Revisting the Steady-State Equilibrium Conditions of Neoclassical Growth Models

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REVISITING THE STEADY-STATE EQUILIBRIUM CONDITIONS OF NEOCLASSICAL GROWTH MODELS

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Abstract: Since the publication of Uzawa (1961), it has been widely accepted that technical change must be purely labor-augmenting for a growth model to exhibit steady-state path. But in this paper, we argue that such a constraint is unnecessary. Further, our model shows that, as long as the sum of the growth rate of *marginal efficiency of capital accumulation* and the rate of *capital-augmenting technological progress* equals zero, steady-state growth can be established without constraining the direction of technological change. Thus Uzawa's theorem represents only a special case, and the explanatory power of growth models would be greatly enhanced if such a constraint is removed.

Keywords: Neoclassical Growth Model; Uzawa's Steady-state Growth Theorem; Direction of Technical Change;Adjustment Cost

JEL Classifications: E13, O33, O40

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1 Introduction

In one of his celebrated articles, Uzawa (1961) proved that, for a growth model to exhibit steady-state path (that is, long-run equilibrium), the direction of technological change should be exclusively Harrod-neutral, i.e. purely labor augmenting. Since then, it has been widely cited as the steady-state growth theorem, or just Uzawa's theorem, and an overwhelming majority of the growth literature makes restrictive Harrod-neutral assumption about the direction of technical change.[1] However, why should technical change along steady-state path be exclusively purely labor-augmenting? Considering that technical enhancement can also be Hicks-neutral or Solow-neutral in reality, there is no compelling reason for us to believe that steady-state technical change should be exclusively Harrod-neutral.

Over the last a few decades, researchers have delved into this theorem by either providing a more simplified proof (see Barro and Sala-i-Martin 2004; Schlicht 2006; Acemoglu 2009), or seeking for a more satisfactory justification (see Fellner 1961; Kennedy 1964; Samuelson 1965; Drandakis and Phelps 1966; Acemoglu 2003; Jones 2005; Jones and Scrimgeour 2008). However, none of these studies escaped from the premise laid out by Uzawa (1961), nor did they explore whether steady-state growth can actually be achieved without *ex-ante* defining the direction of technical change. To this end, the present paper attempts to revisit the steady-state equilibrium conditions of neoclassical growth models without assuming Harrod neutrality. Our purpose is to identify the general conditions needed for a neoclassical growth model to exhibit steady-state equilibrium, which also sheds light on the validity of Uzawa's theorem.

It may be noted that although the discussions about steady-state equilibrium conditions are still far from satisfactory, the key point has been clarified in all versions of proofs of Uzawa's theorem. For instance, based on the work of Schlicht (2006), Acemoglu (2009, Chapter 2) proves that balanced growth rates between capital and output in steady state can immediately result from the assumed capital accumulation equation,[2] and also lead to the derivation of Uzawa's theorem directly. This is to say that the capital accumulation equation is served as a critical condition for Uzawa's theorem to hold.

However, the standard capital accumulation equation ignores the adjustment costs that are typically associated with the replacements for worn-out equipments, the

[1] Alternatively, the production function is assumed to be Cobb-Douglas for a neoclassical model to exhibit steady-state growth. Since technological progress can always be shown as labor-augmenting when the production function is in Cobb-Douglas shape, it should not be regarded as a separate assumption.

[2] That is, $\dot{K} = Y - C - \delta K$, which means that the dynamics in aggregate capital stock is governed by the difference between output Y net of consumption C (or investment), and the depreciation amount of capital stock (δK). In the equation, a dot over a variable denotes its differentiation with respect to time.

installation of new machines, the cost of learning itself, and sometimes the cost related to the purchase of machines from capital goods producers (Eisner and Strotz 1963, Lucas 1967, Foley and Sidrauski 1970, Mussa 1977, Bailey and Scarth 1980, 1983). When adjustment costs are taken into account, it takes more than one unit of net investment (in the form of the final product) to get one additional unit of capital stock. Accordingly, the capital accumulation function should be reformulated as $\dot{K} \leq Y - C - \delta K$ rather than $\dot{K} = Y - C - \delta K$. Although Abel and Blanchard (1983) did analyze a neoclassical growth model with adjustment costs, they made no implications about the direction of steady-state technical change.

