Hesitant fuzzy sets: The Hurwicz approach to the analysis of project evaluation problems

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Abstract

We provide a methodology to perfom an extensive and systematized analysis of problems where experts voice their opinions on the attributes of projects through a hesitant fuzzy decision matrix. A weighted average of specific *parametric* expressions for two tenable indices of satisfaction permits to give a profuse picture of the relative performance of the projects. When the parameter grows, these indices tend to replicate the evaluation by respective simplistic expressions that only depend on the least, resp., the largest, evaluation and the number of evaluations in each cell. This provides the decision-maker with ample information on which he or she can rely in order to make the final decision.

Index Terms

Hesitant fuzzy set; Group decision making; Project evaluation

I. Introduction

It has long been recognized that fuzzy sets (FS) and fuzzy logic provide useful tools for the management of human subjectivity in decision-making contexts (see [1], [2] and [3] as a sample). However in some practical problems, imprecise human knowledge (and especially group knowledge) cannot be suitably represented by fuzzy sets and some generalizations are needed. This was established as early as in Zadeh [4]. In this paper we are interested in a methodology that permits to perfom an extensive and systematized analysis of problems that are better modelled by Torra's [5] hesitant fuzzy sets (HFSs, originally considered by Grattan-Guinness [6]), which incorporate many-valued sets of memberships.

We focus on the following common situation. We need to compare some alternatives or projects, and some experts evaluate their performance with respect to a set of attributes or characteristics. In this context the group knowledge on each project must be naturally represented by set-valued memberships, instead of just membership degrees as in fuzzy sets. Henceforth not only we permit imprecision or vagueness, but also a touch of uncertainty since we do not attach more value to a voiced opinion than to another one. Then the question arises: How do we analyze the problem of prioritizing these projects?

The formal statement of this question refers to hesitant fuzzy decision matrices, i.e., matrices whose cells contain hesitant fuzzy elements (HFEs). These HFEs collect the opinions voiced by the experts on each attribute of the succesive projects. In our description rows are associated with projects and can be assimilated with HFSs. Thus we want to compare rows in these matrices on the basis of their relative performance (as alternatives or projects).

The problem posed above has received attention from varios authors recently. Xia and Xu [7] and Farhadinia [8] propose to use aggregating operators in order to associate a single HFE with each project. Then score functions give rankings of the aggregate HFEs. Xu and Xia [9] rank the projects according to a direct appeal to distances. Finally, Zhou and Li [10] design a lexicographic ranking that refines the proposal in [7].

In order to make a broader analysis of these decision-making situations we draw inspiration from two sources. In the first place, we observe that the relative fitness of the projects (i.e., of their associated HFSs)

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can be estimated either by the ‘distance’ to an ideal HFS or the ‘similarity’ to an anti-ideal HFS. Here we suggest respective novel *parametric* indicators for such proxies that incorporate the relative importance of the attributes through *ex-ante* allocations of weights. Their asymptotic behavior, i.e., the role of the parameter, is disclosed: when the parameter goes to infinity these indicators tend to provide an evaluation by respective simplistic expressions that only depend on the least, resp., the largest, evaluation and the number of evaluations on each attribute. In the second place, we draw inspiration from the Hurwicz approach to decision making under uncertainty (cf., e.g., Luce and Raiffa [11]), which advocates for the combined use of ‘best and worst outcomes’ to assess the value of uncertain decisions. Thus the Hurwicz approach permits us to combine our two plausible parametric indices by their weighted sums, which includes both indices as extreme cases. Their limit behavior replicates the case of the original indicators. Now for each project we obtain a segment instead of a single number, which can provide a richer analysis of the decision problem. Obviously, for any choice of the averaging aggregator a concrete ranking of projects arises.

This paper is organized as follows. Section II establishes some basic definitions. Section III introduces our proposals for ranking hesitant fuzzy sets, as well as results concerning the asymptotic behavior of our indices. In Section IV we put in practice the methodology that permits to study the hierarchization of projects characterized by hesistant fuzzy sets. We also visualize their asymptotic behavior in a fully developed example. We conclude in Section V.

II. Notation and definitions

For any set A, $\mathcal{P}^*(A)$ denotes the set of non-empty subsets of A, and $\mathcal{F}^*(A)$ denotes the set of non-empty finite subsets of A.

Definition 1: A hesitant fuzzy element (HFE) is a non-empty, finite subset of $[0, 1]$. The set of HFEs is denoted by $\mathcal{F}^*([0, 1])$.

Henceforth we refer to X, a fixed set of alternatives.

