A geometrical approach to structural change modeling

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A GEOMETRICAL APPROACH TO STRUCTURAL CHANGE MODELING

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Abstract: We propose a model for studying the dynamics of economic structures. The model is based on qualitative information regarding structural dynamics, in particular, (a) the information on the geometrical properties of trajectories (and their domains) which are studied in structural change theory and (b) the empirical information from stylized facts of structural change. We show that structural change is path-dependent in this model and use this fact to restrict the number of future structural change scenarios significantly. We focus on labour-allocation-dynamics in a tree-sector-economy. However, our approach can be applied to other types of structural change (e.g. income-distribution-dynamics).

Keywords: structure, dynamics, qualitative, geometrical, simplex, path-dependency.

JEL-codes: C61, C69, D30, O14, O41.

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1. INTRODUCTION

Many economic theories study the structure of the economy, e.g. the structure of: GDP (break-down by types of expenditure/income), employment (labour allocation across sectors), wages (wage distribution across agents) and income (income distribution across households). Most of these theories study the *dynamics* of this structure, i.e. structural change. In this paper we provide a qualitative approach for studying structural change. We focus on the dynamics of employment structure, i.e. dynamics of labour reallocation across sectors. However, our approach can be applied to other structural change theories; see Stijepic (2014a,b) for examples.

Recently, the dynamics of employment structure (in three-sector models) have been analysed in many theoretical papers[1]. While these models feature specific mathematical and economic assumptions, our qualitative model is very general in this respect. It combines the information from empirical evidence (stylized facts) and the information on the geometrical properties of typical trajectories studied in structural change theory.[2] We show that structural change is path-dependent in our model and use this fact to reduce the number of future structural change scenarios significantly.

The toolkit of the modern economist contains different tools for dynamic analysis, e.g. vector auto regression, simulation and phase-diagram analysis. Most of these tools combine theoretical and empirical information on the subject of analysis for deriving some predictions regarding future dynamics or policy responses. We do exactly the same. However, in contrast to many quantitative and/or predominantly theoretical approaches we do not use specific economic assumptions and, thus, do not follow specific economic doctrines, but use the mathematical assumptions which are common to most

[1]For example, Kongsamut et al. (2001), Meckl (2002), Ngai and Pissarides (2007), Acemoglu and Guerrieri (2008), Foellmi and Zweimueller (2008) and Buera and Kaboski (2009).

[2] Our approach has some similarity with "qualitative simulation algorithms" used in physics; see Kuipers (1986) and Lee and Kuipers (1988).

economic models. This idea results in a qualitative yet very general model of structural change.

We approach as follows. First, we show that the dynamics of a three-sector economy (agriculture, manufacturing, services) can be modelled on a 2-simplex; i.e. the 2-simplex is the domain of the structural change trajectory. Second, we provide an economic interpretation of points and trajectories on the 2-simplex. Third, we collect some information from widely-accepted stylized facts of structural change and translate this information into dynamics on the 2-simplex. Fourth, the structural change trajectories which arise in structural change theory have some specific geometrical properties. In particular, we use the fact that they are continuous and do not intersect themselves (on the 2-simplex). Fifth, by combining the information from the previous steps we obtain a qualitative meta-model of structural change. This model implies that today's structural change depends on past structural change. That is, there is some sort of path-dependency in structural change. Sixth, the path-dependency restricts the set of feasible future structural change scenarios significantly. (For a summary of these scenarios see Section 9.) This fact can be used in structural change predictions.

The rest of the paper is set up as follows. We show in Section 2 that structural change in the three-sector-model can be depicted on a 2-simplex. In Section 3 we show how to interpret the points and trajectories on the 2-simplex. In Section 4 we discuss briefly the stylized facts of labour allocation dynamics. In Section 5 we present our model of structural change. In Section 6 we derive the scenarios of structural change and show "path-dependency". Section 7 is devoted to a discussion of model assumptions. In Section 8 we show that most structural change models assume implicitly continuous and non-self-intersecting trajectories. Some concluding remarks are provided in Section 9.

2. STRUCTURAL DYNAMICS AND STANDARD SIMPLEXES

Most aspects of the structure of an economy can be described by a set of shares of an aggregate construct. This fact is illustrated in the following examples. For a general discussion see Stijepic (2014a).

Example 1: *If we are interested in the distribution of income across households* $j = 1,2,...h$*, we may study the shares of households j in aggregate income y. That is, we study the system* y_j / y*,* $j = 1,2,...h$*, where* y_j *is the income of household j.*

Example 2: *Labour allocation across sectors. Assume that E is some measure of aggregate employment (e.g. the number of hours worked in the whole economy). Furthermore, let* E_i *denote the employment in sector i (e.g. hours worked in sector i). The literature on labour allocation (cited in Section 1) studies the dynamics of the system* $\ell_i := E_i / E$*,* $i = 1,2,...n$*, where n is the number of sectors and* ℓ_i *is the employment share of sector i.*

We focus on Example 2. There are two facts which allow us to model labour allocation (Example 2) on a standard simplex: (I) Standard structural change literature (see Section 1) assumes that $\ell_1 + \ell_2 + ...\ell_n = 1$. That is, all labour available is employed in sectors $i = 1,2,...n$ or, equivalently, E is the aggregate of sectors $i = 1,...n$, i.e. $E := E_1 + E_2 + ...E_n$. This definition is not crucial for any result, since, if $E \neq E_1 + ...E_n$, we can always define an auxiliary variable E_{n+1} such that $E = E_1 + ...E_n + E_{n+1}$ and study the dynamics of the system $\ell_i := E_i / E$, $i = 1,...n+1$. (II) The definitions $\ell_i := E_i / E$, $i = 1,...n$, and $E := E_1 + ...E_n$ imply $0 \leq \ell_i \leq 1$ for $i = 1,...n$.

These facts reduce the set of all feasible ℓ_i drastically: the ℓ_i are located on a standard simplex of dimension (n-1).[3] Let $\Delta_{(n-1)}$ denote this simplex; thus, $\Delta_{(n-1)} := \{(\ell_1, \ell_2, ... \ell_n) \in \mathbf{R}^{\mathbf{n}} : \ell_i \geq 0,\ i = 1,...n;\ \ell_1 + \ell_2 + ... \ell_n = 1\}$.

In the main part of the paper we will study a lower-dimensional case of Example 2, as defined in the following assumption.

ASSUMPTION 1: a) *We study an economy divided into three sectors: agriculture (a), manufacturing (m) and services (s).* ***b)*** ℓ_i *denotes the share of labour devoted to sector i,* $i = a, m, s$. ***c)*** *The employment shares* ℓ_i *satisfy the following relations:*

(1) $\ell_a + \ell_m + \ell_s = 1$

(2) $0 \leq \ell_i \leq 1$ *for* $i = a, m, s$.

According to the discussion above, the employment shares of the economy defined in Assumption 1 are located on a subset of a two-dimensional standard simplex (Δ_2), where Δ_2 is given by the following definition:

(3) $\Delta_2 := \{(\ell_a, \ell_m, \ell_s) \in \mathbf{R}^3 : \ell_i \geq 0,\ i = a, m, s;\ \ell_a + \ell_m + \ell_s = 1\}$

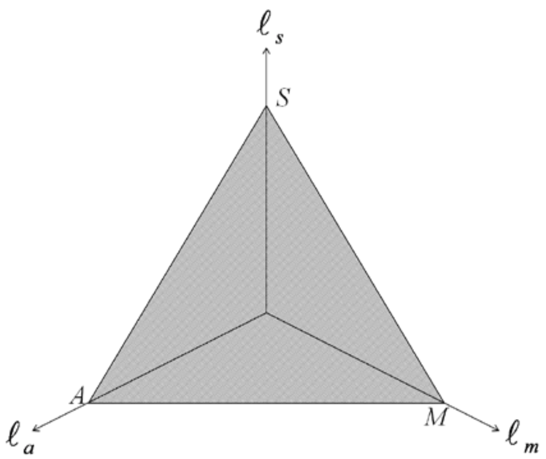
where (ℓ_a, ℓ_m, ℓ_s) is a vector of Cartesian coordinates indicating the labour allocation and **R** is the set of real numbers.

In the remaining part of this section we recapitulate some standard geometrical concepts for the analysis of the system defined in Assumption 1.

The simplex Δ_2 is a triangle. In Figure 1 we depict Δ_2 in a three-dimensional Cartesian coordinate system with the coordinates (ℓ_a, ℓ_m, ℓ_s).

[3]On simplexes see e.g. Border (1985), p.20, and Munkres (1984), p.2.

Figure 1: *The simplex* Δ_2 *in the Cartesian coordinate system (* ℓ_a, ℓ_m, ℓ_s *).*



The vertices (*A*, *M*, *S*) and the origin (*O*) of the coordinate system in Figure 1 can be expressed in Cartesian coordinates (ℓ_a, ℓ_m, ℓ_s) as follows:

(4) $A := (1,0,0)$

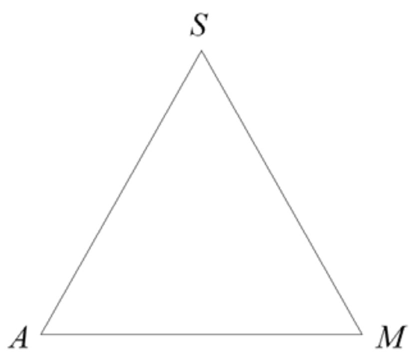
(5) $M := (0,1,0)$

(6) $S := (0,0,1)$

(7) $O := (0,0,0)$

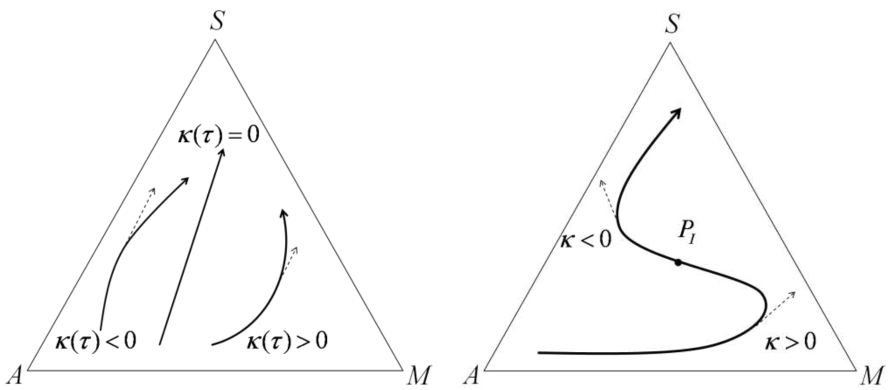
Since the depiction in Figure 1 is inconvenient and unnecessary, we depict, henceforth, Δ_2 in the plane, as shown in Figure 2.

Figure 2: *The simplex* Δ_2 *in the plane.*



Imagine a trajectory τ on Δ_2 describing a movement from vertex *A* to vertex *S*, cf. Figure 3. To abbreviate discussion, we define here the concept of "signed curvature" (κ) of a trajectory in a very simple and somewhat restrictive way; for a rigorous mathematical definition see any book on introductory differential geometry. We say that the signed curvature of τ is uniformly positive ($\kappa(\tau) > 0$), if the tangential vector is always on the right-hand-side of τ; i.e. τ describes a counter-clockwise movement. The signed curvature of τ is uniformly negative ($\kappa(\tau) < 0$), if the tangential vector is always on the left-hand-side of τ; i.e. the movement described by τ is clockwise. If the curvature of τ is uniformly zero ($\kappa(\tau) = 0$), then τ is an oriented line-segment. These definitions imply that, if the curvature of τ is "uniform", there are no inflection points on τ. See Figure 3.

Figure 3: *Curvature of trajectories on* Δ_2 *and an inflection point* P_I.



3. INTERPRETATION OF POINTS AND TRAJECTORIES ON THE 2-SIMPLEX

In this section we elaborate the interpretation of points and vectors/trajectories on Δ_2.

Remark 1: First, we turn to the interpretation of *points* on Δ_2. (4)-(6) imply that at the vertices *A*, *M* and *S* all labour is employed in agriculture, manufacturing and services, respectively; cf. Figure 1. Thus, we define:

DEFINITION 1: a) *An economy situated in vertex A of* Δ_2 *is named "pure agricultural economy".* ***b)*** *An economy situated in vertex M of* Δ_2 *is named "pure manufacturing economy".* ***c)*** *An economy situated in vertex S of* Δ_2 *is named "pure services economy".*

LEMMA 1: a) *All points on the* $\overline{MS}$ *-edge of* Δ_2 *feature* $\ell_a = 0$. ***b)*** *All points on the* $\overline{SA}$*-edge of* Δ_2 *feature* $\ell_m = 0$. ***c)*** *All points on the* $\overline{AM}$ *-edge of* Δ_2 *feature* $\ell_s = 0$. *See also Figure 2.*

PROOF: This property of the standard 2-simplex is well-known. For a proof see APPENDIX A. ■

Remark 2: Thus, an economy situated on the $\overline{MS}$-edge of Δ_2 or moving along this edge does not employ any labour in the agricultural sector. Analogously, on the $\overline{SA}$-edge labour is not employed in the manufacturing sector and on the $\overline{AM}$ -edge labour is not employed in the services sector.

Remark 3: The following three lemmas show how a movement (directional/tangential vector or trajectory) on Δ_2 can be interpreted in terms of sectoral employment share dynamics.

LEMMA 2: *Assume that (I) P is an arbitrary point in the interior of* Δ_2*, (II)* $\vec{v}$ *is the directional vector associated with point P indicating the direction of movement from point P along* Δ_2*, (III)* l_s^p *is a line going through P, (IV)* l_s^p *is* *parallel to the $\overline{AM}$ -edge of Δ_2 and (V) σ is the angle between l_s^p and $\vec{v}$ (in point P); cf. Figure 4. Under these assumptions the following is true:*

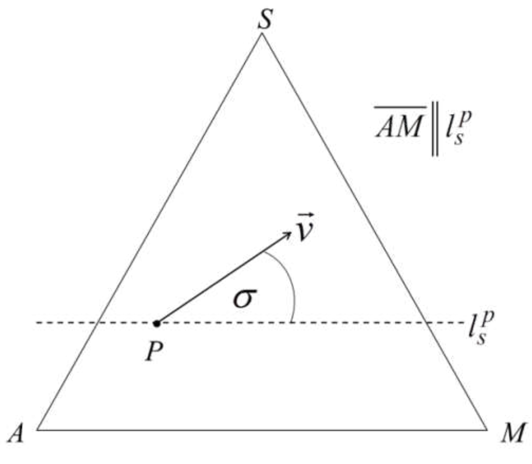
a) *If $0° < \sigma < 180°$, the movement from P in direction indicated by $\vec{v}$ is associated with an increase in ℓ_s.*

b) *If $180° < \sigma < 360°$, the movement from P in direction indicated by $\vec{v}$ is associated with a decrease in ℓ_s.*

c) *If $\sigma = 180°$ or if $\sigma = 0°$, the movement from P in direction indicated by $\vec{v}$ is not associated with a change in ℓ_s.*

PROOF: This property of the standard simplex is well-known. For a proof see APPENDIX A. ■

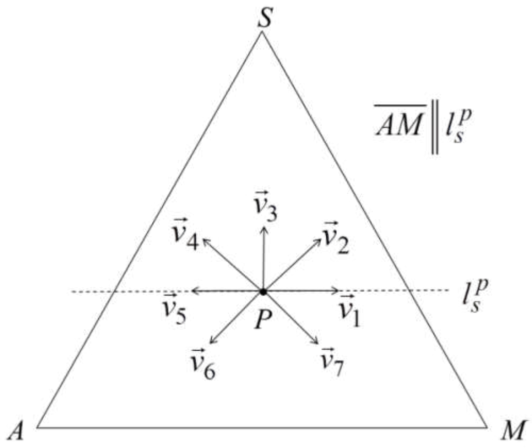
Figure 4: *A directional vector on Δ_2 in relation to an $\overline{AM}$ -parallel (l_s^p).*



Remark 4: Figure 5 illustrates Lemma 2. The vectors $\vec{v}_1$ and $\vec{v}_5$ are parallel to $\overline{AM}$. Thus, a movement along these vectors is not associated with a change in the service-share (ℓ_s). Along vectors $\vec{v}_2$, $\vec{v}_3$ and $\vec{v}_4$ the service-share increases. Along vectors $\vec{v}_6$ and $\vec{v}_7$ the service-share decreases. Overall, we can easily determine whether the movement along a trajectory-segment on Δ_2

is associated with an increase/decrease in ℓ_s: if all tangential vectors on this segment have an angle between 0 and 180 degrees to the edge $\overline{AM}$, we know that ℓ_s increases steadily along this trajectory segment.

