Optimal Use of Put Options in a Stock Portfolio

Bell, Peter N

University of Victoria

13 March 2014

Optimal Use of Put Options in a Stock Portfolio

Peter N. Bell

Department of Economics, University of Victoria

March 29, 2014

This research was supported by the Joseph-Armand Bombardier Canada Graduate Scholarship – Doctoral through the Social Sciences and Humanities Research Council from the Government of Canada.

Abstract

I analyze a portfolio optimization problem where an agent holds an endowment of stock and is allowed to buy some quantity of a put option on the stock. My model rephrases a fundamental question from insurance economics: how much coverage should a risk averse agent buy? Classic studies of rational insurance purchasing use exact algebraic analysis with a binomial probability model of portfolio value to explore this problem. In contrast, I use numerical techniques to approximate the probability distributions for key variables. Using large-sample, asymptotic analysis, I identify the optimal quantity of put options for three types of preferences over the distribution of portfolio value. The location of the optimal quantity varies with preferences and provides examples of important concepts from the rational insurance purchasing literature: coinsurance for log utility ($q^*<1$), full-insurance for quantile-based preferences ($q^*=1$), and over-insurance for mean-variance utility ($q^*>1$). I use resampling analysis to show that the optimal quantity is well defined for mean-variance and quantile-based preferences, but the optimal quantity for log utility is not stable. Although my analysis corroborates the classic result that coinsurance is optimal for log utility, I show that the specific amount of coinsurance is not well defined. In addition, the optimal quantity for mean-variance utility in my model is not allowed in a classic insurance model. By matching and extending the set of results for basic rational insurance purchasing, my research demonstrates the value of using numerical techniques to analyze the optimal use of financial derivatives in a continuous setting.

Keywords: Portfolio optimization, financial derivative, put option, quantity, expected utility, numerical analysis

Optimal Use of Put Options in a Stock Portfolio

Introduction

In this working paper I explore how to *use* financial derivatives. I present a portfolio optimization problem where an agent holds one unit of stock and is allowed to buy a put option on the stock. What quantity of put options should he buy? Although this question receives brief attention from Philip Jorion in terms of the minimum variance hedge ratio (2007, p. 296), I frame my research in context of a topic in insurance economics: rational choice of insurance coverage. The classic approach to rational insurance purchasing uses an exact, analytic solution to the portfolio optimization problem (Mossin, 1968). I update this approach with modern perspectives on financial derivatives and numerical analysis.

Jan Mossin is an influential scholar in the economic theory of risk taking. His 1968 article provides the phrase rational insurance purchasing to describe the analysis of insurance from the buyer's perspective. The article attempts to "illustrate the power of the expected utility approach to problems of risk taking" (1968, p. 553) by exploring three questions: the maximum premium an agent would pay to buy insurance, the optimal amount of insurance coverage at a given premium, and the optimal deductible amount. I focus on Mossin's contribution to this second question, the optimal amount of insurance coverage, because it can be rephrased today as: how best to use financial derivatives to mitigate the risk of loss.

Mossin separates wealth into several different terms (safe wealth, risky wealth, and size of loss). Although this definition of wealth requires cumbersome notation, given in Equation (1), it produces a parsimonious model for optimal coverage (1968, p. 557). Mossin uses the model to generate testable implications about risk aversion based on insurance choices (1968, p. 564).

$$Y = A + L - X + (C/L)X - p\,C \qquad (1)$$

The key variable in Mossin's model is total wealth or portfolio value (Y). The other variables are defined as follows: safe wealth (A), risky wealth (L), size of loss (X), price of insurance (p), and amount of coverage for loss (C). Since the size of loss (X) is the only random variable in the model, total wealth can be seen as the random value for an asset (A+L– X) plus the net payoff for a financial derivative ((C/L)X – p C). This separation of total value into two terms is a simple and powerful idea that I will use in my model.

