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Abstract

This study presents an approach based on a perturbation technique to construct global solutions to dynamic stochastic general equilibrium models (DSGE). The main idea is to expand a solution in a series of powers of a small parameter scaling the uncertainty in the economy around a solution to the deterministic model, i.e. the model where the volatility of the shocks vanishes. If a deterministic path is global in state variables, then so are the constructed solutions to the stochastic model, whereas these solutions are local in the scaling parameter. Under the assumption that a deterministic path is already known the higher order terms in the expansion are obtained recursively by solving linear rational expectations models with time-varying parameters. The present work proposes a method rested on backward recursion for solving this type of models.

Key words: DSGE, perturbation method, rational expectations models with time-varying parameters, asset pricing model

1 INTRODUCTION

Perturbation methods are the most widely-used approach to solve nonlinear DSGE models owing to their ability to deal with medium and large-size models for reasonable computational time. Perturbations applied in macroeconomics are used to expand the exact solution around a deterministic steady state in powers of state variables and a parameter scaling the uncertainty in the economy. The solutions based on the Taylor series expansion are intrinsically local, i.e. they are accurate in some neighborhood (presumably small) of the deterministic steady state. Out of the neighborhood, for example, in the case of sufficiently

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large shocks (or under the initial conditions that are far away from the steady state) the approximated solution can imply explosive dynamics, even if the original system is still stable for the same shocks (or initial conditions) (Kim, Kim, Schaumburg, and Sims (2008); Den Haan, and De Wind (2012)).

This study presents an approach based on a perturbation technique to construct global solutions to DSGE models. The proposed solutions are represented as a series in powers of a small parameter σ scaling the covariance matrix of the shocks. The zero order approximation corresponds to the solution to the deterministic model, because all shocks vanish as $\sigma = 0$. Global solutions to deterministic models can be obtained reasonably fast by effective numerical methods[1] even for large size models (Hollinger (2008)). For this reason the next stages of the method are carried out under the assumption that the solution to the deterministic model under given initial conditions is known.

The higher-order systems depend only on quantities of lower orders, therefore can be solved recursively. The homogeneous part of these systems is the same for all orders and depends on the deterministic solution. Consequently, each system can be represented as a rational expectation model with time-varying parameters. In the case of rational expectations models with constant parameters the stable block of equations can be isolated and solved forward. This is not possible for models with time-varying parameters. The present work proposes a method for solving this type of models. The method starts with finding a finite-horizon solution by using backward recursion. Next we prove that under certain conditions as the horizon tends to infinity the finite-horizon solutions approach to a limit solution that is bounded for all positive time.

If the parameter σ is small enough, then the solutions obtained are close to the deterministic solution. At the same time, whenever the deterministic solution is global in state variables so is the approximate solution to the stochastic problem. For this reason, we shall call this approach semi-global, whereas the perturbation methods based on series expansion around the steady state will be referred to as local. In contrast to the solutions obtained by the local perturbation methods, the solutions provided by the semi-global method inherit "global" properties, such as monotonicity and convexity, from the exact solution and thus cannot explode by construction.

We apply the method to the asset pricing model of Burnside (1998). Since the model has a closed-form solution we can check the accuracy of an approximate solution against the exact one. We compare the accuracy of the second order solution of the semi-global method with the local Taylor series expansion of order two (Schmitt-Grohé, and Uribe (2004)). The semi-global approach indicates superior performance in accuracy and inherits global properties from the exact solution.

This paper contributes to a growing literature on using the perturbation technique for solving DSGE models. The perturbation methodology in eco-

[1] The algorithms incorporated in the widely-used software such as Dynare (and less available Troll) find a stacked-time solution and are based on Newton's method combined with sparse-matrix techniques (Adjemian, Bastani, Juillard, Karamé, Mihoubi, Perendial, Pfeifer, Ratto, and Villemot (2011)).

nomics has been advanced by Judd and co-authors as in Judd (1998); Gaspar, and Judd (1997); Judd, and Guu (1997). Jin, and Judd (2002) give a theoretical basis for using perturbation methods in DSGE modeling; namely, applying the implicit function theorem, they prove that the perturbed rational expectations solution continuously depends on a parameter and therefore tends to the deterministic solution as the parameter tends to zero.

Almost all of the literature is concerned with the approximations around the steady state as in Collard, and Juillard (2001); Schmitt-Grohé, and Uribe (2004); Kim, Kim, Schaumburg, and Sims (2008); Gomme, and Klein (2011). Lombardo (2010) uses series expansion in powers of σ to find approximations to the exact solution recursively. Borovička, and Hansen (2013) employ Lombardo's approach to construct shock-exposure and shock-price elasticities, which are asset-pricing counterparts to impulse response functions. This approach has some similarity with that employed in the current paper. However, both papers apply the expansion only around the deterministic steady state, therefore the solution obtained remains local. Lombardo's approach can be treated as a special case of the method proposed in this study, namely a deterministic solution around which the expansion is used is only the steady state.

Judd (1998, Chapter 13) outlines how to apply perturbations around the known entire solution, which is not necessarily the steady state. He considers the simple continuous and discrete-time stochastic growth models in the dynamic programming framework. This paper develops a rigorous approach to construct solutions to DSGE models in general form by using the perturbation method around a global deterministic path.

The rest of the paper is organized as follows. The next section presents the model. Section 3 provides a detailed exposition of series expansions for DSGE models. In Section 4 we transform the model into a convenient form to deal with. Section 5 presents the method for solving rational expectations models for time-varying parameters. The proposed method is applied to an asset pricing model in Section 6, where it also compared with the local perturbation method in terms of accuracy. Conclusions are presented in Section 7.

2 The Model

DSGE models usually have the form

$$E_t f(y_{t+1}, y_t, x_{t+1}, x_t, z_{t+1}, z_t) = 0, \tag{2.1}$$

$$z_{t+1} = \Lambda z_t + \sigma \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim N(0, \Omega), \tag{2.2}$$

where E_t denotes the conditional expectations operator, x_t is an $n_x \times 1$ vector containing the t-period endogenous state variables; y_t is an $n_y \times 1$ vector containing the t-period endogenous variables that are not state variables; z_t is an $n_z \times 1$ vector containing the t-period exogenous state variables; ε_t is the vector with the corresponding innovations; $\sigma\Omega$ is the $n_z \times n_z$ covariance matrix of the innovations; f maps $\mathbb{R}^{n_y} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_z}$ into $\mathbb{R}^{n_y} \times \mathbb{R}^{n_x}$ and is assumed to be sufficiently smooth. The scalar σ ($\sigma > 0$) is a scaling parameter for the disturbance terms ε_t. We assume that all mixed moments of ε_t are finite. All eigenvalues of the matrix Λ have modulus less than one.

The solution to (2.1) and (2.2) is of the form:

$$y_t = h(x_t, z_t), \tag{2.3}$$

where h maps $\mathbb{R}^{n_x} \times \mathbb{R}^{n_z}$ into $\mathbb{R}^{n_y}$. Another way of stating the problem to solve is to say: for a given initial condition (x_0, z_0) find the initial condition y_0 for the vector y such that the solution (x_t, y_t) to (2.1)–(2.2) will be bounded for all $t > 0$.

3 Series Expansion

In this section we shall follow the perturbation methodology (see, for example, Holmes (2013)) to derive an approximate solution to the model (2.1)–(2.2). For small σ, we assume that the solution has a particular form of expansions

$$y_t = \sum_{n=0}^{\infty} \sigma^n y^{(n)}(x_t, z_t) \tag{3.1}$$

$$x_t = \sum_{n=0}^{\infty} \sigma^n x_t^{(n)}, \tag{3.2}$$

where $y^{(i)}(x_t, z_t)$ and $x_t^{(i)}$, $i = 0, 1, 2, \ldots$, are the i-order of approximation to the solution (2.3) and the variable x_t, respectively. The exogenous process z_t can also easily be represented in the form of expansion in σ

$$z_t = z_t^{(0)} + \sigma z_t^{(1)}. \tag{3.3}$$

Indeed, plugging (3.3) into (2.2) gives

$$z_{t+1} = z_{t+1}^{(0)} + \sigma z_{t+1}^{(1)} = \Lambda(z_t^{(0)} + \sigma z_t^{(1)}) + \sigma \varepsilon_{t+1}.$$

Collecting the terms of like powers of σ and equating them to zero, we get

$$z_{t+1}^{(0)} = \Lambda z_t^{(0)}, \tag{3.4}$$

$$z_{t+1}^{(1)} = \Lambda z_t^{(1)} + \varepsilon_{t+1}. \tag{3.5}$$

Since the expansion (3.3) must be valid for all σ at the initial time $t = 0$, the initial conditions are

$$z_0^{(0)} = z_0 \quad \text{and} \quad z_0^{(1)} = 0. \tag{3.6}$$

Note that the arguments of the functions $y^{(i)}$ are expansions in powers of σ. Substituting (3.2) and (3.3) into (3.1) yields

$$y_t = \sum_{i=0}^{\infty} \sigma^i y^{(i)} \left(\sum_{j=0}^{\infty} \sigma^j x_t^{(j)}, z_t^{(0)} + \sigma z_t^{(1)} \right). \tag{3.7}$$