Figure 5: *Representative vectors regarding ℓ_s-changes.*



LEMMA 3: *Assume that (I) P is an arbitrary point in the interior of Δ_2, (II) $\vec{v}$ is the directional vector associated with point P indicating the direction of movement from point P along Δ_2, (III) l_m^p is a line going through P, (IV) l_m^p is parallel to the $\overline{SA}$-edge of Δ_2 and (V) μ is the angle between l_m^p and $\vec{v}$ (in point P); cf. Figure 6. Under these assumptions the following is true:*

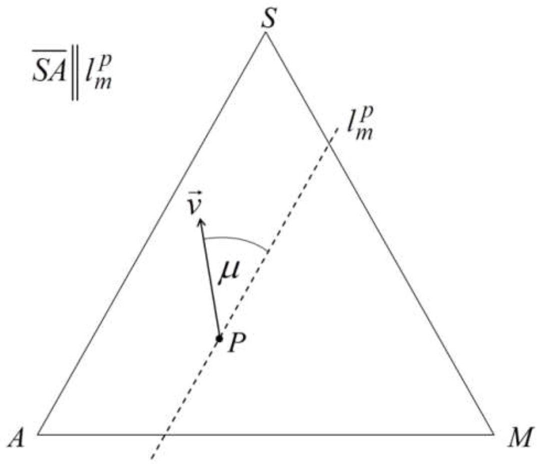
a) *If $0° < \mu < 180°$, the movement from P in direction indicated by $\vec{v}$ is associated with a decrease in ℓ_m.*

b) *If $180° < \mu < 360°$, the movement from P in direction indicated by $\vec{v}$ is associated with an increase in ℓ_m.*

c) *If $\mu = 180°$ or if $\mu = 0°$, the movement from P in direction indicated by $\vec{v}$ is not associated with a change in ℓ_m.*

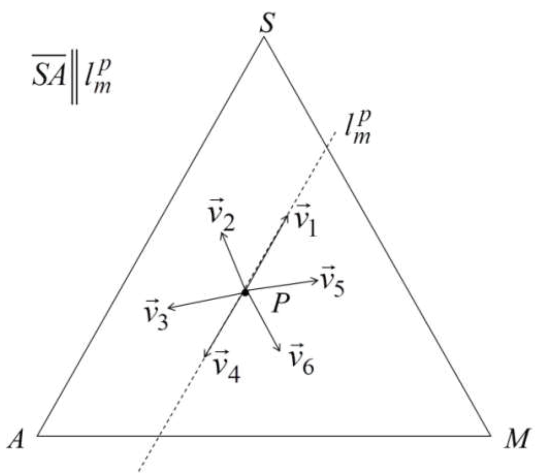
PROOF: Analogous to the Proof of Lemma 2. ■

Figure 6: *A directional vector on* Δ_2 *in relation to a* $\overline{SA}$*-parallel (* l_m^p *).*



Remark 5: Figure 7 illustrates Lemma 3: movement along vectors $\vec{v}_1$ and $\vec{v}_4$ is not associated with a change in the manufacturing-share (ℓ_m); ℓ_m declines along vectors $\vec{v}_2$ and $\vec{v}_3$; ℓ_m increases along vectors $\vec{v}_5$ and $\vec{v}_6$.

Figure 7: *Representative vectors regarding* ℓ_m*-changes.*



LEMMA 4: *Assume that (I) P is an arbitrary point in the interior of* Δ_2*, (II)* $\vec{v}$ *is the vector associated with point P indicating the direction of movement from point P along* Δ_2*, (III)* l_a^p *is a line going through P, (IV)* l_a^p *is parallel to the* $\overline{MS}$*-edge of* Δ_2 *and (V)* α *is the angle between* l_a^p *and* $\vec{v}$ *(in point P); cf. Figure 8. Under these assumptions the following is true:*

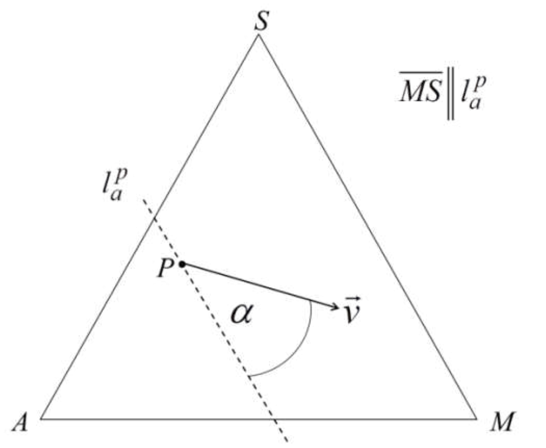
a) *If* $0° < \alpha < 180°$*, the movement from P in direction indicated by* $\vec{v}$ *is associated with a decrease in* ℓ_a*.*

b) *If* $180° < \alpha < 360°$*, the movement from P in direction indicated by* $\vec{v}$ *is associated with an increase in* ℓ_a*.*

c) *If* $\alpha = 180°$ *or if* $\alpha = 0°$*, the movement from P in direction indicated by* $\vec{v}$ *is not associated with a change in* ℓ_a*.*

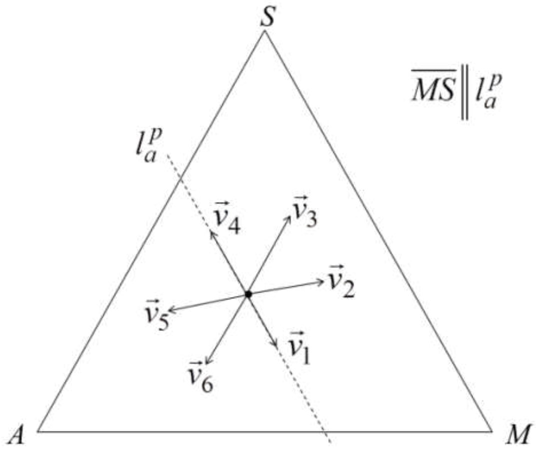
PROOF: Analogous to the Proof of Lemma 2. ■

Figure 8: *A directional vector on* Δ_2 *in relation to a* $\overline{MS}$*-parallel (*l_a^p*).*



Remark 6: Figure 9 illustrates Lemma 4: movement along vectors $\vec{v}_1$ and $\vec{v}_4$ is not associated with a change in the agriculture-share (ℓ_a); ℓ_a declines along vectors $\vec{v}_2$ and $\vec{v}_3$; ℓ_a increases along vectors $\vec{v}_5$ and $\vec{v}_6$.

Figure 9: *Representative vectors regarding* ℓ_a*-changes.*



Remark 7: Lemmas 2-4 show how to interpret a movement (a directional vector or trajectory) on Δ_2. The angle set (α, μ, σ) associated with a directional vector gives us all the necessary information about the structural change associated with the movement along the vector.

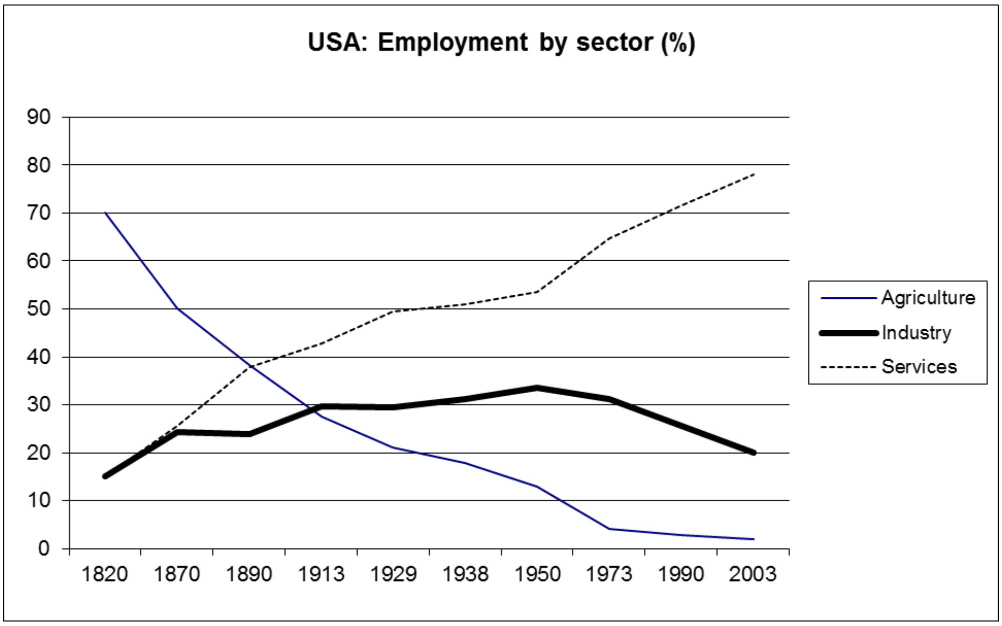
Remark 8: The interpretation of directional vectors derived in this section allows us to interpret trajectories on Δ_2 directly, i.e. without analysis of three-dimensional Cartesian coordinates of trajectories. That is, we can analyse structural change as dynamics of a system on a bounded subset of the plane. This allows us to benefit from the nice properties of such systems mentioned in Section 1; cf., e.g., Guckenheimer and Holmes (1990), p.42f.

4. STYLIZED FACTS OF STRUCTURAL CHANGE

Theories of long-run labour reallocation across sectors have a long tradition in economics. Some classical contributions are, for example: Clark (1940), Baumol (1967) and Kuznets (1969). For a detailed review of structural change literature see e.g.: Schettkat and Yocarini (2006), Krüger (2008) or Silva and

Teixeira (2008). Most studies focus on a three-sector framework (agriculture, manufacturing, services). Some newer empirical evidence on labour dynamics in this framework is provided by Maddison (1989, 1995a,b, 2007), Kongsamut et al. (2001) and Raiser et al. (2004). Some of this evidence is depicted in APPENDIX F. Throughout the paper we will use the US-dynamics as an example (cf. Figure 10).

Figure 10: *Past structural change dynamics in the USA (standard diagram).*



Data source: Maddison, Angus (2007): Contours of the World Economy I-2030 AD, Essays in Macro-economic History, Oxford University-Press, New York, p.384

We use the following widely-accepted stylized facts in our model.

Stylized Fact 1: *At early stages of development the economy is dominated by agriculture ("agricultural economy").* This is one of the best known facts of development economics. For evidence see any contribution on structural change, e.g. Maddison (1989, 1995a,b, 2007), Kongsamut et al. (2001) and Raiser et al. (2004). Figure 10 implies that 70% of labour has been employed in agriculture in 1820 in the USA.

Stylized Fact 2: *At later stages of development the economy is dominated by services ("services economy").* "Later stages of development" refers here to industrialized countries (e.g. OECD core-countries). For evidence on high service-share in industrialized countries see e.g.: Schettkat and Yocarini (2006), Maddison (1989, 1995a,b, 2007) and Raiser et al. (2004); see also the figures in APPENDIX F. For example, Figure 10 implies that nearly 80% of labour has been employed in the services sector in 2003 in the USA.

Stylized Fact 3 *(Long-run trend): The employment share of agriculture declines over the development process.* See also Kongsamut et al. (2001). This stylized fact is an implication of Stylized Facts 1 and 2. For evidence see e.g.: the regression results presented by Kongsamut et al. (1997, 2001), the evidence presented by Maddison (1989, 1995a,b, 2007), Figure 10 and the figures in APPENDIX F.

Stylized Fact 4 *(Long-run trend): The employment share of services grows over the development process.* See also Kongsamut et al. (2001). This stylized fact is an implication of Stylized Facts 1 and 2. For evidence see e.g.: the regression results presented by Kongsamut et al. (1997, 2001), the evidence presented by Maddison (1989, 1995a,b, 2007), Figure 10 and the figures in APPENDIX F.

These well-known and widely accepted stylized facts are necessary for our results. For the sake of completeness, we add a further stylized fact, which is not necessary for our results.

Stylized Fact 5 *(optional): The employment share of manufacturing grows at early stages of development ("industrialization") and declines at later stages of development ("tertiarisation").* This stylized fact implies that the curve which describes the dynamics of manufacturing sector employment is concave; cf. Figure 10. This result has been emphasized by Ngai and Pissarides (2007). It may be questioned whether this stylized fact applies to all countries. Furthermore, we are dealing here with long-run growth modelling; thus, we are only interested in trends. For example, Kongsamut et al. (2001)

could not find any trend in manufacturing employment share; therefore, they postulate the following stylized fact:

Stylized Fact 5' *(optional): The manufacturing employment share is "constant" over the development process.*

Our model covers all these cases (Fact 5 and 5'): our results are consistent with a inclining, declining, constant, concave or convex manufacturing-share (curve).

Note that the dynamics depicted in the standard diagram (Figure 10) can be translated into dynamics on Δ_2 by using Lemmas 1-4; see also APPENDIX B.

5. A QUALITATIVE MODEL OF STRUCTURAL CHANGE

In this section we define a qualitative model which satisfies the stylized facts postulated in Section 4. In the next section we use this model to elaborate scenarios of future structural change.

ASSUMPTION 2: a) *The dynamics of the sector structure* (ℓ_a, ℓ_m, ℓ_s) *on* Δ_2 *are described by the curve* $\phi(t)$, $\underline{t} < t < \bar{t}$, $t \in \mathbf{R}$, *where:* $\phi(t)$ *is a coordinate vector determining the position of the economy on* Δ_2 *at time t;* **R** *is the set of Real numbers.* ***b)*** $\phi(t) \in \Delta_2$ *for* $\underline{t} < t < \bar{t}$. ***c)*** $\phi(t)$ *is continuous (in t) on* Δ_2 *for* $\underline{t} < t < \bar{t}$.

DEFINITION 2: a) $\phi(t_0) =: P_0$ *and* $\phi(t_T) =: P_T$, *where* $\underline{t} < t_0 < t_T < \bar{t}$.

b) $\tau := \{\phi(t) \in \Delta_2 : \underline{t} < t < \bar{t}\}$ *is the trajectory on* Δ_2 *describing the dynamics of sector structure for* $\underline{t} < t < \bar{t}$.

c) $\tau_{0T} := \{\phi(t) \in \Delta_2 : t_0 \le t \le t_T\}$ *is the segment of trajectory* τ *connecting the points* P_0 *and* P_T.

d) $\tau_{T+} := \{\phi(t) \in \Delta_2 : t > t_T\}$ *is the* τ*-segment following after point* P_T.

e) $\tau_{0-} := \{\phi(t) \in \Delta_2 : t < t_0\}$ *denotes the* τ *-segment preceding point* P_0.

f) $[\tau_{0T}] := \bigcup_{t_0 \le t \le t_T} \phi(t)$ is the set of all points covered by trajectory-segment τ_{0T} .

Remark 9: (i) Implicitly, we assume here some economic model which generates a continuous trajectory (τ) on Δ_2, cf. Assumption 2 and Definition 2b. For discussion and examples of mathematical and economic models which generate continuous trajectories of structural change see Section 8. **(ii)** We partition the trajectory τ in segments, cf. Definition 2c-e. **(iii)** We assume the existence of a *continuous* trajectory (cf. Assumption 2c), since we analyse long-run structural change. It is hard to imagine that sector-employment-shares jump (i.e. change non-marginally) at some point in time, since (a) changes in sector-employment-shares require labour reallocation across sectors and (b) significant numbers of workers cannot be reallocated instantly.

