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Heller, Yuval

Tel-Aviv University

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Yuval Heller

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School of Mathematical Sciences, Tel-Aviv University, Tel-Aviv 69978, Israel.

Phone: 972-3-640-5386. Fax: 972-3-640-9357. Email: helleryu@post.tau.ac.il

Abstract

A strong correlated equilibrium is a correlated strategy profile that is immune to joint deviations. Existing solution concepts assume that players receive simultaneously correlated recommendations from the mediator. An *ex-ante* strong correlated equilibrium (Moreno D., Wooders J., 1996. Games Econ. Behav. 17, 80-113) is immune to deviations that are planned before receiving the recommendations. In this note we focus on mediation protocols where players may get their recommendations at several stages, and show that an *ex-ante* strong correlated equilibrium is immune to deviations at all stages of the protocol.

Key words: coalition-proofness, strong correlated equilibrium, common knowledge, incomplete information, non-cooperative games. JEL classification: C72, D82.

1 Introduction

In the mid-90s, a series of papers considered the following question for normal-form games: what happens when players are allowed to correlate their strategies (using a correlation device or a mediator) but some players may jointly deviate from potential correlated outcomes? Quite a few solution concepts with similar names emerged: strong and coalition-proof correlated equilibrium (Einy and Peleg, 1995; Milgrom and Roberts, 1996; Moreno and Wooders, 1996; Ray 1996, 1998). The different concepts can be characterized (see Ray, 1996) according to three main parameters: (1) When players are allowed to discuss deviations: before correlation (*ex-ante* equilibrium) or

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after correlation (*ex-post* equilibrium)? (2) Can deviators transmit private information truthfully and construct new correlation devices? (3) Does the equilibrium have to be immune to all joint deviations (strong equilibrium, as in Aumann, 1959) or only to self-enforcing deviations (coalition-proof equilibrium, as in Bernheim et al., 1987). Recently, Bloch and Dutta (2009) revived this research agenda and discussed the issue of transmission of information in an "admissible" way.

All the concepts mentioned above assume that players receive simultaneously correlated recommendations from the mediator. A natural question that arises is what happens if the recommendations are not received simultaneously: players may receive the recommendations sequentially (see, e.g., Heller, 2009), possibly at a random order, or each recommendation may be transmitted in several "pieces". What are the proper solution concepts in this case, and what are the relationships between these concepts?

The contribution of this note is twofold. First, we introduce a new solution concept that captures joint deviations at different stages of the recommendation transmission protocol, and formally model it by an incomplete information model à la Aumann (1987). We define an *all-stage strong correlated equilibrium* as a correlated strategy profile that is immune to all joint deviations at all stages.[2] Second, we show that this new notion coincides with Moreno and Wooders (1996)'s notion of *ex-ante* strong correlated equilibrium (which is immune to deviations only before recieving the recomendations). This implies that this *ex-ante* notion is much more "robust" than originally presented, and that this set of *ex-ante* equilibria is included in all other sets of strong correlated equilibria (see Figure 1 in Section 3).

Holmstrom and Myerson (1983, Sections 4-5)'s classical result, adapted to our framework, shows that resistance to deviations of the grand coalition at the *ex-ante* stage implies resistance to such deviations at the *ex-post* stage. Our result extends it in two ways: proving the resistance at all stages (not only at the *ex-post* stage), and against deviations of all coalitions.

The note is organized as follows. Section 2 presents the model. Section 3 presents the result and the proof. In Section 4 we demonstrate the intuition behind the result.

2 Model and Definitions

A game in strategic form is a tuple: $G = \left(N, (A^i)_{i\in N}, (u^i)_{i\in N}\right)$, where N is the finite and non-empty set of players. For each $i \in N$, A^i is player i's finite and non-empty set of actions, and u^i is player i's payoff function, a real-valued function on $A = \prod_{i\in N} A^i$. Given a coalition $S \subseteq N$, let $A^S = \prod_{i\in S} A^i$, and let $-S = \{i \in N \mid i \notin S\}$ denote the complementary coalition.

It is convenient to use Aumann (1987)'s model of incomplete information in modeling a mediation protocol. A *state space* is a finite probability space $(\Omega, \mathbb{P})$. Each state $\omega \in \Omega$ describes all parameters that may be the object of uncertainty on the part of the

[2] Deviators are allowed to transmit private information and construct correlation devices.

players: signals that are received from the mediator, messages that deviating players can send each other, and realizations of random devices that can be used to correlate joint deviations. The distribution $\mathbb{P}$ is the common prior belief over Ω. Finiteness of Ω is assumed to simplify presentation, but it plays no role in the result.