Expanding y_t for small σ and collecting the terms of like powers, we have

$$y_t = \sum_{n=0}^{\infty} \sigma^n y^{*(n)}(x_t^{(0)}, x_t^{(1)}, ..., x_t^{(n)}, z_t^{(0)}, z_t^{(1)}), \tag{3.8}$$

where

$$y^{*(0)}(x_t^{(0)}, z_t^{(0)}) = y^{(0)}(x_t^{(0)}, z_t^{(0)}),$$
$$y^{*(1)}(x_t^{(0)}, x_t^{(1)}, z_t^{(0)}, z_t^{(1)}) = y^{(1)}(x_t^{(0)}, z_t^{(0)}) + y_{1,0;t}^{(0)} x_t^{(1)} + y_{0,1;t}^{(0)} z_t^{(1)},$$

and

$$y^{*(n)}(x_t^{(0)}, x_t^{(1)}, ..., x_t^{(n)}, z_t^{(0)}, z_t^{(1)}) = y^{(n)}(x_t^{(0)}, z_t^{(0)}) + y_{1,0;t}^{(0)} x_t^{(n)} + p_{n,t}, \tag{3.9}$$

where the mapping $p_{n,t} = p_n(x_t^{(0)}, x_t^{(1)}, \ldots, x_t^{(n-1)}, z_t^{(0)}, z_t^{(1)})$ has arguments with superscript less than n and is defined as

$$p_{n,t} = \sum_{l=0}^{n} \frac{1}{l!} \sum_{j=0}^{n-l} \sum_{k=1}^{n-j-l} \frac{1}{k!} y_{k,l;t}^{(j)} \left[\sum_{i_1+i_2+\cdots+i_k=n-j-l} \binom{n-j-l}{i_1; i_2; ...; i_k} x_t^{(i_1)}, x_t^{(i_2)}, ..., x_t^{(i_k)}, \left(z_t^{(1)}\right)^l \right]$$

Here $y_{k,l;t}^{(j)}$ denotes the mixed partial derivative of $y^{(j)}$ of order k and l with respect to x_t and z_t, respectively, at the point $(x_t^{(0)}, z_t^{(0)})$, and $\left(z_t^{(1)}\right)^l = (z_t^{(1)}, \ldots, z_t^{(1)})$ (l times). In other words, $y_{k,l;t}^{(j)}$ is a $(k+l)$-multilinear mapping (see, for example, Abraham, Marsden, and Ratiu (2001, p. 55)) depending on $(x_t^{(0)}, z_t^{(0)})$ (and hence on t). Substituting (3.9) into (3.8), we can rewrite (3.8) as

$$y_t = \sum_{n=0}^{\infty} \sigma^n \left[y^{(n)}(x_t^{(0)}, z_t^{(0)}) + y_{1,0;t}^{(0)} x_t^{(n)} + p_{n,t} \right]. \tag{3.10}$$

Then substituting (3.2), (3.3) and (3.10) into (2.1), collecting the terms of like powers of σ and setting their coefficients to zero, we have

Coefficient of σ^0

$$f\left(y^{(0)}(x_{t+1}^{(0)}, z_{t+1}^{(0)}), y^{(0)}(x_t^{(0)}, z_t^{(0)}), x_{t+1}^{(0)}, x_t^{(0)}, z_{t+1}^{(0)}, z_t^{(0)} \right) = 0, \tag{3.11}$$

The requirement that (3.2) and (3.3) must hold for all arbitrary small σ implies that the initial conditions for (3.11) are

$$z_0^{(0)} = z_0 \quad \text{and} \quad x_0^{(0)} = x_0. \tag{3.12}$$

The terminal condition is the steady state. The system of equations (3.4) and (3.11) is a deterministic model since it corresponds to the model (2.1) and

(2.2), where all shocks vanish. The deterministic model (3.4) and (3.11) with the initial conditions (3.12) can be solved globally by a number of effective algorithms, for example the extended path method (Fair, and Taylor (1983)) or a Newton-like method (for example, Juillard (1996)). As this study is primarily concerned with stochastic models, in what follows we suppose that the solution $(x_t^{(0)}, y^{(0)}(x_t^{(0)}, z_t^{(0)}))$ for $t > 0$ to the deterministic model is already known.

Coefficient of σ^n, $n > 0$

$$\begin{aligned} E_t\Big\{ & f_{1,t+1} \cdot y_{t+1}^{(n)} + f_{2,t+1} \cdot y_t^{(n)} + \Big[f_{1,t+1} \cdot y_{1,0;t+1}^{(0)} + f_{3,t+1}\Big] x_{t+1}^{(n)} \\ & + \Big[f_{2,t+1} \cdot y_{1,0;t}^{(0)} + f_{4,t+1}\Big] x_t^{(n)} + \eta_{t+1}^{(n)} \Big\} = 0, \end{aligned} \tag{3.13}$$

where $y_t^{(n)} = y^{(n)}(x_t^{(0)}, z_t^{(0)})$. The requirement that (3.2) must hold for all arbitrary small σ implies that the initial condition for (3.13) is

$$x_0^{(n)} = 0. \tag{3.14}$$

The matrices

$$f_{i,t+1} = f_i\left(y^{(0)}(x_{t+1}^{(0)}, z_{t+1}^{(0)}), y^{(0)}(x_t^{(0)}, z_t^{(0)}), x_{t+1}^{(0)}, x_t^{(0)}, z_{t+1}^{(0)}, z_t^{(0)}\right), i = 1, \ldots, 6,$$

are the Jacobian matrices of the mapping f with respect to y_{t+1}, y_t, x_{t+1}, x_t, z_{t+1}, and z_t, respectively, at the point $\left(y^{(0)}(x_{t+1}^{(0)}, z_{t+1}^{(0)}), y^{(0)}(x_t^{(0)}, z_t^{(0)}), x_{t+1}^{(0)}, x_t^{(0)}, z_{t+1}^{(0)}, z_t^{(0)}\right)$.

The mapping $E_t\eta_t^{(n)}$ is of the form:

$$E_t\eta_{t+1}^{(n)} = E_t\eta^{(n)}\left(x_{t+1}^{(0)}, x_t^{(0)}, \ldots, x_{t+1}^{(n-1)}, x_t^{(n-1)}, z_{t+1}^{(0)}, z_t^{(0)}, z_{t+1}^{(1)}, z_t^{(1)}\right),$$

where $\eta^{(n)}$ is some mapping for which the set of arguments includes only quantities of order less than n. The vector $z_{t+1}^{(1)}$ enters the expectations $E_t\eta_{t+1}^{(n)}$ in the form of the mixed moments of order n or less. The subscript $t+1$ in $f_{i,t+1}$ and $\eta_{t+1}^{(n)}$ reflects their dependence on $t+1$ through $x_{t+1}^{(0)}$ and $z_{t+1}^{(0)}$.

The expectation $E_t\eta_{t+1}^{(n)}$ is bounded if all mixed moments of $z_{t+1}^{(1)}$ are bounded up to order n and the vectors

$$\left(y_{t+1}^{(0)}, y_t^{(0)}, x_{t+1}^{(0)}, x_t^{(0)}, \ldots, y_{t+1}^{(n-1)}, y_t^{(n-1)}, x_{t+1}^{(n-1)}, x_t^{(n-1)}, z_{t+1}^{(0)}, z_t^{(0)}, z_{t+1}^{(1)}, z_t^{(1)}\right)$$

are bounded for all $t \geq 0$.

Equation (3.13) with the initial conditions (3.14) is a linear rational expectations model with time-varying coefficients. To solve the problem (3.13)–(3.14) is equivalent to find a bounded solution $(x_t^{(n)}, y_t^{(n)})$ for $t > 0$ under the assumption that the bounded solutions to the problems of all orders less than n are already known. It is worth noting that the homogeneous part of (3.13) is the same for all $n > 0$ and the difference is only in the non-homogeneous terms $E_t\eta_{t+1}^{(n)}$. In Section 5 we present a method for solving such types of model and prove the convergence of the solutions implied by the method to the exact solution. In the next section we transform equation (3.13) in a more convenient form to deal with.

4 Transformation of the Model

Define the deterministic steady state as vectors $(\bar{y}, \bar{x}, 0)$ such that

$$f(\bar{y}, \bar{y}, \bar{x}, \bar{x}, 0, 0) = 0. \tag{4.1}$$

We can represent $f_{i,t+1}$ in (3.13) as $f_{i,t+1} = f_i + \hat{f}_{i,t+1}$, $i = 1, \ldots, 6$, where $f_i = f_i(\bar{y}, \bar{y}, \bar{x}, \bar{x}, 0, 0)$ are the Jacobian matrices of the mapping f with respect to y_{t+1}, y_t, x_{t+1}, x_t, z_{t+1}, and z_t, respectively, at the steady state, and

$$\hat{f}_{i,t+1} = f_{i,t+1}(y^{(0)}_{t+1}, y^{(0)}_t, x^{(0)}_{t+1}, x^{(0)}_t, z^{(0)}_{t+1}, z^{(0)}_t) - f_i(\bar{y}, \bar{y}, \bar{x}, \bar{x}, 0, 0). \tag{4.2}$$

Note also that $\hat{f}_{i,t+1} \to 0$ as $t \to \infty$, because a deterministic solution must tend to the deterministic steady state as t tends to infinity. Consequently, $f_{i,t+1}$ can be thought of as a perturbation of f_i.

