) = 0$. Hence the value of d^* is then given by $A(\mu, d^*) = \mu$ and the value of m^* is given by $B[\mu - i(m^*)] = \mu d^*$. Then the third point to be considered is given by: $P_3^* = (r_3^*, m_3^*, d_3^*)$.

In order to study the stability of equilibrium point $P^* = P_3^* = (r_3^*, m_3^*, d_3^*)$, we compute the Jacobian matrix $J(P^* =)$ of the system (6) – (8). The signs of the real parts of the eigenvalues of $J(P^*)$ evaluated at a given equilibrium point $P^* = (r^*, m^*, d^*)$ determine its stability. Hence by using the Routh-Hurwitz criteria of stability we get:

Theorem 1: The interior solution $P^* = (r^*, m^*, d^*) > 0$ for the system (6) – (8) is locally asymptotically stable if the following conditions hold:

$$(i)\ \frac{\partial A}{\partial r}(\mu, d^*) < 1 \quad \text{and} \quad (ii)\ B'(\mu - i(m^*)) < d^*.$$

Foley (1987, p. 372) defines the variable $\varepsilon = \frac{\partial A}{\partial r}(\mu, d^*) - 1$. By considering that β is the elasticity of the rate of borrowing with respect to money and

π the elasticity of the rate of growth of capital outlays with respect to total liquidity of the enterprise, Foley shows that the system will be stable if: $\beta\pi < (\pi - \varepsilon)(1 - \varepsilon)$.

The feedback structure of our version of the Foley model can best be discussed by looking at the Jacobian matrix of the dynamics at the interior steady state $P^* = P_3^*$ from the qualitative point of view:

$$
\begin{aligned}
J(P^*) &= \begin{pmatrix} (A_r(P^*) - 1)\mu & 0 & A_d(P^*)\mu \\ -m^* & 0 & 0 \\ B'(P^*) - 1)(\mu - i(m^*)) - d^* & -B'(P^*)i'(m^*) & -\mu \end{pmatrix} \\
&= \begin{pmatrix} \pm & 0 & + \\ - & 0 & 0 \\ \pm & + & - \end{pmatrix}
\end{aligned}
$$

It is obvious that $\det J(P^*) < 0$ holds true, i.e., eigenvalues can only cross the imaginary axis away from its zero. This is the precondition for a cyclical loss of stability by way of a Hopf-bifurcation if this happens from the left to the right and if the third eigenvalue is negative (and must stay negative). Sufficiently large values of both A_r, B will moreover always make the system locally repelling, while small values of them together with a sufficiently small value of A_d (a small determinant) will make the steady state a local attractor. Theorem 1 provides more details on the latter situation and in particular shows that the last condition is not needed to ensure local asymptotic stability.

The forces that create local instability are thus the impact of profitability on debt accumulation as well as on output expansion, which when both large ensure this result from the purely qualitative perspective. These two supply side forces, a real and a financial accelerator process, therefore shape Foley's profit-rate liquidity cycle.

Let us now then study the possibility of the existence of a limit cycle in the system (6) – (8) by using the Hopf bifurcation analysis. In contrast to Foley (1987), we choose the growth rate of the money supply μ as a bifurcation parameter for system (6) – (8).[1] We assume for the initially given situation that theorem 1 holds true or more generally that the steady state $P^* = (r^*, m^*, d^*)$ is asymptotically stable for a given $\mu > 0$. We would like to know if P^*, which depends on μ, will lose its stability when the parameter μ changes. We consider the characteristic polynomial of the matrix $J(P^*)$, as given by $\lambda^3 + a_1(\mu)\lambda^2 + a_2(\mu)\lambda + a_3(\mu)$. The Routh-Hurwitz conditions for local asymptotic stability state this stability if and only if the parameters of the characteristic polynomial are all positive and if $E = a_1(\mu)a_2(\mu) - a_3(\mu) > 0$ holds true.