We now fix a game G and a state space $(\Omega, \mathbb{P})$. Given coalition S, a *correlated strategy S-tuple* is a function $\mathbf{f}^S = (f^i)_{i \in S}$ from Ω into A^S, and an *S-information structure* $(\mathcal{F}^i)_{i \in S}$ is an S-tuple of partitions of Ω. We interpret $\mathcal{F}^i$ as the information partition of player i at some stage of the mediation protocol; that is, if the true state is $\omega \in \Omega$ then player i is informed of that element $F^i(\omega)$ of $\mathcal{F}^i$ that contains ω.

A mediation protocol is modeled as follows. The correlated strategy N-tuple $\mathbf{f} = \mathbf{f}^N$ describes the vector of recommended actions. At the beginning of the mediation protocol (the *ex-ante* stage), the players are completely ignorant: their N-information structure is the coarsest one ($\forall i \in N$, $\mathcal{F}^i = \{\Omega\}$). As the protocol goes on, the players receive signals from the mediator, and the information partition of each player $i \in N$ becomes finer. At the end of the protocol (the *ex-post* stage), each player i knows his recommended action: each $\mathbf{f}^i$ is measurable with respect to $\mathcal{F}^i$.

A joint deviation of coalition S is a pair $\left(\mathcal{G}^S, \mathbf{g}^S\right)$, where $\mathcal{G}^S$ denotes the information the deviators have at the stage of the mediation protocol in which they agree to deviate, and $\mathbf{g}^S$ denotes the actions that S members will play in G. Specifically, each $\mathcal{G}^i$ describes the information that player $i \in S$ deduced from: the signals he received from the mediator so far; the messages he received from the other deviators; and the unanimous agreement of S members to deviate. Like the existing notions of strong correlated equilibrium (and in contrast with the coalition-proof notions), we assume that joint deviations are binding (à la Moulin and Vial, 1978): when the members of S unanimously agree to deviate, they are bound to follow the deviation even if new information received at a later stage makes it unprofitable.

The deviation is played with the assistance of a new mediator. Each deviator sends the new mediator all the signals he has received during the original mediation protocol (both before and after the unianimous agreement to deviate). After the new mediator receives the recommended actions of all the deviators, $\mathbf{f}^S$, it sends each deviator $i \in S$ a new recommended action $\mathbf{g}^i$. We assume that the deviators (and the new mediator) have no information about the actions recommended to the non-deviating players, except the conditional probability given the information they have on their own recommended actions. That is, we assume that $\left(\mathcal{G}^S, \mathbf{g}^S\right)$ is conditionally independent of $\mathbf{f}^{-S}$ given $\mathbf{f}^S$. Formally:

Definition 1 A *joint deviation* of coalition S from a correlated strategy N-tuple $\mathbf{f}$ is a pair $\left(\mathcal{G}^S, \mathbf{g}^S\right)$, where $\mathcal{G}^S$ is an S-information structure, $\mathbf{g}^S$ is a correlated strategy S-tuple, and both are conditionally independent of $\mathbf{f}^{-S}$, given $\mathbf{f}^S$. That is: $\forall \omega \in \Omega$, $E^S = (E^i)_{i \in S} \subseteq \Omega^S$, $b^S \in A^S$, $a \in A$, $\mathbb{P}\left(\mathcal{G}^S(\omega) = E^S, \mathbf{g}^S = b^S, \mathbf{f}^{-S} = a^{-S} | \mathbf{f}^S = a^S\right)$
$= \mathbb{P}\left(\mathcal{G}^S(\omega) = E^S, \mathbf{g}^S = b^S | \mathbf{f}^S = a^S\right) \cdot \mathbb{P}\left(\mathbf{f}^{-S} = a^{-S} | \mathbf{f}^S = a^S\right)$.

$\left(\mathcal{G}^S, \mathbf{g}^S\right)$ is an *ex-ante joint deviation* if $\mathcal{G}^S$ is coarsest: $\forall i \in S$, $\mathcal{G}^i = \{\Omega\}$; it is an *ex-post joint deviation* if $\forall i \in S$, $\mathbf{f}^i$ is measurable with respect to $\mathcal{G}^i$.