In this paper, we incorporate adjustment costs into the investment function of a typical firm. The purpose is to derive the general steady-state equilibrium conditions for neoclassical growth models without resorting to the Harrod-neutral assumption, and to compare our findings with Uzawa's theorem that hinges on Harrod neutral assumption. According to our results, the true condition for a neoclassical growth model to exhibit steady-state equilibrium is that the sum of the growth rate of *marginal efficiency of capital accumulation* and the rate of *capital-augmenting technological progress* equals zero. This implies that *capital-augmenting technological progress* may exist along the steady state path as long as the *marginal efficiency of capital accumulation* does not remain constant.[3] In other words, neoclassical steady-state path can allow for non-Harrod neutral types of technical change, as opposed to the statement of Uzawa's theorem.

The rest of the paper is organized as follows. Section 2 sets up the model with adjustment costs and derives steady-state equilibrium conditions in a general sense. Section 3 analyzes whether or not steady-state growth is dependent upon labor-augmenting technological progress, and the relationship between Uzawa's theorem and our steady-state equilibrium conditions. Section 4 concludes. Appendix contains the alternative proof of the steady-state equilibrium conditions (similar methods can be found in Schlicht (2006)).

2 The Model

2.1 Formulation of the Model

Consider a representative consumer in the economy with the usual constant relative risk aversion (CRRA) preferences. Then, the lifetime utility of the representative consumer can be expressed as

$$\int_{t=0}^{\infty} \frac{C(t)^{1-\theta}}{1-\theta} e^{-\rho t} dt, \qquad (1)$$

[3] In the absence of adjustment costs, *marginal efficiency of capital accumulation* remains constant.

where $C(t)$ is the consumption at the period t, θ is the coefficient of relative risk aversion, and ρ is the rate of time preferences.

The production function satisfies the standard neoclassical properties,[4] and allows for both capital-augmenting and labor-augmenting technology

$$Y(t) = F[B(t)K(t), A(t)L(t)], \tag{2}$$

where $Y(t), K(t), L(t)$ denotes output, capital stock and labor at the time point t, $B(t)$ and $A(t)$ refer to the capital-augmenting and labor-augmenting technologies, respectively. Thus, the interaction terms $B(t)K(t)$ and $A(t)L(t)$ represent, respectively, the effective capital and effective labor at the time t. Further, the initial endowment of technology and labor is greater than one, i.e. $A(0), B(0), L(0) \geq 1$. In addition, the growth rates of labor L and both technologies are assumed to be exogenous, that is, $\dot{B}(t)/B(t) = b \geq 0$, $\dot{A}(t)/A(t) = a \geq 0$, and $\dot{L}(t)/L(t) = n \geq 0$.

The budget constraint of the representative consumer is given by[5]

$$Y(t) = C(t) + I(t), \text{ where } C(t), I(t) > 0. \tag{3}$$

The investment function $I(t)$ has two parts, including the purchase of new capital goods $I_k(t)$ and the additional adjustment cost $h[I_k(t)]$ incurred for the new capital to be used in the production:

$$I(t) = I_k(t) + h[I_k(t)], \tag{4}$$

where $h[0] = 0, \partial h/\partial I_k \geq 0, \partial^2 h/\partial I_K^2 \geq 0$. So the adjustment cost has increasing marginal costs with regard to new capital goods I_k.

The net increase in the stock of capital at a point in time is the difference between the level of investment $I_k(t)$ and the depreciation $\delta K(t)$. To be more accurate, our capital accumulation function can be formulated as follows:

$$\dot{K}(t) = I_k(t) - \delta K(t), \tag{5}$$

where $K(0) > 0, \delta \geq 0,$ and $I_k(t) > 0$.