For $\mu = \mu^*$, the Jacobian matrix $J(P^*)$ has a pair of complex eigenvalues with zero real part if and only if:

$$E = a_1(\mu^*)\, a_2(\mu^*) - a_3(\mu^*) = 0 \tag{12}$$

[1] This differs – despite the use of the same symbol – from the parameter μ which Foley uses in his paper.

or, equivalently, $\left[\lambda^2(\mu^*) + a_2(\mu^*)\right]\left[\lambda(\mu^*) + a_1(\mu^*)\right] = 0$, which has three roots $\lambda_1(\mu^*) = i\sqrt{a_2(\mu^*)}$, $\lambda_2(\mu^*) = -i\sqrt{a_2(\mu^*)}$ and $\lambda_3(\mu^*) = -a_1(\mu^*) < 0$. For all μ, the roots are in general of the form $\lambda_1(\mu) = u(\mu) + iv(\mu)$ $\lambda_2(\mu) = u(\mu) - iv(\mu)$ and $\lambda_3(\mu) = -a_1(\mu)$.

The matrix $J(P^*)$ shows that $E = 0$ holds for $\mu = 0$ and $E > 0$ at the initially given situation. Since $a_1(\mu)a_2(\mu)$ is a quadratic function of μ times a term that depends on the moving equilibrium, while for $a_3(\mu)$ this only holds in a linear fashion (if the dependence on the remaining coefficients of the matrix $J(P^*)$ is ignored) we can assume that the functions A, B, i can easily be chosen such that the value of E becomes negative if the growth rate of money supply μ is decreased towards zero. The existence of the parameter μ^* considered above is therefore not difficult to show.

To apply the Hopf bifurcation theorem at $\mu = \mu^*$, we need to verify the transversality condition, namely:

$$\left[\frac{d\,Re(\lambda_i)(\mu)}{d\mu}\right]_{\mu=\mu^*} \neq 0, \quad i = 1,2 \tag{13}$$

This is the content of:

Theorem 2. If the following assumptions hold true, namely

$$a_3'(\mu^*) > \left[a_1(\mu)\, a_2(\mu)\right]'_{\mu=\mu^*}$$

,

$$E(\mu^*) = a_1(\mu^*)\, a_2(\mu^*) - a_3(\mu^*) = 0, \quad \lambda_3(\mu^*) = -a_1(\mu^*) < 0,$$

then at $\mu = \mu^*$, there exists a one-parameter family of periodic solutions bifurcating from the equilibrium point $P^* = (r^*, m^*, d^*)$ with period T, where $T \to T_0$ as $\mu \to \mu^*$, where $T_0 = 2\pi/\sqrt{a}_2$, with a_2 given by $a_1(\mu^*)\, a_2(\mu^*) - a_3(\mu^*) = 0$.

Remarks:

1. If this family of periodic orbits exists for $\mu \to \mu^*$ from above the Hopf-bifurcation is called subcritical while the opposite is called supercritical. In the latter case a stable limit cycle is generated when μ passes μ^* from below, while a stable 'corridor' – bounded by a periodic orbit – gets lost in the first case as the parameter μ reaches the bifurcation point μ^* from below. The character of the occurring Hopf-bifurcation is however difficult to determine in dimension three and thus must be a matter of numerical simulations of the model.
2. Similar results as for th parameter μ can also be shown for example for the function $i(\cdot)$ if this function is parameterized in an appropriate way.
3. For the mathematics employed in this paper and the subsequent simulations the reader is referred to M.C. Almeida and H.N. Moreira, A Mathematical Model of Immune Response in Cutaneous Leishmaniosis, Journal of Biological Systems, 15: 313-354, 2007. J. Guckenheimer and P.Holmes, Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, 3rd ed., Springer-Verlag, New York, 1990. M.W. Hirsch and S.Smale, Differential Equations, Dynamical Systems, and Linear Algebra, Academic press, New York, 1974. B.D. Hassard,

N.D.Kazarinoff, and Y-H. Wan, Theory and Applications of Hopf Bifurcation, London Mathematical Society, Lecture Note Series 41, Cambridge University Press, Cambridge, 1981. S.Lynch, Dynamical Systems with Applications Using MAPLE, 2rd ed.,Birkhauser, Boston, 2010. S.Wiggins, Introduction to Applied Nonlinear Dynamical Systems and Chaos, 2nd ed., Springer-Verlag, New York, 2003.