To shorten notation, further on we omit the superscript (n) when no confusion can arise. Therefore Equations (3.13) can be written in the vector form

$$\Phi_{t+1} E_t \begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \Lambda_{t+1} \begin{bmatrix} x_t \\ y_t \end{bmatrix} + E_t \eta_{t+1}, \tag{4.3}$$

where $\Phi_t = \left[f_3 + \hat{f}_{3,t}, f_1 + \hat{f}_{1,t}\right]$ and $\Lambda_t = \left[f_4 + \hat{f}_{4,t}, f_2 + \hat{f}_{2,t}\right]$. We assume that the matrices Φ_t are invertible for all $t \geq 0$. This assumption holds if, for example, the Jacobian $[f_3, f_1]^{-1}$ at the steady state is invertible[2] and the terms $\hat{f}_{1,t}$ and $\hat{f}_{3,t}$ are small enough for all $t \geq 0$. Pre-multiplying (4.3) by Φ^{-1}_{t+1}, we get

$$E_t \begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = L \begin{bmatrix} x_t \\ y_t \end{bmatrix} + M_{t+1} \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \Phi^{-1}_{t+1} E_t \eta_{t+1}, \tag{4.4}$$

where $L = [f_3, f_1]^{-1} [f_4, f_2]$ and

$$M_{t+1} = \left[f_3 + \hat{f}_{3,t+1}, f_1 + \hat{f}_{1,t+1}\right]^{-1} \left[f_4 + \hat{f}_{4,t+1}, f_2 + \hat{f}_{2,t+1}\right] - [f_3, f_1]^{-1} [f_4, f_2].$$

Notice that $lim_{t\to\infty} M_t = 0$. As in the case of rational expectations models with constant parameters it is convenient to transform (4.4) using the spectral property of L. Namely, the matrix L is transformed into a block-diagonal one using the block-diagonal Schur factorization[3]

$$L = ZPZ^{-1}, \tag{4.5}$$

where

$$P = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}; \tag{4.6}$$

[2] This assumption is made for ease of exposition. If $[f_3, f_1]$ is a singular matrix, then further on we must use a generalized Schur decomposition for which derivations remain valid, but become more complicated.

[3] The function bdschur of Matlab Control System Toolbox performs this factorization.

where A and B are quasi upper-triangular matrices with eigenvalues larger and smaller than one (in modulus), respectively; and Z is an invertible matrix[4]. We also impose the conventional Blanchard-Kan condition (Blanchard, and Kahn (1980)) on the dimension of the unstable subspace, i.e., $dim(B) = n_y$.

After introducing the auxiliary variables

$$[s_t, u_t]' = Z^{-1}[x_t, y_t]' \tag{4.7}$$

and pre-multiplying (4.4) by Z^{-1}, we have

$$E_t s_{t+1} = As_t + Q_{11,t+1}s_t + Q_{12,t+1}u_t + \Psi_{1t+1}E_t\eta_{t+1}, \tag{4.8}$$

$$E_t u_{t+1} = Bu_t + Q_{21,t+1}s_t + Q_{22,t+1}u_t + \Psi_{2t+1}E_t\eta_{t+1}, \tag{4.9}$$

where $[\Psi_{1,t+1}, \Psi_{2,t+1}] = Z\Phi_{t+1}^{-1}$ and

$$\begin{bmatrix} Q_{11,t+1} & Q_{12,t+1} \\ Q_{21,t+1} & Q_{22,t+1} \end{bmatrix} = ZM_{t+1}Z^{-1}. \tag{4.10}$$

System (4.8)-(4.9) is a linear rational expectations model with time-varying parameters, thus we cannot apply the approaches used in the case of models with constant parameters (Blanchard, and Kahn (1980); Anderson and Moor (1985); Sims (2001); Uhlig (1999), etc.). In Subsection 5.2 we devevop a method for solving this type of models.

5 Solving the Rational Expectations Model with Time-Varying Parameters

5.1 Notation

This subsection introduces some notation that will be necessary further on. By $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^n$. The induced norm for a real matrix D is defined by

$$\|D\| = \sup_{|s|=1} |\ Ds|.$$

The matrix Z in (4.5) can be chosen in such a way that

$$\|A\| < \alpha + \gamma < 1 \text{ and } \|B^{-1}\| < \beta + \gamma < 1, \tag{5.1}$$

where α and β are the largest eigenvalues (in modulus) of the matrices A and B^{-1}, respectively, and γ is arbitrarily small. This follows from the same arguments as in Hartmann (1982, §IV 9), where it is done for the Jordan matrix decomposition. Note also that $\|B\|^{-1} < 1$ for sufficiently small γ. Let

$$B_t = B + Q_{22,t}$$

[4] A simple generalized Schur factorization is also possible to employ here, but at the cost of more complicated derivations.

and $A_t = A + Q_{11,t}$.(5.2)By definition, put

$$a = \sup_{t=0,1,\dots} \|A_t\|, \quad b = \sup_{t=0,1,\dots} \left\|B_t^{-1}\right\|, \tag{5.3}$$

$$c = \sup_{t=0,1,\dots} \|Q_{12,t}\|, \quad d = \sup_{t=0,1,\dots} \|Q_{21,t}\|. \tag{5.4}$$

Here and in what follows we assume that all the matrices B_t, $t = 0, 1, \dots,$ are invertible. The numbers a, b, c and d depend on the initial conditions $(x_0^{(0)}, z_0^{(0)})$. From the definitions of A_t, A, B_t, B, $Q_{12,t}$ and $Q_{21,t}$ and the condition $lim_{t\to\infty}(x_t^{(0)}, z_t^{(0)}) = (\bar{x}, 0)$, it follows that

$$\lim_{t\to\infty} c(x_t^{(0)}, z_t^{(0)}) = 0, \quad \lim_{t\to\infty} d(x_t^{(0)}, z_t^{(0)}) = 0, \tag{5.5}$$

$$\lim_{t\to\infty} a(x_t^{(0)}, z_t^{(0)}) = \|A\| < 1, \quad \lim_{t\to\infty} b(x_t^{(0)}, z_t^{(0)}) = \left\|B^{-1}\right\| < 1.$$

This means that c and d can be arbitrary small and

$$a < 1 \quad \text{and} \quad b < 1 \tag{5.6}$$

by choosing $(x_0^{(0)}, z_0^{(0)})$ close enough to the steady state.

5.2 Solving the transformed system (4.8)–(4.9)

Taking into account notation (5.1), we can rewrite (4.8)–(4.9) in the form

$$E_t s_{t+1} = A_{t+1} s_t + Q_{12,t+1} u_t + \Psi_{1,t+1} E_t \eta_{t+1}, \tag{5.7}$$

$$E_t u_{t+1} = B_{t+1} u_t + Q_{21,t+1} s_t + \Psi_{2,t+1} E_t \eta_{t+1}. \tag{5.8}$$

In this subsection we construct a bounded solution to (5.7)–(5.8) for $t \geq 0$ with an arbitrary initial condition $s_0 \in \mathbb{R}^{n_x}$ and find under which conditions this solution exists. For this purpose, we first start with solving a finite-horizon model with a fixed terminal condition using backward recursion. Then, we prove the convergence of the obtained finite-horizon solutions to a bounded infinite-horizon one as the terminal time T tends to infinity.

Fix a horizon $T > 0$. Using the invertibility of $B_{T=1}$ and solving Equation (5.8) backward, we can obtain u_T as a linear function of s_T, the terminal condition $E_T u_{T+1}$ and the “exogenous” term $\Psi_{2,T+1} E_T \eta_{T+1}$

$$u_T = -B_{T+1}^{-1} Q_{21,T+1} s_T - B_{T+1}^{-1} \Psi_{2,T+1} E_T \eta_{T+1} + B_{T+1}^{-1} E_T u_{T+1}.$$

Proceeding further with backward recursion, we shall obtain finite-horizon solutions for each $t = 0, 1, 2, \dots, T$. For doing this we need to define the following recurrent sequence of matrices:

$$K_{T,T-i-1} = L_{T+1,T-i}^{-1} \left(Q_{21,T-i} + K_{T,T-i} A_{T-i}\right), \quad i = 0, 1, \dots, T, \tag{5.9}$$

where

$$L_{T,T-i} = B_{T-i} + K_{T,T-i} Q_{12,T-i}, \tag{5.10}$$

with the terminal condition $K_{T,T+1} = 0$. In (5.9) and (5.10) the first subscript T defines the time horizon, while the second subscript defines all times between 0 and $T+1$. Let $u_{T,T-i}$, $i = 0, 1, \ldots, T$, denote the $(T-i)$-time solution obtained by backward recursion that starts at the time T.

Proposition 5.1. *Suppose that the sequence of matrices* (5.9) *and* (5.10) *exists; then the solution to* (5.7)–(5.8) *has the following representation:*

$$u_{T,T-i} = -K_{T,T-i}s_{T-i} + g_{T,i} + \left(\prod_{k=1}^{i+1} L_{T,T-i+k}^{-1}\right) E_{T-i}\left(u_{T+1}\right), \tag{5.11}$$

where $i = 0, 1, \ldots, T$; *and*

$$g_{T,i} = -\sum_{j=1}^{i+1}\prod_{k=1}^{j} L_{T,T-i+k}^{-1}(\Psi_{2,T-i+j} + K_{T,T-i+j}\Psi_{1,T-i+j})E_{T-i}\eta_{T-i+j}. \tag{5.12}$$

For the proof see Appendix A. The sequence of matrices (5.9) exists if all matrices $L_{T,T-i}$, $i = 0, 1, \ldots, T$, are invertible. For this we need, in addition, some boundedness condition on the matrices $B_{T-i}^{-1}K_{T,T-i+1}Q_{12,T-i}$.

Proposition 5.2. *If for a, b, c and d from* (5.3)–(5.4) *the inequality*

$$cd < \frac{1}{4}\left(\frac{1}{b} - a\right)^2 = \left(\frac{1-ab}{2b}\right)^2 \tag{5.13}$$

holds, then

$$\left\|B_{T-i}^{-1}\right\| \cdot \left\|K_{T,T-i+1}\right\| \cdot \left\|Q_{12,T-i}\right\| < 1, \quad i = 0, 1, 2, \ldots T. \tag{5.14}$$

For the proof see Appendix A.