Definition 2: A *hesitant fuzzy set* (HFS) on X is a function from X to $\mathcal{P}^*([0, 1])$. A *typical hesitant fuzzy set* on X is a function from X to $\mathcal{F}^*([0, 1])$. **HFS**(X) means the set of HFSs on X, and the set of typical HFSs on X is denoted by HFS(X).

Unless otherwise stated, HFSs are assumed to be typical.

From a formal point of view, a (typical) HFS is a subset $M \subseteq X \times \mathcal{F}^*([0, 1])$ such that for each $x \in X$, there is exactly one element $h_M(x) \in \mathcal{F}^*([0, 1])$ such that $(x, h_M(x)) \in M$.

Each HFS on X defines a set of membership values for each element of X, and in the case that the HFS is typical such set is always finite. HFEs represent the set of possible membership values of a typical hesitant fuzzy set at an alternative.

By restricting ourselves to either $\mathcal{F}^*([0, 1])$ or $\mathcal{P}^*([0, 1])$, i.e., non-empty HFEs, we disregard ‘nonsense elements’ in each HFS: on each alternative, at least one assessment must be made.

From a practical point of view, the hesitant fuzzy set M can be represented as $M = \{(x, h_M(x)) \mid x \in X\}$. For example, following Torra [5] we define

$$M^* = \{(x, 1) \mid x \in X\}$$

as the *ideal* or *full* HFS on X, and

$$M^- = \{(x, 0) \mid x \in X\}$$

as the *anti-ideal* or *empty* HFS on X.

Clearly, when all HFEs involved in the definition of an HFS on X are singletons we can identify such HFS with a fuzzy set (FS) on X. That is to say,

HFEs of the form

$$M = \{(x, h_M(x)) \mid x \in X, h_M(x) = \{M_x\}\}$$

can be identified with the FS on X whose membership function is

$$\begin{array}{rcl} \mu_M : X & \longrightarrow & [0,1] \\ x & & \mu_M(x) = M_x \end{array}$$

For each typical hesitant fuzzy set M on X, we denote

$$h_M(x) = \{h_M^1(x), ..., h_M^{l_M(x)}(x)\}$$

where indexes are chosen so that $h_M^1(x) < \ldots < h_M^{l_M(x)}(x)$. In particular, the cardinality of the HFE $h_M(x)$ is $l_M(x) = |h_M(x)|$. Observe that if the set of membership values at an element is not finite (i.e., if we refer to a non-typical HFS) then such arrangement in increasing order cannot be made in general. In any case, because $h_M(x)$ is a set, repetitions are excluded by definition.

Now we proceed to formalize the general concepts of distance and similarity between HFSs.

Definition 3: A distance measure between HFSs on X is a function $d : \mathbf{HFS}(X) \longrightarrow [0,1]$ that satisfies the following properties: for every $M, N \in \mathbf{HFS}(X)$,

1) $0 \leqslant d(M,N) \leqslant 1$;
2) $d(M,N) = 0$ if and only if $M = N$;
3) $d(M,N) = d(N,M)$.

Definition 4: A similarity measure between HFSs on X is a function $s : \mathbf{HFS}(X) \longrightarrow [0,1]$ that satisfies the following properties: for every $M, N \in \mathbf{HFS}(X)$,

1) $0 \leqslant s(M,N) \leqslant 1$;
2) $s(M,N) = 1$ if and only if $M = N$;
3) $s(M,N) = s(N,M)$.

There are similitudes between the latter concepts. When d is a distance measure between HFSs on X, the expression $s = 1 - d$ defines a similarity measure between HFSs on X. Conversely, when s is a similarity measure between HFSs on X, the expression $d = 1 - s$ defines a distance measure between HFSs on X.

Example 1: A very simple example of a distance measure between HFSs on X is the trivial distance function $d_t : \mathbf{HFS}(X) \longrightarrow [0,1]$ defined as follows: for every $M, N \in \mathbf{HFS}(X)$,

$$d_t(M,N) = \begin{cases} 0 & \text{if } M = N \\ 1 & \text{otherwise} \end{cases}$$

This is a distance that does not discriminate among unequal HFSs.

Of course, one can also produce a very simple example of a similarity measure between HFSs on X that we denote by $s_t : \mathbf{HFS}(X) \longrightarrow [0,1]$ and is defined as follows: for every $M, N \in \mathbf{HFS}(X)$,

$$s_t(M,N) = \begin{cases} 1 & \text{if } M = N \\ 0 & \text{otherwise} \end{cases}$$

This trivial similarity function does not discriminate among unequal HFSs either.