4 Numerical Simulations

We now present some numerical simulation results to verify the local asymptotically stability of equilibrium point and Hopf bifurcation of the system (6) – (8). In what follows let us assume that $A(r,d) = f(r)\,g(d)$. Besides, let us consider the system (6) – (8) with the following specification:

$$f(r) = \left(1+r^2\right)^2 \quad g(d) = 1 - \exp(-d)$$

$$B(r - i(m)) = b(r - i(m)), \quad i(m) = -m.$$

In this case $A(r,d) = f(r)\,g(d) = \left(1 + \left(1+r^2\right)^2\right)(1-\exp(-d))$. By substituting these functions in the system (6) – (8) we obtain:

$$r^{'} = r\left[-r + \left(1+r^2\right)^2(1-\exp(-d))\right], m^{'} = m(\mu - r), d^{'} = b(r+m) - rd.$$

The interior solution is given by: $P^* = (r^*, m^*, d^*)$, where

$$r^* = \mu, m^* = \frac{(d^* - b)\mu}{b}, d^* = \ln\left(\mu^4 + 2\mu^2 + 1\right) - \ln\left(\mu^4 + 2\mu^2 - \mu + 1\right),$$

with $d^* > b$. By inserting these results in the Jacobian we obtain:

$$J(P^*) = \begin{pmatrix} \mu\left(\frac{3\mu^2-1}{\mu^2+1}\right) & 0 & \mu\left(-\mu + \left(1+\mu^2\right)^2\right) \\ -m^* & 0 & 0 \\ b - d^* & b & -\mu \end{pmatrix},$$

The characteristic equation is given by: $\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0$, where:

$$\begin{aligned} a_1 &= tr\,J(P^*) = 2\mu\frac{\left(\mu^2-1\right)}{\mu^2+1}, \\ a_2 &= \min J(P^*) = -\mu^2\frac{\left(3\mu^2-1\right)}{\mu^2+1} + \mu\left(d^* - b\right)\left(-\mu + \left(1+\mu^2\right)^2\right), \\ a_3 &= \det J(P^*) = -\mu^2\left(d^* - b\right)\left(-\mu + \left(1+\mu^2\right)^2\right). \end{aligned}$$

Moreover,

$$
\begin{aligned}
E(\mu,b) &= \left[-\mu^2\left(d^*-b\right)\left(-\mu+\left(1+\mu^2\right)^2\right)\right] \\
&- \left[2\mu\frac{\left(\mu^2-1\right)}{\mu^2+1}\right]\left[-\mu^2\frac{\left(3\mu^2-1\right)}{\mu^2+1}+\mu\left(d^*-b\right)\left(-\mu+\left(1+\mu^2\right)^2\right)\right].
\end{aligned}
$$

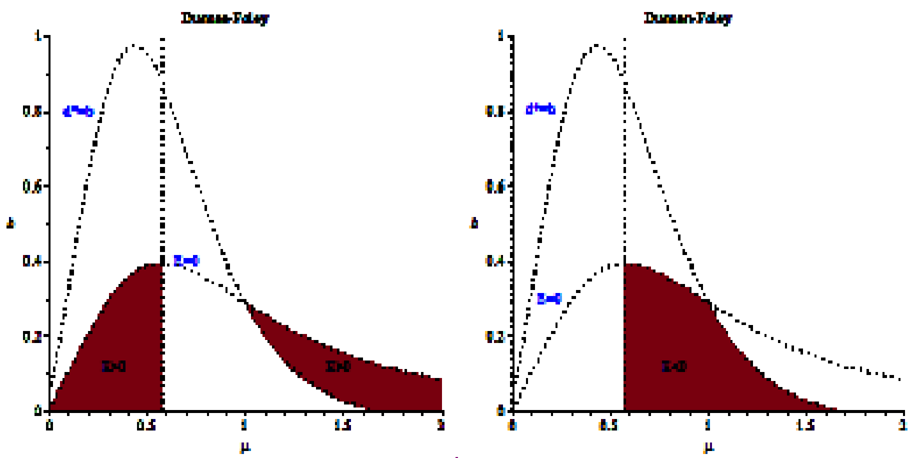

Figure 1: The region in the plane (μ, b), below the curve on the left, corresponds to a stable spiral steady state, that is, complex eigenvalues having a negative real part, and $E > 0$. The region in the plane (μ, b) , below the curve on the right, corresponds to a saddle spiral with unstable plane focus, with complex eigenvalues having a positive real part, and $E < 0$. The curve $E(\mu, b) = 0$, represents the Hopf bifurcations in the plane (μ, b).