ASSUMPTION 3: *Let the Cartesian coordinates of points* P_0 *and* P_T *(cf. Definition 2a) be given as follows:* $P_0 = (\ell_a^0, \ell_m^0, \ell_s^0)$ *and* $P_T = (\ell_a^T, \ell_m^T, \ell_s^T)$*. The trajectory segment* τ_{0T} *satisfies the following qualitative requirements:*

a) P_0 *is relatively close to vertex A, i.e.* $\ell_a^0 > 1/2$ *; cf. Figure 2.*

b) P_T *is relatively close to vertex S, i.e.* $\ell_s^T > 1/2$ *; cf. Figure 2.*

c) τ_{0T} *has uniform signed curvature (* $\kappa(\tau_{0T}) > 0$ *or* $\kappa(\tau_{0T}) < 0$ *or* $\kappa(\tau_{0T}) = 0$ *), i.e.* τ_{0T} *has no inflection points; cf. Section 2 and Figure 3.*

d) *The economy approaches* P_T *monotonously along* τ_{0T} *; i.e. all tangential vectors on* τ_{0T} *satisfy the following (vector-angle-)conditions:* $0° \le \sigma \le 180°$ *(cf. Lemma 2 and Figure 4) and* $0° \le \alpha \le 180°$ *(cf. Lemma 4 and Figure 8).*

Remark 10: Remember that Definition 2a/c implies that trajectory-segment τ_{0T} describes a movement from point P_0 to point P_T; the economy is in P_0 at time t_0 and in P_T at time t_T. Thus, Assumption 3 can be explained as follows:

a) Assumption 3a is due to Stylized Fact 1. Stylized Fact 1 states that agriculture is dominant at the beginning of the development process (t_0); cf. Section 4. Assumption 3a implies that more than 50% of labour is employed in the agricultural sector at t_0; thus, Stylized Fact 1 is satisfied. Note that Lemma 4b implies: the closer a point to vertex *A*, the greater the employment share of agriculture (ℓ_a); cf. Remark 6.

b) Assumption 3b is due to Stylized Fact 2. Stylized Fact 2 states that the services sector is dominant at later stages of development (t_T); cf. Section 4. Assumption 3b implies that more than 50% of labour is employed in the services sector at t_T; thus, Stylized Fact 2 is satisfied. Note that Lemma 2a implies: the closer a point to vertex *S*, the greater the employment share of services (ℓ_s); cf. Remark 4. We discuss Assumption 3b in Section 7.1 as well.

c) Assumption 3c follows from our growth-theoretical approach to structural change; cf. Section 4. We are interested in long-run trends not fluctuations. Thus, in general, the assumption of a linear trajectory $(\kappa(\tau_{0T}) = 0)$ is sufficient for our analysis; for example, the stylized facts and the model elaborated by Kongsamut et al. (2001) imply a linear trajectory of structural change on Δ_2. We allow for uniformly positive $(\kappa(\tau_{0T}) > 0)$ and uniformly negative $(\kappa(\tau_{0T}) < 0)$ signed curvature of τ_{0T} for reasons of generality. If τ_{0T} had one or many inflection points in reality, its trend could be approximated by a trajectory with uniform signed curvature; see also the discussion of Stylized Fact 5 in Section 4. Furthermore, our results remain valid, if there are inflection points, provided that τ_{0T} does not feature strong fluctuations; we discuss this aspect in Section 7.2.

d) Assumption 3d is due to Stylized Facts 3 and 4. Stylized Facts 3 and 4 refer to long-run trends and state that services employment share (ℓ_s) increases over the development process and agricultural employment share (ℓ_a) decreases over the development process. Assumption 3d (and Assumption 3a/b) implies that ℓ_s increases monotonously over time ($0° \leq \sigma \leq 180°$, cf. Lemma 2 and Figure 4) and ℓ_a decreases monotonously over time ($0° \leq \alpha \leq 180°$, cf. Lemma 4 and Figure 8). The assumption of monotonous dynamics is due to the fact that we analyse here long-run trends; cf. Remark 10c. In Section 7 we show that this assumption is not necessary for our results.

LEMMA 5: *If Assumptions 1-3 are satisfied, the dynamics described by trajectory-segment τ_{0T} are consistent with Stylized Facts 1-4 (cf. Section 4).*
PROOF: For a proof of Lemma 5, see Remark 10a/b/d. ■

ASSUMPTION 4: *The trajectory τ does not intersect itself. In particular, $\phi(t) \notin [\tau_{0T}]$ for $t > t_T$ (cf. Definition 2f).*

Remark 11: Assumption 4 implies that the economy never returns to a state in which it has been previously. This assumption is widespread in structural change analysis and in mathematical literature. We discuss it in Section 8.

Remark 12: Assumption 4 does not allow for closed trajectories. A closed trajectory (or: closed orbit) in the plane is a Jordan curve. For example, the circle is a Jordan curve. The economy moving along such a closed trajectory repeats the cycle (infinitely) many times. Thus, if an economy satisfies Assumption 1-3 and moves along a closed trajectory, at some point in time $t > t_T$ the economy enters the set $[\tau_{0T}]$ and moves along it (from point P_0 to point P_T). All our results remain valid if we allow for closed trajectories; see also APPENDIX C and D. On closed trajectories, see any introductory book on differential equation systems.

DEFINITION 3: a) *An economy situated in point* $P = (\ell_a^p, \ell_m^p, \ell_s^p)$ *at time* $t_D > t_T$ *"has undergone a process of* ***relative deindustrialization*** *since* t_T*", if there exists a point* $P_{0T} = (\ell_a^{0T}, \ell_m^{0T}, \ell_s^{0T}) \in \tau_{0T}$ *such that* $\ell_s^p = \ell_s^{0T}$ *and* $\ell_m^p < \ell_m^{0T}$. ***b)*** *An economy situated in point* $P = (\ell_a^p, \ell_m^p, \ell_s^p)$ *at time* $t_I > t_T$ *"has undergone a process of* ***relative industrialization*** *since* t_T*", if there exists a point* $P_{0T} = (\ell_a^{0T}, \ell_m^{0T}, \ell_s^{0T}) \in \tau_{0T}$ *such that* $\ell_s^p = \ell_s^{0T}$ *and* $\ell_m^p > \ell_m^{0T}$.

Remark 13: a) If we want to know whether an economy at time $t_D > t_T$ "has undergone a process of relative deindustrialization since t_T", we have to do the following. First, find data on services employment share (ℓ_s^p) at time $t_D > t_T$. Second, find a data-point (P_{0T}) in the past – exactly speaking, in the period $[t_0, t_T]$ – which satisfies $\ell_s^p = \ell_s^{0T}$, i.e. the services share associated with P_{0T} is equal to the services share at t_D. Third, compare the manufacturing employment share associated with P_{0T} to the manufacturing employment share at t_D. If $\ell_m^p < \ell_m^{0T}$, then the economy "has undergone a process of relative deindustrialization since t_T". **b)** Definition 3b implies that the concept of "relative industrialization" is antipodal to the concept of "relative deindustrialization": while "relative deindustrialization" implies that today's manufacturing-share is relatively small, "relative industrialization" implies that today's manufacturing-share is relatively great. **c)** Remarks 13a/b imply that the concept of "relative (de)industrialization" (Definition 3) is based on comparing the today's manufacturing-share to the manufacturing-share in a "comparable" situation from the past, where a past situation is "comparable" to today's situation if the service-share in the past situation is equal to today's service-share.

DEFINITION 4: *Let the Cartesian coordinates of points* P_0 *and* P_T *(cf. Definition 2a) be given as follows:* $P_0 = (\ell_a^0, \ell_m^0, \ell_s^0)$ *and* $P_T = (\ell_a^T, \ell_m^T, \ell_s^T)$. *Furthermore, let the Cartesian coordinates of a point* $P_{0T} \in [\tau_{0T}]$ *be given as follows:* $P_{0T} = (\ell_a^{0T}, \ell_m^{0T}, \ell_s^{0T})$. *We define the following partitions of* Δ_2 *(cf. Figure 11):*

(8) $\mathbf{A} := \{(\ell_a, \ell_m, \ell_s) \in \Delta_2 : \ell_a \geq \ell_a^0\}$

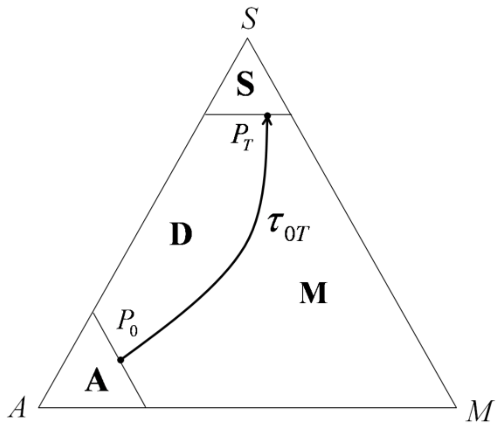
(9) $\mathbf{S} := \{(\ell_a, \ell_m, \ell_s) \in \Delta_2 : \ell_s \geq \ell_s^T\}$

(10) $\mathbf{D} := \mathbf{L} \cap (\Delta_2 \backslash (\mathbf{A} \cup \mathbf{S}))$

(11) $\mathbf{M} := \Delta_2 \backslash (\mathbf{A} \cup \mathbf{S} \cup \mathbf{D} \cup [\tau_{0T}])$

where $\mathbf{L}$ *is given by* $l_{0T} := \{(\ell_a, \ell_m, \ell_s) \in \Delta_2 : \ \ell_s = \ell_s^{0T}, \ \ell_m < \ell_m^{0T}\}$ *and* $\mathbf{L} := \{l_{0T} : P_{0T} \in [\tau_{0T}^p]\}$.

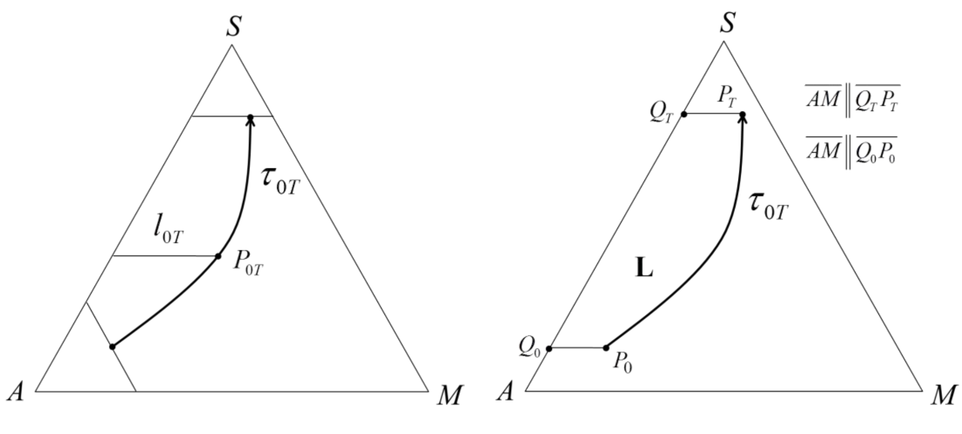
Figure 11: *A partitioning of* Δ_2 *according to Definition 4.*



Remark 14: a) For an explanation of partitions **A** and **S** see (Proof of) Lemma 6. Note that the geometrical properties of partitions **A** and **S** (cf. Figure 11)

follow from Definition 4 and Lemmas 4 and 2. **b)** The explanation of partition **D** is a little bit more complicated. Imagine an arbitrary point $P_{0T} = (\ell_a^{0T}, \ell_m^{0T}, \ell_s^{0T})$ on the trajectory τ_{0T}. The set of all points (ℓ_a, ℓ_m, ℓ_s) on Δ_2 which satisfy $\ell_s = \ell_s^{0T}$ and $\ell_m < \ell_m^{0T}$ is given by $\{(\ell_a, \ell_m, \ell_s) \in \Delta_2 : \ell_s = \ell_s^{0T},\ \ell_m < \ell_m^{0T}\} \equiv l_{0T}$. The basic knowledge of calculus implies that (cf. Figure 12): l_{0T} is a line-segment on Δ_2; l_{0T} is parallel to the $\overline{AM}$-edge of Δ_2 (cf. Lemma 2c and Figures 4); l_{0T} is bounded on the right-hand side by the point P_{0T} and on the left-hand side by the $\overline{AS}$-edge of Δ_2 (cf. Lemma 3a/c). Thus, l_{0T} is the set of all points on Δ_2 which are situated on the left-hand side of point P_{0T}; cf. Figure 12. If we do this procedure with every point on τ_{0T} and join all the line-segments l_{0T} which are created by this procedure, we obtain the set $\mathbf{L} := \{l_{0T} : P_{0T} \in [\tau_{0T}]\}$; cf. Figure 12. **D** is given by $\mathbf{D} := \mathbf{L} \cap (\Delta_2 \setminus (\mathbf{A} \cup \mathbf{S}))$, cf. Definition 4. For economic interpretation of partition **D** see (Proof of) Lemma 6. **c)** The partition **M** is the part of Δ_2 which is not assigned to the other partitions (**A**, **S**, **D**, $[\tau_{0T}]$). It contains all the points which are on the right-hand side of τ_{0T}. The proof of this fact is analogous to the proof in Remark 14b.

Figure 12: *The sets* l_{0T} *and* $\mathbf{L}$ *on* Δ_2.



DEFINITION 5: a) *In an "agricultural economy" the greatest share of labour is employed in the agricultural sector, i.e.* $\ell_a > 1/2$. ***b)*** *In a "manufacturing economy" the greatest share of labour is employed in the manufacturing sector, i.e.* $\ell_m > 1/2$. ***c)*** *In a "services economy" the greatest share of labour is employed in the services sector, i.e.* $\ell_s > 1/2$.

LEMMA 6: a) *An economy situated in partition* **A** *is an agricultural economy (cf. Definition 5a).* ***b)*** *An economy situated in partition* **S** *is a "services economy" (cf. Definition 5c).* ***c)*** *An economy situated in partition* **D** *"has undergone a process of relative deindustrialization since* t_T*" (cf. Definition 3a).* ***d)*** *Partition* **M** *contains all the points which satisfy Definition 3b ("relative industrialization").*

PROOF: a) Definition 4 and Assumption 3a imply that all points in partition **A** satisfy the following inequality: $\ell_a \geq \ell_a^0 > 1/2$. This fact implies Lemma 6a; cf. Definition 5a. **b)** Definition 4 and Assumption 3b imply that all points in **S** satisfy the following condition: $\ell_s \geq \ell_s^T > 1/2$. This fact implies Lemma 6b; cf. Definition 5c. **c)** The definition of l_{0T} (cf. Definition 4) implies that all points which belong to l_{0T} satisfy Definition 3a, i.e. if an economy is situated on l_{0T}, then the economy "has undergone a process of relative deindustrialization since t_T". (Note that l_{0T} exists only for $t > t_T$; cf. Definition 4). Definition 4 implies that **L** consists of such line-segments, i.e. line-segments which satisfy Definition 3a. Thus, **L** satisfies Definition 3a as well. **D** is a subset of **L**. Thus, **D** satisfies Definition 3a. That is, an economy situated in partition **D** "has undergone a process of relative deindustrialization since t_T"; cf. Definition 3a, which proves Lemma 6c. **d)** The proof of Lemma 6d is analogous to the proof of Lemma 6c. Note, however, that **M** does not only contain points which satisfy Definition 3b but also points which do not satisfy Definition 3a/b. The latter points do not satisfy Definition 3a/b, since they are not “comparable” (cf. Remark 13c). We omit the discussion of this fact, since it has no relevance for our results. ■

LEMMA 7: *Assume an economy which satisfies Assumptions 1-4. Let this economy be situated in partition* **S** *at time* t_S*, i.e.* $\phi(t_S) \in \mathbf{S}$*, where* $t_T \le t_S < \bar{t}$*. If this economy leaves* **S***, then it must enter* **D** *or* **M***. That is: if* $\phi(t) \notin \mathbf{S}$ *for* $t_S < t < t_x$*, then either* $\phi(t) \in \mathbf{D}$ *for* $t_S < t < t_D \le t_x$ *or* $\phi(t) \in \mathbf{M}$ *for* $t_S < t < t_M \le t_x$*, where* $t_x > t_S$*,* $t_D > t_S$ *and* $t_M > t_S$ *are points in time.*

PROOF: Note that Definition 4 implies that **S** is a closed set; thus, (i) the boundary between **S** and **M** belongs to **S** and (ii) the boundary between **S** and **D** belongs to **S**; cf. Figure 11. The proof of Lemma 7 can be divided into the following parts.