III. Ranking typical HFSs: the Hurwicz approach

A. Statement of the problem

In this Section we consider the analysis of the following problem. There are m alternatives or projects whose performance with regard to n criteria or attributes is evaluated by a team of experts (in a range from 0 to 1). Each expert can be hesitant on the performance of the projects, therefore he or she can emit any *finite* number of evaluations to express his or her doubts. For each project, all evaluations by the experts on each criteria are collected into a set of values. This presumes anonymity of the experts: all opinions are equally considered in this process. Formally, this produces an HFS associated with the project: for each attribute, a finite set of values in $[0,1]$ is given. We face a problem under complete uncertainty: the importance of each particular appraisal is totally unknown.

The opinions of the experts can be captured by a hesitant fuzzy decision matrix (HFDM), i.e., an $m \times n$ matrix whose cells contain HFEs, in such way that its rows trivially define HFSs (one for each project). Columns correspond to respective evaluations of the projects by fixed criteria.

Suppose that we need to rank or prioritize the projects. The problem arises: How do we analyze the decision problem posed?

The next example illustrates the notation and terminology that we have presented.

Example 2: Two experts express their opinions on two projects A_1, A_2. Their assessments on each project concern two attributes P_1, P_2. They are allowed to be hesitant. Their opinions are the following.
Opinion of the first agent on A_1: assessment 0.7 for attribute P_1 and assessment 0.8 for attribute P_2.
Opinion of the first agent on A_2: assessment either 0.3 or 0.4 for attribute P_1 and assessment either 0.8 or 0.9 for attribute P_2.
Opinion of the second agent on A_1: assessment either 0.6 or 0.7 for attribute P_1 and assessment either 0.75 or 0.8 for attribute P_2.
Opinion of the second agent on A_2: assessment either 0.4 or 0.5 for attribute P_1 and assessment either 0.7 or 0.75 for attribute P_2.
Their assessments are collected by a HFDM whose values are given in Table I.
Let $X = \{P_1, P_2\}$. Then project A_1 is characterized by the typical HFS on X defined by

$$\begin{array}{rcl} M_1 : X & \longrightarrow & \mathcal{F}^*([0,1]) \\ P_1 & & h_{M_1}(P_1) = \{0.6, 0.7\} \\ P_2 & & h_{M_1}(P_2) = \{0.75, 0.8\} \end{array}$$

and project A_2 is characterized by the typical HFS on X defined by

$$\begin{array}{rcl} M_2 : X & \longrightarrow & \mathcal{F}^*([0,1]) \\ P_1 & & h_{M_2}(P_1) = \{0.3, 0.4, 0.5\} \\ P_2 & & h_{M_2}(P_2) = \{0.7, 0.75, 0.8, 0.9\} \end{array}$$

Furthermore, $l_{M_1}(P_1) = l_{M_1}(P_2) = 2$, $l_{M_2}(P_1) = 3$, and $l_{M_2}(P_2) = 4$. And $h^1_{M_1}(P_1) = 0.6$, $h^2_{M_1}(P_1) = 0.7$, $h^1_{M_1}(P_2) = 0.75$, $h^2_{M_1}(P_2) = 0.8$, $h^1_{M_2}(P_1) = 0.3$, $h^2_{M_2}(P_1) = 0.4$, $h^3_{M_2}(P_1) = 0.5$, $h^1_{M_2}(P_2) = 0.7$, $h^2_{M_2}(P_2) = 0.75$, $h^3_{M_2}(P_2) = 0.8$, $h^4_{M_2}(P_2) = 0.9$.

	P_1	P_2
A_1	$\{0.6, 0.7\}$	$\{0.75, 0.8\}$
A_2	$\{0.3, 0.4, 0.5\}$	$\{0.7, 0.75, 0.8, 0.9\}$

TABLE I: Hesitant fuzzy decision matrix

B. Analysis of the problem: the Hurwicz approach

Several contributions have dealt with the problem posed above. Xia and Xu [7] start by using aggregating operators in order to associate an HFE with each project, and then use a score function to rank them. Farhadinia [8] proposes a variation with a different score function. Xu and Xia [9] proceed in a more direct way: they rank the projects according to their distance to the ideal HFS. Finally, Zhou and Li [10] do not produce evaluations of projects but give a lexicographic ranking that refines the proposal in [7].

Our proposal intends to make a broader analysis. It has two sources of inspiration.