We now present numerical simulations to verify the situation of a Hopf-bifurcation for the system (6) – (8) in the special formulation of this section. Here, we consider $\mu \in [0.5773502691,\ 0.5773502693]$, and $b = 0.3$. If we increase the values of μ from 0.5773502691 to 0.5773502693, we can show that $E\left(0.5773502691,\ 0.3\right) = 8.\,10^{-11} > 0$ and $E\left(0.5773502693,\ 0.3\right) = -3.\,10^{-11} < 0$. By the Intermediate Value Theorem, applied to function $E(\mu, 0.3)$, there is at least one $\mu^* \epsilon\,[0.5773502691,\ 0.5773502693]$ such that $E\left(\mu^*,\ 0.3\right) = 0$, that is, the complex eigenvalues are purely imaginary. So it is clear that system (6) – (8) enters into a Hopf bifurcation for increasing μ, and $r\left(t\right), m\left(t\right)$ and $d(t)$ show oscillations when $\mu = \mu^*$.

To see this, let us consider

(i) $\quad \mu = 0.5773502691,$ system (6) – (8) has an unique equilibrium $P^* = (0.5773502691,\ 0.1783750492,\ 0.3926863944)$, which is a stable spiral equilibrium, with eigenvalues

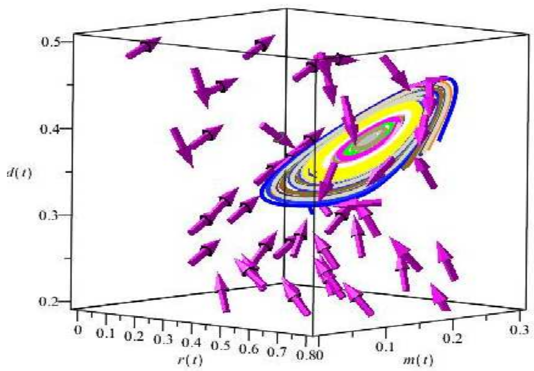

Figure 2: Various solutions in the space (r, m, d) when $\mu = \mu^{*}= 0.5773502692$, b = 0.3.

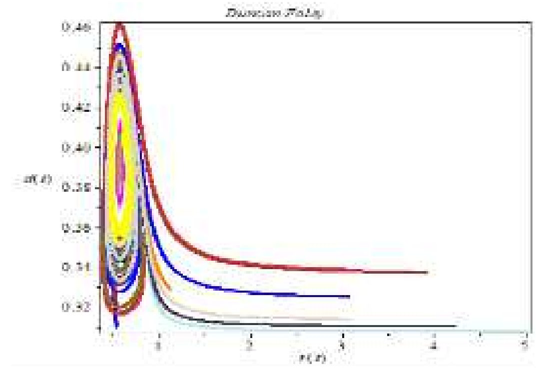

Figure 3: Projections of the solutions in the plane (r,d) when $\mu = \mu^{*}$=0.5773502692, b = 0.3.

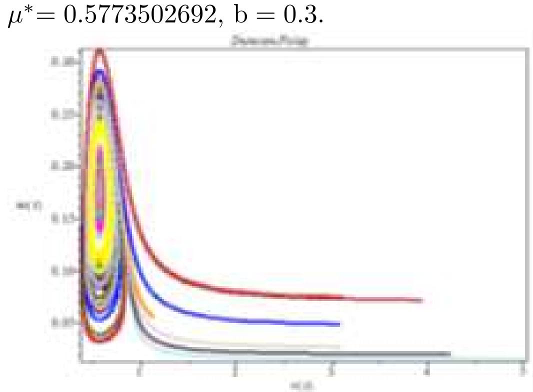

Figure 4: Projections of the solutions in the plane (r,m) when $\mu = \mu^{*}$=0.5773502692, b = 0.3.

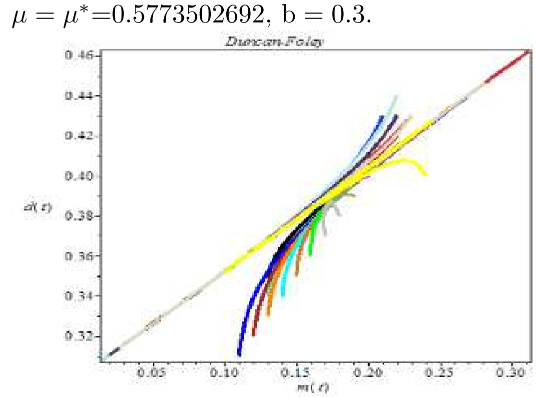

Figure 5: Projections of the solutions in the plane (m, d) when $\mu = \mu^{*}$=0.5773502692, b = 0.3.

