# General Equilibrium with Endogenous Trading Constraints

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# GENERAL EQUILIBRIUM WITH ENDOGENOUS TRADING CONSTRAINTS

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Abstract. We build a general equilibrium model where agents are subject to endogenous trading constraints, making the access to financial trade dependent on prices and consumption decisions. Besides, our framework is compatible with the existence of endogenous financial segmentation and credit markets' exclusion. Two results of equilibrium existence are shown. In the first one, we assume individuals can *super-replicate* financial payments buying durable commodities and investing in assets that give liquidity to all agents. In the second result, under strict monotonicity of preferences, we suppose there are agents that may compensate with increments in present demand the losses of well-being generated by reductions of future consumption.

Keywords. Incomplete Markets; General Equilibrium; Endogenous Trading Constraints.

JEL Classification. D52, D54.

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## 1. Introduction

The differentiated access to commodity or asset markets endogenously emerges due to regulatory or institutional considerations. As a consequence, several kinds of trading restrictions are observed in financial markets: margin calls, collateral requirements, short-sale constraints, consumption quotas or income-based access to funding, among others. With the aim of understanding the effects of those restrictions in competitive markets, a vast literature of general equilibrium has been developed. That research has given consideration to models where financial trade is restricted by fixed, price-dependent, or consumption-dependent portfolio constraints. Nevertheless, channels connecting prices with both portfolio constraints and consumption possibilities, have not thoroughly been addressed by the literature. The objective of this paper is to contribute in this direction.

The existence of competitive equilibria was deeply studied in incomplete markets models where agents are subject to exogenous portfolio constraints. The case of portfolio restrictions determined by linear equality constraints is addressed by Balasko, Cass and Siconolfi (1990) for nominal assets, and by Polemarchakis and Siconolfi (1997) for real assets. When portfolio restrictions are determined by convex and closed sets containing zero, the case of nominal or numéraire assets is studied by Cass (1984, 2006), Siconolfi (1987), Cass, Siconolfi and Villanacci (2001), Martins-da-Rocha and Triki (2005), Won and Hahn (2007, 2012), Aouani and Cornet (2009, 2011), and Cornet and Gopalan (2010). In the same context, the case of real assets is analyzed by Radner (1972), Angeloni and Cornet (2006), and Aouani and Cornet (2011). In general terms, these authors prove equilibrium existence under non-redundancy hypotheses over financial structures and/or financial survival requirements. Under these assumptions, individuals' allocations and asset prices can be endogenously bounded without inducing frictions in the model.

There are also several results that include price-dependent portfolio constraints in nominal or real assets markets. These models assume that financial constraints are determined by a finite number of inequalities, and use differentiable techniques to ensure the existence of equilibrium and to analyze its stability and local-uniqueness. In this context, equilibrium existence is addressed by Carosi, Gori and Villanacci (2009) for numéraire asset markets with portfolio constraints, by Gori, Pireddu and Villanacci (2013) for numéraire and real asset markets with borrowing constraints, and by Hoelle, Pireddu, and Villanacci (2012) for real asset markets with wealth-dependent credit limits.

In addition, Seghir and Torres-Martínez (2011) propose a model where trading constraints restrict the access to debt in terms of first-period consumption. Financial survival conditions are not required, and the relationship between financial access and individual consumption allows to include financial practices as collateralized borrowing. In order to prove equilibrium existence, they assume individuals may compensate with increments in present demand the losses on well-being generated by reductions of future consumption.

In this paper, we analyze the existence of equilibria in incomplete financial markets when agents are subject to price-dependent trading constraints that affect the access to commodities and financial contracts. Furthermore, we make the financial segmentation and exclusion of debt markets compatible with the existence of a competitive equilibrium. Our approach is general enough to be compatible with incomplete market economies where there exist wealth-dependent financial access, investment-dependent credit access, borrowing constraints precluding bankruptcy, security exchanges, commodity options with deposit requirements, or assets that are backed by financial collateral.

We consider a two-period economy with uncertainty about the realization of a state of nature in the second period. There is a finite number of agents that are able to smooth consumption across states of nature by trading financial contracts. Moreover, the access to physical and financial markets is determined by price-dependent trading constraints.