Firstly, we draw inspiration from the approach in Xu and Xia [9, Example 1]. In order to analyze the relative performance of the projects (or of the HFSs that characterize them) we build on two relevant indicators, namely the 'distance' to the ideal HFS and the 'similarity' to the anti-ideal HFS. Both seem tenable indices of fitness for an HFS although of course, many distance and similarity indices can be used in analogy with the many proposals of distances between HFSs in the literature. In order to avoid confusions here we develop the model with a single concrete specification, namely, Definition 5 below that slightly echoes the use of the generalized hesitant weighted distance [9, Eq. (11)]. We leave the details of possible variations to the interested reader, e.g., specifications that replace our indicators in Definition 5 by expressions inspired on the generalized hesitant weighted Hausdorff distance or the generalized hybrid hesitant weighted distance.

We assume that each of the attributes has associated a weight w_i such that $w_1 + \ldots + w_n = 1$. Weights are indicative of the relative importance of the attributes, hence a zero weight would mean a dispensable criteria that can be omitted in the analysis. This means that we do not lose generality if we assume $w_i > 0$ for each i henceforth.

Definition 5: Given $\lambda > 0$ and $\mathbf{w} = (w_1, \ldots, w_n)$ with $w_i > 0$ for each i and $w_1 + \ldots + w_n = 1$, the *λ-adjusted hesitant weighted distance to the ideal HFS* is defined as

$$\Delta_{ahw}^{\lambda,\mathbf{w}}(M) = \sum_{i=1}^{n} \frac{w_i}{l_M(x_i)} \left(\sum_{j=1}^{l_M(x_i)} (1 - h_M^j(x_i))^{\lambda} \right)^{\frac{1}{\lambda}}$$

for each $M \in \text{HFS}(M)$ and the *λ-generalized hesitant weighted similarity to the anti-ideal HFS* is defined as

$$\Sigma_{ahw}^{\lambda,\mathbf{w}}(M) = 1 - \sum_{i=1}^{n} \frac{w_i}{l_M(x_i)} \left(\sum_{j=1}^{l_M(x_i)} (h_M^j(x_i))^{\lambda} \right)^{\frac{1}{\lambda}}$$

for each $M \in \text{HFS}(M)$

Observe $\Delta_{ahw}^{\lambda,\mathbf{w}}(M) = 0$ if and only if $M = M^*$, and $\Sigma_{ahw}^{\lambda,\mathbf{w}}(M) = 0$ if and only if $M = M^*$. Therefore both indicators share the characteristic that the higher the evaluation of a project, the worse its performance. In the case of [9, Example 1], only the analog of the first indicator is used. In fact both indicators coincide in the focal instance $\lambda = 1$:

Lemma 1: If $\lambda = 1$ and $\mathbf{w} = (w_1, \ldots, w_n)$ verifies $w_i > 0$ for each i and $w_1 + \ldots + w_n = 1$, then $\Delta_{ahw}^{\lambda,\mathbf{w}}(M) = \Sigma_{ahw}^{\lambda,\mathbf{w}}(M)$ for every $M \in$HFS(X).

Proof: For every $M \in$HFS(X),

$$\Delta_{ahw}^{1,\mathbf{w}}(M) = \sum_{i=1}^{n} \frac{w_i}{l_M(x_i)} \sum_{j=1}^{l_M(x_i)} (1 - h_M^j(x_i)) =$$

$$= \sum_{i=1}^{n} \frac{w_i}{l_M(x_i)} \left(l_M(x_i) - \sum_{j=1}^{l_M(x_i)} h_M^j(x_i) \right) =$$

$$= \sum_{i=1}^{n} w_i - \sum_{i=1}^{n} \left(\frac{w_i}{l_M(x_i)} \sum_{j=1}^{l_M(x_i)} h_M^j(x_i) \right) =$$

$$= 1 - \sum_{i=1}^{n} \frac{w_i}{l_M(x_i)} \left(\sum_{j=1}^{l_M(x_i)} h_M^j(x_i) \right) = \Sigma_{ahw}^{1,\mathbf{w}}(M)$$

Secondly, we draw inspiration from the Hurwicz approach to decision making under uncertainty, which is very popular in Economics since its introduction in 1950 (cf., e.g., Luce and Raiffa [11]). In spirit it postulates the use of weighted sums of best and worst outcomes to assess the value of decisions. We can adapt it to the structure of our problem. In order to evaluate the acceptability of an HFS, both the ‘distance’ to the ideal HFS and the ‘similarity’ to the anti-ideal HFS are potentially useful. Instead of discarding one indicator in the benefit of the other, the Hurwicz approach permits us to combine both plausible indices. To be precise, in order to evaluate the hesitant fuzzy set M we define a value

$$\Lambda_{\alpha}^{\lambda,\mathbf{w}}(M) = \alpha\Delta_{ahw}^{\lambda,\mathbf{w}}(M) + (1-\alpha)\Sigma_{ahw}^{\lambda,\mathbf{w}}(M)$$