By equation (4), the investment $I(t)$ is surely a monotonically increasing function of $I_k(t)$ as $dI(t)/dI_k(t) = 1 + dh/dI_k(t) \geq 1$. Solving for the inverse function of equation (4) yields:

$$I_k(t) = G[I(t)] \leq I(t), \tag{6}$$

[4] That is, constant returns to scale (CRS), positive but diminishing marginal products, Inada conditions, and essentiality of each input（Barro and Sala-i-Martin,2004）.

[5] Since the focus of this paper is on the possible existence of capital-augmenting technological progress in steady state, we ignore the differences between individual consumption and social consumption.

where $G[I(t)]$ is the efficiency function of capital accumulation, which reflects the degree to which investment is converted to new capital goods.

By inserting formula (6) into (5), we obtain the capital accumulation equation with investment adjustment costs:

$$\dot{K}(t) = G[I(t)] - \delta K(t). \tag{7}$$

It is evident from equations (6) and (7) that $\dot{K}(t) = G[I(t)] - \delta K(t) \leq I(t) - \delta K(t)$, which shows that the speed of capital accumulation depends not only on the level of investment $I(t)$, but also on the conversion efficiency from investment to capital. By the property of the inverse function, we obtain the following relations:

$$\begin{cases} G_I \equiv \frac{\partial G[I(t)]}{\partial I(t)} = \frac{1}{\partial I(t)/\partial I_k(t)} = \frac{1}{1+\partial h/\partial I_k(t)} > 0 \\ G_{II} \equiv \frac{\partial^2 G[I(t)]}{\partial I(t)^2} = \frac{-\partial^2 h/\partial I_k(t)^2}{[1+\partial h/\partial I_k(t)]^2} \leq 0 \end{cases}, \tag{8}$$

where G_I and G_{II} refer to the *marginal efficiency of capital accumulation* and its first-order derivative respectively. Equation group (8) shows that the *marginal efficiency of capital accumulation* diminishes with additional investment that incurs adjustment costs.

Finally, the usual transversality condition is expressed as:

$$\lim_{t \to \infty} K(t) e^{-\rho t} = 0. \tag{9}$$

2.2 Steady-state Equilibrium Conditions

To solve the dynamic optimization problem, we write the current-value Hamiltonian as follows:

$$H(C, K, \lambda) = \frac{C(t)^{1-\theta}}{1-\theta} e^{-\rho t} + \lambda \{G[Y(t) - C(t)] - \delta K(t)\}. \tag{10}$$

The first-order conditions thus are:

$$\begin{cases} \frac{\partial H}{\partial C} = C(t)^{-\theta} e^{-\rho t} - \lambda G_I = 0 \\ \dot{\lambda} = -\frac{\partial H}{\partial K} = -\lambda \left(G_I \frac{\partial Y}{\partial K} \right) - \delta \end{cases}, \tag{11}$$

After some mathematical manipulation on the first-order conditions, we obtain the Euler equation:

$$\theta \frac{\dot{C}(t)}{C(t)} = G_I \frac{\partial Y}{\partial K} - \frac{\dot{G}_I}{G_I} - \rho - \delta. \tag{12}$$

Substituting $\partial Y/\partial K = B[\partial Y/\partial(BK)]$ derived from the production function (2) into equation (12), we can further derive the following necessary condition for consumers to achieve dynamic optimality:

$$\theta\frac{\dot{C}(t)}{C(t)} = G_I B\frac{\partial Y}{\partial(BK)} - \frac{\dot{G}_I}{G_I} - \rho - \delta. \qquad (13)$$

Let k be the ratio of effective capital to effective labor (i.e.$k \equiv BK/AL$), the intensive form of the production function becomes $f(k) = F(BK/AL, 1)$. This implies that the marginal product of effective capital is $f'(k) = \partial Y/\partial(BK)$. Define $c \equiv C/AL$ as the consumption per effective labor. After using equations (7) and (13), we get:

$$\begin{cases} \dfrac{\dot{k}(t)}{k(t)} = b + \dfrac{G[I(t)]}{K(t)} - \delta - a - n \\ \dfrac{\dot{c}(t)}{c(t)} = \dfrac{1}{\theta}\left[G_I B f'(k) - \dfrac{\dot{G}_I}{G_I} - \rho - \delta\right] - a - n \end{cases}. \qquad (14)$$