Two results of equilibrium existence are developed. First, we prove that a competitive equilibrium exists when individuals can *super-replicate* financial payments buying durable commodities and investing in assets that give liquidity to all agents (Theorem 1). As particular cases, we obtain results of equilibrium existence in markets where financial survival conditions hold or where assets are backed by physical collateral. Secondly, under strict monotonicity of preferences, we prove that there is an equilibrium when there are agents that can increase their present demand to compensate any loss of utility generated by a reduction on future consumption (Theorem 2). In particular, we extend the model and the results of Seghir and Torres-Martínez (2011) to be able to allow price-dependent trading constraints that affect the access to both debt and investment.

Our model is described in the next section. In Sections 3-5 we characterize the assumptions on trading constraints. Section 6 gives examples that illustrate our trading rules, and Section 7 is devoted to state our main results. The proofs of our results are given in appendices.

## 2. Model

We consider a two-period economy with uncertainty about the realization of a state of nature in the second period, which belongs to a finite set $S$. Let $\mathcal{S} = \{0\} \cup S$ be the set of states of nature in the economy, where $s = 0$ denotes the unique state at the first period.

There is a finite set $\mathcal{L}$ of perfectly divisible commodities, which are subject to transformation between periods and that can be traded in spot markets at prices $p = (p_s)_{s \in \mathcal{S}} \in \mathbb{R}_+^{\mathcal{L} \times \mathcal{S}}$. We model the transformation of commodities between periods by linear technologies $(Y_s)_{s \in S}$. Thus, a bundle $y \in \mathbb{R}_+^{\mathcal{L}}$ demanded at the first period is transformed, after its consumption and the realization of a state of nature $s \in S$, into the bundle $Y_s y \in \mathbb{R}_+^{\mathcal{L}}$.

There is a finite set $\mathcal{J}$ of financial contracts available for trade at the first period that make promises contingent to the realization of uncertainty. Let $q = (q_j)_{j\in\mathcal{J}} \in \mathbb{R}^{\mathcal{J}}_+$ be the vector of asset prices and denote by $R(p) = (R_{s,j}(p_s))_{(s,j)\in S\times\mathcal{J}} \in \mathbb{R}^{S\times\mathcal{J}}_+$ the vector of assets' payments.[1]

For notation convenience, let $\mathbb{P} := \mathbb{R}^{\mathcal{L}\times\mathcal{S}}_+ \times \mathbb{R}^{\mathcal{J}}_+$ be the space of commodity and asset prices, and let $\mathbb{E} := \mathbb{R}^{\mathcal{L}\times\mathcal{S}}_+ \times \mathbb{R}^{\mathcal{J}}$ be the space of consumption and portfolio allocations.

There is a finite set $\mathcal{I}$ of consumers that trade assets in order to smooth their consumption. Each agent $i \in \mathcal{I}$ is characterized by a utility function $V^i : \mathbb{R}^{\mathcal{L}\times\mathcal{S}}_+ \to \mathbb{R}$, physical endowments $w^i = (w^i_s)_{s\in\mathcal{S}} \in \mathbb{R}^{\mathcal{L}\times\mathcal{S}}_+$, and trading constraints determined by a correspondence $\Phi^i : \mathbb{P} \twoheadrightarrow \mathbb{E}$.

Given prices $(p,q) \in \mathbb{P}$, each agent $i$ chooses a consumption bundle $x^i = (x^i_s)_{s\in\mathcal{S}}$ and a portfolio $z^i = (z^i_j)_{j\in\mathcal{J}}$ in her choice set $C^i(p,q)$, which is characterized by the set of allocations $(x^i, z^i) \in \mathbb{E}$ satisfying the trading constraint $(x^i, z^i) \in \Phi^i(p,q)$ and the following budget restrictions:

$$\begin{aligned} p_0 \cdot x^i_0 + q \cdot z^i &\leq p_0 \cdot w^i_0; \\ p_s \cdot x^i_s &\leq p_s \cdot (w^i_s + Y_s x^i_0) + \sum_{j\in\mathcal{J}} R_{s,j}(p_s) z^i_j, \qquad \forall s \in S. \end{aligned}$$