which is a weighted sum of the distance to the ideal HFS and the similarity to the anti-ideal HFS; the weight $\alpha \in [0,1]$ can be conceived of as an index of 'enviness' because when $\alpha = 1$, the indicator coincides with $\Delta_{ahw}^{\lambda,\mathbf{w}}$, i.e., with the selected distance to the ideal HFS. When $\alpha = 0$, the indicator coincides with $\Sigma_{ahw}^{\lambda,\mathbf{w}}$, i.e., with the selected similarity to the anti-ideal HFS. The higher the evaluation of an HFS by $\Lambda_{\alpha}^{\lambda,\mathbf{w}}$, the worse its suitability. Therefore for each HFS we obtain a segment (as a function of α) instead of a single number, which can provide a more extensive analysis of the decision situation to the decision-maker. Obviously, for any fixed α a ranking of HFSs arises, although in general this ranking is dependent on the choice of the parameter. The decision maker can observe from a single drawing for which values of the parameter a given alternative is ranked first.

Remark 1: As a consequence of Lemma 1, when $\lambda = 1$ a unique ranking is obtained independently of the value of the parameter α because when $\mathbf{w} = (w_1, \ldots, w_n)$ verifies $w_i > 0$ for each i and $w_1 + \ldots + w_n = 1$, then $\Lambda_{\alpha}^{1,\mathbf{w}}(M) = \Delta_{ahw}^{1,\mathbf{w}}(M) = \Sigma_{ahw}^{1,\mathbf{w}}(M)$ for every $M \in$HFS(X).

C. Asymptotic behavior of the indicators: interpretations

We proceed to check that using our indicators with 'large' values of the λ parameter produces evaluations that are increasingly similar to those that derive from very simple indicators. Such indicators are crude evaluations that only rely on the number of different evaluations for each attribute and either the maximum or the minimum of such respective values. To this purpose let us define

$$A(M) = \sum_{i=1}^{n} \frac{w_i}{l_M(x_i)} \max_{j=1,\ldots,l_M(x_i)} (1 - h_M^j(x_i)) =$$

$$= \sum_{i=1}^{n} \frac{w_i}{l_M(x_i)} (1 - \min_{j=1,\ldots,l_M(x_i)} h_M^j(x_i))$$

for each $M \in$ HFS(M) and

$$B(M) = 1 - \sum_{i=1}^{n} \frac{w_i}{l_M(x_i)} \max_{j=1,\ldots,l_M(x_i)} h_M^j(x_i)$$

for each $M \in$ HFS(M). Then our claim boils down to the following statement:

Proposition 1: For every $M \in \text{HFS}(X)$,

$$\lim_{\lambda\to\infty} \Delta^{\lambda,\mathbf{w}}_{ahw}(M) = A(M)$$

and also

$$\lim_{\lambda\to\infty} \Sigma^{\lambda,\mathbf{w}}_{ahw}(M) = B(M)$$

Therefore,

$$\lim_{\lambda\to\infty} \Lambda^{\lambda,\mathbf{w}}_{\alpha}(M) = \alpha A(M) + (1-\alpha)B(M)$$

for every $M \in$HFS(X) and $\alpha \in [0,1]$.

Proof: We appeal to some basic properties of the l_p norms on any $\mathbb{R}^t$, defined as $||(x_1,\cdots,x_t)||_p = \left(\sum_{j=1}^t |x_j|^p\right)^{\frac{1}{p}}$ for every $p \geqslant 1$.[1] We first observe that when $M \in$HFS(X),

$$\Delta^{\lambda,\mathbf{w}}_{ahw}(M) =$$

$$= \sum_{i=1}^n \frac{w_i}{l_M(x_i)} ||(1, \overset{l_M(x_i)}{\cdots\cdots}, 1) - (h^1_M(x_i), \cdots, h^{l_M(x_i)}_M(x_i))||_\lambda$$

Now it is easy to deduce the consequence $\lim_{\lambda\to\infty} \Delta^{\lambda,\mathbf{w}}_{ahw}(M) = A(M)$: for each $i =, ..., n$, when λ approaches infinity the l_λ norm on $\mathbb{R}^{l_M(x_i)}$ approaches the l_∞ or maximum norm defined as $||(x_1,\cdots,x_t)||_\infty = \max(|x_1|,\cdots,|x_t|)$ (cf., Fabala et al. [13, Exercise 1.9]).

The proof of the second claim is almost identical to the one above. The final statement can be trivially derived from the former ones.