Assuming that, after some time point t_0, $\dot{c}(t)/c(t) = 0$ and $\dot{k}(t)/k(t) = 0$, which corresponds to the steady-state equilibrium path. Then we have:

$$\begin{cases} \dfrac{G[I(t)]}{K(t)} = a + n + \delta - b \\ G_I B f'(k) - \dfrac{\dot{G}_I}{G_I} = \rho + \delta + \theta(a + n) \end{cases}. \qquad (15)$$

Let $G_I(t)B(t) = G_I(t_0)B(t_0)\exp\left\{\int_{t_0}^{t}\left[\dot{G}_I(\tau)/G_I(\tau) + \dot{B}(\tau)/B(\tau)\right]d\tau\right\}$, and insert it into equation (15), we obtain:

$$f'(k^*) = \frac{\rho + \delta + \theta(a + n) + \dot{G}_I/G_I}{G_I(t_0)B(t_0)\exp\left\{\int_{t_0}^{t}\left[\dot{G}_I(\tau)/G_I(\tau) + \dot{B}(\tau)/B(\tau)\right]d\tau\right\}}. \qquad (16)$$

Since $\dot{k}(t)/k(t) = 0$ after time point t_0, the left-hand side of equation (16) is a constant and $f'(k^*) > 0$. And the right-hand side of (16) must be a constant too, which requires $\dot{G}_I/G_I = -\dot{B}/B = -b$. Thus, the condition for (15) to hold after t_0 is [6]

$$\dot{G}_I(t)/G_I(t) + \dot{B}(t)/B(t) = 0. \qquad (17)$$

According to (17), for our neoclassical growth model to be in steady-state equilibrium, the sum of the growth rate of G_I (*marginal efficiency of capital*

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[6] The same condition can be obtained by using the capital accumulation equation, as in Schlict (2006), Jones and Scrimgeour (2008), and Acemoglu (2008, chapter 2). This alternative proof is relegated to the Appendix.

accumulation) and the growth rate of B (*capital-augmenting technological change*) must equal zero. Substituting (17) into (16), we have

$$f'(k^*) = [\rho + \delta + \theta(a + n) - b]/G_I(t_0)B(t_0). \tag{18}$$

Equation (18) shows that, when (17) is satisfied, the ratio of effective capital to effective labor k turns out to be a constant. By equation (7), we obtain the steady-state growth rate of capital

$$\dot{K}^*/K^* = G(I)/K - \delta = a + n - b. \tag{19}$$

Similarly, by equations (2) and (3) and $c = C/AL$, we obtain the steady-state growth rate of the other three endogenous variables as follows:

$$\dot{Y}^*/Y^* = \dot{I}^*/I^* = \dot{C}^*/C^* = a + n. \tag{20}$$

Clearly, our neoclassical growth model exhibits steady-state growth when condition (17) is satisfied. This is independent of either Harrod-neutral technical change or the Cobb-Douglas production function.

Further, if we take the first-order Taylor expansion of equation (14) around the steady-state (c^*, k^*), we get

$$\begin{pmatrix} \frac{\dot{k}(t)}{k(t)} \\ \frac{\dot{c}(t)}{c(t)} \end{pmatrix} \approx \begin{pmatrix} \frac{\partial[\dot{k}(t)k(t)]}{\partial k}\Big|_{\substack{k=k* \\ c=c*}}, & -\frac{G_I}{k^*} \\ \frac{1}{\theta}G_I(t_0)B(t_0)f''(k^*), & 0 \end{pmatrix} \binom{k}{c}, \tag{21}$$

with the coefficient determinant shown as follows:

$$\det\begin{bmatrix} \frac{\partial[\dot{k}(t)k(t)]}{\partial k}\Big|_{\substack{k=k* \\ c=c*}}, & -\frac{G_I}{k^*} \\ \frac{1}{\theta}G_I(t_0)B(t_0)f''(k^*), & 0 \end{bmatrix} = \frac{1}{\theta}G_I(t_0)B(t_0)f''(k^*)\frac{G_I}{k^*} < 0. \tag{22}$$

As can be seen from coefficient determinant (22), steady-state growth of our model actually implies the stable saddle path when $\dot{G}_I/G_I + \dot{B}/B = 0$.