The objective in Mossin's model is to maximize expected utility over wealth (E(U(Y))). The agent's choice variable is the level of coverage (C), which is constrained because an agent cannot buy more coverage than the value of the asset ($0 \leq C \leq L$). Mossin solves the optimization problem algebraically. He shows that the first order conditions evaluated at the boundaries for the choice variable (C=0 or L) justify an interior solution (1968, p. 557), but he is not able to produce an exact formula for the solution in general.

To gain analytic traction, Mossin specifies a model that yields an analytic solution (log utility and binomial probability model for loss). He uses comparative statics to show how the optimal coverage changes with wealth, which provides the foundation for the testable predictions mentioned above (1968, p. 558). The binomial model that Mossin uses is a classic part of insurance economics because it characterizes a situation where the agent suffers either no loss or the complete loss of an asset. The binomial model provides an exact solution, which is valuable in modern mathematical economics, but the results are limited because they only consider losses of one size.

Mossin's 1968 article was influential. It was extended by Razin (1976) to consider the minimax regret function from Leonard Savage's decision theory and again by Briys & Loubergé (1985) to consider bounded rationality, an important extension to rational choice theory. Since both subsequent articles extended Mossin's model by changing the objective function, I use three types of preferences in my model. I use the familiar log and mean-variance utility functions, and a quantile-based objective function inspired by Value at Risk (VaR). Although the quantile-based objective is not a utility function, it is relevant to risk management decisions.

Both Razin (1976) and Briys & Loubergé (1985) kept the binomial probability model that Mossin introduced. My model maintains the basic structure of wealth as a random asset plus a financial derivative, but I abandon the binomial model. Instead, I focus on numerical analysis of continuous probability models. I specify an interval of values for the choice variable, simulate the distribution of wealth for each value as a statistical ensemble, and then compare the distributions to identify the optimal quantity of put options.

Model Setup

To develop my model, I briefly discuss assumptions about the agent and his portfolio. I assume that the agent cares only about the portfolio value when the derivative expires, as in Mossin (1968). The portfolio value at expiry is random, but the agent knows the probability distribution for the value. The agent also knows how the derivative affects the distribution of portfolio value; thus, he can rank different quantities of put options. This model is classic decision theory: portfolio optimization with perfect information.

I assume the portfolio is composed of one unit of stock and some quantity of European put options. The agent knows the initial value of the stock and the distribution of the future value. The put option is infinitely divisible, the strike price is equal to the initial stock price (at the money), and the stock expires in one time step (one year). Equation (2) represents the agent's portfolio optimization problem in this simple setting.

(2) $\max_{q>0} V(W(q)) \quad \text{s.t.} \quad W(q) = S + N(S,q)$

The choice variable in Equation (2) is the quantity of derivatives (q). The objective function is denoted as V(), which can be thought of as expected utility. For robustness, I use three different forms for V(). The forms represent important preferences in the literature (expected log utility, mean-variance utility[1], and 5% quantile[2]). The value of the portfolio at expiry is denoted W(q), which is a random variable. The value of the stock at expiry is denoted as S and net payoff for the derivative is N(S,q).

(3) $N(S,q) = q\,[\,(K-S)^{+} - O\,]$

Equation (3) defines the net payoff for a put option. The quantity (q) appears as a linear, multiplicative term. The term $(K-S)^{+}$ is the intrinsic value of the put at expiry. As above, I assume the strike (K) is equal to the stock price when the agent makes the initial decision for the quantity (q). I calculate the option price (O) using the Black-Scholes formula because the stock price is log-normal. When the agent picks q, they do not know the net payoff of the put option because the value of N(S,q) depends on the future value of the stock (S), which is random.

---

[1] I use a standard value for the risk aversion coefficient ($\lambda=0.1$) for the mean-variance utility ($U(X)=\mu-(\lambda/2)\sigma^2$).
[2] Defined by the distribution of portfolio value (W(q)). It is *w* such that: $\Pr(W(q)<w)=0.05$. Note that VaR is the 5% quantile from the loss distribution; a larger VaR is bad, but a larger quantile for wealth is good.