An intuitive interpretation is in order. $\Delta^{\lambda,\mathbf{w}}_{ahw}(M)$ refers to similarity to an ideal HFS, and a proxy of that idea is given by the worst evaluation on each attribute, which is the information from which $A(M)$ is designed. Similarly, $\Sigma^{\lambda,\mathbf{w}}_{ahw}(M)$ refers to similarity to an anti-ideal HFS, and a proxy of that idea is given by the best evaluation on each attribute, which is the information on which $B(M)$ is designed.

IV. Analysis of the hierarchization of projects

In order to illustrate our proposal for the analysis of the hierarchization of projects characterized by hesistant fuzzy decision matrices, we build on the discussion of Xu and Xia [9, Example 1], which is adapted from Kahraman and Kaya [14].

We need to compare five energy projects, denoted by alternatives A_i ($i = 1, \ldots, 5$). Some experts evaluate the performance of the five alternatives with respect to four attributes: P1 (technological), P2 (environmental), P3 (socio-political) and P4 (economic). The attribute weight vector is $\mathbf{w} = (0.15, 0.3, 0.2, 0.35)$. The hesitant fuzzy decision matrix that arises is immediate from the data in Table II.

In order to analyze the relative performance of the projects we first need to produce the 'distance' to the ideal HFS and the 'similarity' to the anti-ideal HFS of each project, as measured by a realization of λ in Definition 5. Tables III, IV, and V show the results of these computations for different values of the parameter, namely, $\lambda = 1, 2, 20$. The tables also show the respective values of the compromise index $\Lambda^{\lambda,\mathbf{w}}_{\alpha}$ as a function of $\alpha \in [0,1]$. As proven in Lemma 1, the evaluations when $\lambda = 1$ are coincident hence the conclusion $A5 \succ A3 \succ A4 \succ A1 \succ A2$ irrespective of which compromise index and value of α we use in that case. The consequences of Lemma 1 are apparent in Figure 1.

In order to compare the projects under a given choice of λ, the corresponding five segments $\Lambda^{\lambda,\mathbf{w}}_{\alpha}$ can be drawn as in Figure 2. As mentioned above, when $\lambda = 1$ these segments are horizontal. Futhermore,

[1]When $0 < p < 1$ such expression does not define a norm, although $||(x_1,\cdots,x_t)||_p = \sum_{j=1}^t |x_j|^p$ does (Maddox [12, p. 31]).

because of the asymptotic behavior of the indices the conclusions from these drawings when λ grows are increasingly similar to the conclusion from the comparison of the segments

$$I_\alpha(A_i) = \alpha A(A_i) + (1-\alpha)B(A_i)$$

These segments –one for each project– are uniquely determined by the hesitant fuzzy decision matrix. Both $I_\alpha(A_i)$, $A(A_i)$ and $B(A_i)$ are shown in Table VI and the $I_\alpha(A_i)$ segments are plotted in Figure 3. We recall that they only depend on the least and the largest evaluation and the number of evaluations in each cell.

With respect to the asymptotic behavior it can be checked that the evaluations of the projects by the $\Delta^{\lambda,\mathbf{w}}_{ahw}$ indicator are identical to the respective evaluations by A when $\lambda = 55$, and the evaluations of the projects by the $\Sigma^{\lambda,\mathbf{w}}_{ahw}$ indicator are identical to the respective evaluations by B when $\lambda = 75$ (with a 10^{-6} precision).

	P_1	P_2	P_3	P_4
A_1	$\{0.5, 0.4, 0.3\}$	$\{0.9, 0.8, 0.7, 0.1\}$	$\{0.5, 0.4, 0.2\}$	$\{0.9, 0.6, 0.5, 0.3\}$
A_2	$\{0.5, 0.3\}$	$\{0.9, 0.7, 0.6, 0.5, 0.2\}$	$\{0.8, 0.6, 0.5, 0.1\}$	$\{0.7, 0.4, 0.3\}$
A_3	$\{0.7, 0.6\}$	$\{0.9, 0.6\}$	$\{0.7, 0.5, 0.3\}$	$\{0.6, 0.4\}$
A_4	$\{0.8, 0.7, 0.4, 0.3\}$	$\{0.7, 0.4, 0.2\}$	$\{0.8, 0.1\}$	$\{0.9, 0.8, 0.6\}$
A_5	$\{0.9, 0.7, 0.6, 0.3, 0.1\}$	$\{0.8, 0.7, 0.6, 0.4\}$	$\{0.9, 0.8, 0.7\}$	$\{0.9, 0.7, 0.6, 0.3\}$