Definition 1. *A vector* $((\overline{p},\overline{q}),(\overline{x}^i,\overline{z}^i)_{i\in\mathcal{I}}) \in \mathbb{P}\times\mathbb{E}^{\mathcal{I}}$ *is a competitive equilibrium for the economy with endogenous markets segmentation when the following conditions hold:*

(i) *Each agent* $i \in \mathcal{I}$ *maximizes her preferences,* $(\overline{x}^i,\overline{z}^i) \in \operatorname*{argmax}_{(x^i,z^i)\in C^i(\overline{p},\overline{q})} V^i(x^i)$.

(ii) *Individuals' plans are market feasible,*

$$\sum_{i\in\mathcal{I}} (\overline{x}^i_0, (\overline{x}^i_s)_{s\in S}, \overline{z}^i) = \sum_{i\in\mathcal{I}} (w^i_0, (w^i_s + Y_s w^i_0)_{s\in S}, 0).$$

One of our objectives is to determine conditions that make price-dependent trading constraints $\{\Phi^i\}_{i\in\mathcal{I}}$ compatible with equilibrium existence. Another one is to have within our findings equilibrium existence results for economies where *financial market segmentation* and *exclusion of credit markets* is observed.

More precisely, we say that there is financial market segmentation when there are contracts that not all agents can trade, i.e., $\{j \in \mathcal{J} : \exists i \in \mathcal{I},\ (x^i,z^i) \in \Phi^i(p,q) \implies z^i_j = 0,\ \forall (p,q) \in \mathbb{P}\} \neq \emptyset$. Moreover, there exists exclusion of credit markets when there are agents without access to liquidity through financial contracts, i.e., $\{i \in \mathcal{I} :\ (x^i,z^i) \in \Phi^i(p,q) \implies z^i \geq 0,\ \forall (p,q) \in \mathbb{P}\} \neq \emptyset$.

[1]Our financial structure is general enough to be compatible with several types of assets. For instance, to include a nominal asset $j$ it is sufficient to assume that there is $(N_{s,j})_{s\in S} \in \mathbb{R}^S_+$ such that $R_{s,j} \equiv N_{s,j}, \forall s \in S$. To include a real asset $k$ we can define payments $R_{s,k}(p_s) = p_s A_{s,k}$, $\forall s \in S$, where $(A_{s,k})_{s\in S} \in \mathbb{R}^{\mathcal{L}\times S}_+$.

Notice that these situations are incompatible with the existence of *financial survival*, which requires that all agents have access to some amount of liquidity by short-selling any financial contract.

We impose the following assumptions about agents' characteristics and financial payments:

Assumption (A1)

*For any $i \in \mathcal{I}$, the following properties hold:*

*(i) $V^i$ is continuous and strongly quasi-concave.*[2]

*(ii) $V^i$ is strictly increasing in at least one commodity at any state of nature.*

*(iii) $W^i = (W^i_s)_{s\in\mathcal{S}} := (w^i_0, (w^i_s + Y_s w^i_0)_{s\in S}) \gg 0$.*

*Furthermore, for each $(s,l) \in \mathcal{S}\times\mathcal{L}$, there is an agent whose utility function is strictly increasing in commodity $l$ at state of nature $s$.*

Assumption (A2)

*For each $j \in \mathcal{J}$, $\{R_{s,j}\}_{s\in S}$ are continuous and satisfy $(R_{s,j}(p_s))_{s\in S} \neq 0$, $\forall p \in \mathbb{R}^{\mathcal{L}\times\mathcal{S}}_{++}$.*

The requirements imposed in Assumption (A1) are classical. Assumption (A2) guarantees that asset payments are non-trivial and do not compromise the continuity of choice set correspondences.

## 3. Basic Assumptions on Trading Constraints

In this section we introduce the basic assumptions over trading constraints. We depart with hypotheses that ensure that the well behavior of choice sets is not affected by trading constraints. To shorten notations, given $j \in \mathcal{J}$, let $\widehat{e}_j \in \mathbb{E}$ be the plan composed by one unit of investment in $j$.