Numerical Analysis

In the numerical analysis of my model, I assume specific values for all parameters. For example, I assume that the initial price of the stock is $100 and the returns are normally distributed with 0% average and 10% volatility for one time step. This model for asset value satisfies the random walk hypothesis and provides a basis for pricing the put option with Black-Scholes. I provide further details on these assumptions in an appendix, which contains Matlab code that can reproduce all of my results.

Asymptotic Analysis

For each value of q in an interval ([0.00,2.00] with step size 0.01), I simulate the stock price a large number of times (1,000,000) to estimate the distribution of portfolio value for that quantity. I calculate utility over the distribution and analyze it in several different ways. To begin, I report the optimal quantity (q*) for each type of preferences in Table 1.

Table 1: Optimal quantity of put options for different preferences

Preferences	Optimal Quantity (q*)
Log Utility	0.63
Mean-Variance Utility	1.57
5% Quantile	1.00

Table 1 shows that the optimal quantity differs across the three preferences. The standard question for the insurance literature is whether it is better to have full insurance (q*=1) or coinsurance (q*<1) and Table 1 shows that both are optimal under different preferences in my model. As in prior research, the optimal choice for log utility is coinsurance (q*=0.63). However, the optimal choice for quantile-based preferences is and full insurance (q*=1.00), which is a striking result in a numerical setting because it is on a knife-edge. Table 1 also shows that the optimal quantity for mean-variance utility is well above one (q*>1), which is inadmissible in the classical model of rational insurance purchasing. Table 1 suggest that my modelling framework can generate results that match and extend the classic results from a rational insurance purchasing model.

Now that I have identified the optimal choice across different preferences, I briefly characterize each optimum. I do this by estimating the shape of the objective function over a range of values for the choice variable. This is straightforward because the objective function and choice variable are each 1-dimensional in my model. In Table 2 I report a money-metric associated with the expected utility for each value of the choice variable and utility measure. For the log and mean-variance utility, this money-metric is the certainty equivalent. For the quantile-based objective function, the money-metric is the 5% quantile from the distribution of portfolio value. For each type of preference, a higher value is preferred to a lower value.