TABLE II: Hesitant fuzzy decision matrix

Index	A_1	A_2	A_3	A_4	A_5
$\Delta^{\lambda,\mathbf{w}}_{ahw} = \Sigma^{\lambda,\mathbf{w}}_{ahw} = \Lambda^{\lambda,\mathbf{w}}_{\alpha}$	0.477917	0.502667	0.4025	0.429167	0.35575

TABLE III: Elements for the analysis when $\lambda = 1$

Index	A_1	A_2	A_3	A_4	A_5
$\Delta^{\lambda,\mathbf{w}}_{ahw}$	0.283547	0.298098	0.286277	0.287185	0.198659
$\Sigma^{\lambda,\mathbf{w}}_{ahw}$	0.70767	0.715997	0.581673	0.6353	0.655254
$\Lambda^{\lambda,\mathbf{w}}_{\alpha}$	$0.70767 - \alpha 0.424123$	$0.715997 - \alpha 0.417899$	$0.581673 - \alpha 0.295396$	$0.6353 - \alpha 0.348115$	$0.655254 - \alpha 0.456595$

TABLE IV: Elements for the analysis when $\lambda = 2$

Index	A_1	A_2	A_3	A_4	A_5
$\Delta^{\lambda,\mathbf{w}}_{ahw}$	0.217176	0.227353	0.241676	0.242988	0.15326
$\Sigma^{\lambda,\mathbf{w}}_{ahw}$	0.795055	0.786808	0.660709	0.714421	0.773713
$\Lambda^{\lambda,\mathbf{w}}_{\alpha}$	$0.795055 - \alpha 0.577879$	$0.786808 - \alpha 0.559455$	$0.660709 - \alpha 0.419033$	$0.714421 - \alpha 0.471433$	$0.773713 - \alpha 0.620453$

TABLE V: Elements for the analysis when $\lambda = 20$

Index	A_1	A_2	A_3	A_4	A_5
A	0.217083	0.227167	0.241667	0.242917	0.15325
B	0.795417	0.786833	0.660833	0.715	0.77425
I_α	$0.795417 - \alpha 0.578333$	$0.786833 - \alpha 0.559667$	$0.660833 - \alpha 0.419167$	$0.715 - \alpha 0.472083$	$0.77425 - \alpha 0.621$

TABLE VI: Limit values of the indicators. I_α denotes $\alpha A + (1-\alpha)B$.

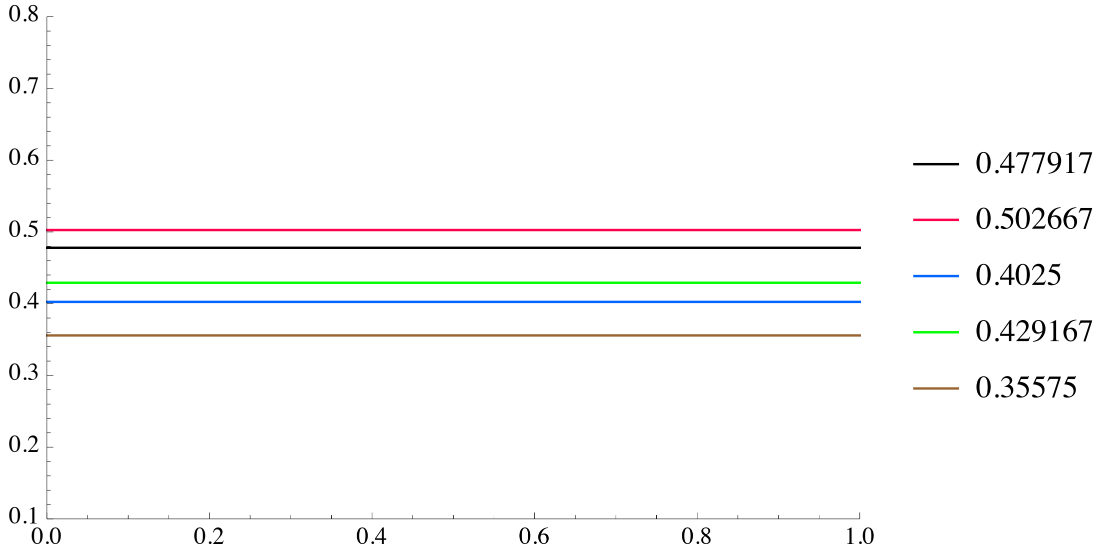
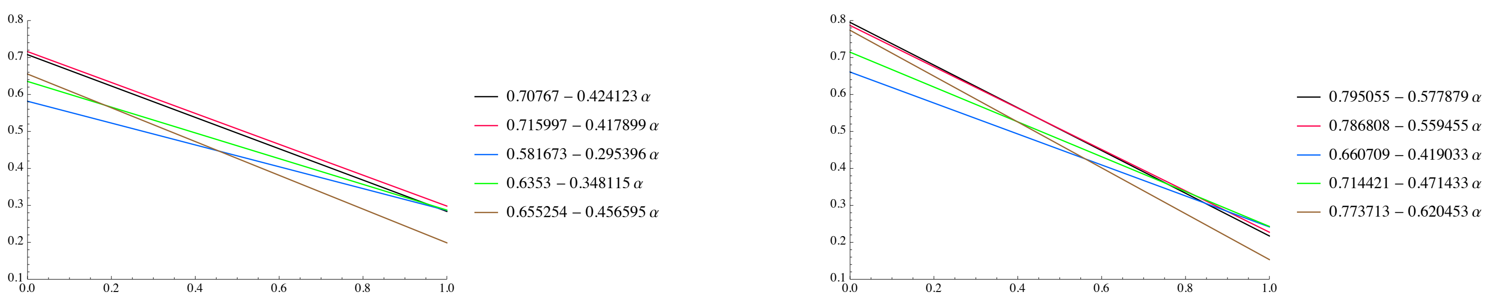