Assumption (A3)

*For any agent $i \in \mathcal{I}$, $\Phi^i$ is lower hemicontinuous with closed graph and convex values.*

Assumption (A4)

*For any $i \in \mathcal{I}$ and $(p,q) \in \mathbb{P}$ the following properties hold:*

*(i) If $(x^i, z^i) \in \Phi^i(p,q)$, then $(y^i, z^i) \in \Phi^i(p,q)$, $\forall y^i \geq x^i$. Also, $(0,0) \in \Phi^i(p,q)$.*

*(ii) For every $j \in \mathcal{J}$ there is an agent $h \in \mathcal{I}$ such that $\Phi^h(p,q) + \widehat{e}_j \subseteq \Phi^h(p,q)$.*

[2]Strongly quasi-concavity of $V^i$ requires that $V^i(\lambda x^i + (1-\lambda)y^i) > \min\{V^i(x^i), V^i(y^i)\}$ when $V^i(x^i) \neq V^i(y^i)$.

Under Assumption (A3) trading constraints do not compromise the continuity or the convexity of choice set correspondences. Moreover, agents are not required to trade financial contracts if they want to demand a portion of initial endowments or increase consumption departing from a trading feasible allocation (Assumption (A4)(i)). Assumption (A4)(ii) requires that for any financial contract there is at least one agent that can increase her long position on it.[3]

Example 1 (Exogenous Portfolio Constraints)

Assume that, for every $i \in \mathcal{I}$, $\Phi^i(p,q) = \mathbb{R}_+^{\mathcal{L}\times\mathcal{S}} \times Z^i$, $\forall (p,q) \in \mathbb{P}$, where $Z^i \subseteq \mathbb{R}^{\mathcal{J}}$. Then, Assumption (A3) is satisfied if and only if $\{Z^i\}_{i\in\mathcal{I}}$ are closed and convex sets. Assumption (A4)(i) holds if and only if $0 \in \bigcap_{i\in\mathcal{I}} Z^i$. Assumption (A4)(ii) holds if and only if, for each asset $j$ there is an agent $i$ such that $Z^i + \vec{e}_j \subseteq Z^i$, where $\vec{e}_j$ is the $j$-th canonical vector of $\mathbb{R}^{\mathcal{J}}$.

Notice that, (A3) and (A4) are satisfied when $\{Z^i\}_{i\in\mathcal{I}}$ are linear spaces and $\{\vec{e}_j\}_{j\in\mathcal{J}} \subset \bigcup_{i\in\mathcal{I}} Z^i$. Also, if trading constraints only restrict the access to credit (i.e., $Z^i + \mathbb{R}_+^{\mathcal{J}} \subseteq Z^i$, $\forall i \in \mathcal{I}$), then (A3) and (A4) hold if and only if $\{Z^i\}_{i\in\mathcal{I}}$ are closed and convex sets containing zero. □

Example 2 (Price-Dependent Borrowing Constraints)

Assume that, for every $i \in \mathcal{I}$,

$$\Phi^i(p,q) = \{(x^i,z^i) \in \mathbb{E} :\ z^i + g_k^i(p,q) \geq 0,\ \forall k \in \{1,\ldots,m_i\}\}, \quad \forall (p,q) \in \mathbb{P},$$

where $m_i \in \mathbb{N}$ and $g_k^i = (g_{k,j}^i) : \mathbb{P} \to \mathbb{R}_+^{\mathcal{J}}$, $\forall k \in \{1,\ldots,m_i\}$. In this context, Assumptions (A3) and (A4) are satisfied if and only if $\{g_k^i\}_{1\leq k\leq m_i}$ are continuous for every $i \in \mathcal{I}$. □

## 4. Bounds on Attainable Allocations

Restrictions on trading constraints are also imposed by assumptions over the correspondence of *attainable allocations* $\Omega : \mathbb{P} \twoheadrightarrow \mathbb{E}^{\mathcal{I}}$, defined as the set-valued mapping that associates prices with market feasible allocations satisfying individuals' budget and trading constraints, i.e.,

$$\Omega(p,q) := \left\{((x^i,z^i))_{i\in\mathcal{I}} \in \prod_{i\in\mathcal{I}} C^i(p,q) : \sum_{i\in\mathcal{I}}(x^i,z^i) = \sum_{i\in\mathcal{I}}(W^i,0)\right\}.$$

Notice that, any element of $\Omega(p,q)$ satisfies budget constraints with equality.