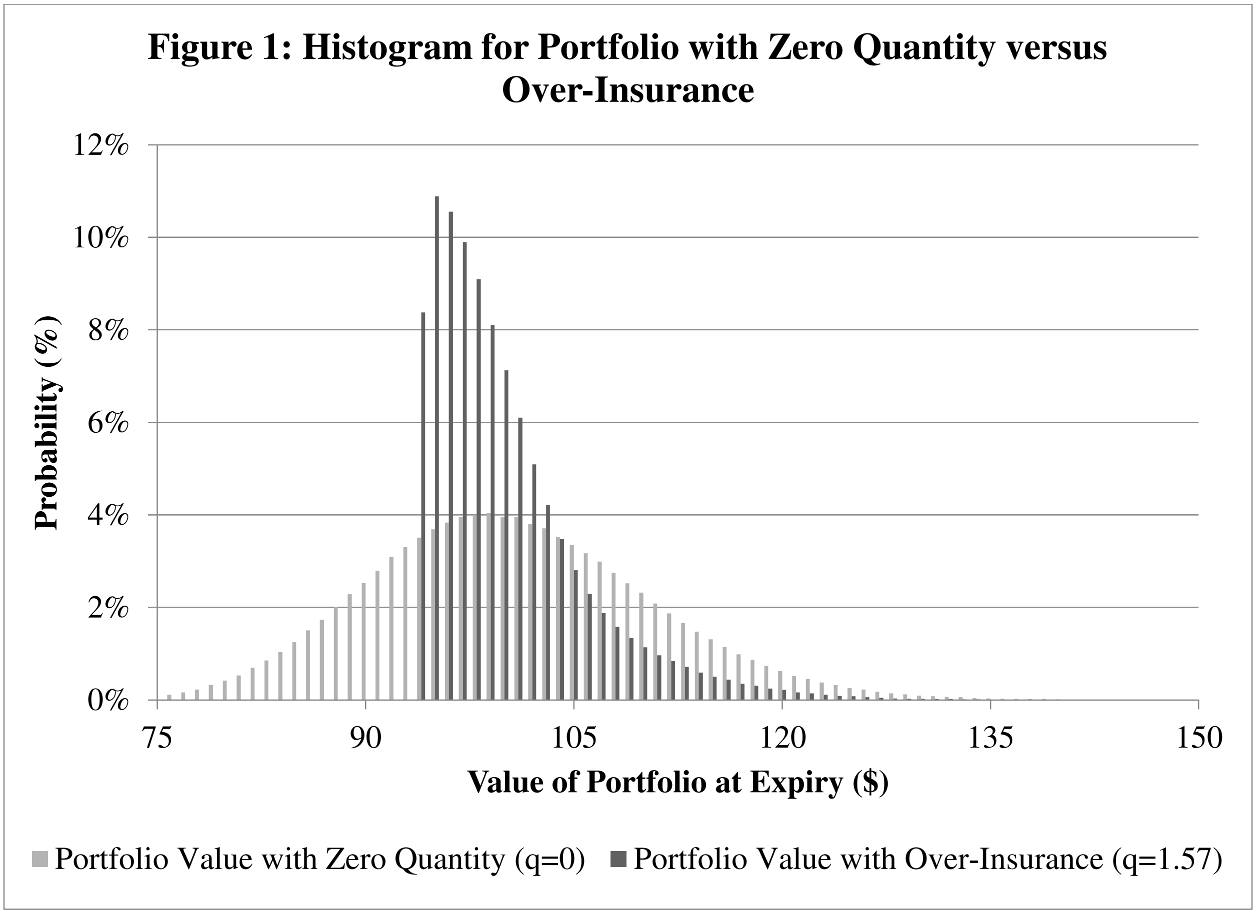
Table 2: Shape of objective function for range of values for choice variable

	0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
Log Utility	100.00	100.03	100.07	***100.08***	***100.08***	100.08	100.04	100.00	99.98	99.92	99.84
Mean-Variance	49.83	58.16	65.30	71.25	76.10	79.89	82.49	***83.90***	***84.07***	83.13	81.25
5% Quantile	84.87	87.07	89.31	91.53	93.78	***96.01***	***95.42***	94.77	94.09	93.38	92.65

Table 2 provides a basic sense of the shape of the objective function using a sparse sampling rate for the choice variable (0.2 units apart). I highlight two data points around the optimal quantity for each type of preferences in ***bold and italics***. Table 2 shows that the optimal quantities for the mean-variance utility and the quantile-based objective functions are both well-defined because the objective functions are concave around the optimums. In contrast, the results for the log utility raise concerns because there is little variation in the agent's valuation of different portfolios. The results suggest that optimal quantity for log utility may not be robust to sample selection. Although Table 2 uses sparse sampling points, the results give confidence in optimum under the mean-variance utility and the 5% quantile preferences, but not the log utility.

When an agent with mean-variance utility holds zero put options (q=0), Table 2 shows that he would trade the portfolio for \$50. Note that the initial value of the stock is \$100. The large difference between valuations speaks to the negative effect of risk on a risk-averse agent. If the same agent buys a close approximation to his optimal quantity of put options (q*=1.6), then he would be much better off and would trade the portfolio for \$84.07. This significant increase shows the value of risk management in a basic portfolio context.

Figure 1 shows how a put option changes the probability distribution of portfolio value. The figure shows a discrete approximation for the continuous density function. The shape marked by light bars represents the portfolio with zero put options and the shape marked by dark bars represent the portfolio with over-insurance. When an agent decides what quantity of put option to buy, he is effectively picking which distributions he likes best. As such, visualizing these distributions can tell us a lot about how preferences are driving decisions.