A player $i \in S$ will agree to be a part of a joint deviation, only if his expected payoff when deviating, conditional on his information and on the unanimous agreement of all members of S to deviate, is larger than when playing the original correlated strategy N-tuple. His agreement to participate in the joint deviation is a public signal to all the deviators about that fact. This implies that if the players in S unanimously decide to deviate at some state $\omega \in \Omega$, then it is common knowledge among them (at ω) that each player believes that he will profit by this deviation (as demonstrated in Section 4). In that case we say that the deviation is profitable. Formally:

Definition 2 (Aumann 1976) Let S be a coalition, $(\mathcal{G}^i)_{i\in S}$ an S-information structure, and $\omega \in \Omega$ a state. An event $E \in \mathcal{B}$ is *common knowledge* among the members of S at ω if E includes that member of $\mathcal{G}^{meet} = \bigwedge_{i\in S} \mathcal{G}^i$ that contains ω.

Definition 3 Let $\mathbf{f}$ be a correlated strategy profile, and S a coalition. A joint deviation $\left(\mathcal{G}^S, \mathbf{g}^S\right)$ is a *profitable joint deviation from* $\mathbf{f}$, if there exists $\omega \in \Omega$ such that it is common knowledge among the members of S at ω that $\forall i \in S$, $\mathbf{E}\left(u^i\left(\mathbf{f}\right)|\mathcal{G}^i\left(\omega\right)\right) < \mathbf{E}\left(u^i\left(\mathbf{g}^S, \mathbf{f}^{-S}\right)|\mathcal{G}^i\left(\omega\right)\right)$. [3]

We end this section by defining an all-stage (resp., *ex-ante*, *ex-post*) strong correlated equilibrium as an N-tuple that is immune to joint deviations at all stages (resp., *ex-ante* stage, *ex-post* stage).

Definition 4 A correlated strategy N-tuple $\mathbf{f}$ is an *all-stage* (resp., *ex-ante*, *ex-post*) *strong correlated equilibrium* if no coalition has a (resp., ex-ante, ex-post) profitable joint deviation from $\mathbf{f}$.

One can verify the following facts: (1) An all-stage strong correlated equilibrium is also a strong correlated equilibrium according to all the definitions in the existing literature (referred to below). (2) $\left(\mathcal{G}^S, \mathbf{g}^S\right)$ is *a* profitable *ex-ante* joint deviation from $\mathbf{f}$ if and only if $\forall i \in S$, $\mathbf{E}\left(u^i\left(\mathbf{f}\right)\right) < \mathbf{E}\left(u^i\left(\mathbf{g}^S, \mathbf{f}^{-S}\right)\right)$. (3) Our definition of ex-ante strong correlated equilibrium is equivalent to Moreno and Wooders (1996)'s definition, and it is more restrictive than all other existing *ex-ante* definitions. [4] (4) Our definition of *ex-post* strong correlated equilibrium is also an *ex-post* equilibrium according to all other existing *ex-post* definitions. [5]

3 Result

We now show that the *ex-ante* notion and the all-stage notion coincide.

[3] $\left(\mathbf{g}^S, \mathbf{f}^{-S}\right)$ denotes the N-tuple where its i-th component is $\mathbf{g}^i$ if $i \in S$ and $\mathbf{f}^i$ if $i \in -S$.

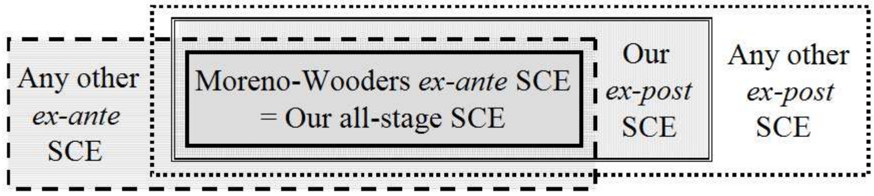
[4] Other *ex-ante* definitions in the literature impose restrictions on deviating coalitions: in Ray (1996) coalitions cannot construct new correlation devices; in Milgrom and Roberts (1996) only some of the coalitions can coordinate deviations.

[5] Other existing *ex-post* definitions impose restrictions on deviating coalitions: Einy and Peleg (1995) require deviations to be strictly profitable at all states; Ray (1998) allow coalitions to use only pure deviations; Bloch and Dutta (2008) restrict the information structure to represent only *credible* information sharing.