Proposition 5.3. *If inequality* (5.14) *holds, then the matrices $L_{T,T-i}$, $i = 0, 1, 2. \ldots, T$, are invertible.*

Proof. From (5.10) and the invertibility of B_{T-i} it follows that

$$L_{T,T-i} = B_{T-i}\left(I + B_{T-i}^{-1}K_{T,T-i}Q_{12,T-i}\right). \tag{5.15}$$

The matrices $L_{T,T-i}$ are invertible if and only if the matrices $\left(I + B_{T-i}^{-1}K_{T,T-i}Q_{12,T-i}\right)$ are invertible. From the norm property and (5.14) we have

$$\left\|B_{T-i}^{-1}K_{T,T-i+1}Q_{12,T-i}\right\| \leq \left\|B_{T-i}^{-1}\right\| \cdot \left\|K_{T,T-i+1}\right\| \cdot \left\|Q_{12,T-i}\right\| < 1.$$

Now the invertibility of $\left(I + B_{T-i}^{-1}K_{T,T-i}Q_{12,T-i}\right)$ follows from Golub, and Van Loan (1996, Lemma 2.3.3) ☐

For $i = T$ from (5.11) we have

$$u_{T,0} = -K_{T,0}s_0 + g_{T,T} + \left(\prod_{k=1}^{T+1} L_{T,k}^{-1}\right) E_0\left(u_{T+1}\right). \tag{5.16}$$

This is a finite-horizon solution to the rational expectations model with time-varying coefficients (5.7)–(5.8) and with a given initial condition s_0. What is left is to show that the solution $u_{T,0}$ of the form (5.16) converges to some limit as $T \to \infty$.

Proposition 5.4. *If inequality (5.13) holds, then the limit*
$\lim_{T\to\infty} K_{T,j} = K_{\infty,j}$ *for* $j = 0, 1, 2, \ldots$
exists in the matrix space defined in Subsection 5.1.

For the proof see Appendix A.

Proposition 5.5. *If inequality (5.14) holds, then*

$$\lim_{T\to\infty} \prod_{k=1}^{T+1} L_{T,k}^{-1} = 0 \tag{5.17}$$

and

$$\lim_{T\to\infty} g_{T,T} = g_{\infty}, \tag{5.18}$$

where g_∞ *is some vector in* $\mathbb{R}^{n_y}$.

Proof. From (5.10) and Proposition 5.4 it follows that

$$\lim_{T\to\infty} L_{T,k} = B_k + K_{\infty,k} Q_{12,k} = L_{\infty,k}.$$

Then the limit in (5.17) can be represented as

$$\lim_{T\to\infty} \prod_{k=1}^{T+1} L_{T,k}^{-1} = \lim_{T\to\infty} \prod_{k=1}^{T+1} L_{\infty,k}^{-1}. \tag{5.19}$$

Since $K_{\infty,k}$ is bounded (it follows from formula (A.7) in Appendix A) and

$$\lim_{k\to\infty} Q_{12,k} = 0, \quad \text{and} \quad \lim_{k\to\infty} B_k^{-1} = B^{-1},$$

we have $lim_{k\to\infty} L_{\infty,k}^{-1} = B^{-1}$. Therefore, if $\delta > 0$ is arbitrary small, there is an $N = N_\delta \in \mathbb{N}$ such that

$$\|L_{\infty,k}^{-1}\| \le \beta + \delta = \rho < 1, \tag{5.20}$$

for $k > N$, where β is the largest eigenvalue (in modulus) of the matrix B^{-1}. From this, the norm property and (5.19) we obtain

$$\lim_{T\to\infty} \left\| \prod_{k=1}^{T+1} L_{T,k}^{-1} \right\| \le \lim_{T\to\infty} \prod_{k=1}^{T+1} \left\| L_{\infty,k}^{-1} \right\| \le \lim_{T\to\infty} C_1 \rho^{T-K} = 0,$$

where C_1 is some constant. Therefore, (5.17) is proved.

By (5.20) the products in (5.12) decay exponentially with the factor ρ as $j \to \infty$. From this and the boundedness of the terms $K_{T,k}$, $\Psi_{2,k}$, $\Psi_{1,k}$ and $E_0\eta_k$, $T \in \mathbb{N}$ and $k = 1, 2, \ldots, T+1$, it follows that the series

$$g_{T,T} = -\sum_{j=1}^{T+1} \prod_{k=1}^{j} L_{T,k}^{-1}(\Psi_{2,j} + K_{T,j}\Psi_{1,j})E_0\eta_j.$$

converges to some g_∞ as $T \to \infty$. □

From Proposition 5.4 and Proposition 5.5 it may be concluded that as T tends to infinity Equation (5.16) takes the form:

$$u_0 = -K_{\infty,0}s_0 + g_\infty. \tag{5.21}$$

Formula (5.21) gives us the unique bounded solution to the transformed rational expectation model with time-varying parameters (5.7)–(5.8). Note also that the proofs of Propositions 5.2–5.5 are based on inequality (5.13) that is a spectral gap condition for the unstable and stable parts of the system (5.7)–(5.8), and in a sense substitutes for the Blanchard-Kahn condition for rational expectations models with time-varying parameters. It follows from (5.3)–(5.6) that inequality (5.13) always holds if initial conditions $(x_t^{(0)}, z_t^{(0)})$ is close enough to the steady state. Nonetheless, the condition (5.13) is not local by itself.

5.3 Restoring the original variables $x_t^{(n)}$ and $y_t^{(n)}$

Recall that we deal with the n-order problem (3.13)–(3.14), and now we put the superscript (n) back in notation. To find the bounded solution in terms of the original variables $x_t^{(n)}$ and $y_t^{(n)}$ we need to obtain the initial values $u_0^{(n)}$ and $s_0^{(n)}$ that correspond to that of the problem (4.4), i.e. $x_0^{(n)} = 0$. From (4.7) and (5.21) we have

$$\begin{bmatrix} s_0^{(n)} \\ -K_{\infty,0}^{(n)}s_0^{(n)} + g_\infty^{(n)} \end{bmatrix} = Z^{-1}\begin{bmatrix} 0 \\ y_0^{(n)} \end{bmatrix},$$

where Z^{-1} is a matrix that is involved in the block-diagonal Schur factorization (4.5) and has the following block-decomposition:

$$Z^{-1} = \begin{bmatrix} Z^{11} & Z^{12} \\ Z^{21} & Z^{22} \end{bmatrix}.$$

Hence

$$s_0^{(n)} = Z^{12}y_0^{(n)}, \tag{5.22}$$

$$-K_{\infty,0}^{(n)}s_0^{(n)} + g_\infty^{(n)} = Z^{22}y_0^{(n)}. \tag{5.23}$$

Substituting (5.22) into (5.23) and assuming that the matrix $Z^{22} + K_{\infty,0}^{(n)}Z^{12}$ is invertible, we get

$$y_0^{(n)} = (Z^{22} + K_{\infty,0}^{(n)}Z^{12})^{-1}g_\infty^{(n)}. \tag{5.24}$$

The left-hand side of (5.24) corresponds to $y^{(n)}(x_0, z_0)$ in (3.1). The dependence of $y_0^{(n)}$ on (x_0, z_0) follows from $K_{\infty,0}^{(n)}$ and $g_\infty^{(n)}$. Therefore, formula (5.24) determines the solution to the original rational expectations model with time-varying parameters (3.13) and with the initial condition $x_0^{(n)} = 0$. By assumption, the solutions of lower order are already computed, thus the policy function approximation is of the form[5]

$$y_t = \sum_{i=0}^{n} \sigma^i y^{(i)}(x_t, z_t).$$

The matrix $(Z^{22} + K_{\infty,0}Z^{12})$ is invertible if (i) the matrix Z^{22} is square and invertible, and (ii) the norm of the matrix $K_{\infty,0}$ is small enough. The condition (i) corresponds to Proposition 1 of Blanchard, and Kahn (1980); at the same time, the condition (ii) can be always attained if initial conditions $(x_t^{(0)}, z_t^{(0)})$ are close enough to the steady state, which follows from (A.6) and (A.7) of Appendix A. Notice again that these conditions are not local by themselves.

If we are interested in finding dynamics, for example, impulse response functions; then knowing $y_0^{(n)}$ and using (5.22)–(5.24) we can recover initial conditions $(s_0^{(n)}, u_0^{(n)})$ in terms of the variables $s^{(n)}$ and $u^{(n)}$, solve equations (5.7)–(5.8) with these initial conditions , and finally obtain the solution to (4.4), using the transformation Z. This provides the solution to (4.4) in the form

$$x_t^{(n)} = Z_{11}s_t^{(n)} + Z_{12}u_t^{(n)},$$

$$y_t^{(n)} = Z_{21}s_t^{(n)} + Z_{22}u_t^{(n)},$$

where Z_{ij}, $i = 1, 2$, $j = 1, 2$, are blocks of the block-decomposition of the matrix Z.