3 The Direction of Steady-State Technical Change

3.1 The Relevance of the Marginal Efficiency of Capital Accumulation in Steady State

By incorporating adjustment costs into the firm's investment function, we have shown that the neoclassical steady-state growth requires the condition $\dot{G}_I/G_I + \dot{B}/B = 0$ to be satisfied. The immediate implication is that, unless *marginal efficiency of capital accumulation* G_I remains unchanged, the possibility of capital-augmenting technological progress $\dot{B}$ cannot be ruled out. As discussed in section 2.1, marginal efficiency of capital accumulation will gradually decline as the investment adjustment costs increase (i.e. $G_{II} < 0$).This means that the case of $\dot{G}_I/G_I<0$ cannot be excluded, and the possibility of having capital-augmenting technical change on steady state path (i.e. $\dot{B}/B > 0$) should not be ruled out either.

To further illustrate the point, let us consider a specific form of adjustment cost function $h[I_K(t)] = I_K(t)[B(t) - 1]$, where $B(t) \geq 1$. This function shows that the adjustment cost correlates positively with the levels of investment $I_K(t)$ and capital-augmenting technology $B(t)$. In addition, the marginal adjustment cost increases with $B(t)$. Substitute this function into equation (4) and divide both sides by $B(t)$, we get

$$I_K(t) = I(t)/\, B(t). \tag{23}$$

Combining equation (23) with equation (5) yields the following capital accumulation equation:

$$\dot{K}(t) = I(t)/B(t) - \delta K(t). \tag{24}$$

According to the capital accumulation equation (24), the marginal efficiency of capital accumulation is derived to be $G_I(t) = 1/B(t)$. This turns out to satisfy the condition $\dot{G}_I/G_I + \dot{B}/B = 0$. In this case, our steady-state equilibrium condition shown in (17) will always be met so long as $\dot{A}(t)/A(t) = a \geq 0$ and $\dot{B}(t)/B(t) = b \geq 0$. In the meantime, k^* in steady state is determined by $f'(k^*) = [\rho + \delta + \theta(a + n) - b]$ when $G_I(t) = 1/B(t)$, and the growth rates of Y, I, C and K are determined by equations (19) and (20). As a result, there is no restriction placed on the direction of technical change in steady state.[7]

3.2 A Comparison between Uzawa's Theorem and Our Steady-state Equilibrium Conditions

Uzawa's theorem says that technical change has to be purely labor-augmenting rather than capital-augmenting along the steady-state path. In other words, the implicit requirement of Uzawa's theorem is $\dot{B}/B = 0$. However, our analysis above shows that $\dot{B}/B = 0$ becomes possible if and only if marginal efficiency of capital accumulation holds unchanged (i.e. $\dot{G}_I/G_I = 0$). It is also worth noting that the existing researches aiming to prove Uzawa's theorem have unexceptionally assumed $\dot{K} = I - \delta K$, where $I = Y - C$. Obviously, this assumption leads to $G_I = 1$ and $\dot{G}_I/G_I = 0$. Thus, Uzawa's theorem only represents a very special condition for

[7] In steady state, the growth of labour is exogenously given and the growth of capital is endogenously determined. Although accumulations of capital and labour are asymmetric, the technical change can still be of any type. This contradicts the view of Acemoglu (2003) and Jones and Scrimgeour (2008), which holds that Uzawa theorem is drawn from such asymmetry.

neoclassical growth model to exhibit steady-state growth. In a more general situation where the marginal efficiency of capital accumulation G_I varies, the validity of Uzawa's theorem is questionable.