LEMMA 8: If $\phi(t) \notin \mathbf{S}$*, then* $\phi(t) \in \mathbf{A} \cup \mathbf{M} \cup \mathbf{D} \cup \mathrm{int}([\tau_{0T}])$. PROOF: Definition 4, which defines a partitioning of Δ_2, and Assumption 2b imply Lemma 8. See also Definition 2f. ◊

In the following we discuss which of the partitions (**A**, **M**, **D**, int([τ_{0T}])) the economy can enter at the instant at which it leaves **S**.

LEMMA 9: Let **Z** *denote a partition of* Δ_2*, i.e.* $\mathbf{Z} \in \{\mathbf{A}, \mathbf{M}, \mathbf{D}, \mathbf{S}, \mathrm{int}([\tau_{0T}])\}$. *Assume that* **Z** *and* **S** *are separated. Then the following is true: if* $\phi(t_y) \in \mathbf{S}$ *then* $\phi(t_z) \notin \mathbf{Z}$ *for* $t_z \to t_y$*, where* $t_z > t_y$. PROOF: Two sets $\mathbf{X} \subset \Delta_2$ and **Y** $\subset \Delta_2$ are separated if **X***$\cap$**Y**=**X**$\cap$**Y***=∅, where **X*** is the closure of **X** and **Y*** is the closure of **Y**; see e.g. Flegg (1974), p.163f. Thus, Lemma 9 is implied by the fact that $\phi(t)$ is continuous (cf. Assumption 2c) and **S** and **Z** are separated. The economy cannot “jump” from **S** to **Z**, since a “jump” contradicts the continuity assumption. ◊

LEMMA 10: Partitions **S** *and* **A** *are separated.* PROOF: Lemma 9 is implied by Definition 4 and Assumption 3a/b. In particular, the fact that Assumption 3a/b requires that P_0 is “close” to vertex *A* and P_T is “close” to vertex *S* implies that partitions **S** and **A** are separated by a non-empty set $(\mathbf{D} \cup \text{int}([\tau_{0T}]) \cup \mathbf{M})$; cf. Definition 4 and Figure 11. This vague statement can be specified as follows. If $\ell_a^0 > 1/2$ and $\ell_s^T > 1/2$ (cf. Assumptions 3a/b) **S** and **A** are separated, as shown in the following proof. If **S** and **A** are not separated then there must exist a point $P = (l_a^P, l_m^P, l_s^P) \in \Delta_2$ which satisfies (i) $P \in \mathbf{A}^*$ and $P \in \mathbf{S}$ or (ii) $P \in \mathbf{A}$ and $P \in \mathbf{S}^*$; cf. Proof of Lemma 9. (8) and (9) imply that in the cases (i) and (ii) the following is true: $l_a^P \geq l_a^0$ and $l_s^P \geq l_s^T$. Thus, since we assume $\ell_a^0 > 1/2$ and $\ell_s^T > 1/2$, the following is true: $\ell_a^P > 1/2$ and $\ell_s^P > 1/2$ and, thus, $\ell_a^P + l_s^P > 1$. This contradicts (1). Thus, **S** and **A** are separated if $\ell_a^0 > 1/2$ and $\ell_s^T > 1/2$. ◊

LEMMA 11: In general, (i) **S** *and* **M** *are not separated and (ii)* **S** *and* **D** *are not separated.* PROOF: Lemma 11 is implied by Definition 4 and Assumption 3; cf. Figure 11. For the case where **S** is separated from **M** and **D** see the discussion at the end of this proof. ◊

LEMMA 12: $\phi(t) \notin \text{int}([\tau_{0T}])$ *for* $t > t_S$. PROOF: Lemma 12 is implied by the fact that $t_S \geq t_T$ (as assumed in Lemma 7) and by Assumption 4. ◊

LEMMA 13: Assume that **S** *is not separated from* **M** *and/or* **D**. *Then the following is true: if* $\phi(t_S) \in \mathbf{S}$ *and* $\phi(t_z) \notin \mathbf{S}$ *then* $\phi(t_z) \in \mathbf{M}$ *or* $\phi(t_z) \in \mathbf{D}$ *for* $t_z \to t_S$, *where* $t_z > t_S$. PROOF: This lemma is implied by Lemmas 8-12. ◊

Lemma 13 completes the proof of Lemma 7. Note that, if **S** *is* separated from **M** and **D**, the economy cannot leave **S** (due to Assumption 2c) and, thus, stays in **S**. In this case the premise of Lemma 7 ("...*if* $\phi(t) \notin \mathbf{S}$ *for* $t_S < t < t_x$...") is not satisfied. That is, this case is not relevant for Lemma 7. Furthermore, note that Lemma 7 would hold, even if we allowed that τ is a closed trajectory (cf. Remark 12); see APPENDIX C for a proof. ■

6. MODELL-PREDICTIONS OF STRUCTURAL CHANGE

We will assume now that t_0 corresponds to a point in time in the early history of an industrialized country (e.g. the year 1820 in the history of the USA; cf. Figure 10). Furthermore, we assume that t_T corresponds to now. Thus, the trajectory-segment τ_{0T} corresponds to the past structural change (in the USA) and the trajectory-segment τ_{T+} corresponds to the future structural change (in the USA). We translate now the properties of τ_{T+} into structural change scenarios. First, we show that there are three scenarios of future development. Then, we discuss how these scenarios can be continued.

LEMMA 14: *The economy which satisfies Assumptions 1-4 is situated in partition* $\mathbf{S}$ *at time* t_T.

PROOF: Definition 2a implies that the economy is in point P_T at time t_T. Definition 4 implies that the point P_T is located in partition $\mathbf{S}$. ■

THEOREM 1: *Assume an economy which satisfies Assumptions 1-4. Let this economy be situated in partition* $\mathbf{S}$ *at time* $t = t_T$, *i.e.* $\phi(t_T) \in \mathbf{S}$ *(cf. Lemma 14). There are only three alternative scenarios regarding the development of this economy in the future (* $t > t_T$ *):*

Scenario I: *The economy stays in* $\mathbf{S}$ *for* $t > t_T$, *i.e.* $\phi(t) \in \mathbf{S}$ *for* $t > t_T$*; cf. Figure 13.*

Scenario II: *At some point in time* $t_{SD} > t_T$ *the economy departs from* $\mathbf{S}$ *and enters* $\mathbf{D}$. *That is,* $\phi(t) \in \mathbf{S}$ *for* $t_T < t \le t_{SD}$ *and* $\phi(t) \in \mathbf{D}$ *for* $t_{SD} < t < t_x$, *where* $t_x > t_{SD}$ *is some point in time. See also Figure 14.*

Scenario III: *At some point in time* $t_{SM} > t_T$ *the economy departs from* $\mathbf{S}$ *and enters* $\mathbf{M}$. *That is,* $\phi(t) \in \mathbf{S}$ *for* $t_T < t \le t_{SM}$ *and* $\phi(t) \in \mathbf{M}$ *for* $t_{SM} < t < t_y$, *where* $t_y > t_{SM}$ *is some point in time. See also Figure 15.*

PROOF: The economy being in **S** can stay in **S** forever. This outcome is possible if, for example, the economy converges to a fixed point in **S**. If the economy does not stay in **S**, i.e. if the economy leaves **S**, the economy must enter **D** or **M**; cf. Lemma 7. Note that Definition 4 implies that **S** is a closed set; thus, (i) the boundary between **S** and **M** belongs to **S** and (ii) the boundary between **S** and **D** belongs to **S**. ■

Figure 13: *Scenario I.*

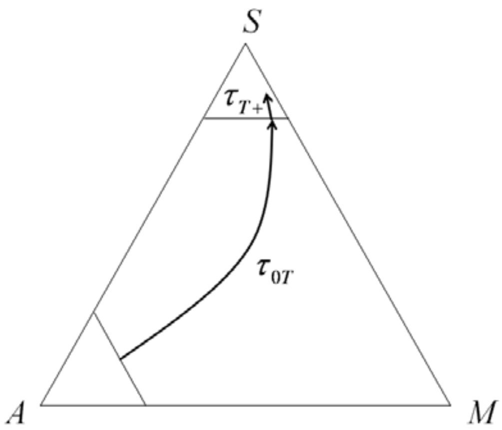


Figure 14: *Scenario II.*

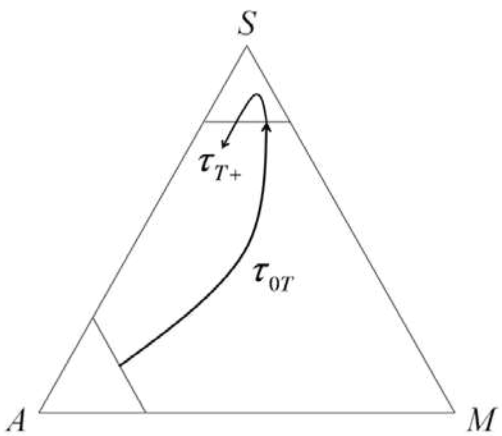
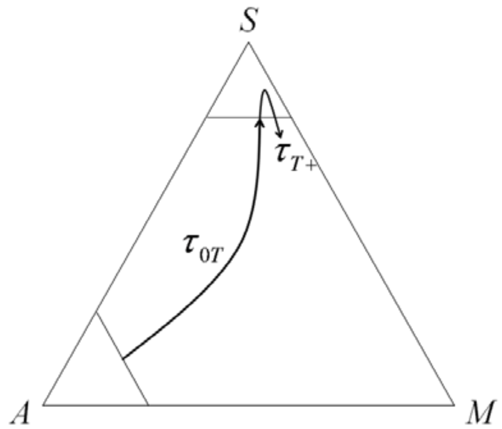


Figure 15: *Scenario III.*



Remark 15: Of course, when the economy is in **D** (Scenario II) or in **M** (Scenario III), it can move further to **A** or go back to **S**. These "secondary steps" (sub-scenarios) will be discussed later.

COROLLARY 1 (Economic Interpretation of Theorem 1): *Assume an economy which (a) satisfies Assumptions 1-4 and (b) is dominated by the services sector today. Then, there are only three alternative scenarios regarding future development of this economy. (I) The economy remains a "services economy" forever (cf. Definition 5c). (II) At some future point in time the economy starts a process of "relative deindustrialization" (cf. Definition 3a). (III) At some future point in time the economy starts a process of "relative industrialization" (cf. Definition 3b).*

PROOF: a) Theorem 1 postulates that in Scenario I the economy stays in **S** forever. Lemma 6b shows that the economy situated in **S** is a services economy. **b)** Theorem 1 postulates that in Scenario II the economy enters **D**. Lemma 6c implies that the economy situated in partition **D** "has undergone a process of relative deindustrialization since t_T" (cf. Definition 3a). **c)** Theorem 1 postulates that in Scenario III the economy enters **M**. Lemma 6d implies that the economy situated in partition **M** "has undergone a process of relative industrialization since t_T" (cf. Definition 3b). ■

Remark 16: a) In Scenario I the service-share (ℓ_s) does not fall below its today's level, i.e. $\ell_s \geq \ell_s^T$ for $t > t_T$. **b)** Scenario I implies that there is some fixed point or limit cycle in **S**; cf. Stijepic (2014a), "Theorem 1", on the limit-properties of trajectories satisfying Assumptions 1-4. **c)** The structural change models discussed in Section 1 predict Scenario I (fixed point).

Remark 17: Scenario II implies that at some future point in time the service-share (ℓ_s) starts shrinking (cf. Definition 4, Lemma 2b and Figure 14) while the manufacturing sector remains below "comparable" past levels; cf. Definition 3a and Remark 13c.

Remark 18: Scenario III implies that at some future point in time the service-share (ℓ_s) starts shrinking (cf. Definition 4, Lemma 2b and Figure 15) while the manufacturing sector remains above "comparable" past levels; cf. Definition 3b and Remark 13c.

Remark 19: Now we turn to the secondary steps (sub-scenarios), i.e. we analyse what happens after the economy has entered partition **D** or **M**. Of course, the economy can stay in one of these partitions (if there is a fixed point or limit cycle). In this case, the economy remains relatively (de)industrialized forever. In the following we discuss what happens if the economy departs from partitions **D** and **M**.

THEOREM 2 (Continuation of Scenario II): *Assume an economy which satisfies Assumptions 1-4. Furthermore, let this economy be situated in partition* **D** *at time* t_D*, i.e.* $\phi(t_D) \in \mathbf{D}$*, where* $t_T < t_D < \bar{t}$*. If this economy* *leaves* **D** *at time* t_x*, then it enters* **A** *or* **S** *at* t_x*, where* $t_D < t_x < \bar{t}$*. That is: if* $\phi(t) \in \mathbf{D}$ *for* $t_D \le t < t_x$ *and* $\phi(t_x) \notin \mathbf{D}$*, then either* $\phi(t_x) \in \mathbf{A}$ *or* $\phi(t_x) \in \mathbf{S}$*.*

PROOF: See Figure 11. Note that Definition 4 implies that **S** and **A** are closed sets; thus, (i) the boundary between **S** and **D** belongs to **S** and (ii) the boundary between **A** and **D** belongs to **A**. The proof of Theorem 2 is analogous to the Proof of Lemma 7. The proof can be divided into following parts.

LEMMA 15: If $\phi(t) \notin \mathbf{D}$*, then* $\phi(t) \in \mathbf{A} \cup \mathbf{M} \cup \mathbf{S} \cup \text{int}([\tau_{0T}])$. PROOF: Lemma 15 is implied by Definition 4, which defines a partitioning of Δ_2, and Assumption 2b. See Definition 2f. ◊

In the following we discuss which of the partitions (**A**, **M**, **S**, int([τ_{0T}])) the economy can enter at the instant at which it leaves **D**.

LEMMA 16: Partitions **D** *and* **M** *are separated. In particular,* **D** *and* **M** *are separated by* $[\tau_{0T}]$. PROOF: Lemma 16 is implied by Definition 4. As discussed in Remark 14b/c: **D** contains all the points on the left-hand side of τ_{0T}; **M** contains all the points on the right-hand side of τ_{0T}. For a definition of "separated", see Proof of Lemma 9. ◊

LEMMA 17: If $\phi(t) \in \mathbf{D}$ *for* $t_D \le t < t_x$*, then* $\phi(t_x) \notin \mathbf{M}$. PROOF: Lemma 17 is implied by the fact that **D** and **M** are separated (cf. Lemma 16) and $\phi(t)$ is continuous (cf. Assumption 2c). The "jump" from **D** to **M** violates the continuity assumption of $\phi(t)$; cf. (Proof of) Lemma 9. ◊

LEMMA 18: In general, **D** *is not separated from* **S** *and* **A**. PROOF: Lemma 18 is implied by Definition 4 and Assumption 3. For a discussion of the case in which **D** is separated from **S** and **A**, see the end of this proof. ◊

LEMMA 19: If $\phi(t) \in \mathbf{D}$ *for* $t_D \le t < t_x$*, then* $\phi(t_x) \notin \text{int}([\tau_{0T}])$. PROOF: Lemma 19 is implied by the fact that $t_x > t_D > t_T$ (as assumed in Theorem 2) and by Assumption 4. ◊

LEMMA 20: Assume that **D** *is not separated from* **S** *and/or* **A***. Then, the following is true: if* $\phi(t) \in \mathbf{D}$ *for* $t_D \le t < t_x$ *and* $\phi(t_x) \notin \mathbf{D}$*, then either*

$\phi(t_x) \in \mathbf{A}$ *or* $\phi(t_x) \in \mathbf{S}$. PROOF: This lemma is implied by Lemmas 15 and 17-19. ◊

Lemma 20 proves Theorem 2. Note that, if **D** is separated from **S** and **A**, the economy must stay in **D**. Thus, the premise Theorem 2 ("...*if*... $\phi^p(t_x) \notin \mathbf{D}$...") is not satisfied. That is, this case is not relevant for Theorem 2. Furthermore, note that Theorem 2 would hold, even if we allowed that τ is a closed trajectory (cf. Remark 12); see APPENDIX D for a proof. ■

COROLLARY 2 (Economic Interpretation of Theorem 2): *Assume that an economy is "relatively deindustrialized" (cf. Definition 3a) at time* t_D *(Scenario II), where* $t_D > t_T$ *(cf. Definition 2a). A necessary condition among others for entering a path of relative industrialization (cf. Definition 3b) and becoming a (pure) manufacturing economy (cf. Definition 1b) at some point in time* $t_M > t_D$ *is that prior to that (i.e. at some point in time* $t_x < t_M$ *) the economy becomes an agricultural economy (cf. Definition 5a) or a services economy (cf. Definition 5c) (where* $t_x > t_D$ *).*

PROOF: PART 1: Corollary 2 refers to an economy which is situated in partition **D**; cf. Lemma 6c. Definition 4 and Assumption 3 imply that **M** contains vertex *M* and some neighbourhood of vertex *M*; cf. Figure 11. In particular, Assumption 3c ensures that τ_{0T} does not contain vertex *M*; thus, **M** contains the vertex *M* and some (eventually small) neighbourhood of vertex *M*. Thus, if the economy cannot reach any point in **M**, it cannot reach *M* (and some, potentially small, neighbourhood of *M*), and, thus, the economy cannot become a (pure) manufacturing economy; cf. Definition 1b. Lemma 6a (6b) implies that an economy situated in **A** (**S**) is a(n) agricultural economy (services economy); cf. Definition 5a (5c). Furthermore, Lemma 6c implies that an economy situated in **M** has entered a path of "relative industrialization" (cf. Definition 3b).