$\lambda_1 = -0.577350269119614, \lambda_{2,3} = -4.90193108326764\,10^{-10} \pm 0.253451957477561\,i.$

Here:$\lambda_1 < 0, Re\lambda_2 < 0;\quad a_1 < 0,\ a_2 > 0,\ a_3 < 0,\ E(0.5773502691,\ 0.3) = 8.\,10^{-11} > 0;$

(ii) $\mu = 0.5773502693$, system $(6)-(8)$ has an unique equilibrium

$P^{*} = (0.5773502693,\ 0.1783750492,\ 0.3926863944),$

which is a saddle spiral with unstable plane focus, with eigenvalues:

$\lambda_1 = -0.577350269287299,\quad \lambda_{2,3} = 4.93649315869149\,10^{-10} \pm 0.253451957678457\,i.$

Here: $\lambda_1 < 0, Re\lambda_2 > 0;\quad a_3 < 0,\ E\,(0.5773502693,\ 0.3) = -3.10^{-11} < 0.$

Geometrically, we give an example in which $\mu = \mu^{*} \cong 0.5773502692$ and $b = 0.3$. System has an unique equilibrium point: $P^{*} = (0.5773502692,\ 0.1783750492,\ 0.3926863944)$, with eigenvalues

$$\lambda_1 = -0.577350269203456,$$

$$\lambda_{2,3} = 4.93649315869149\, 10^{-10} \pm 0.253451957678457\, i.$$

Here $\lambda_1 < 0$, Re $\lambda_2 > 0$; $\quad a_3 < 0,\ E\,(0.5773502692,\, 0.3) = 2.10^{-11} \cong 0.$
To see if the Hopf bifurcation occurs at $\mu = \mu^*$, we need to verify the transversality condition, namely:

$$\left[\frac{d\,Re\,(\lambda_i)\,(\mu)}{d\mu}\right]_{\mu=\mu^*} = \frac{\left[a_1\,(\mu)\,a_2\,(\mu)\right]'_{\mu=\mu^*} - a_3'\,(\mu^*)}{2\left[a_1^2\,(\mu^*) + a_2\,(\mu^*)\right]} \neq 0. \tag{14}$$

Calculating the derivative, we have

$$\left[\frac{d\,Re(\lambda_i)(\mu)}{d\mu}\right]_{\mu=\mu^*} = -0.7499999996 < 0. \tag{15}$$

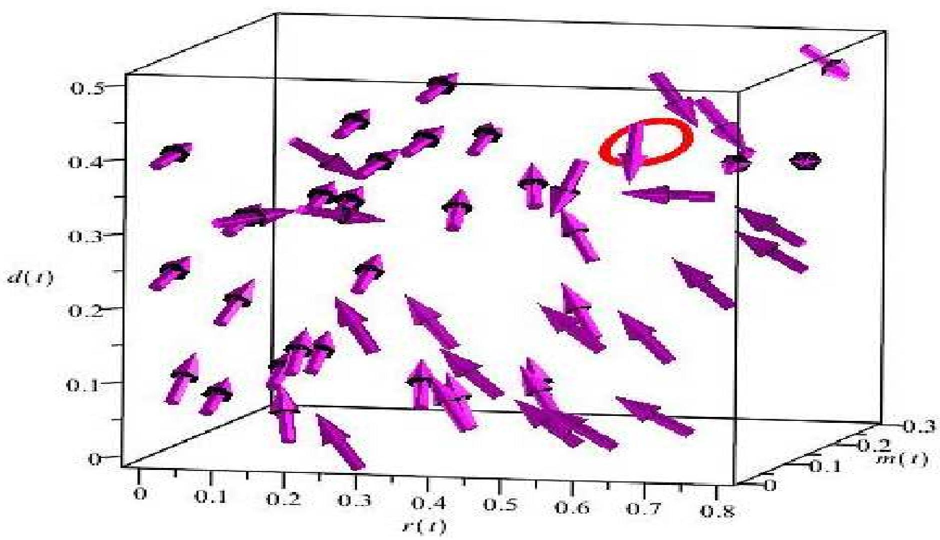

Figure 6: Phase portrait of a solution in space (r, m, d), when $\mu = \mu^*$=0.5773502692, b $= 0.3$. Initial condition $r\,(0) = 0.62, m\,(0) = 0.20, d\,(0) = 0.40$; $t = 0 \cdots 100$.