6 An Asset Pricing Model

In this section we apply the presented method to an nonlinear asset pricing model proposed by Burnside (1998) and analyzed by Collard, and Juillard (2001). The representative agent maximizes the lifetime utility function

$$\max \left(E_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^\theta}{\theta} \right)$$

subject to

$$p_t e_{t+1} + C_t = p_t e_t + d_t e_t,$$

where $\beta > 0$ is a subjective discount factor, $\theta < 1$ and $\theta \neq 0$, C_t denotes consumption, p_t is the price at date t of a unit of the asset, e_t represents units

[5] In fact, it is not hard to prove that in the case of symmetric distribution of ε_t for all odd n the unique bounded solution is $x_t^{(n)} \equiv 0$ and $y_t^{(n)} \equiv 0$. We will show this for a simple asset pricing model in Section 6 for $i = 1$.

of a single asset held at the beginning of period t, and d_t is dividends per asset in period t. The growth of rate of the dividends follows an AR(1) process

$$\mathrm{x_t} = \ (1-\ \rho)\ \mathrm{\bar{x}}\ +\ \rho x_{t-1} + \sigma\varepsilon_{t+1}, \tag{6.1}$$

where $x_t = ln(d_t/d_{t-1})$, and $\varepsilon_{t+1} \sim NIID(0,1)$. The first order condition and market clearing yields the equilibrium condition

$$y_t = \beta E_t \left[\exp(\theta x_{t+1})\,(1 + y_{t+1})\right], \tag{6.2}$$

where $y_t = p_t/d_t$ is the price-dividend ratio. This equation has an exact solution of the form (Burnside (1998))

$$y_t = \sum_{i=1}^{\infty} \beta^i \exp\left[a_i + b_i(x_t - \bar{x})\right], \tag{6.3}$$

where

$$a_i = \theta\bar{x}i + \frac{1}{2}\left(\frac{\theta\sigma}{1-\rho}\right)^2 \left[i - \frac{2\rho(1-\rho^i)}{1-\rho} + \frac{\rho^2(1-\rho^{2i})}{1-\rho^2}\right] \tag{6.4}$$

and

$$b_i = \frac{\theta\rho(1-\rho^i)}{1-\rho}.$$

It follows from (6.2) that the deterministic steady state of the economy is

$$\bar{y} = \frac{\beta\exp(\theta\bar{x})}{1-\beta\exp(\theta\bar{x})}.$$

6.1 Solution

We now obtain a solution to the system (6.1)–(6.2) as an expansion in powers of the parameter σ using the second-order approximation method developed in Sections 3–5. Specifically, we are seeking for the solution of the form:

$$y_t = y^{(0)}(x_t) + \sigma y^{(1)}(x_t) + \sigma^2 y^{(2)}(x_t) \tag{6.5}$$

$$x_t = x_t^{(0)} + \sigma x_t^{(1)}. \tag{6.6}$$

Substituting (6.6) into (6.1) and collecting the terms containing σ^0 and σ^1, we obtain the representation (6.6) for x_t

$$x_{t+1}^{(0)} = (1-\ \rho)\ \mathrm{\bar{x}}\ + \rho x_t^{(0)} \tag{6.7}$$

$$x_{t+1}^{(1)} = \rho x_t^{(1)} + \varepsilon_{t+1}. \tag{6.8}$$

Since the expansion (6.6) must be valid for all σ at the initial time $t = 0$, the initial conditions are

$$x_0^{(0)} = x_0 \quad \text{and} \quad x_0^{(1)} = 0. \tag{6.9}$$

Substituting (6.5) and (6.6) into (6.2), then collecting the terms of like powers of σ and setting the coefficients of like powers of σ to zero, we have (for details see Appendix B)

Coefficient of σ^0

$$y_t^{(0)} = \beta \exp(\theta x_{t+1}^{(0)})(1 + y_{t+1}^{(0)}), \tag{6.10}$$

$$x_{t+1}^{(0)} = \rho x_t^{(0)}. \tag{6.11}$$

Coefficient of σ^1

$$\begin{aligned} y_{1;t}^{(0)} x_t^{(1)} + y_t^{(1)} = & \\ & + \exp(\theta x_{t+1}^{(0)}) \beta E_t \left[\theta x_{t+1}^{(1)} (1 + y_{t+1}^{(0)}) + y_{1;t+1}^{(0)} x_{t+1}^{(1)} + y_{t+1}^{(1)} \right], \end{aligned} \tag{6.12}$$

$$x_{t+1}^{(1)} = \rho x_t^{(1)} + \varepsilon_{t+1}.$$

Coefficient of σ^2

$$\begin{aligned} & y_t^{(2)} = -y_{1;t}^1 x_t^{(1)} - \tfrac{1}{2} y_{2;t}^{(0)} \left(x_t^{(1)} \right)^2 \\ & + \tfrac{1}{2} \beta \left[\theta^2 \left(1 + y_{t+1}^{(0)} \right) + 2\theta y_{1;t+1}^{(0)} + y_{2,t+1}^{(0)} \right] \exp(\theta x_{t+1}^{(0)}) E_t \left(x_{t+1}^{(1)} \right)^2 \\ & + \beta \exp(\theta x_{t+1}^{(0)}) E_t \left[y_{t+1}^{(1)} + x_{t+1}^{(1)} \left(y_{1;t+1}^{(1)} + \theta y_{t+1}^{(1)} \right) + E_t \left(y_{t+1}^{(2)} \right) \right], \end{aligned} \tag{6.13}$$

where $y_{j;t}^{(i)}$, $i = 0, 1, j = 1, 2$, are derivatives of $y^{(i)}$ of order j at the point $x_t^{(0)}$. Here and further on, for the simplicity of notation, we write $y_t^{(i)}$ instead of $y^{(i)}(x_t^{(0)})$, $i = 0, 1, 2$.

The system (6.10)-(6.1) is a deterministic model. Its solution can be obtained by taking $\sigma = 0$ in (6.3) and (6.4)

$$y_t^{(0)} = \sum_{i=1}^{\infty} \beta^i \exp \left\{ \theta \left[\bar{x} i + \frac{\rho(1-\rho^i)}{1-\rho} (x_t - \bar{x}) \right] \right\}, \tag{6.14}$$

For the first order approximation we can rewrite (6.12) in the form

$$\begin{aligned} y_{1;t}^{(0)} x_t^{(1)} + y_t^{(1)} = {} & \beta \exp(\theta x_{t+1}^{(0)}) \left[\theta (1 + y_{t+1}^{(0)}) + y_{1;t+1}^{(0)} \right] E_t x_{t+1}^{(1)} \\ & + \beta \exp(\theta x_{t+1}^{(0)}) E_t y_{t+1}^{(1)} \end{aligned} \tag{6.15}$$

Under the assumption that $y_t^{(0)}$ and $x_t^{(0)}$ are known for $t \geq 0$, Equations (6.15) and (6.8) constitute a forward looking model. Since $x_0^{(1)} = 0$, from (6.8) we have $E_0 x_t^{(1)} = 0$ for $t > 0$. It is easily shown that the only bounded solution of (6.15) is $y_t^{(1)} \equiv 0$ for $t \geq 0$.

Equation (6.13) is a linear forward-looking equation with time varying deterministic coefficients. This equation can be solved by the backward recursion considered in Section 5. Taking into account that the initial value of $x_t^{(1)}$ is zero, it can be easily checked that the solution of (6.13) has the form

$$\begin{aligned} y_t^{(2)} = \frac{1}{2}\sum_{n=1}^{\infty} \beta^n \exp\left[\theta\left(x_{t+1}^{(0)} + x_{t+2}^{(0)} + \cdots + x_{t+n}^{(0)}\right)\right] \cdot \left[\theta^2\left(1 + y_{t+n}^{(0)}\right)\right. \\ \left. + 2\theta y_{1;t+n}^{(0)}\right] \cdot E_t\left(x_{t+n}^{(1)}\right)^2 \end{aligned} \tag{6.16}$$

Here $y_{1;t+n}^{(0)}$ can be obtained by differentiating (6.3) with respect to x_t and is given by

$$y_{1;t}^{(0)} = \sum_{i=1}^{\infty} \beta^i \frac{\rho(1-\rho^i)}{1-\rho} \exp\left\{\theta\left[\bar{x}i + \frac{\rho(1-\rho^i)}{1-\rho}(x_t - \bar{x})\right]\right\}.$$

From (6.8) and (6.9) we have the moving-average representation for $x_{t+1}^{(1)}$:

$$x_{t+n}^{(1)} = \varepsilon_{t+n} + \rho\varepsilon_{t+n-1} + \ldots + \rho^{n-1}\varepsilon_{t+1}.$$

Since the sequence of innovations ε_t, $t > 0$, is independent it follows that

$$\begin{aligned} E_t\left(x_{t+n}^{(1)}\right)^2 &= E_t\left(\varepsilon_{t+n} + \rho\varepsilon_{t+n-1} + \cdots + \rho^{n-1}\varepsilon_{t+1}\right)^2 \\ &= 1 + \rho^2 + \cdots + \rho^{2(n-1)} = \frac{1-\rho^{2n}}{1-\rho^2}. \end{aligned} \tag{6.17}$$

From (6.7) we have

$$\begin{aligned} & x_{t+1}^{(0)} + x_{t+2}^{(0)} + \cdots + x_{t+n}^{(0)} = \bar{x} + \rho(x_t^{(0)} - \bar{x}) + \bar{x} + \rho^2(x_t^{(0)} - \bar{x}) \\ & + \bar{x} + \rho^n(x_t^{(0)} - \bar{x}) = n\bar{x} + \tfrac{\rho(1-\rho^n)}{1-\rho}(x_t^{(0)} - \bar{x}). \end{aligned} \tag{6.18}$$

Finally, inserting (6.17) and (6.18) into (6.16) gives

$$y_t^{(2)} = \frac{\theta^2}{2}\sum_{n=1}^{\infty} \beta^n \frac{1-\rho^{2n}}{1-\rho^2} \exp\left\{\theta\left[n\bar{x} + \theta\frac{\rho(1-\rho^n)}{1-\rho}(x_t^{(0)} - \bar{x})\right]\right\}\left[\theta^2(1 + y_{t+n}^{(0)}) + 2\theta y_{1;t+n}^{(0)}\right].$$

To summarize, we find the policy function approximation in the form

$$y(x) = y_t^{(0)}(x) + \sigma^2 y_t^{(2)}(x).$$

The solutions for the higher orders $y_t^{(i)}(x)$, $i > 2$, can be obtained in much the same way as for $y_t^{(2)}(x)$. Note also that it is easily shown that for all odd i the unique bounded solution is $y_t^{(i)} \equiv 0$.