PART 2: As Theorem 2 shows, the economy, which is situated in partition **D**, cannot reach any point in **M**, unless the economy traverses **S** or **A**. Definition 4 implies that, in general, **S** and **A** are not separated from **M**; for a definition of "separated", see Proof of Lemma 9. Thus, in general, an economy which is in **S** or **A** can enter **M** at the next instant.

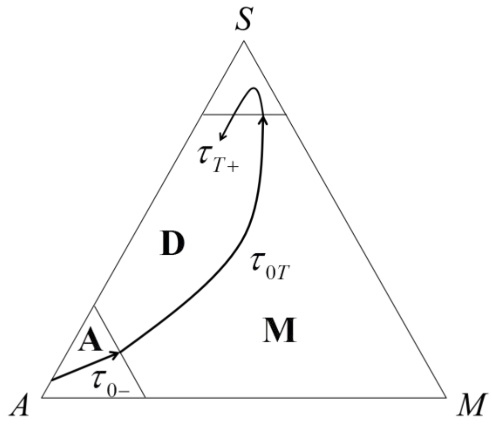
PART 3: However, this is not always guaranteed; i.e. even if the economy enters **A** or **S**, it may not always be possible that the economy can go from there to **M**, as shown in the following example. Assume that the economy which moves along τ satisfies Assumption 4. Furthermore, assume that τ_{0-} (cf. Definition 2e) touches the boundary of Δ_2 somewhere in **A**. Under these conditions, trajectory-segment $\tau_{T+} \subset \tau$ cannot go from **D** to **M** via **A**, since on this way it intersects the trajectory-segment $\tau_{0-} \subset \tau$. Assumption 4 prohibits self-intersections of τ. See also Figure 16.

PART 4: The following example demonstrates that an economy which is situated in **D** may not be able to enter **S** or **A**. Assume that $P_T = S$ (cf. (6) and Definition 2a); i.e. the trajectory-segment τ_{0T} ends in vertex *S*. In this case $\mathbf{S} = S = P_T \subset \tau_{0T}$; cf. Definition 4. Thus, the trajectory-segment τ_{T+} cannot enter **S**, since in this way it violates Assumption 4. A similar example can be constructed to show that in some cases τ_{T+} cannot enter **A**.

PART 5: For the reasons postulated in Parts 3 and 4 of this proof, Corollary 2 contains the formulation "*…A necessary condition among others…*". That is, when the economy is situated in **D**, traversing **S** or **A** is one necessary conditions among many necessary conditions for entering **M**.

These facts imply Corollary 2. ■

Figure 16: *A case in which the way from* **D** *to* **M** *via* **A** *is not possible.*



COROLLARY 3 (Economic Interpretation of Theorem 2): *Assume that an economy is "relatively deindustrialized" (cf. Definition 3a) at time t_D (Scenario II), where $t_D > t_T$ (cf. Definition 2a). A necessary condition among others for entering a path of "relative industrialization" (cf. Definition 3b) and becoming a (pure) manufacturing economy (cf. Definition 1b) at some point in time $t_M > t_D$ is that prior to that (i.e. at some point in time $t_x < t_M$) the agriculture-share ℓ_a grows beyond ℓ_a^0 (cf. Definition 4) or the service-share ℓ_s grows beyond ℓ_s^T (cf. Definition 4) (where $t_x > t_D$).*

PROOF: The proof of Corollary 3 consists of the following parts.

LEMMA 21: If the economy moves along τ_{T+} from **D** *to* **M**, *then the economy muss cross the interior of the line-segment $\overline{AP_0}$, or vertex A, or the interior of the line-segment $\overline{P_T S}$, or vertex S.* PROOF: $P_0 \in \Delta_2$, $P_T \in \Delta_2$ and $[\tau_{0T}] \subset \Delta_2$; cf. Definition 2. P_0 is closer to vertex *A* than P_T is; P_T is closer to vertex *S* than P_0 is; cf. Assumption 3a/b. P_0 and P_T are connected by the interior of $[\tau_{0T}]$; cf. Definition 2a/c/f. These facts imply that vertex *A* and vertex *S* are connected by the curve $c_\Delta := A \cup \text{int}(\overline{AP_0}) \cup P_0 \cup \text{int}([\tau_{0T}]) \cup P_T \cup \text{int}(\overline{P_T S}) \cup$

S. c_Δ is a one-dimensional connected subset of Δ_2. It separates Δ_2 into two disjoint parts: the left-hand side and the right-hand side. **D** is on the left-hand side (of $[\tau_{0T}]$); **M** is on the right-hand side (of $[\tau_{0T}]$); cf. Definition 4 and Remark 14 b/c. Thus, if the economy moves from **D** to **M** (along a continuous trajectory) it must cross the curve c_Δ. τ_{T+} is continuous on Δ_2; cf. Assumption 2c and Definition 2d. The economy which moves along τ_{T+} cannot cross $[\tau_{0T}] \subset c_\Delta$; cf. Assumption 4. Thus, the economy which moves along τ_{T+} from **D** to **M** must cross $A \cup \text{int}(\overline{AP_0})$ or $\text{int}(\overline{P_T S}) \cup S$. Remember that $[\tau_{0T}] = P_0 \cup \text{int}([\tau_{0T}]) \cup P_T$; cf. Definition 2a/c/f. ◊

LEMMA 22: If the economy, which moves from **D** *to* **M** *along* τ_{T+}, *crosses the line-segment* $A \cup \text{int}(\overline{AP_0})$ *at time* t_a, *then* $\ell_a > \ell_a^0$ *at time* t_a. *If the economy, which moves from* **D** *to* **M** *along* τ_{T+}, *crosses the line-segment* $\text{int}(\overline{P_T S}) \cup S$ *at time* t_s, *then* $\ell_s > \ell_s^T$ *at time* t_s. PROOF: If the economy moves from P_0 towards vertex *A*, ℓ_a grows; cf. Lemma 4. Thus, the economy situated in $A \cup \text{int}(\overline{AP_0})$ features a greater ℓ_a than the economy situated in point P_0. If the economy moves from P_T towards vertex *S*, ℓ_s grows; cf. Lemma 2. Thus, the economy situated in $\text{int}(\overline{P_T S}) \cup S$ features a greater ℓ_s than the economy situated in point P_T. $P_0 = (\ell_a^0, \ell_m^0, \ell_s^0)$ and $P_T = (\ell_a^T, \ell_m^T, \ell_s^T)$; cf. Definition 4. ◊

As shown in the Proof of Corollary 1, **M** contains all the points which feature "relative industrialization" and "pure manufacturing economy". Corollary 3 refers to an economy which is situated in partition **D** at time t_D; cf. Lemma 6c. Corollary 3 refers to an economy which moves along trajectory-segment τ_{T+}, since Corollary 3 refers to $t_D > t_T$; cf. Definition 2d. These facts and Lemmas 21 and 22 imply Corollary 3. ■

THEOREM 3 (Continuation of Scenario III): *Assume an economy which satisfies Assumptions 1-4. Furthermore, let this economy be situated in partition* $\mathbf{M}$ *at time* t_M, *i.e.* $\phi(t_M) \in \mathbf{M}$, *where* $t_T < t_M < \bar{t}$. *If this economy leaves* $\mathbf{M}$ *at time* t_x, *then it enters* $\mathbf{A}$ *or* $\mathbf{S}$ *at* t_x, *where* $t_M < t_x < \bar{t}$. *That is: if* $\phi(t) \in \mathbf{M}$ *for* $t_M \le t < t_x$ *and* $\phi(t_x) \notin \mathbf{M}$, *then either* $\phi(t_x) \in \mathbf{A}$ *or* $\phi(t_x) \in \mathbf{S}$.

PROOF: The proof is analogous to the proof of Theorem 2. ■

COROLLARY 4 (Economic Interpretation of Theorem 3): *Assume that an economy is "relatively industrialized" (cf. Definition 3b) at time* t_M *(Scenario III), where* $t_M > t_T$. *A necessary condition among others for entering a path of relative deindustrialization (cf. Definition 3a) at some point in time* $t_D > t_M$ *is that prior to that (i.e. at some point in time* $t_x < t_D$*) the economy becomes an agricultural economy (cf. Definition 5a) or a services economy (cf. Definition 5c) (where* $t_M < t_x$*).*

PROOF: Note that this time the economy is situated in partition **M** and Corollary 4 is about entering partition **D**. The rest of the proof is obvious, since analogous to the proof of Corollary 2. ■

COROLLARY 5 (Economic Interpretation of Theorem 3): *Assume that an economy is "relatively industrialized" (cf. Definition 3b) at time* t_M *(Scenario III), where* $t_M > t_T$. *A necessary condition among others for entering a path of relative deindustrialization (cf. Definition 3a) at some point in time* $t_D > t_M$ *is that prior to that (i.e. at some point in time* $t_x < t_D$*) the agriculture-share* ℓ_a *grows beyond* ℓ_a^0 *(cf. Definition 4) or the service-share* ℓ_s *grows beyond* ℓ_s^T *(cf. Definition 4) (where* $t_x > t_M$*).*

PROOF: Note that this time the economy is situated in partition **M** and Corollary 5 is about entering partition **D**. The rest of the proof is obvious, since analogous to the proof of Corollary 3. ■

Example 3 (Application of Corollaries 3 and 5 to the USA): We can use the data from Figure 10 to illustrate the meaning of Corollary 3 (5). Assume that that the US-economy starts a process of relative deindustrialization (industrialization) in the year 2020; cf. Definition 3. Thus, Scenario II (Scenario III) applies here. Corollary 3 (5) implies that after 2020 the US economy cannot become relatively industrialized (deindustrialized), cf. Definition 3, unless it increases its services employment share beyond 80% (see year 2003 in Figure 10) or increases the employment share of agriculture beyond 70% (see year 1820 in Figure 10). Note that these levels are only lower bounds. That is, a stronger increase in services/agricultural employment share may be necessary for starting a process of relative industrialization (deindustrialization); cf. Part 5 of Proof of Corollary 2.

COROLLARY 6 (Interpretation of Theorems 2 and 3; Path-dependency): *If the economy which satisfies Assumptions 1-4 leaves partition* **S** *(i.e. starts a process of relative (de)industrialization), it is restricted in its future development-possibilities (τ_{T+}), unless it returns to the beginning of the path (***S***) or to the beginning of the development process (***A***).*

Remark 20: a) Path-dependency means here: if the economy takes a certain path (e.g. path X) it cannot take (a path from some subset of) other paths, unless it returns to the beginning (of path X). **b)** Consider the USA as an example. The development path of the USA is comparable to the path depicted in Figure 13, where trajectory-beginning stands for the year 1820 and trajectory-end stands for today. (See APPENDIX B for an explicit proof). In 1820 the USA have all options open (from the mathematical point of view): they can take a *straight* path (i.e. a linear trajectory) to any point on Δ_2 while satisfying Assumption 4, because τ_{0T} does not exist at this point in time. Today, the USA have still many options open: even if Assumption 4 is satisfied, they can take a straight path to any partition (**A**, **M**, **D**) – i.e. become an agricultural economy or “relatively (de)industrialized” – or remain a services economy (**S**). The future development of the USA is, however, relatively restricted if they leave partition **S** and, thus, enter partition **D** or **M** (i.e. start a process of relative (de)industrialization). Then, the USA cannot take a straight path from **D** to **M** or from **M** to **D**. That is, if the USA enter a path of relative (de)industrialization, they remain relatively (de)industrialized unless they return to the beginning of the path (i.e. to an employment structure comparable to the structure in 2003) or to the beginning of their development process (i.e. to an employment structure comparable to the structure in 1820); cf. Example 3.

7. DISCUSSION AND GENERALIZATIONS

7.1 Closeness of P_T to Vertex *S*

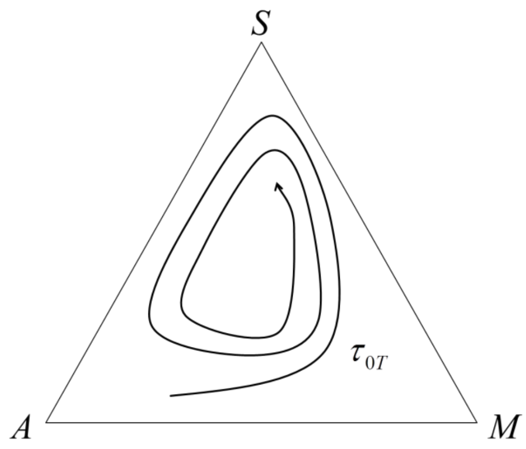
We assume in our paper that P_T is close to vertex *S*; cf. Assumption 3b. As discussed in Remark 10b, this assumption is satisfied by today’s industrialized countries, since, in general, they employ more than 50% of labour in the services sector. If this assumption is not satisfied (which is the case in developing economies), the economy under consideration is not a “services economy”. Thus, for example Scenario I in Corollary 1 must be reformulated correspondingly. Nevertheless, Theorems 2 and 3 and Corollaries 2-6 remain valid. However, they must be interpreted cautiously, since in this case the “partition” **S** does not contain only points which are labelled “services economy” but also points which are labelled “manufacturing economy” (i.e. states in which the manufacturing sector is dominant).

In general, the key to our results (e.g. “path dependency”) is the fact that τ_{0T} partitions Δ_2. The longer the development path (τ_{0T}), the longer the barrier between **D** and **M** and, thus, the stronger the restriction of future development possibilities (which is given by the set of prohibited τ_{T+}-shapes).

7.2 Curvature, Inflection Points and Monotonous Approach to Vertex *S*

As argued previously (cf. Remark 10c/d), the assumption of uniform curvature of τ_{0T} (cf. Assumption 3c) and monotonous dynamics along τ_{0T} (cf. Assumption 3d) is motivated by the fact that structural change is about long-run dynamics. That is, τ_{0T} describes a trend; thus, fluctuations are neglected. In general, if we relax Assumption 3c/d, our results remain valid, since (a) we interpret the partitions of Δ_2 in relation to τ_{0T} (cf. Definition 3, Definition 4 and Lemma 6) and (b) the postulates in our paper are conditional (cf. e.g. Lemma 7). Only in some extreme cases of violation of Assumption 3c/d our results must be interpreted cautiously. For example, in the case depicted in Figure 17 our concept of relative (de)industrialization (cf. Definition 3) is not sufficient for describing the future development scenarios; some new/additional concepts are necessary in this case.

Figure 17: *An extreme case of non-monotonous dynamics from* t_0 *to* t_T.