The oscillatory coexistence of $r(t), m(t)$ and $d(t)$ for $\mu = \mu^* \cong 0.5773502692,\ b = 0.3$, are illustrated in $Figs.2-5$, corresponding to the 25 initial conditions,[2] see also the appendix.

[2] These initial conditions are: $[x\,(0)\,, y\,(0)\,, z\,(0)] = [r\,(0)\,, m\,(0)\,, d(0)] =$

$[0.55, 0.13, 0.34]$ $[0.54, 0.13, 0.35]$ $[0.54, 0.11, 0.31]$ $[0.54, 0.12, 0.32]$ $[0.55, 0.13, 0.33]$

$[0.56, 0.14, 0.34]$ $[0.57, 0.15, 0.35]$ $[0.58, 0.16, 0.36]$ $[0.59, 0.17, 0.37]$ $[0.60, 0.18, 0.38]$

$[0.61, 0.19, 0.39]$ $[0.62, 0.20, 0.40]$ $[0.63, 0.21, 0.41]$ $[0.64, 0.22, 0.42]$ $[0.65, 0.23, 0.43]$

$[0.63, 0.20, 0.40]$ $[0.62, 0.21, 0.40]$ $[0.62, 0.22, 0.42]$ $[0.63, 0.23, 0.42]$ $[0.63, 0.21, 0.43]$

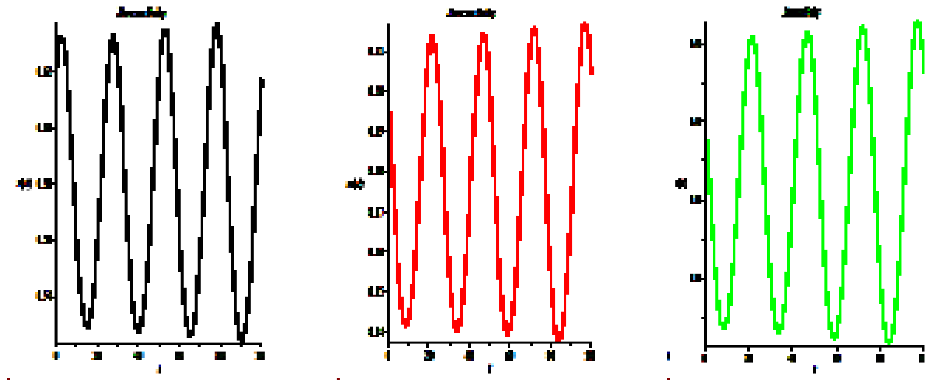

Figure 7: Time series solutions in the planes t, r, t, m and t, d when $\mu = \mu^* = 0.5773502692$, $b = 0.3$. Initial conditions $r(0) = 0.62$, $m(0) = 0.20$, $d(0) = 0.40$ and $t = 0...100$.

5 Concluding Remarks

In this paper we have studied the existence of limit cycles in a modified version of Foley's model, in which the interest rate is assumed to be a function of the amount of money per unit of capital, whereas Foley assumes that the profit / interest gap is a function of the amount of money per unit of capital. The outcome is a model that emphasizes the role of money, besides capital outlays, as an important determinant of the economic dynamics. It was shown that the model displays a stable equilibrium growth path that can be destabilized into a limit cycle due to a decrease in the growth rate of the supply of money. Like in the original Foley's model there was also here no explicit study of price level changes, since the profit margins on sales was taken as given. A more satisfactory approach would include an explicit treatment of the determination of profit margins, and hence of movements of the aggregate price level, which is here left for future research.

References

Foley, D. (1986), Liquidity-profit rate cycles in a capitalist economy. Journal of Economic Behavior and Organization, 8, 363 – 376.

Goodwin, R.M., 1982, Essays in Economic Dynamics (Macmillan, London).

Hicks, J. (1950), A Contribution to the Theory of the Trade Cycle (Oxford University Press, Oxford)

$[0.63, 0.22, 0.44]$ $[0.63, 0.22, 0.43]$ $[0.62, 0.23, 0.43]$ $[0.61, 0.23, 0.40]$ $[0.60, 0.24, 0.40]$

Kalecki, M., 1969, Studies in the Theory of Business Cycles: 1933-1939 (Irwin, Homewood, IL).