In addition, Uzawa's theorem indicates that the steady-state growth for a neoclassical model is a knife-edge one (Growiec 2008). However, our results suggest that the prerequisite for the attainment of steady-state growth neither limits the directions of technological progress nor is a knife-edge condition. More specifically, our steady-state equilibrium requires the sum of $\dot{G}_I/G_I$ and $\dot{B}/B$ to be zero, which means that $\dot{G}_I/G_I$ and $\dot{B}/B$ are interdependent of each other. Howevre, the existing growth literature surrounding Uzawa's theorem assumes the independency between $\dot{G}_I/G_I$ and $\dot{B}/B$, and it follows that the steady-state condition $\dot{G}_I/G_I + \dot{B}/B = 0$ has to be a knife-edge one.

4 Conclusions

Since Uzawa (1961), economists have long believed that technical change should be exclusively purely labor augmenting for the attainment of steady-state growth. By incorporating adjustment costs into the firm's investment function, this paper re-examines the steady-state equilibrium conditions of neoclassical growth models without assuming Harrod neutrality or Cobb-Douglas production function. We show that, for a neoclassical growth model to exhibit steady-state growth, it is just required that the sum of the growth rate of marginal efficiency of capital accumulation and the rate of capital-augmenting technical change equals zero.

Our results suggest that no restrictions need to be placed on the direction of technical change, for the sake of steady state growth in neoclassical growth models. Indeed, the directions of steady-state technological progress can be of any type, in particular when the marginal efficiency of capital accumulation is variable. Thus, Uzawa's theorem only represents a special case and lacks the generality. In a more general situation where the marginal efficiency of capital accumulation varies, the validity of Uzawa's theorem does not hold any more.

Although most of existing growth models is constrained by either the direction of technological progress or the shape of production function, the present paper shows that it is not necessary. In fact, the scope of neoclassical growth models will be greatly expanded if these constraints are removed. Future studies could concentrate on the factors determining the direction of steady-state technological progress, and the choice of ideal direction of technological progress in accordance with the specific needs.

Appendix: An Alternative Derivation of Steady-State Equilibrium Condition (Similar to the Methodology in Schlicht (2006))

Consider an economy with a neoclassical production function $\tilde{F}$. This function relates, at any point in time t, the quantity produced, denoted by $Y(t)$, to labor input *L*(t) and capital input *K*(t). The production function is assumed to exhibit, at any point in time, constant returns to scale. Due to technological progress, the production function shifts over time. We have:

$$Y(t) = \tilde{F}[K(t), L(t), t], \tag{A1}$$

which is with constant return to K(t) and L(t), so

$$\tilde{F}[\lambda K(t), \lambda L(t), t] = \lambda \tilde{F}[K(t), L(t), t] \text{ for all } [K(t), L(t), t, \lambda] \in R_+^4. \tag{A2}$$

Labor input *L* grows exponentially at rate *n*, that is

$$L(t) = L(0)e^{nt}. \tag{A3}$$

Proposition: If the system $Y(t) = C(t) + I(t)$, $\dot{K}(t) = G[I(t)] - \delta K(t)$, (A1) - (A3) and $g_\varphi \equiv \dot{G}_I / G_I \neq 0, G_I \equiv \partial G / \partial I$ possess a solution where *Y(t), C(t)*, and *K(t)* are all nonnegative and grow with constant growth rates g_y, g_c and g_k, respectively, then for any t ≥0, there exists a function $F: R_+^2 \to R_+$, which is homogeneous of degree one in its both arguments, such that the aggregate production function can be represented as

$$Y(t) = F[B(t)K(t), A(t)L(t)] \tag{A4}$$

and $A(t) \in R_+$ and $B(t) \in R_+$, $B(t) = \exp(-g_\varphi t)$ or $\dot{G}_I / G_I + \dot{B}/B = 0$, $A(t) = \exp[(g_y - n)t]$.