There are two important features in Figure 1. First, the probability of low values for the portfolio is less when the agent buys put options; this is because put options are designed to offset losses. Notice how the value of the portfolio with derivatives is never less than $90. Second, the probability of high values for the portfolio is also less when the agent buys put options; this is because the agent has to pay for the put options in the good times. Notice how the right tail is lower when the agent buys the option. These two features show that over-insurance reduces the frequency and severity of both high and low values for the portfolio, which reduces the variance of the portfolio value at both ends and benefits a risk-averse agent.

Robustness to Small Sample

The analysis so far uses a single, large sample to establish all results. This asymptotic analysis may hide variability that develops as the sample changes. To detect such variability I conduct further simulations with resampling. Basically, I repeat the analysis above to estimate the distribution for the optimal value, but I use a smaller sample size (1,000 draws of S) and loop the calculation many times (10,000 estimates of q*). I present the distribution of the optimal values in Table 3.

Table 3: Frequency for location of optimal quantity (q*) by preferences

	0-0.2	0.2-0.4	0.4-0.6	0.6-0.8	0.8-1.0	1.0-1.2	1.2-1.4	1.4-1.6	1.6-1.8	1.8-2.0
Log-Utility	18%	29%	22%	15%	9%	4%	2%	1%	0%	0%
Mean-Variance Utility	0%	0%	0%	0%	0%	0%	3%	43%	49%	5%
5% Quantile	0%	0%	0%	0%	0%	100%	0%	0%	0%	0%

Table 3 shows the distribution of the optimal quantity across each objective function. The columns in Table 3 are closed on the lower side and open on the upper side (0-0.2 denotes the interval [0,0.2]). Table 3 shows that the distribution of the optimal value varies greatly by preferences. Again, the results for the quantile-based preferences are striking because the optimum is always located at the special point of full insurance (q*=1.00 with 100% frequency). The results for the mean-variance utility are encouraging because they show a symmetric, narrow distribution around the optimal value. Thus, the optimal quantity for the quantile-based and mean-variance preferences are, in some sense, well behaved.

Table 3 shows the optimal quantity for log utility is not well behaved. The optimal quantity is broadly dispersed between 0 and 1, indicating that the optimal quantity for log utility is not robust to sample selection. This echoes the potential problem I noted in Table 2 for log utility. A classic result from Mossin is that coinsurance is optimal for log utility (1968, p. 558). My analysis shows that coinsurance is generally the optimal choice for log utility, but not always; the optimal quantity for log utility is actually greater than 1.0 with 7% frequency.

Conclusion

This paper analyzes a portfolio optimization problem where an agent holds one unit of stock and is allowed to buy put options on the stock. This is a specific example of the general situation where an agent endowed with an asset is allowed to trade derivatives on the asset. I position this research in relation to the question of optimal coverage in the literature on rational insurance purchasing. The rational insurance purchasing literature uses algebraic analysis to identify an exact solution under a binomial probability model for asset values (Mossin, 1968). In contrast, I use numerical analysis to approximate a solution under a log-normal probability model. This different analytic perspective allows me to develop rich insight into the optimization problem by approximating continuous variables of interest.

The results show that the character of the optimal quantity depends on the agent's preferences. I find it is always optimal to have full insurance for quantile-based preferences, which is a striking numerical result that deserves further attention because the quantile-based preferences are designed after the VaR risk measure. As in the classic insurance literature, I find it is generally optimal to have coinsurance under log utility. However, I find the specific amount of coinsurance is not robust to resampling. Sometimes the optimal quantity is greater than one, which means over-insurance is optimal under log utility. These results reflect problems with log utility that are related to issues raised by Ole Peters (2011).