Appendix

The Jacobian matrix at any singular point $P_i^* = (r_i^*, m_i^*, d_i^*), i = 1, .., 3$ is given by $(p(r_i^*, m_i^*) = r_i^* - i(m_i^*))$:

$$J(P_i^*) = \begin{pmatrix} A(r_i^*, d_i^*) + r_i^* \frac{\partial A}{\partial r}(r_i^*, d_i^*) - 2r_i^* & 0 & r_i^* \frac{\partial A}{\partial d}(r_i^*, d_i^*) \\ -m_i^* & \mu - r_i^* & 0 \\ B'(p(r_i^*, m_i^*)) \frac{\partial p}{\partial r}(r_i^*, m_i^*) - d_i^* & B'(p(r_i^*, m_i^*)) \frac{\partial p}{\partial m}(r_i^*, m_i^*) & -r_i^* \end{pmatrix}$$

The characteristic equation of the Jacobian matrix is given by: $\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$, where

$$\begin{aligned} a_1 &= -\left[A(r_i^*, d_i^*) + r_i^* \frac{\partial A}{\partial r}(r_i^*, d_i^*) - 4r_i^* + \mu\right] \\ a_2 &= (\mu - 2r_i^*)\left[A(r_i^*, d_i^*) + r_i^* \frac{\partial A}{\partial r}(r_i^*, d_i^*) - 2r_i^*\right] \\ &- r_i^* \frac{\partial A}{\partial d}(r_i^*, d_i^*)\left[B'(p(r_i^*, m_i^*)) \frac{\partial p}{\partial r}(r_i^*, m_i^*) - d_i^*\right] - r_i^*(\mu - r_i^*) \\ a_3 &= -r_i^*(\mu - r_i^*)\left[A(r_i^*, d_i^*) + r_i^* \frac{\partial A}{\partial r}(r_i^*, d_i^*) - 2r_i^*\right] \\ &- r_i^* \frac{\partial A}{\partial d}(r_i^*, d_i^*)\left[B'(p(r_i^*, m_i^*)) \frac{\partial p}{\partial r}(r_i^*, m_i^*) - d_i^*\right] \\ &- r_i^* m_i^* B'(p(r_i^*, m_i^*)) \frac{\partial p}{\partial m}(r_i^*, m_i^*) \frac{\partial A}{\partial d}(r_i^*, d_i^*) \end{aligned}$$

Proof of Theorem 1

From Routh-Hurwitz criteria, the system (6) – (8) is stable around the positive equilibrium point $P_3^* = (r_3^*, m_3^*, d_3^*)$ if the conditions $a_1 > 0, a_2 > 0, a_3 > 0, a_1 - a_2 a_3 > 0$ are satisfied. But we know that: $a_1 = \mu\left[2 - \frac{\partial A}{\partial r}(\mu, d_3^*)\right]$, since it is assumed that $\frac{\partial A}{\partial r}(\mu, d_3^*) < 1$, hence $\frac{\partial A}{\partial r}(\mu, d_3^*) < 2$, which yields $a_1 > 0$.

The value of a_2 is given by:
$a_2 = -\mu\left\{\mu\left[\frac{\partial A}{\partial r}(\mu, d_3^*) - 1\right] + \frac{\partial A}{\partial d}(\mu, d_3^*)\left[B'(p(\mu, m_3^*)) - d_3^*\right]\right\}.$
Then, if $\frac{\partial A}{\partial r}(\mu, d^*) < 1$ and $B'(p(\mu, m_3^*)) < d_3^*$, we get

$$\mu \frac{\partial A}{\partial r}(\mu, d_3^*) - \mu + \frac{\partial A}{\partial d}(\mu, d_3^*)\left[B'(p(\mu, m_3^*)) - d_3^*\right] < 0.$$

The value of a_3 is given by: $a_3 = \mu m^* \frac{\partial A}{\partial d}(\mu, d_3^*) B'(p(\mu, m_3^*)) \frac{\partial p}{\partial m}(\mu, m_3^*)$.
Note this condition is easily satisfied, since: $\frac{\partial A}{\partial d}(r, d) > 0$, $B'[.] > 0$ and $\frac{\partial p}{\partial m} =$