Proof. By assumption we have

$$Y(t) = Y(0)e^{g_y t}, \ C(t) = C(0)e^{g_c t} \text{ and } K(t) = K(0)e^{g_k t}. \tag{A5}$$

From $\dot{K}(t) = G[I(t)] - \delta K(t)$ and equations (A5) we obtain

$$(g_k + \delta)K(t) = G[Y(t) - C(t)], \tag{A6}$$

or

$$(g_k + \delta)K(0)e^{g_k t} = G[Y(0)e^{g_y t} - C(0)e^{g_c t}]. \tag{A7}$$

For all t. Taking time derivatives yields

$$g_k(g_k + \delta)K(0)e^{g_k t} = G_I\left[g_y Y(0)e^{g_y t} - g_c C(0)e^{g_c t}\right], \tag{A8}$$

or

$$g_k(g_k+\delta)K(0) = G_I\left[g_y Y(0)e^{(g_y-g_k)t} - g_c C(0)e^{(g_c-g_k)t}\right]. \qquad \text{(A9)}$$

Replacing G_I with $G_I(0)\exp\left[\int_{\tau=0}^{t} g_\varphi(\tau)d\tau\right]$ in equation (A9), we can obtain

$$\frac{g_k(g_k+\delta)K(0)}{G_I(0)} = g_y Y(0)\exp\left[(g_y-g_k)t + \int_{\tau=0}^{t} g_\varphi(\tau)d\tau\right] - g_c C(0)\exp\left[(g_c-g_k)t + \int_{\tau=0}^{t} g_\varphi(\tau)d\tau\right]. \qquad \text{(A10)}$$

Taking time derivatives again yields

$$\left\{(g_y-g_k+g_\varphi)g_y Y(0)\exp\left[(g_y-g_k)t + \int_{\tau=0}^{t} g_\varphi(\tau)d\tau\right]\right\} - \left\{(g_c-g_k+g_\varphi)g_c C(0)\exp\left[(g_c-g_k)t + \int_{\tau=0}^{t} g_\varphi(\tau)d\tau\right]\right\} = 0, \qquad \text{(A11)}$$

which implies

$$(g_y-g_k+g_\varphi)g_y Y(0)\exp\left[(g_y-g_c)t\right] - (g_c-g_k+g_\varphi)g_c C(0) = 0. \qquad \text{(A12)}$$

This equation can hold for all *t* if any of the following four conditions is true:

(i) $g_y = g_c = g_k - g_\varphi$

(ii) $g_y = g_c$ and $Y(0) = C(0)$;

(iii) if $g_y = g_k - g_\varphi$ and $C(0) = 0$;

(iv) $g_c = g_k - g_\varphi$ and $Y(0) = 0$.

The latter three possibilities contradict, respectively, that $g_k > 0$, that $g_c>0$ (which implies $C(0) > 0$ and $K(0) > 0$, and hence $Y(0) > C(0)$), and that $Y(0) > 0$. Therefore (i) must apply and $g_y = g_c = g_k - g_\varphi$.

Define

$$F[K(t), L(t)] \equiv \tilde{F}[K(t), L(t), t]. \quad \text{(A13)}$$

As $Y(0) = F[K(0), L(0)]$, $Y(t) = Y(0)\exp(g_y t)$, $K(0) = K(t)\exp(-g_k t)$, and $L(0) = L(t)\exp(-nt)$, and *F* is linear homogeneous, we can write

$$Y(t) = F\{\exp[(g_y - g_k)t]\, K(t), \exp\,[(g_y - n)t]L(t)\}. \quad \text{(A14)}$$

Replacing $g_y = g_k - g_\varphi$ in equation (A14), we can obtain

$$Y(t) = F\{\exp(-g_\varphi t)\, K(t), \exp\,[(g_y - n)t]L(t)\}. \quad \text{(A15)}$$

Define $B(t) \equiv \exp\,(-g_\varphi t)$ and $A(t) \equiv \exp[(g_y - n)t]$, we can obtain

$$Y(t) = F[B(t)K(t), A(t)L(t)]. \quad \text{(A16)}$$

From$B(t) \equiv \exp\,(-g_\varphi t)$, we can obtain the Steady-State Equilibrium Conditions as follow:

$$\dot{G}_I/G_I + \dot{B}/B = 0. \quad \text{(A17)}$$

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