The rational insurance purchasing literature explicitly disallows over-insurance (Mossin, 1968, p. 557). In contrast, my results show that over-insurance is the optimal choice for the mean-variance utility function. To show how over-insurance affects an agent, I compare the probability distribution of portfolio value for zero insurance against over-insurance. I find that, in bad times, over-insurance decreases the severity and frequency of low values for the portfolio because the agent receives the option payoff. In good times, over-insurance decreases the severity and frequency of high values for the portfolio because the agent pays the option premium. Thus, over-insurance reduces the frequency and severity for both high and low values of the portfolio, which reduces the variance and benefits a risk-averse agent. My analysis demonstrates that numerical techniques are valuable in this setting because they allow us to identify optimal values with continuous probability models and continuous choice variables, which is not possible in the classic analytic model of Mossin.

There are, of course, some limitations to my research. By using numerical techniques, I have picked arbitrary values for many parameters and it is possible that different values for parameters may change the qualitative nature of the results. Interested readers could investigate the parameters in the stock price, the level of risk aversion, or the percentile used in the quantile-based preferences. Another limitation is my simple assumptions about the agent's portfolio. It is possible that different values for the strike price of the option or the timing of cash flows could change the results further. The model also takes a simple view of randomness; I use known randomness, not Knightian uncertainty. It may be possible to extend the analysis to a Bayesian setting with subjective beliefs about probability distributions and preferences, which may be a useful guide for design of experiments with human subjects. Finally, all of my analysis has used ensemble averages and the results may be very different with time averages; Ole Peters (2011) has shown that time averages resolve misconceptions at the heart of expected utility theory associated with the St. Petersburg paradox.

References

Briys, E.P., & Loubergé, H. (1985). On the Theory of Rational Insurance Purchasing: A Note. *The Journal of Finance, 40*(2), 577-581.

Jorion, P. (2007). *Financial Risk Manager* (4th ed.). Hoboken, NJ: John Wiley and Sons Inc.

Mossin, J. (1968). Aspects of Rational Insurance Purchasing. *Journal of Political Economy, 76*(4), 553-568.

Peters, O. (2011). The time resolution of the St. Petersburg paradox. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences, 369*(1956). 4913-4931.

Razin, A. (1976). Rational Insurance Purchasing. *The Journal of Finance, 31*(1), 133-137.