$-i'(m) > 0$, because of $i^{'}(m) < 0$. The value of $a_1a_2{-}a_3$ is given by:

$$\begin{gathered} a_1a_2{-}a_3 = \\ \mu\frac{\partial A}{\partial d}(\mu, d_3^*)\{\mu\left[\frac{\partial A}{\partial r}(\mu, d_3^*) - 1\right][B'(p(\mu, m_3^*)) - d_3^*] \\ +(-m_3^*)B'(p(\mu, m_3^*))\frac{\partial p}{\partial m}(\mu, m_3^*)\}. \end{gathered}$$

Note that if $\frac{\partial A}{\partial r}(\mu, d^*) < 1$ and $B'(p(\mu, m_3^*) < d_3^*$, then $a_1 - a_2a_3 > 0$, since $B^{'}(p(\mu, m_3^*))\frac{\partial p}{\partial m}(\mu, m_3^*) < 0$. Hence, from the Routh-Hurwitz criteria, the system $(6) - (8)$ is stable around the positive equilibrium point $P_3^* = (r_3^*, m_3^*, d_3^*)$.

Proof of Theorem 2

Substituting $\lambda_i(\mu) = u(\mu) \pm iv(\mu)$ into (13) and calculating the derivative, we get:

$$A(\mu)u^{'}(\mu) - B(\mu)v^{'}(\mu) = -C(\mu) \tag{16}$$

$$B(\mu)u^{'}(\mu) + A(\mu)v^{'}(\mu) = -D(\mu), \tag{17}$$

where

$A(\mu) = 3u^2(\mu) + 2a_1(\mu)u(\mu) + a_2(\mu) - 3v^2(\mu)$

$B(\mu) = 6u(\mu)v(\mu) + 2a_1(\mu)v(\mu)$

$C(\mu) = a_1^{'}(\mu)u^2(\mu) + a_2^{'}(\mu)u(\mu) + a_3^{'}(\mu) - a_1^{'}(\mu)v^2(\mu)$

$D(\mu) = 2a_1^{'}(\mu)u(\mu)v(\mu) + a_2^{'}(\mu)v(\mu)$

By considering that $u(\mu^*) = 0, v(mu^*) = \sqrt{a_2(\mu^*)}$ we have:

$A(\mu^*) = -2a_2(\mu^*)$,

$B(\mu^*) = 2a_1(\mu^*)\sqrt{a_2(\mu^*)}$,

$C(\mu^*) = a_3^{'}(\mu^*) - a_1^{'}(\mu^*)a_2(\mu^*)$

$D(\mu^*) = a_2^{'}(\mu^*)\sqrt{a_2(\mu^*)}$

Solving for $u'(\mu^*)$ from (16) ,(17), we obtain

$\left[\frac{d\,Re(\lambda_i)(\mu)}{d\mu}\right]_{\mu=\mu^*} = u^{'}(\mu^*) = -\left[\frac{A(\mu^*)C(\mu^*)+B(\mu^*)D(\mu^*)}{A^2(\mu^*)+B^2(\mu^*)}\right]$

From the equations $A(\mu^*), .., D(\mu^*)$ we then get:

$$\begin{aligned} \left[\frac{d\,Re(\lambda_i)(\mu)}{d\mu}\right]_{\mu=\mu^*} &= \frac{a_3^{'}(\mu^*) - a_1(\mu^*)a_2^{'}(\mu^*) - a_1^{'}(\mu^*)a_2(\mu^*)}{2\left[a_1^2(\mu^*) + a_2(\mu^*)\right]} \\ &= \frac{a_3^{'}(\mu^*) - \left[a_1(\mu)a_2(\mu)\right]^{'}_{\mu=\mu^*}}{2\left[a_1^2(\mu^*) + a_2(\mu^*)\right]} > 0 \end{aligned}$$

if the following conditions hold:

$$a_3^{'}(\mu^*) > \left[a_1(\mu)a_2(\mu)\right]^{'}_{\mu=\mu^*}, a_1(\mu^*)a_2(\mu^*){-}a_3(\mu^*) = 0, \lambda_3(\mu^*) = -a_1(\mu^*) < 0.$$

This implies that a Hopf-bifurcation occurs at $\mu{=}\ \mu^*$ and that it is non-degenerate.