Theorem 5 *A correlated strategy N-tuple* is *an ex-ante strong correlated equilibrium if and only if it is an all-stage strong correlated equilibrium.*

Theorem 5 implies inclusion relations among the different notions of strong correlated equilibria, which are described in Figure 1. [6]

Figure 1. Relations Among Different Notions of Strong Correlated Equilibria (SCE)



PROOF. The definitions imply that an all-stage equilibrium is also an *ex-ante* equilibrium. We only have to prove the converse. Let $\mathbf{f}$ be a correlated strategy N-tuple that is not an all-stage strong correlated equilibrium. We will show that $\mathbf{f}$ is not an *ex-ante* strong correlated equilibrium.

Let $S \subseteq N$, $\left(\mathcal{G}^S, \mathbf{g}^S\right)$ a profitable joint deviation from $\mathbf{f}$, and $\omega_0 \in \Omega$ a state, such that it is common knowledge in ω_0 that $\forall i \in S$, $\mathbf{E}\left(u^i\left(\mathbf{f}\right)|\mathcal{G}^i\left(\omega_0\right)\right) < \mathbf{E}\left(u^i\left(\mathbf{g}^S, \mathbf{f}^{-S}\right)|\mathcal{G}^i\left(\omega_0\right)\right)$. That is:

$$\mathcal{G}^{meet}(\omega_0) \subseteq \left\{\omega \mid \forall i \in S, \mathbf{E}\left(u^i\left(\mathbf{f}\right)|\mathcal{G}^i\left(\omega\right)\right) < \mathbf{E}\left(u^i\left(\mathbf{g}^S, \mathbf{f}^{-S}\right)|\mathcal{G}^i\left(\omega\right)\right)\right\} \tag{1}$$

For each deviating player $i \in S$, write $G^{meet} = G^{meet}(\omega_0) = \dot{\bigcup}_j G^i_j$ where the G^i_j are disjoint members of $\mathcal{G}^i$, and let $\omega^i_j \in G^i_j$ be a state in G^i_j. We now construct an *ex-ante* profitable joint deviation $\left(\mathcal{G}^S_{ea}, \mathbf{g}^S_{ea}\right)$ as follows: $\forall i \in S$, $\mathcal{G}^i_{ea} = \{\Omega\}$, and

$$\mathbf{g}^S_{ea}\left(\omega\right) = \begin{cases} \mathbf{g}^S(\omega) & \omega \in F^{meet} \\ \mathbf{f}^S(\omega) & \omega \notin F^{meet} \end{cases}$$

Observe that $\mathbf{g}^S_{ea}$ and $\mathbf{f}^{-S}$ are conditionally independent given $\mathbf{f}^S$, thus $\mathbf{g}^S_{ea}$ is well-defined. We finish the proof by showing that $\forall i \in S$, $\mathbf{E}\left(u^i\left(\mathbf{g}^S_{ea}, \mathbf{f}^{-S}\right)\right) > \mathbf{E}\left(u^i\left(\mathbf{f}\right)\right)$, which implies that $\mathbf{g}^S_{ea}$ is an *ex-ante* profitable joint deviation:

$$\begin{aligned} & \mathbf{E}\left(u^i\left(\mathbf{g}^S_{ea}, \mathbf{f}^{-S}\right)\right) - \mathbf{E}\left(u^i\left(\mathbf{f}\right)\right) \\ = & \int_\Omega \left(u^i\left(\left(\mathbf{g}^S_{ea}, \mathbf{f}^{-S}\right)(\omega)\right) - u^i\left(\mathbf{f}(\omega)\right)\right) d\mu \\ = & \int_{F^{meet}} \left(u^i\left(\left(\mathbf{g}^S_{ea}, \mathbf{f}^{-S}\right)(\omega)\right) - u^i\left(\mathbf{f}(\omega)\right)\right) d\mu \end{aligned} \tag{2}$$

[6] Moreno and Wooders (1996, Section 4) and Bloch and Dutta (2009, Example 1) demonstrate that there are no similar inclusion relations among the different notions of coalition-proof correlated equilibria. The first paper also presents an example of an *ex-post* strong correlated equilibrium that is not an *ex-ante* equilibrium.