7.3 Assumption of Low-dimensional Structure

The fact that the three-sector economy moves on a two-dimensional bounded set (Δ_2) is essential to our results. A trajectory (a one-dimensional manifold) can partition the 2-simplex (a bounded subset of the plane) such that a large set of future trajectories is prohibited and, thus, the number of structural change scenarios is reduced significantly. In contrast, a trajectory in a three-dimensional space does not partition the space and the set of prohibited trajectories is more or less irrelevant. Thus, our approach requires that higher-dimensional structures are reduced to three-dimensional structures by e.g. defining groups, which is, anyway, often done in economics. For example, a sector is a group of many similar industries. Another example are quintiles associated with distribution functions. As shown in Section 2, three-dimensional structures (e.g. three-sector models) can be depicted on the 2-simplex. Note that two-dimensional structures can be depicted on the 2-simplex as well. (In this case the economy is moving along an edge of the 2-simplex).

8. NON-SELF-INTERSECTING STRUCTURAL CHANGE TRAJECTORIES IN THE LITERATURE

The key to all our results is the assumption that the (continuous) structural change trajectory τ does not intersect itself; cf. Assumptions 2c and 4. In general, the literature on structural change satisfies this assumption. For example, the models cited in Section 1 (see Footnote 1) generate continuous non-self-intersecting trajectories, as we show in this section. Each of these models provides a set of *economic* assumptions which ensure that the structural change trajectory does not intersect itself. In APPENDIX E we discuss some *mathematical* conditions which ensure that the structural change trajectory does not intersect itself.

A continuous structural change trajectory τ intersects itself on Δ_2, if there exist three points in time $t_x, t_y, t_z \in (\underline{t}, \bar{t})$ and three points on τ ($\phi(t_x) \in \tau$, $\phi(t_y) \in \tau$ and $\phi(t_z) \in \tau$) which satisfy the following conditions: $t_x < t_y < t_z$

and $\phi(t_x) = \phi(t_z) \neq \phi(t_y)$. That is, the following equations hold at the point of intersection:

(12) $\ell_a(t_x) = \ell_a(t_z) \neq \ell_a(t_y)$

(13) $\ell_m(t_x) = \ell_m(t_z) \neq \ell_m(t_y)$

(14) $\ell_s(t_x) = \ell_s(t_z) \neq \ell_s(t_y)$

(15) $t_x < t_y < t_z$.

where $\ell_i(t)$ denotes the employment share of sector i at time t, $i = a, m, s$. These facts imply that, if any of the three employment shares (ℓ_a, ℓ_m or ℓ_s) increases/decreases monotonously over time, then τ does not intersect itself (since in this case at least one of the equations (12)-(14) is not satisfied if (15) is satisfied). Thus, we can easily check whether a model predicts self-intersections of the structural change trajectory on Δ_2: if the model predicts that ℓ_a and/or ℓ_m and/or ℓ_s increases/decreases monotonously over time, then there is no self-intersection. The most structural change models and, in particular, the models discussed in Section 1 (cf. Footnote 1) predict continuous and monotonous dynamics of ℓ_a and/or ℓ_s; thus, these models do not predict self-intersections and are, thus, covered by our paper.

9. CONCLUDING REMARKS

Many theories of economic dynamics share common mathematical properties. We focus in our paper on theories which deal with structures (e.g. labour allocation, income distribution and investment structures) and structural change. Our paper has two goals. Our first goal is to show how qualitative information from empirical and theoretical/mathematical sources on structural change can be used to derive qualitative statements regarding the nature of structural change, i.e. we propose a qualitative approach to structural change modelling. Our second goal is to apply our approach to the analysis of long-run labour allocation dynamics.

Our qualitative approach to structural change modelling is based on three facts: (1) three-dimensional structural change is defined on a two-dimensional bounded set (a 2-simplex), (2) the trajectory of past structural change (τ_{0T}) partitions the 2-simplex into economically interpretable sections and (3) the non-self-intersection rule (cf. Section 8) prohibits some movements from one section to another. Jointly, these facts imply path-dependency of structural change, which can be used to reduce the number of feasible structural change scenarios. The advantage of this approach is that it is less dependent on specific economic assumptions associated with specific schools of economic thought in comparison to standard approaches to structural change modelling. The results of our qualitative model of structural change are presented in Theorems 1-3.

Our approach is applicable to many types of structural change; see Stijepic (2014a,b) for definitions and examples. Such an application requires that partitions and trajectories in our model are interpreted correspondingly. To demonstrate this fact we provided throughout the paper an application of our approach to a specific type of structural change: long-run labour reallocation across agriculture, manufacturing and services. These results are presented in Corollaries 1-6.

Our model shows that the overwhelmingly large set of potential structural change paths can be reduced significantly by using our methods. We have shown that today's industrialized economies have only three alternative scenarios regarding their future structural change. They can (I) remain services economies forever, (II) enter a path of relative deindustrialization or (III) enter a path of relative industrialization. If they enter a path of relative (de)industrialization, their future options are even more restricted. They can leave this path only if they return to its beginning (i.e. become services economies again) or return to the beginning of their development process (i.e. become agricultural economies again).

Note that the terms which we use to describe the economy and to label the partitions – e.g. "services economy", "(de)industrialization" and "agricultural economy" – are strictly defined in our paper. Nevertheless, our definitions of these terms coincide with the common sense of these terms.

As discussed in Section 7, our results are relatively robust.

Overall, we add a further tool to the toolkit of the modern economist. Usually, the understanding of a dynamic phenomenon requires using many different approaches of dynamic analysis, since most dynamic models feature some weaknesses. For example, on the one hand, some (econometric) models lack sufficient microfoundation, on the other hand, the established microfoundation is often "unrealistic" or does not yield quantitatively satisfying predictions. Thus, the more tools we have to double-check the results achieved by other tools, the better.

Further research could deal with the following aspects. First, our approach could be applied to other fields of structural change (e.g. to income-distribution dynamics), which requires deriving relevant stylized facts and reinterpreting the partitions created by the trajectory of past structural change (τ_{0T}). Second, additional models of qualitative structural change could be developed. For example, Stijepic (2014a,b) studies the limit-properties of qualitative structural change models and applies the results to neoclassical growth theory. In contrast, we study here the "transitional dynamics" of structural change, i.e. the question what happens before the limit.

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APPENDIX A: Proof of Lemmas 1 and 2

Proof of Lemma 1

In the following we use the Cartesian coordinates (ℓ_a, ℓ_m, ℓ_s) to identify the points on Δ_2. The set of points on the $\overline{SA}$-edge of Δ_2 is given by the following set of linear-combinations: $\overline{SA} = \{\gamma S + (1-\gamma)A : 0 \leq \gamma \leq 1\}$; cf. Fig. 2. By using (4) and (6) we obtain the Cartesian coordinates (ℓ_a, ℓ_m, ℓ_s) of this set: $\overline{SA} = \{((1-\gamma),\ 0,\ \gamma) : 0 \leq \gamma \leq 1\}$. This equation implies that the second coordinate (ℓ_m) is equal to zero for all γ, i.e. $\ell_m = 0$ on the whole line-segment $\overline{SA}$. This proves part b of Lemma 1. The proof of parts a and c is analogous. ■

Proof of Lemma 2

Again, we use Cartesian coordinates (ℓ_a, ℓ_m, ℓ_s) to identify points on Δ_2.

Let the Cartesian coordinates of point *P* be given as follows: $P = (\ell_a^P, \ell_m^P, \ell_s^P)$. The line l_s^P intersects the $\overline{SA}$-edge of Δ_2 at the coordinates $((1-\ell_s^P),\ 0, \ell_s^P)$; cf. (1), Lemma 1, Fig. 4 and Fig. 1. Analogously, line l_s^P intersects the $\overline{MS}$-edge of Δ_2 at the coordinates $(0,\ (1-\ell_s^P), \ell_s^P)$. By calculating the set of linear-combinations of these two intersection-points (the same approach as in the Proof of Lemma 1) we obtain the set of coordinates of the intersection between l_s^P and Δ_2: $l_s^P \cap \Delta_2 = \{((1-\lambda)(1-\ell_s^P),\ \lambda(1-\ell_s^P), \ell_s^P) : 0 \leq \lambda \leq 1\}$. We can see now that on the whole line-segment $l_s^P \cap \Delta_2$ the services-employment-share is equal to the services-employment-share in point *P*, i.e. $\ell_s = \ell_s^P$ on $l_s^P \cap \Delta_2$. Thus, a movement from point *P* along the line-segment $l_s^P \cap \Delta_2$ is not associated with a change in ℓ_s. That is, if the directional vector $\vec{v}$ lies in

$l_s^p \cap \Delta_2$ (i.e. if $\sigma = 0°$ or $\sigma = 180°$), ℓ_s does not change during the movement. This proves part c of Lemma 2.

Let l_s' be another line. Assume that l_s' satisfies the following assumptions: (1) l_s' is parallel to l_s^p and (2) l_s' intersects Δ_2. According to the arguments above, the Cartesian coordinates of the intersection of l_s' and Δ_2 are: $l_s' \cap \Delta_2 = \{((1-\delta)(1-\ell_s'),\ \delta(1-\ell_s'),\ \ell_s') : 0 \le \delta \le 1\}$, where ℓ_s' is the Cartesian ℓ_s-coordinate of l_s'.

Let $d(U,V)$ be the Euclidean distance between two arbitrary points *U* and *V* of the three-dimensional Cartesian coordinate system. We define the Euclidean distance between the vertex *S* and the line l_s^p in the three-dimensional Cartesian coordinate system ($d(S,l_s^p)$) as follows: $d(S,l_s^p) \equiv d(S,\ (l_s^p \cap \Delta_2)_{\lambda=1/2})$, where *S* is given by (6) and $(l_s^p \cap \Delta_2)_{\lambda=1/2}$ stands for the Cartesian coordinates of the point on the line-segment $l_s^p \cap \Delta_2$ corresponding to $\lambda = 1/2$, i.e. $(l_s^p \cap \Delta_2)_{\lambda=1/2} = (1/2(1-\ell_s^p), 1/2(1-\ell_s^p),\ \ell_s^p)$; cf. Fig. A1. Thus, $d(S,l_s^p) = \sqrt{[0-1/2(1-\ell_s^p)]^2 + [0-1/2(1-\ell_s^p)]^2 + (1-\ell_s^p)^2}$ $= \sqrt{3/2}(1-\ell_s^p)$. It can be shown, analogously, that $d(S,l_s')$, i.e. the Euclidean distance between the vertex *S* and the line l_s', is given by $d(S,l_s') \equiv d(S,\ (l_s' \cap \Delta_2)_{\delta=1/2}) = \sqrt{3/2}(1-\ell_s')$; cf. Fig. A1. These facts imply that $d(S,l_s') = \sqrt{3/2}(1-\ell_s') < \sqrt{3/2}(1-\ell_s^p) = d(S,l_s^p)$, if and only if $\ell_s' > \ell_s^p$. That is, l_s' is closer to the vertex *S* than l_s^p is, if and only if the Cartesian ℓ_s-coordinate of l_s' is greater than the Cartesian ℓ_s-coordinate of l_s^p.

Let us summarize these facts, as follows: (i) *P* is a point on l_s^p, (ii) *S*, l_s^p and l_s' lie in the plane (on Δ_2), (iii) l_s^p and l_s' are parallel to $\overline{AM}$ and (iv) $d(S,l_s') < d(S,l_s^p)$, if and only if $\ell_s' > \ell_s^p$, i.e. (iv) implies: the closer an $\overline{AM}$-parallel on Δ_2 to the vertex *S*, the greater the service-share (ℓ_s) associated with the points on this parallel. Facts (i)-(iv) imply that, if $\ell_s' > \ell_s^p$, a movement starting from *P* can reach the line l_s' if and only if the angle (σ) between the corresponding vector $\vec{v}$ and l_s^p is greater than 0° and smaller than 180°; cf. Fig. A2. That is, a movement from *P* brings the system to a state/point (on l_s') with a greater ℓ_s, if and only if the corresponding vector $\vec{v}$ has an angle $0° < \sigma < 180°$. Vice versa, only if $0° < \sigma < 180°$ the movement along the vector $\vec{v}$ is associated with a transition to a point on a line l_s' which lies closer to *S* and, therefore, features a greater ℓ_s; cf. Fig. A2. That is, only if $0° < \sigma < 180°$ the movement along the vector $\vec{v}$ is associated with an increase in ℓ_s. This proves part a of Lemma 2. Part b can be proven analogously: if $\ell_s' < \ell_s^p$, l_s' is further away from *S* than l_s^p is; thus, only a vector-angle $180° < \sigma < 360°$ can bring us to a point on l_s' which features a lower ℓ_s. ■

Figure A1

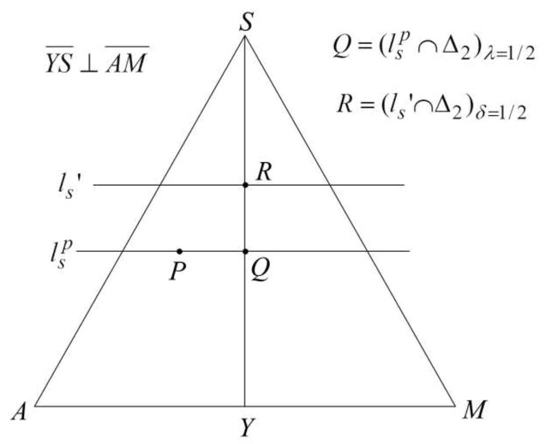
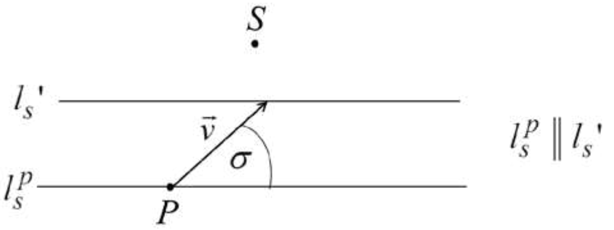


Figure A2



$$\ell_s' > \ell_s^p \leftrightarrow d(S, l_s') < d(S, l_s^p)$$

APPENDIX B: Translation of a Standard Diagram into a Δ_2-Diagram

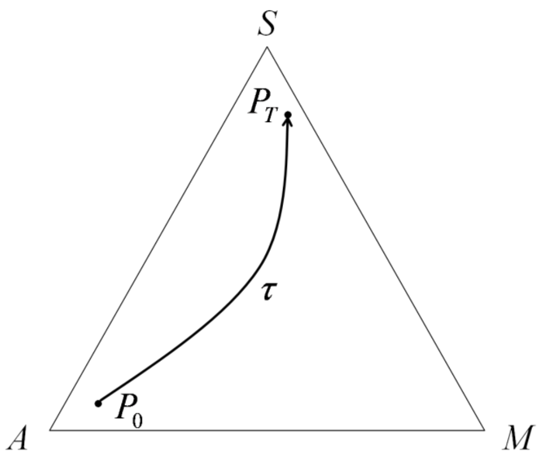
We translate now the dynamics depicted in Fig. 10 into (stylized) dynamics on Δ_2.

Fig. 10 implies that in 1820 the greatest share of labour is employed in agriculture. Thus, in 1820 the economy is on Δ_2 relatively close to vertex *A*; cf. (4), Remark 1 and Lemma 4a. Furthermore, in 1820 services and manufacturing have non-trivial employment shares. Thus, the trajectory on Δ_2 is in the interior of Δ_2; cf. Lemma 1. By using this information we can depict the starting point P_0 of the trajectory on Δ_2; cf. Fig. B1.

Now we turn to the situation in 2003. We can see that the greatest share of labour is employed in services and that the employment shares of manufacturing and agriculture are relatively small. Thus, in 2003 the trajectory is relatively close to the vertex *S*; cf. (6), Remark 1, and Lemma 2b. Furthermore, in 2003 the manufacturing-employment-share is greater than the agriculture-employment-share. Thus, the economy is close to the $\overline{MS}$-edge of Δ_2; cf. Lemmas 1-3. By using this information we can depict the trajectory-end (P_T) on Δ_2 in Fig. B1.