6.2 Accuracy Check

This subsection compares the accuracy of the second order of the presented method with the local Taylor series expansions of order 2 (Schmitt-Grohé, and

Uribe (2004)). The following three criteria are used to check the accuracy of the approximation methods:

$$E_{0,\infty} = 100 \cdot \max_i \left\{ \left| \frac{y(x_i) - \tilde{y}(x_i)}{y(x_i)} \right| \right\},$$

$$E_{1,\infty} = 100 \cdot \max_i \left\{ \left| \frac{\Delta y(x_i) - \Delta \tilde{y}(x_i)}{\Delta y(x_i)} \right| \right\},$$

$$E_{2,\infty} = 100 \cdot \max_i \left\{ \left| \frac{\Delta^2 y(x_i) - \Delta^2 \tilde{y}(x_i)}{\Delta^2 y(x_i)} \right| \right\},$$

where $y(x_i)$ denotes the closed-form solution, $\tilde{y}(x_i)$ is an approximation of the true solution by the method under study, $\Delta y(x_i) = y(x_i) - y(x_i - \Delta x)$ and $\Delta x = x_i - x_{i-1}$ are the first difference of y and x, respectively, $\Delta^2 y(x_i)$ is the second difference of y, i.e $\Delta^2 y(x_i) = \Delta y(x_i) - \Delta y(x_{i-1})$. The criterion $E_{0,\infty}$ is the maximal relative error made using an approximation rather than the true solution. The criteria $E_{1,\infty}$ and $E_{2,\infty}$ capture the accuracy of the characteristics of the shape of an approximate policy function, namely the slope and convexity, by comparing the maximal relative first and second differences of an approximate and the closed-form solutions. All criteria are evaluated over the interval $x_i \in [\bar{x} - \Delta \cdot \sigma_x, \bar{x} + \Delta \cdot \sigma_x]$, where σ_x is the unconditional volatility of the process x_t and $\Delta = 5$.

The parameterization follows Collard, and Juillard (2001), where the benchmark parameterization is chosen as in Mehra, and Prescott (1985). We therefore set the mean of the rate of growth of dividend to $\bar{x} = 0.0179$, its persistence to $\rho =$ -0.139 and the volatility of the innovations to $\sigma = 0.0348$. The parameter θ was set to -1.5 and β to 0.95. We investigate the implications of larger curvature of the utility function, higher volatility and more persistence in the rate of growth of dividends in terms of accuracy.

Table 1 shows that the maximal relative error for the benchmark parameterization is three times larger for the approximation of the Taylor series expansion than for the semi-global method, however the errors for both these methods are very small - 0.06% and 0.02%, respectively. The increase in the conditional volatility of the rate of growth of dividends to $\sigma = 0.1$ yields the higher approximation errors of 2% and 1% for the local Taylor series expansion and semi-global method, respectively. Increasing the curvature of the utility function to $\theta = -10$ yields the maximal approximation error 8.4% for the Taylor series expansion approximation and about two times smaller for the semi-global method.

The semi-global method becomes considerably more accurate than the local Taylor series expansion if the persistence of the exogenous process increases. For the parameterization $\rho = 0.5$ and $\sigma = 0.03$ the proposed method gives the maximal relative approximation error six times smaller than the local Taylor series expansion. Increasing the persistence to $\rho = 0.9$ yields the maximal relative approximation error of 193% for the local Taylor series expansion and 9% for the semi-global method. This effect is more pronounced for the criteria $E_{1,\infty}$

Table 1: The relative errors of the approximate solutions

Criterion	$E_{0,\infty}$		$E_{1,\infty}$		$E_{2,\infty}$	
Model	SG[a]	P2[b]	SG	P2	SG	P2
Parameterization						
Benchmark	0.02	0.06	0.02	1.47	0.02	4.53
$\theta = -10$	4.75	8.39	4.66	25.0	4.56	37.6
$\sigma = 0.1$	1.30	2.23	1.29	12.0	1.28	19.3
$\rho = 0.5$	0.26	1.56	0.28	8.72	0.30	26.6
$\rho = 0.5, \theta = -5$	10.3	27.8	11.0	69.4	11.6	71.3
$\rho = 0.9$	9.30	193	11.3	392	12.8	360

a The semi-global method of order two

b The local Taylor series perturbation method of order two (Schmitt-Grohé, and Uribe (2004))

and $E_{2,\infty}$. Furthermore, for any parameterization the semi-global approximation gives at least 5 times more accurate solution in the metrics $E_{1,\infty}$ and 9 times in the metrics $E_{2,\infty}$ than the local Taylor series expansion.

Figure 1 shows the policy functions for the high persistence case, $\rho = 0.9$, and indicates that the semi-global method traces globally the pattern of the true policy function much better than the local Taylor series expansion. Moreover, from Figure 1 we can also see another undesirable property of the the local Taylor series expansion, namely this method can provide impulse response functions with a wrong sign. Indeed, the steady state value of y_t is $\bar{y} = 12.3$. After a positive shock the true impulse response function is negative, whereas the impulse response function implied by the local perturbation method is positive, if the shock is large enough. Note also that the solution produced by the semi-global method is indistinguishable from the true solution for positive shocks (the bottom right corner of the Figure 1).

7 Conclusion

This study proposes a method based on a perturbation around a deterministic path for constructing approximate solutions to DSGE models. The solutions obtained are global in the state space whenever so is the deterministic solution. As by product, an approach to solve linear rational expectations models with deterministic time-varying parameters is developed. This approach might be valuable in itself, for example, it can be used to solve Markov-Switching DSGE models. All results are obtained for DSGE models in general form and proved rigorously.

The advantage of using the local perturbation methods lies in the fact that they can deal with medium and large-size models for reasonable computational time. However, this methods are intrinsically local as they employ perturbations around the steady state. Whereas the global methods used in DSGE

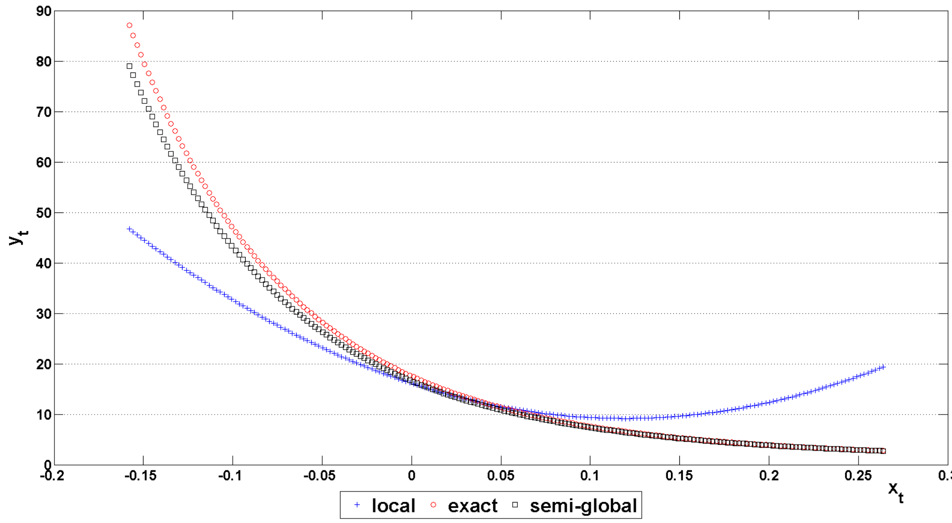

Figure 1: Comparison of the policy functions for $\rho = 0.9$.

modeling, such as projection and stochastic simulations, suffer from the curse of dimensionality, i.e. they can handle only small-dimension models. The proposed approach has a potential to solve high-dimensional models as it shares some preferable properties with the local perturbation methods. Namely, the computational gain may come from calculation of conditional expectations. To compute the conditional expectations using the semi-global method all that we need is to know the moments of the distribution up to the order of approximation, while the use of the global methods mentioned above involves either stochastic simulations or quadratures. The former is time consuming, the later can deal with only low-order integrals. The practical implementation of the approach to larger size models the author leaves for future research.

A Proofs for Section 5

PROOF OF PROPOSITION 5.1: The proof is by induction on i. Suppose that $i = 0$. For the time T from (5.8) we have

$$E_T u_{T+1} = B_{T+1} u_T + Q_{21,T+1} s_T + \Psi_{2,T+1} E_T \eta_{T+1}.$$

As B_{T+1} is invertible, we have

$$u_{T,T} = -K_{T,T} s_T - g_{T,0} + L^{-1}_{T,T,T} E_T u_{T+1},$$

where $K_{T,T} = B^{-1}_{T+1} Q_{21,T+1}$; $g_{T,0} = -B^{-1}_{T+1} \Psi_{2,T+1} E_T \eta_{T+1}$ and $L^{-1}_{T,T+1} = B^{-1}_{T+1}$. From (5.9), (5.10) and (5.12) it follows that the inductive assumption is proved for $i = 0$. Assuming that (5.11) holds for $i > 0$, we will prove it for $i+1$. To this end, consider Equation (5.8) for the time $t = T - i - 1$. As the matrix B_{T-i} is invertible, we obtain

$$u_{T,T-i-1} = -B_{T-i}^{-1}Q_{21,T-i}s_{T-i-1} - B_{T-i}^{-1}\Psi_{2,T-i}E_{T-i-1}\eta_{T-i} + B_{T-i}^{-1}E_{T-i-1}u_{T,T-i}.$$

Substituting the induction assumption (5.11) for $u_{T,T-i}$ yields

$$\begin{aligned}&u_{T,T-i-1} = -B_{T-i}^{-1}Q_{21,T-i}s_{T-i-1} - B_{T-i}^{-1}\Psi_{2,T-i}E_{T-i-1}\eta_{T-i}\\ &+B_{T-i}^{-1}E_{T-i-1}\left[-K_{T,T-i}s_{T-i} + g_{T,i} + \left(\prod_{k=1}^{i+1} L_{T,T-i+k}^{-1}\right) E_{T-i}\left(u_{T+1}\right)\right].\end{aligned}$$