$$
\begin{aligned}
&= \int_{F^{meet}} \left(u^i \left(\left(\mathbf{g}^S, \mathbf{f}^{-S} \right) (\omega) \right) - u^i \left(\mathbf{f}(\omega) \right) \right) d\mu && (3) \\
&= \sum_j \int_{F^i_j} \left(u^i \left(\left(\mathbf{g}^S, \mathbf{f}^{-S} \right) (\omega) \right) - u^i \left(\mathbf{f}(\omega) \right) \right) d\mu && (4) \\
&= \sum_j \left(\mathbf{E} \left(u^i \left(\mathbf{g}^S, \mathbf{f}^{-S} \right) | \mathcal{G}^i \left(\omega^i_j \right) \right) - \mathbf{E} \left(u^i \left(\mathbf{f} \right) | \mathcal{G}^i \left(\omega^i_j \right) \right) \right) > 0
\end{aligned}
$$

Equation (2) holds since $\mathbf{g}^S_{ea} = \mathbf{f}^S$ outside F^{meet}, (3) holds since $\mathbf{g}^S_{ea} = \mathbf{g}^S$ in F^{meet}, (4) follows from $G^{meet} = \dot{\bigcup}_j G^i_j$, and the inequality is implied by (1). □

4 Example

The next example presents an *ex-ante* strong correlated equilibrium, and a specific deviation that is considered by the grand coalition after two players received their recommended actions. At first glance, it seems that all players would unanimously agree to deviate. However, a more thorough analysis reveals that this is not the case, and demonstrates the intuition behind Theorem 5: (1) why unanimous agreement to deviate implies that it is common knowledge that the deviation is profitable; and (2) why the lack of *ex-ante* profitable deviations implies that there are no profitable deviations at later stages.

Table 1 shows the matrix representation of a 3-player game, where player 1 chooses the row, player 2 chooses the column, and player 3 chooses the matrix.

Table 1
A 3-Player Game With an *Ex-Ante* Strong Correlated Equilibrium

	c_1			c_2			c_3		
	b_1	b_2	b_3	b_1	b_2	b_3	b_1	b_2	b_3
a_1	10,10,10	5, 20,5	0,0,0	5,5,20	0,0,0	0,0,0	0,0,0	0,0,0	0,0,0
a_2	20,5,5	0,0,0	0,0,0	0,0,0	0,0,0	0,0,0	0,0,0	0,0,0	0,0,0
a_3	0,0,0	0,0,0	0,0,0	0,0,0	0,0,0	0,0,0	0,0,0	0,0,0	7,11,12

Let q be as follows: $\left(\frac{1}{4} (a_1, b_1, c_1), \frac{1}{4} (a_2, b_1, c_1), \frac{1}{4} (a_1, b_2, c_1), \frac{1}{4} (a_1, b_1, c_2) \right)$, with an expected payoff of $(10, 10, 10)$. One can verify that q is the distribution of an *ex-ante* strong correlated equilibrium.[7] Consider a stage of a mediation protocol, in which player 1 received a recommendation to play a_1, player 2 received a recommendation to play b_1, and player 3 has not received a recommendation yet. No player knows whether the other players received their recommended actions.[8] Assume that at this stage, the players consider a joint deviation $\mathbf{g}$ - always playing (a_3, b_3, c_3). At first glance, it seems

[7] The distribution of a correlated strategy N-tuple $\mathbf{f}$ is a function q_f that assigns to each n-tuple of actions $a \in A$ the number $\Pr \left(f^{-1}(a) \right)$.
[8] To simplify presentation, we assume that it is common knowledge that each player has either received his recommended action or has not received any information.

that they would unanimously agree to deviate: conditioned on his recommended action, player 1 (2) gets a higher payoff if they deviate. Player 3 does not know his recommended action, and the deviation gives him a higher *ex-ante* expected payoff.

However, we now show that $\mathbf{g}$ is not profitable for player 3. Player 1 can only earn from $\mathbf{g}$ if he received a recommendation to play a_1. Thus, if player 1 agrees to deviate, then the other players deduce that he received a_1. The expected payoff of player 2 , conditioned on that player 1 received a_1, is $11\frac{2}{3}$. Thus, if player 2 agrees to deviate (and get only 11), then player 3 deduces that player 2 received b_1. Conditional on player 1 receiving a_1 and player 2 receiving b_1, the deviation is not profitable to player 3.

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