Finally, we construct the connection between the beginning (P_0) and the end (P_T) of the trajectory. Since we are only interested in trends (and trends are only relevant for our results), we connect the points P_0 and P_T by a smooth line; anyway in Fig. 10 the trends of agriculture-employment-share and service-employment-share are monotonous and the trend-line of manufacturing-employment-share is concave. The latter fact implies that trajectory τ has positive signed curvature ($\kappa > 0$, cf. Fig.3): as shown in Fig. 10 the manufacturing-employment-share increases after t_0 for some period of time and decreases afterwards; Lemma 3 implies, therefore, $\kappa > 0$ on Δ_2. This fact completes the information set needed to construct the stylized trend trajectory on Δ_2 in Fig. B1.

Figure B1: *A stylized trajectory of structural change (τ) on Δ_2.*



APPENDIX C: Proof of Lemma 7 in Case of a Closed Trajectory

In case of a closed trajectory substitute Lemma 12 by the following lemma. The rest of the Proof of Lemma 7 remains valid.

LEMMA 12': Assume that τ *is a closed trajectory (cf. Remark 12). Then, the following is true: if* $\phi(t_y) \in \mathbf{S}$*, then* $\phi(t_z) \notin \mathrm{int}([\tau_{0T}])$ *for* $t_z \to t_y$*, where* $t_z > t_y$. PROOF: τ_{0T} is a trajectory-segment which starts in P_0 and ends in P_T (cf. Definition 2a/c), where $P_0 \in \mathbf{A}$ (cf. Definition 4). The definition of a closed trajectory (cf. Remark 12) implies that the economy which moves along closed trajectory moves along $[\tau_{0T}]$ from point P_0 towards point P_T. Thus, if $\phi(t_z) \in \mathrm{int}([\tau_{0T}])$ for $t_z \to t_y$, where $t_z > t_y$, then $\phi(t_y) \in P_0 \cup \mathrm{int}([\tau_{0T}])$. This contradicts $\phi(t_y) \in \mathbf{S}$, since $(P_0 \cup \mathrm{int}([\tau_{0T}])) \cap \mathbf{S} = \emptyset$, cf. Definition 4. Thus, if $\phi(t_y) \in \mathbf{S}$, then $\phi(t_z) \notin \mathrm{int}([\tau_{0T}])$ for $t_z \to t_y$, where $t_z > t_y$. ◊

APPENDIX D: Proof of Theorem 2 in Case of a Closed Trajectory

If τ is a closed trajectory, substitute Lemma 19 by the following lemma. The rest of the Proof of Theorem 2 remains valid.

LEMMA 19': Let τ *be a closed trajectory. Then,* $\phi(t_x) \notin \mathrm{int}([\tau_{0T}])$ *if* $\phi(t) \in \mathbf{D}$ *for* $t_D \leq t < t_x$. PROOF: Note that $t_D > t_T$, as defined in Theorem 2. If τ is a closed trajectory, then the following is true, (cf. APPENDIX C, Proof of Lemma 12'): if $\phi(t_x) \in \mathrm{int}([\tau_{0T}])$, then $\phi(t_z) \in P_0 \cup \mathrm{int}([\tau_{0T}])$ for $t_z \to t_x$, where $t_z < t_x$. This contradicts $\phi(t) \in \mathbf{D}$ *for* $t_D \leq t < t_x$, since $(P_0 \cup \mathrm{int}([\tau_{0T}])) \cap \mathbf{D} = \emptyset$, cf. Definition 4. Thus, if $\phi(t) \in \mathbf{D}$ *for* $t_D \leq t < t_x$, then $\phi(t_x) \notin \mathrm{int}([\tau_{0T}])$. ◊

APPENDIX E: Mathematical Conditions for Non-Self-Intersection

In this appendix we provide some *mathematical* conditions which ensure that the structural change trajectory does not intersect itself. Such conditions may be useful for: constructing new models of structural change, assessing whether a model satisfies the assumptions of our paper and (thus) assessing qualitative dynamics of complicated models ("qualitative simulation").

Since the discussion of mathematical conditions which ensure the non-self-intersection of trajectories is lengthy and mathematically demanding, we restrict our discussion to two aspects: (1) conditions regarding autonomous differential equation systems, which are in our opinion elementary for understanding the non-self-intersection of trajectories, and (2) conditions regarding non-autonomous differential equation systems and, in particular, the concept of "exogenous structural change", which has in our opinion great potential for application in growth theory.

E.1 Conditions regarding Autonomous Differential Equation Systems

Although autonomous differential equation systems are not widespread in structural change modelling (because they require the endogenization of all relevant aspects of structural change), it makes sense discussing them, since: (1) they can be used in structural change modelling, as implied by the following discussion, and (2) their properties are well elaborated in mathematical literature and, thus, useful for understanding the non-self-intersection property of trajectories. In the following we show that all our results remain valid when we model structural change by using a typical autonomous differential equation system. We derive the properties of this system which ensure that the trajectory τ does not intersect itself. These properties are standard assumptions in mathematical literature on differential equation systems.

ASSUMPTION 2a': *The dynamics of the sector structure* (ℓ_a, ℓ_m, ℓ_s) *on* Δ_2 *are described by the following (autonomous) differential equation system:*

(E1) $\dot{C}(t) = \Phi(C(t)),\quad C \in \Delta_2,\quad t \in \mathbf{R}$

where C(t) is a coordinate vector determining the position of the economy on Δ_2 *at time t,* $\Phi(.)$ *is a vector-function and* $\mathbf{R}$ *is the set of real numbers.*

ASSUMPTION 2b': *There exists a solution (*$\phi(t) \in \Delta_2$*) of equation system (E1) on the open time interval* $(\underline{t}, \bar{t})$.

If we substitute the Assumptions 2a and 2b in Section 5 by Assumptions 2a' and 2b' and let all the other assumptions be valid, we have a full model of structural change and all the results in previous sections remain valid. However, in this modified version of our model the non-self-intersection of τ is still per assumption; cf. Assumption 4. Now, we discuss some conditions which can be imposed on the equation system (E1) to ensure the satisfaction of Assumption 4. The mathematical literature provides many examples of such conditions. All of these conditions ensure the satisfaction of Assumption 4 by ensuring that the autonomous system (E1) has unique (continuous) solutions on (some subset of) Δ_2. For discussion, proof and explanation of these conditions we provide some literature references in the following; however, many other textbooks on differential equations and/or dynamic systems can be consulted. Note that the following conditions are sufficient but not necessary for satisfaction of Assumption 4; thus, there are many dynamic systems which do not satisfy any of these conditions and which, nevertheless, satisfy Assumption 4. Furthermore, note that the following condition-sets are, in general, stronger than necessary. Let $\mathbf{X}$ denote a connected subset of Δ_2 containing the trajectory τ, i.e. $\tau \subset \mathbf{X} \subseteq \Delta_2$. The differential equation system (E1) satisfies Assumption 4 if one of the following conditions is satisfied:

(i) $\Phi(.)$ is locally Lipschitz-continuous in C on the (open) set $\mathbf{X}$; cf. Walter (1998), p.110f, Aulbach (2004), p.119f, or Hale (2009), p.18f. and p.38.

(ii) $\Phi(.)$ is continuous on $\mathbf{X}$ and has continuous first partial derivatives with respect to C on $\mathbf{X}$; cf. Andronow et al. (1965), p.263f, Andronow et al. (1969), p.391f, Aulbach (2004), p.119f. and p.77, or Feldman (2012), p.332.

(iii) $\Phi(.)$ is analytic on $\mathbf{X}$; cf. Andronow et al. (1965), p.263f.

In general, we can use these conditions to elaborate whether a specific theoretical model belongs to the class of models studied in our paper.

E.2 Conditions regarding Non-Autonomous Differential Equation Systems: the Concept of "Exogenous Structural Change"

It is possible to elaborate many different condition-sets which can be imposed on non-autonomous differential equation systems to ensure the non-self-intersection of trajectories. We provide here an example which has been introduced by Stijepic (2014a) under the name "exogenous structural change". Stijepic (2014b) provides examples of application of this concept in structural change literature. We show now how this concept can be integrated into our model.

ASSUMPTION 5: a) *Let the curve* $\tau^x := \{(x_1(t), x_2(t)...x_n(t)) \in \mathbf{X} \subseteq \mathbf{R}^n :$ $\underline{t} \le t < \bar{t}\}$ *be given, where* $(x_1(t), x_2(t)...x_n(t))$ *is the vector of variables or parameters describing the state of the economy at time t and* $\mathbf{X}$ *is a connected subset of n-dimensional Real space (*$\mathbf{R}^n$*).* ***b)*** τ^x *is continuous on* $\mathbf{X}$*.* ***c)*** τ^x *does not intersect itself. In particular, for all* $t_x, t_y \in (\underline{t}, \bar{t})$ *the following is true: if* $t_x \neq t_y$*, then* $(x_1(t_x), x_2(t_x)...x_n(t_x)) \neq (x_1(t_y), x_2(t_y)...x_n(t_y))$.

Note that the curve τ^x may result from some model calculations (e.g. utility/profit optimization) or may be generated by assumptions regarding exogenous variables (technology-parameter, labour-growth).

ASSUMPTION 2a#: *The dynamics of the sector structure (ℓ_a, ℓ_m, ℓ_s) on Δ_2 are described by the following homeomorphism:*

(E2) $\phi(t) = \mathrm{H}(x_1(t), x_2(t), \ldots x_n(t)),\ \ \phi \in \Delta_2,\ \ \underline{t} \le t < \bar{t}$

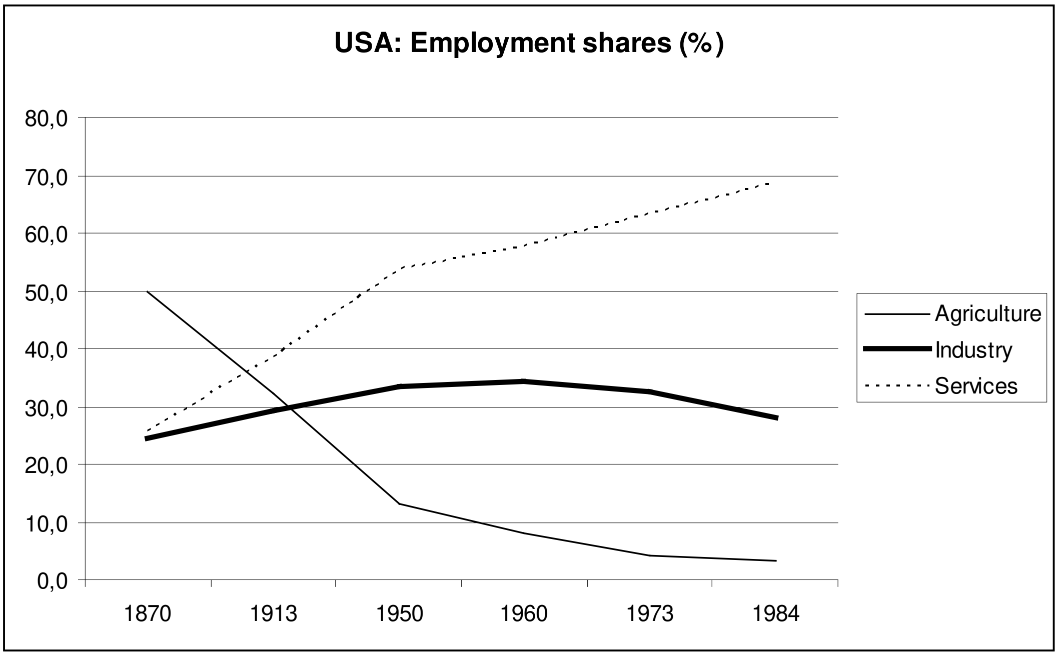
where: $\phi(t)$ *is a coordinate vector determining the position of the economy on* Δ_2 *at time t;* $\mathrm{H}(.) : \mathbf{X} \to \Delta_2$ *is a continuous and bijective vector-function.*

Substitute Assumption 2a in Section 5 by Assumption 2a#. Let all other assumptions from Sections 2-6 be satisfied and assume that Assumption 5 is satisfied. Then all our results remain valid. We name this model “model of exogenous structural change”. If we differentiate (E2) with respect to time, we can see that the differential equation system describing the dynamics of (ℓ_a, ℓ_m, ℓ_s) is non-autonomous.

Note that Assumptions 2c and 4 are not necessary in this modified version of our model, since Assumptions 5 and 2a# ensure that (a) τ is continuous and (b) τ does not intersect itself, as shown in the following. (E2) is a continuous transformation of the continuous curve τ^x. Thus, the resulting curve τ is continuous; cf. (E2) and Definition 2b. Furthermore, since H(.) is continuous and bijective and τ^x does not feature any self-intersections, τ does not feature any self-intersections, as shown in the following. Assume that τ intersects itself. Thus, there must exist two points in time $(t_x$ and $t_y)$ and a point $\phi_I \in \tau$ which satisfy the following conditions: $\phi(t_x) = \phi(t_y) = \phi_I$ and $0 < t_x < t_y$. Thus, $\phi(t_x) = \mathrm{H}(x_1(t_x), \ldots x_n(t_x)) = \phi(t_y) = \mathrm{H}(x_1(t_y), \ldots x_n(t_y))$; cf. (E2). This implies that $(x_1(t_x), \ldots x_n(t_x)) = (x_1(t_y), \ldots x_n(t_y))$, since H is one-to-one (bijective). Assumption 5c implies that $(x_1(t_x), \ldots x_n(t_x)) = (x_1(t_y), \ldots x_n(t_y))$ if and only if $t_x = t_y$, which contradicts $t_x < t_y$. Thus, τ does not intersect itself. See also Stijepic (2014a), “Theorem 3” and “Lemmas 12 and 13”.

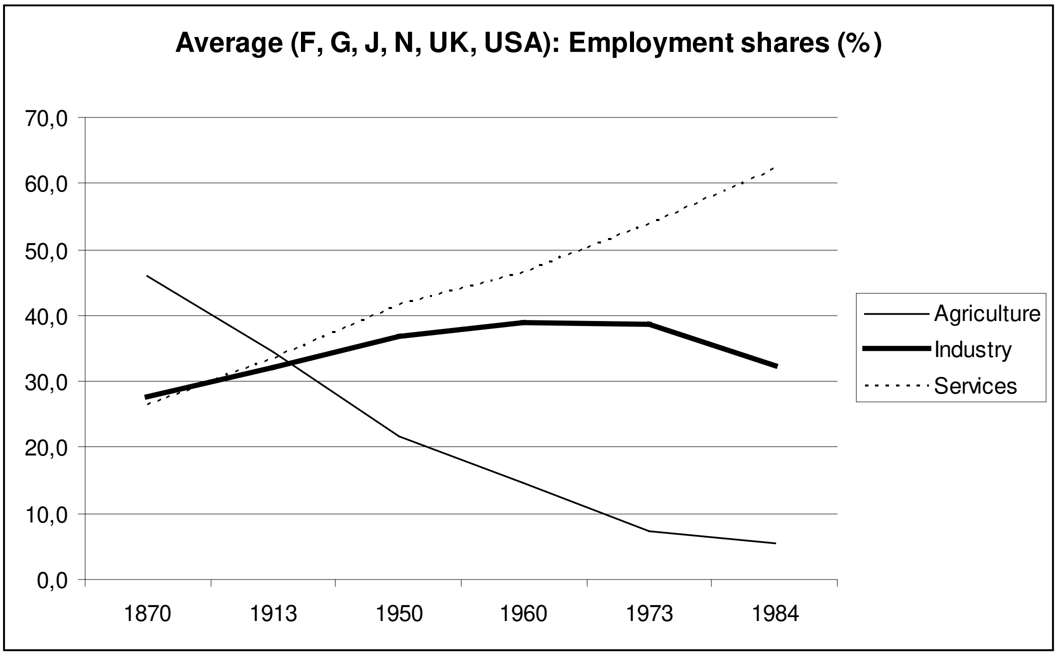
APPENDIX F: Additional Empirical Evidence on Structural Change

Figure F1



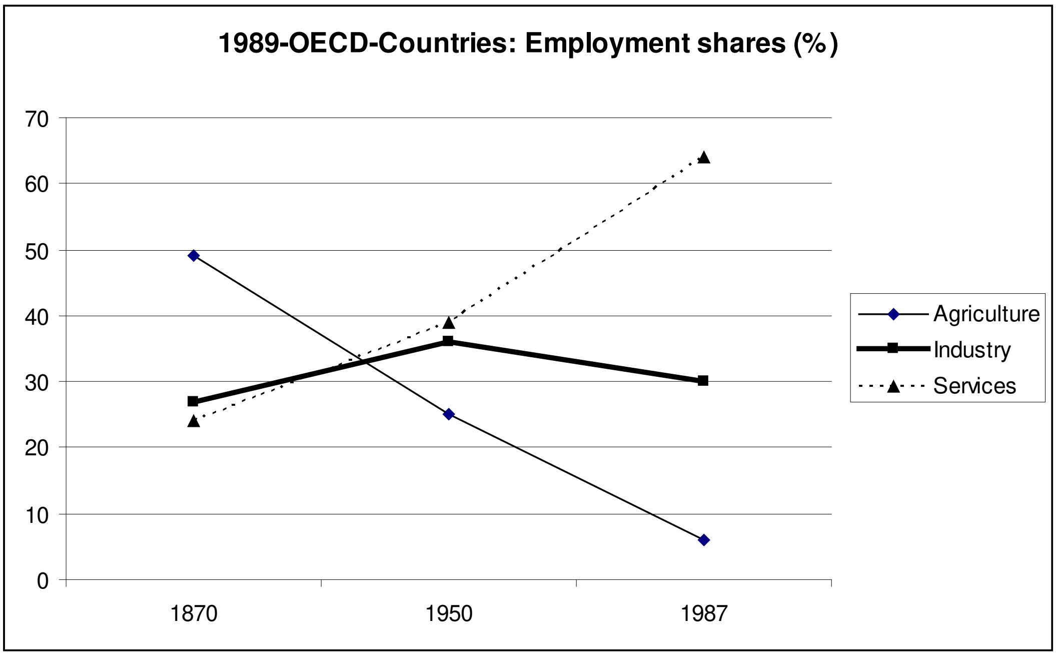
Datasource: Maddison (1995a), p.76.