Substituting (5.7) for $E_{T-i-1}(s_{T-i})$ and using the law of iterated expectations gives

$$\begin{aligned}&u_{T,T-i-1} = -B_{T-i}^{-1}Q_{21,T-i}s_{T-i-1} - B_{T-i}^{-1}\Psi_{2,T-i}E_{T-i-1}\eta_{T-i} + B_{T-i}^{-1}g_{T,i}\\ &+B_{T-i}^{-1}\left(\prod_{k=1}^{i+1} L_{T,T-i+k}^{-1}\right) E_{T-i-1}\left(u_{T+1}\right)\\ &+B_{T-i}^{-1}\left[-K_{T,T-i}\left(A_{T-i}s_{T-i-1} + Q_{12,T-i}u_{T,T-i-1} + \Psi_{1,T-i}E_{T-i-1}\eta_{T-i}\right)\right].\end{aligned}$$

Collecting the terms with $u_{T,T-i-1}$, s_{T-i-1} and η_{T-i}, we get

$$\begin{aligned}&\left(I + B_{T-i}^{-1}K_{T,T-i}Q_{12,T-i}\right)u_{T,T-i-1} = -B_{T-i}^{-1}\big[(Q_{21,T-i} + K_{T,T-i}A_{T-i})s_{T-i-1}\\ &+(\Psi_{2,T-i} + K_{T,T-i}\Psi_{1,T-i})E_{T-i-1}\eta_{T-i} + g_{T,i} + \left(\prod_{k=1}^{i+1} L_{T,T-i+k}^{-1}\right) E_{T-i-1}\left(u_{T+1}\right)\big]\end{aligned}$$

Suppose for the moment that the matrix $Z_{T,T-i} = I + B_{T-i}^{-1}K_{T,T-i}Q_{12,T-i}$ is invertible. Pre-multiplying the last equation by $Z_{T,T-i}^{-1}$, we obtain

$$\begin{aligned}&u_{T,T-i-1} = -Z_{T,T-i}^{-1}B_{T-i}^{-1}\big[(Q_{21,T-i} + K_{T,T-i}A_{T-i})s_{T-i-1}\\ &+(\Psi_{2,T-i} + K_{T,T-i}\Psi_{1,T-i})E_{T-i-1}\eta_{T-i} + g_{T,i}\\ &+\left(\prod_{k=1}^{i+1} L_{T,T-i+k}^{-1}\right) E_{T-i-1}\left(u_{T+1}\right)\big].\end{aligned}$$

Note that $L_{T,T-i} = B_{T-i}Z_{T,T-i}$; then using the definition of $K_{T,T-i-1}$ (5.9), we see that

$$\begin{aligned}&u_{T,T-i-1} = -K_{T,T-i-1}s_{T-i-1}\\ &-L_{T,T-i}^{-1}\left(\Psi_{2,T-i} + K_{T,T-i}\Psi_{1,T-i}\right)E_{T-i-1}\eta_{T-i}\\ &+L_{T,T-i}^{-1}g_{T,i} + L_{T,T-i}^{-1}\left(\prod_{k=1}^{i+1} L_{T,T-i+k}^{-1}\right) E_{T-i-1}\left(u_{T+1}\right).\end{aligned} \tag{A.1}$$

Using the definition of $g_{T,i}$ and $L_{T-i,T-i+j}$ ((5.10) and (5.12)), we deduce that

$$g_{T,i+1} = -L_{T,T-i}^{-1}\left(\Psi_{2,T-i} + K_{T,T-i}\Psi_{1,T-i}\right)E_{T-i-1}\eta_{T-i} + L_{T,T-i}^{-1}g_{T,i}. \tag{A.2}$$

From (A.1) and (A.2) it follows that

$$u_{T,T-i-1} = -K_{T,T-i-1}s_{T-i-1} + g_{T,i+1} + \left(\prod_{k=1}^{i+2} L_{T,T-i-1+k}^{-1}\right) E_{T-i-1}\left(u_{T+1}\right).$$

This proves the proposition.

PROOF OF PROPOSITION 5.2: We begin by rewriting (5.9) as

$$\left(B_{T-i} + K_{T,T-i}Q_{12,T-i}\right) K_{T,T-(i+1)} = \left(Q_{21,T-i} + K_{T,T-i}A_{T-i}\right).$$

Rearranging terms, we have

$$\begin{aligned} K_{T,T-(i+1)} = {} & B_{T-i}^{-1} \cdot \left(Q_{21,T-i} + K_{T,T-i}A_{T-i}\right) \\ & - B_{T-i}^{-1} K_{T,T-i} Q_{12,T-i} K_{T,T-(i+1)}. \end{aligned} \tag{A.3}$$

Taking the norms and using the norm properties gives

$$\left\|K_{T,T-(i+1)}\right\| \leq \left\|B_{T-i}^{-1}\right\| \cdot \|Q_{21,T-i}\| + \left\|B_{T-i}^{-1}\right\| \cdot \|K_{T,T-i}\| \cdot \|A_{T-i}\|$$

$$+ \left\|B_{T-i}^{-1}\right\| \cdot \|K_{T,T-i}\| \cdot \|Q_{12,T-i}\| \cdot \left\|K_{T,T-(i+1)}\right\|.$$

Rearranging terms, we get

$$\left\|K_{T,T-(i+1)}\right\| \leq \frac{\left\|B_{T-i}^{-1}\right\| \cdot \|Q_{21,T-i}\| + \left\|B_{T-i}^{-1}\right\| \cdot \|K_{T,T-i}\| \cdot \|A_{T-i}\|}{1 - \left\|B_{T-i}^{-1}\right\| \cdot \|K_{T,T-i}\| \cdot \|Q_{12,T-i}\|}. \tag{A.4}$$

Inequality (A.4) is a difference inequality with respect to $\|K_{T,T-i}\|$, $i = 0, 1, \ldots, T$, and with time-varying coefficients $\|A_{T-i}\|$, $\left\|B_{T-i}^{-1}\right\|$, $\|Q_{12,T-i}\|$ and $\|Q_{21,T-i}\|$. In (A.4) we assume that

$1 - \left\|B_{T-i}^{-1}\right\| \cdot \|K_{T,T-i}\| \cdot \|Q_{12,T-i}\| \neq 0.$

This is obviously true if $\|K_{T,T-i}\| = 0$. We shall show that if the initial condition $\|K_{T,T+1}\| = 0$, then $\left(1 - \left\|B_{T-i}^{-1}\right\| \cdot \|K_{T,T-i}\| \cdot \|Q_{12,T-i}\|\right) > 0$, $i = 1, 2, \ldots, T$. Indeed, consider the difference equation:

$$s_{i+1} = \frac{bd + bas_i}{(1 - bcs_i)}. \tag{A.5}$$

Lemma A.1. *If inequality (5.13) holds, then the difference equation (A.5) has two fixed points*

$$s_1^* = \frac{2bd}{1 - ba + \sqrt{(1-ba)^2 - 4b^2cd}}, \tag{A.6}$$

$$s_2^* = \frac{1 - ba + \sqrt{(1-ba)^2 - 4b^2cd}}{2bc},$$

where s_1^* is a stable fixed point whereas s_2^* is an unstable one. Moreover, under the initial condition $s_0 = 0$ the solution $s_i, i = 1, 2, \ldots,$ is an increasing sequence and converges to s_1^*.

The lemma can be proved by direct calculation. From (5.4)–(5.3) the values a, b, c and d majorize $\|A_{T-i}\|$, $\left\|B_{T-i}^{-1}\right\|$, $\|Q_{12,T-i}\|$ and $\|Q_{21,T-i}\|$, respectively. If we consider Equation (A.2) and inequality (A.5) as initial value problems with the initial conditions $\|K_{T,T+1}\| = 0$ and $s_0 = 0$, then their solutions obviously satisfy the inequality $\|K_{T,T-i}\| \leq s_{i+1}$, $i = 1, 2, \ldots, T$. In other words, $\|K_{T,T-i}\|$ is majorized by s_i. From the last inequality and Lemma A.1 it may be concluded that

$$\|K_{T,T-i}\| \le s_1^*, \quad i = 0, 1, 2, \ldots, T, \quad T \in \mathbb{N}. \tag{A.7}$$

From (A.6), (A.7) and (5.4) it follows that

$$\left\|B_{T-i}^{-1}\right\| \cdot \|K_{T,T-i}\| \cdot \|Q_{12,T-i}\| \le \frac{2b^2dc}{1 - ba + \sqrt{(1-ba)^2 - 4b^2cd}}. \tag{A.8}$$

From (5.13) we see that $2b^2dc < (1-ab)^2/2$. Substituting this inequality into (A.8) gives

$$\begin{aligned} \left\|B_{T-i}^{-1}\right\| \cdot \|K_{T,T-i}\| \cdot \|Q_{12,T-i}\| &\le \frac{(1-ba)^2}{2(1 - ba + \sqrt{(1-ba)^2 - 4b^2cd)}} \\ &< \frac{(1-ba)^2}{2(1-ba)} = \frac{1-ba}{2} < 1, \end{aligned} \tag{A.9}$$

where the last inequality follows from (5.6). This proves Proposition 2.