Fig. 1: A graphical display of the indicators $\Delta^{1,\mathbf{w}}_{ahw} = \Sigma^{1,\mathbf{w}}_{ahw} = \Lambda^{1,\mathbf{w}}_{\alpha}$



(a) Case $\lambda = 2$

(b) Case $\lambda = 20$

Fig. 2: A graphical display of two indicators $\Lambda^{\lambda,\mathbf{w}}_{\alpha}$

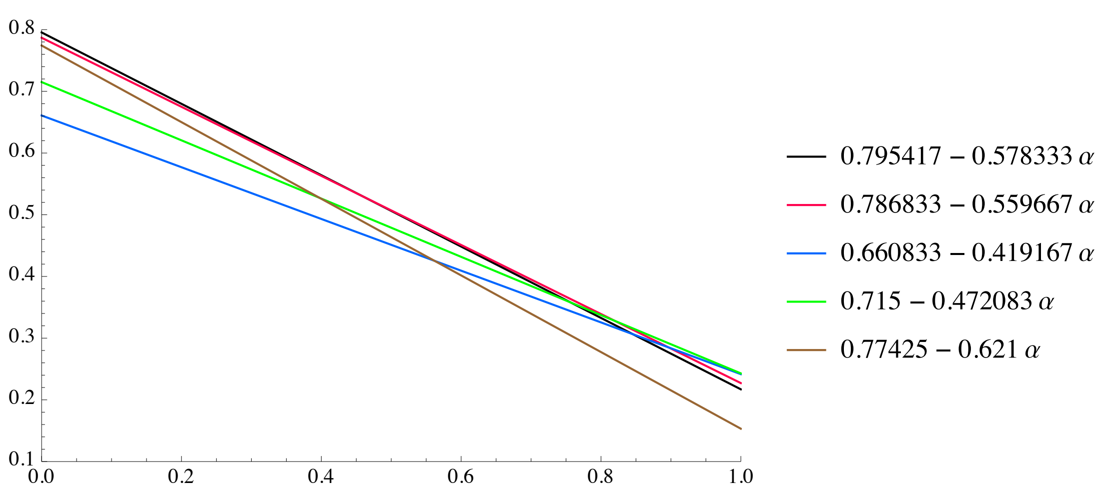


Fig. 3: A graphical display of the indicators $I_\alpha = \alpha A + (1-\alpha)B$

V. Conclusion

We have provided a methodology that permits to perfom an extensive and systematized analysis of problems with a precise specification: experts voice their opinions on the attributes of projects through a hesitant fuzzy decision matrix, that is, an $m \times n$ matrix whose cells contain HFEs. Under a specific *parametric* expression for two reasonable indices of satisfaction, a weighted average permits to give a profuse picture of the relative performance of the projects. The role of the parameter has been disclosed: when it grows the two indices tend to replicate the evaluation by respective simplistic expressions that only depend on the least, resp., the largest, evaluation and the number of evaluations in each cell. All these elements permit the analyst to provide the decision-maker with ample information on which he or she can rely in order to make the final decision.

With respect to related future lines of research, we already mentioned that replacing our indicators with other potentially useful expressions gives direct variations of our proposal. Furthermore, the analysis of the analog problem under hesitant fuzzy linguistic information comes to mind as another natural possibility (cf., e.g., hesitant fuzzy linguistic term sets introduced by Rodríguez, Martínez and Herrera [15], see also Zhu and Xu [16]).

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