Figure F2



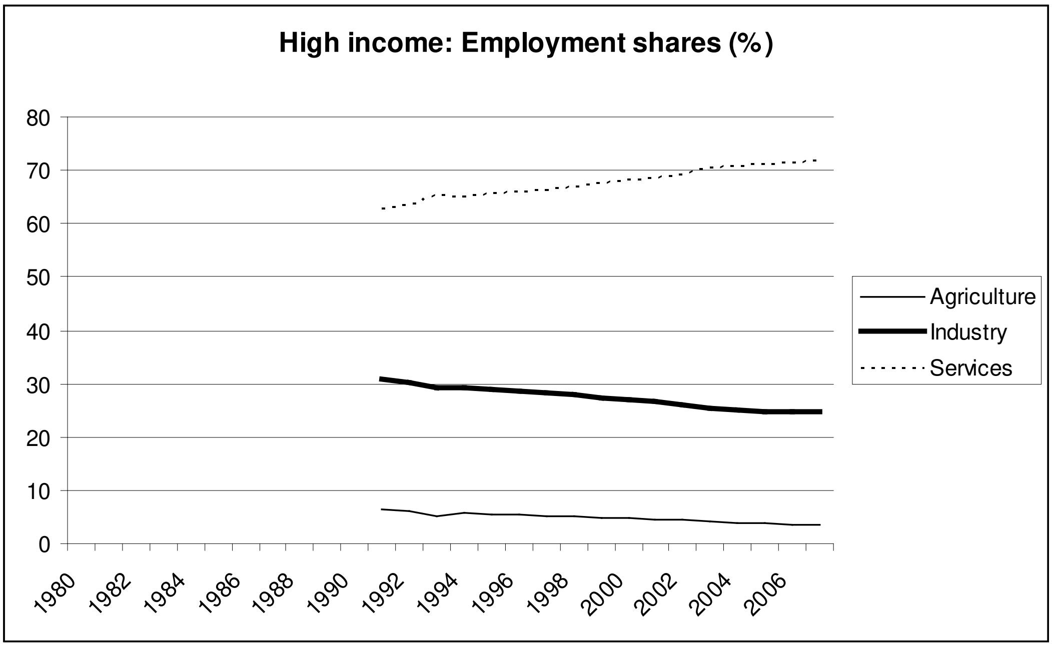
Datasource: Maddison (1995a), p.76.

Figure F3



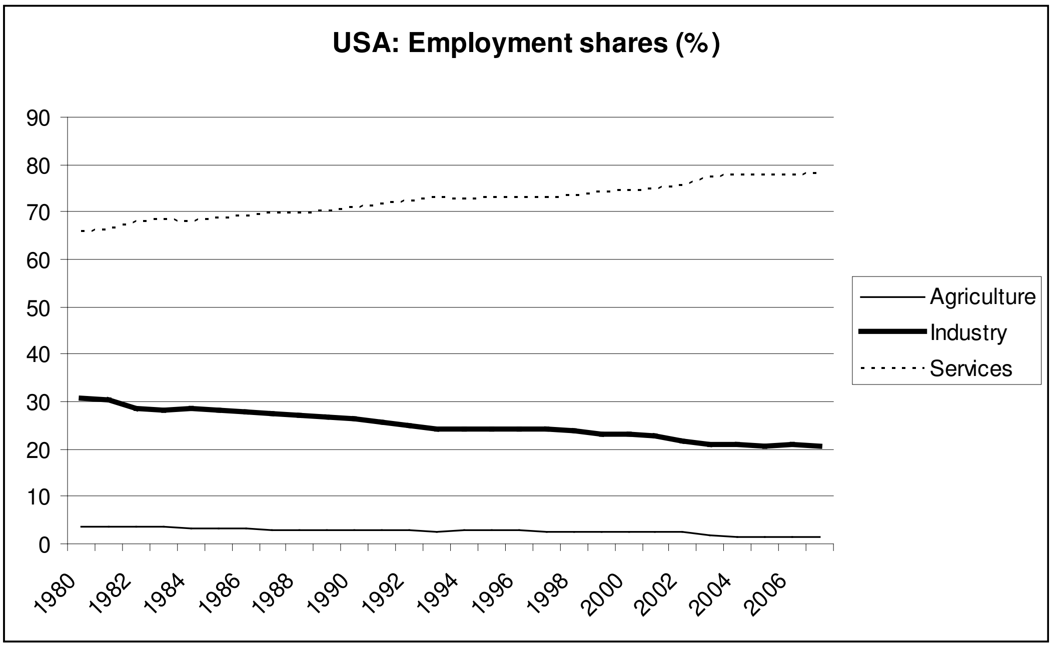
Datasource: Maddison (1995a), p.119.

Figure F4



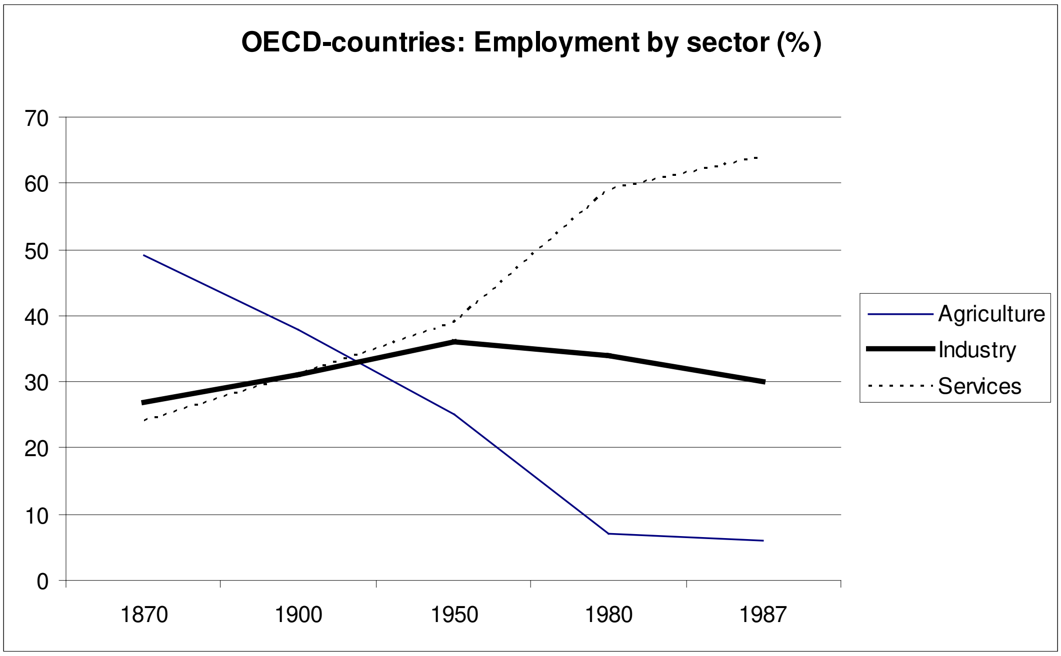
Datasource: The World Bank, World Databank, World Development Indicators.

Figure F5



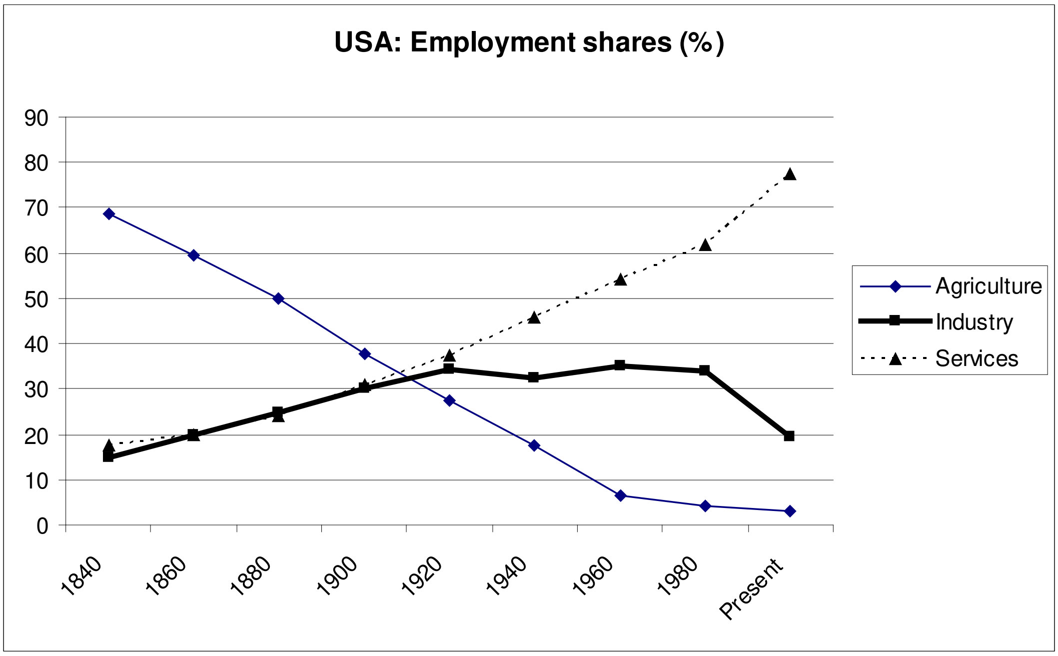
Datasource: The World Bank, World Databank, World Development Indicators.

Figure F6



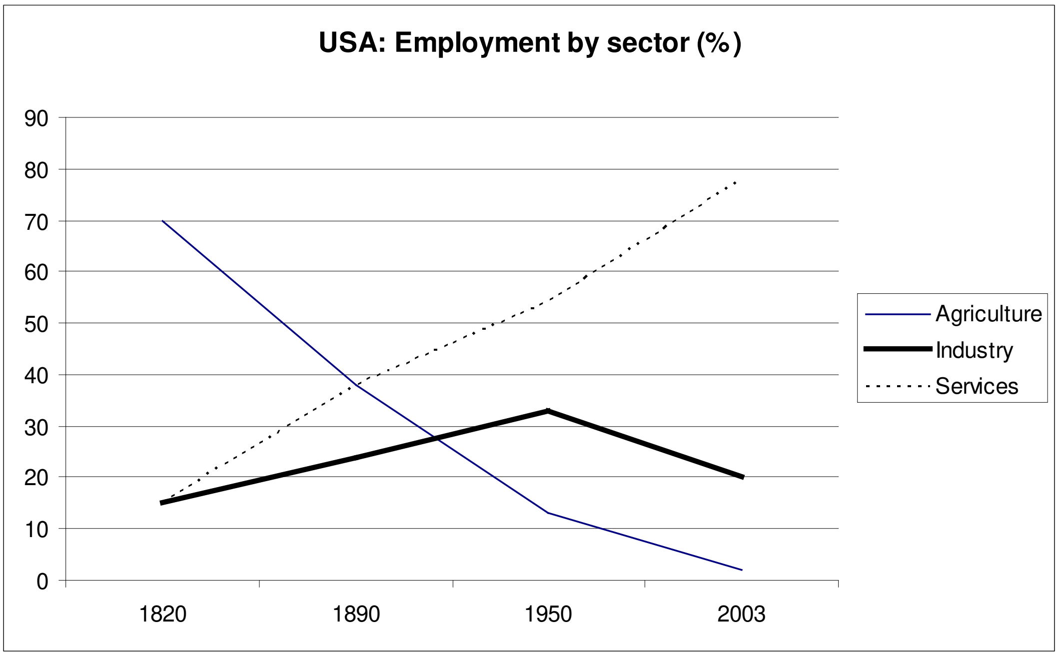
Datasource: Maddison (1989), p.20.

Figure F7



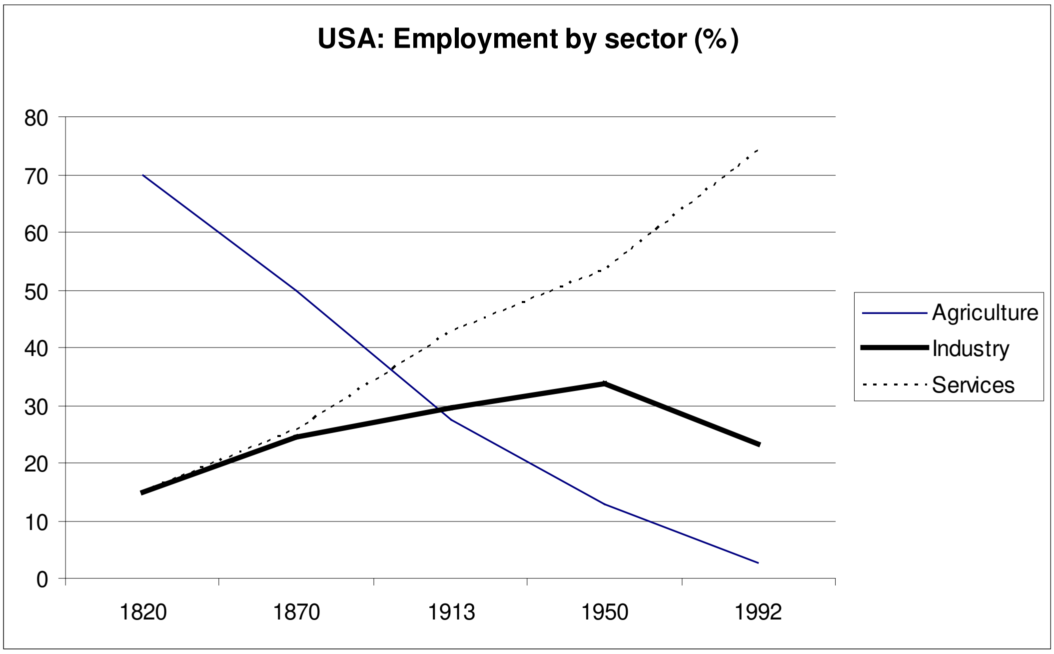
Datasource: Raiser et al. (2004).

Figure F8



Datasource: Maddison (2007), p.76.

Figure F9



Datasource: Maddison (1995b), p.39.