PROOF OF PROPOSITION 5.4: The assertion of the proposition is true if there exist constants M and r such that $0 < r < 1$ and for $T \in \mathbb{N}$

$$\|K_{T,j} - K_{T+1,j}\| \le Mr^{T+1}, \quad j = 0, 1, 2, \ldots. \tag{A.10}$$

Note now that $K_{T,j}$ $(K_{T+1,j})$ is a solution to the matrix difference equation (5.9) at $i = T - j$ $(i = T + 1 - j)$ with the initial condition $K_{T,T+1} = 0$ $(K_{T+1,T+2} = 0)$. Subtracting (A.3) for $K_{T,T-(i+1)}$ from that for $K_{T+1,T-(i+1)}$, we have

$$\begin{aligned} &K_{T,T-(i+1)} - K_{T+1,T-(i+1)} = B_{T-i}^{-1}(K_{T,T-i)} - K_{T+1,T-i})A_{T-i} \\ &- B_{T-i}^{-1}K_{T,T-i)}Q_{12,T-i}K_{T,T-(i+1)} + B_{T-i}^{-1}K_{T+1,T-i}Q_{12,T-i}K_{T+1,T-(i+1)}. \end{aligned}$$

Adding and subtracting $B_{T-i}^{-1} \cdot K_{T,T-i} \cdot Q_{12,T-i} \cdot K_{T+1,T-(i+1)}$ in the right hand side gives

$$\begin{aligned} &K_{T,T-(i+1)} - K_{T+1,T-(i+1)} = B_{T-i}^{-1}(K_{T,T-i)} - K_{T+1,T-i})A_{T-i} \\ &- B_{T-i}^{-1} \cdot K_{T,T-i} \cdot Q_{12,T-i}(K_{T,T-(i+1)} - K_{T+1,T-(i+1)}) \\ &- B_{T-i}^{-1}(K_{T,T-i} - K_{T+1,T-i})Q_{12,T-i} \cdot K_{T+1,T-(i+1)}. \end{aligned}$$

Rearranging terms yields

$$\begin{aligned} &(I + B_{T-i}^{-1}K_{T,T-i}Q_{12,T-i})(K_{T,T-(i+1)} - K_{T+1,T-(i+1)}) \\ &= B_{T-i}^{-1}(K_{T,T-i} - K_{T+1,T-i})A_{T-i} \\ &- B_{T-i}^{-1}(K_{T,T-i} - K_{T+1,T-i})Q_{12,T-i}K_{T+1,T-(i+1)}. \end{aligned}$$

From Proposition 5.3 it follows that the matrix

$$Z_{T,T-i} = (I + B_{T-i}^{-1}K_{T,T-i}Q_{12,T-i})$$

is invertible, then pre-multiplying the last equation by this matrix yields

$$K_{T,T-(i+1)} - K_{T+1,T-(i+1)} = Z_{T,T-i}^{-1}(B_{T-i}^{-1}(K_{T,T-i} - K_{T+1,T-i})A_{T-i}$$
$$- B_{T-i}^{-1}(K_{T,T-i)} - K_{T+1,T-i})Q_{12,T-i}K_{T+1,T-(i+1)}).$$

Taking the norms, using the norm property and the triangle inequality, we get

$$\begin{aligned} &\|K_{T,T-(i+1)} - K_{T+1,T-(i+1)}\| \\ &\leq \|Z_{T,T-i}^{-1}\| \cdot (\|B_{T-i}^{-1}\| \cdot \|K_{T,T-i} - K_{T+1,T-i}\| \cdot \|A_{T-i}\| \\ &+ \|B_{T-i}^{-1}\| \cdot \|K_{T,T-i)} - K_{T+1,T-i}\| \cdot \|Q_{12,T-i}\| \cdot \|K_{T+1,T-(i+1)}\|). \end{aligned} \tag{A.11}$$

From (5.3) and (A.9) we have

$$\begin{aligned} &\|K_{T,T-(i+1)} - K_{T+1,T-(i+1)}\| \\ &\leq \left(ab + \frac{1-ba}{2}\right) \|Z_{T,T-i}^{-1}\| \cdot \|K_{T,T-i} - K_{T+1,T-i}\| \\ &= \frac{1+ba}{2} \|Z_{T,T-i}^{-1}\| \cdot \|K_{T,T-i} - K_{T+1,T-i}\|. \end{aligned} \tag{A.12}$$

From the norm property and Golub, and Van Loan (1996, Lemma 2.3.3) we get the estimate

$$\|Z_{T,T-i}^{-1}\| = \|(I + B_{T-i}^{-1}K_{T,T-i}Q_{12,T-i})^{-1}\| \leq \frac{1}{1 - \|B_{T-i}^{-1}K_{T,T-i}Q_{12,T-i}\|}$$
$$\leq \frac{1}{1 - \|B_{T-i}^{-1}\| \cdot \|K_{T,T-i}\| \cdot \|Q_{12,T-i}\|}$$

By (A.9), we have

$$\|Z_{T,T-i}^{-1}\| =< \frac{1}{1 - \frac{1-ba}{2}} = \frac{2}{1+ba}$$

Substituting the last inequality into (A.12) gives

$$\|K_{T,T-(i+1)} - K_{T+1,T-(i+1)}\| < \|K_{T,T-i} - K_{T+1,T-i}\|. \tag{A.13}$$

Using (A.16) successively for $i = -1, 0, 1, \ldots, T-1$, and taking into account $K_{T,T+1} = 0$ and $K_{T+1,T+1} = B_{T+2}^{-1}Q_{21,T+2}$ results in

$$\begin{aligned} &\|K_{T,j} - K_{T+1,j}\| < \|K_{T,T+1} - K_{T+1,T+1}\| = \|B_{T+2}^{-1}Q_{21,T+2}\| \\ &\leq \|B_{T+2}^{-1}\| \cdot \|Q_{21,T+2}\| \leq b\|Q_{21,T+2}\|, \quad j = 0, 1, 2, \ldots. \end{aligned} \tag{A.14}$$

Recall that $Q_{21,T}$ depends on the solution to the deterministic problem (3.11), i.e. $Q_{21,T} = Q_{21}\left(x_{T+1}^{(0)}, x_T^{(0)}, z_{T+1}^{(0)}, z_T^{(0)}\right)$. From Hartmann (1982, Corollary 5.1) and differentiability of Q_{21} with respect to the state variables it follows that

$$\|Q_{21,T}\| \leq C(\alpha + \theta)^T, \tag{A.15}$$

where α is the largest eigenvalue modulus of the matrix A from (4.6), C is some constant and θ is arbitrary small positive number. In fact, $\alpha + \theta$ determines the speed of convergence for the deterministic solution to the steady state. Inserting (A.15) into (A.16), we can conclude

$$\|K_{T,j} - K_{T+1,j}\| < bC(\alpha+\theta)^{T+2}, \quad j = 0, 1, 2, \dots \tag{A.16}$$

Denoting $M = bC(\alpha + \theta)$ and $r = \alpha + \theta$ we finally obtain (A.10). This proves the proposition.

B Series expansion for Burnside's model

Substituting (6.5) and (6.6) into (6.2) yields

$$\begin{aligned}
&y^{(0)}\left(x_t^{(0)} + \sigma x_t^{(1)}\right) + \sigma y^{(1)}\left(x_t^{(0)} + \sigma x_t^{(1)}\right) + \sigma^2 y^{(2)}\left(x_t^{(0)} + \sigma x_t^{(1)}\right) + \cdots \\
&= \beta E_t\big\{ \exp\left[\theta\left(x_{t+1}^{(0)} + \sigma x_{t+1}^{(1)}\right)\right]\left[1 + y^{(0)}\left(x_{t+1}^{(0)} + \sigma x_{t+1}^{(1)}\right) + \sigma y^{(1)}\left(x_t^{(0)} + \sigma x_t^{(1)}\right)\right. \\
&\left. + \sigma^2 y^{(2)}\left(x_{t+1}^{(0)} + \sigma x_{t+1}^{(1)}\right) + \cdots\right]\big\}
\end{aligned}$$

Expanding y_t for small σ up to order two gives

$$\begin{aligned}
&y_t^{(0)} + \sigma y_{1,t}^{(0)} x_t^{(1)} + \frac{1}{2}\sigma^2 y_{2,t}^{(0)}\left(x_t^{(1)}\right)^2 + \sigma y^{(1)} x_t^{(0)} + \sigma^2 y_{1,t}^{(1)} x_t^{(1)} + \sigma^2 y_t^{(2)} + \cdots \\
&= \beta E_t \exp(\theta x_{t+1}^{(0)})\left[1 + \sigma\theta x_{t+1}^{(1)} + \frac{1}{2}\left(\sigma\theta x_{t+1}^{(1)}\right)^2 + \cdots\right]\left[1 + y_{t+1}^{(0)} + \sigma y_{1,t+1}^{(0)} x_{t+1}^{(1)}\right. \\
&\left. + \sigma^2\frac{1}{2} y_{2,t+1}^{(0)}\left(x_{t+1}^{(1)}\right)^2 + \sigma y_{t+1}^{(1)} + \sigma^2 y_{1,t+1}^{(1)} x_{t+1}^{(1)} + \sigma^2 y_{t+1}^{(2)} + \cdots\right]
\end{aligned}$$

Collecting the terms of like powers of σ of the last equation, we have

$$\begin{aligned}
&y_t^{(0)} + \sigma\left[(y_{1,t}^{(0)} x_t^{(1)} + y^{(1)} x_t^{(0)}\right] + \sigma^2\left[y^{(2)} + y_{2,t}^{(1)} x_t^{(1)} + \frac{1}{2} y_{2,t}^{(0)}\left(x_t^{(1)}\right)^2\right] + \cdots \\
&= \beta\exp\left(\theta x_{t+1}^{(0)}\right) E_t\big\{(1 + y_{t+1}^{(0)}) + \sigma\left[\theta x_{t+1}^{(1)}(1 + y_{t+1}^{(0)}) + y_{1,t+1}^{(0)} x_{t+1}^{(1)} + y_{t+1}^{(1)}\right] \\
&+ \sigma^2\big[\frac{1}{2}(\theta x_{t+1}^{(1)})^2(1 + y_{t+1}^{(0)}) + y_{t+1}^{(2)} + y_{t+1}^{(1)} x_{t+1}^{(1)} + y_{1,t+1}^{(1)} x_{t+1}^{(1)} + \frac{1}{2} y_{2,t+1}^{(0)}\left(x_{t+1}^{(1)}\right)^2 \\
&+ \theta y_{1,t+1}^{(0)}\left(x_{t+1}^{(1)}\right)^2 + \theta x_{t+1}^{(1)} y_{t+1}^{(1)}\big] + \cdots\big\}
\end{aligned}$$

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