Appendix

```
%%  Code Appendix -- Optimal Use of Derivatives
%   © Peter Bell, March 10 2014
%   Written for Matlab to produce all results used in working paper.
%
%%  Section 1:  Global Parameters
%   Set random number generator
clear all
stream = RandStream('mt19937ar','Seed',12);
RandStream.setDefaultStream(stream);

%   Simulate price for stock
numPrice = 10^6;    S0 = 100;   sigma=0.1;

%   Simulate option price
K = 100;    r = 0;
d1 = (1/sigma)*(log(S0/K)+r+sigma^2/2);     d2 = d1 - sigma;
O = cdf('norm',-d2,0,1)*K - cdf('norm',-d1,0,1)*S0;

%   Agent Utility
lambda = 0.5; %

%   Simulations in Section 3 for Asymptotic Setting
numSimOne = 201;  qScaleOne=100;  qStep=1/qScaleOne;
resultTable = zeros(3,numSimOne);
%   numSimOne represents # points for choice variable
%   qScaleOne is parameter to make so that consider q in [0,2]
%   Each simulation has length numPrice (10^6), specified above


%   Simulations in Section 4 for small samples
numSimTwo= 10^4;    numPriceSmall = 1000;
%   numSimTwo represents # of times that identify optimal quantity (q*)
%   numPriceSmall is length of time series, which replaces numPrice


%%  Section 2:  Demo with single value for choice variable
%   Goal: demonstrate how particular quantity affects utility
q = 0.5;
S = S0*exp(randn(numPrice,1)*sigma);
W = S + q*(max(K-S,0)-O);

%   Log Utility
exp(mean(log(S)))
exp(mean(log(W)))

%   Mean-Variance Utility
mean(S) - lambda*var(S)
mean(W) - lambda*var(W)

%   5% Quantile for distribution
temp1 = sort(S);
```

```
temp1(length(temp1)*5/100)
temp2 = sort(W);
temp2(length(temp1)*5/100)


%%  Section 3:  Analyze shape of objective function in asymptotic setting
%   Goal: Calculate material for Table 1, 2, and Figure 1.
%
for numChoice = 1:numSimOne
    qLoop = (numChoice-1)/qScaleOne
    %   Simulate large number of prices for each q, calculate objective
    S = zeros(1,1);     W = zeros(1,1);
    S = S0*exp(randn(numPrice,1)*sigma);
    W = S + qLoop*(max(K-S,0)-O);

    %   Log Utility
    resultTable(1,numChoice) = exp(mean(log(W)));

    %   Mean-Variance Utility
    resultTable(2,numChoice) = mean(W) - lambda*var(W);

    %   5% Quantile for distribution
    temp2 = sort(W);
    resultTable(3,numChoice) = temp2(length(temp2)*5/100);
    resultTable(4,numChoice) = qLoop;
end

%   Table 1:
qTemp = 1:20:220;
tableOne = [(qTemp-1)*qStep;resultTable(1:3, qTemp)];

%   Table 2:    Optimal Choice by Utility
[uMaxLog iMaxLog] = max(resultTable(1,:));
[uMaxMeanVar iMaxMeanVar] = max(resultTable(2,:));
[uMaxQuantile iMaxQuantile] = max(resultTable(3,:));

qStarLog = (iMaxLog-1)*qStep;
qStarMeanVar = (iMaxMeanVar-1)*qStep;
qStarQuantile = (iMaxQuantile-1)*qStep;

tableTwo = [qStarLog qStarMeanVar qStarQuantile ]

%   Figure 1:    Calculate histogram for wealth with optimal derivative
WStarLog = S + qStarLog*(max(K-S,0)-O);
WStarMeanVar = S + qStarMeanVar*(max(K-S,0)-O);

histIndex = 75:1:150;
[nZeroPut xOutOne] = hist(S, histIndex);
[nOptimalPut xOutTwo] = hist(WStarMeanVar, histIndex);

figureOne = [xOutOne' (nZeroPut./numPrice)' (nOptimalPut./numPrice)'];


%%  Section 4:  Robustness of results to resampling with small samples
```

```
%   Goal: Build Table 3 in paper (histogram of q* for each utility)
%
for simCount = 1:numSimTwo
    simCount
    resultTable = zeros(3,numSimOne);
    for numChoice = 1:numSimOne
        qLoop = (numChoice-1)/qScaleOne;
        S = S0*exp(randn(numPriceSmall,1)*sigma);
        W = S + qLoop*(max(K-S,0)-O);

        %   Log Utility
        resultTable(1,numChoice) = exp(mean(log(W)));
        %   Mean-Variance Utility
        resultTable(2,numChoice) = mean(W) - lambda*var(W);
        %   5% Quantile for distribution
        temp2 = sort(W);
        resultTable(3,numChoice) = temp2(length(temp2)*5/100);
    end

    %   Optimal Choice by Utility
    [uMaxLog iMaxLog] = max(resultTable(1,:));
    [uMaxMeanVar iMaxMeanVar] = max(resultTable(2,:));
    [uMaxQuantile iMaxQuantile] = max(resultTable(3,:));

    %   Collect optimal choice (q*) for each run in loop
    qStarLoop(simCount,1) = (iMaxLog-1)*qStep;
    qStarLoop(simCount,2) = (iMaxMeanVar-1)*qStep;
    qStarLoop(simCount,3) = (iMaxQuantile-1)*qStep;
end


%   Calculate histogram for optimal choice q* across resampling
histIndexTwo = 0:0.2:2;
[qStarHistLog xOut] = hist(qStarLoop(:,1), histIndexTwo);
[qStarHistMeanVar xOut] = hist(qStarLoop(:,2), histIndexTwo);
[qStarHistQuantile xOut] = hist(qStarLoop(:,3), histIndexTwo);

tableThree = [(qStarHistLog./numSimTwo); ...
   (qStarHistMeanVar./numSimTwo); (qStarHistQuantile./numSimTwo)];

%%  End of Code.
```