Assumption (A5)(i)

*For every compact set* $\mathbb{P}' \subseteq \mathbb{P}$, $\displaystyle\bigcup_{(p,q)\in\mathbb{P}' :\, (p,q)\gg 0} \Omega(p,q)$ *is bounded.*

[3]Under Assumption (A3), (A4)(i) is equivalent to require that: $\forall j \in \mathcal{J}$, $\exists i \in \mathcal{I}:\ \Phi^i(p,q) + \delta\widehat{e}_j \subseteq \Phi^i(p,q)$, $\forall \delta > 0$.

Assumption (A5)(ii)

*If the projection of* $\mathbb{P}' \subseteq \mathbb{P}$ *on* $\mathbb{R}_+^{\mathcal{L}\times\mathcal{S}}$ *is a compact set, then* $\displaystyle\bigcup_{(p,q)\in\mathbb{P}':\,(p,q)\gg 0} \Omega(p,q)$ *is bounded.*

In our results of equilibrium existence, we require sets of attainable allocations to be uniformly bounded in the sense of (A5)(i) or (A5)(ii). Assumption (A5)(ii), which is stronger than (A5)(i), holds when $\mathcal{J}$ is composed by non-redundant nominal assets, by collateralized assets, or when agents are subject to exogenous short-sale constraints—i.e., when for every $i \in \mathcal{I}$ there exists $m \in \mathbb{R}_+^{\mathcal{J}}$ such that $(x^i, z^i) \in \Phi^i(p,q) \implies z^i \geq -m$, $\forall (p,q) \in \mathbb{P}$.

The existence of upper bounds on attainable allocations is directly related with the non-redundancy of the financial structure. That is, the non-existence of unbounded sequences of trading admissible portfolios that do not generate commitments.

To formalize this relationship, given $(p,q) \in \mathbb{P}$ and $i \in \mathcal{I}$, define

$$\begin{aligned}\mathcal{A}_0^i(p,q) &:= \left\{z^i \in \mathbb{R}^{\mathcal{J}} \setminus \{0\} :\ q \cdot z^i = 0 \ \wedge\ R(p)z^i = 0 \ \wedge\ (W^i, \delta z^i) \in \Phi^i(p,q),\, \forall \delta > 0\right\},\\ \mathcal{A}_1^i(p,q) &:= \left\{z^i \in \mathbb{R}^{\mathcal{J}} \setminus \{0\} :\ R(p)z^i = 0 \ \wedge\ (W^i, \delta z^i) \in \Phi^i(p,q),\, \forall \delta > 0\right\}.\end{aligned}$$

We focus our attention on two non-redundancy conditions, which are defined for every non-empty set $\mathbb{P}' \subseteq \mathbb{P}$. The first one, is a generalization of the requirement imposed by Siconolfi (1987, Assumption (A5)) in nominal asset markets with exogenous portfolio constraints,

$$(\mathrm{NR}_1(\mathbb{P}')) \qquad \bigcup_{i\in\mathcal{I}} \mathcal{A}_1^i(p,q) = \emptyset, \qquad \forall (p,q) \in \mathbb{P}'.$$ [4]

The second one, avoids the existence of unbounded sequences of trading admissible portfolios that do not implement transfers of wealth among states of nature, i.e.,

$$(\mathrm{NR}_0(\mathbb{P}')) \qquad \bigcup_{i\in\mathcal{I}} \mathcal{A}_0^i(p,q) = \emptyset, \qquad \forall (p,q) \in \mathbb{P}'.$$

Since $\mathcal{A}_0^i(p,q) \subseteq \mathcal{A}_1^i(p,q)$, for every non-empty set $\mathbb{P}' \subseteq \mathbb{P}$, $\mathrm{NR}_0(\mathbb{P}')$ is weaker than $\mathrm{NR}_1(\mathbb{P}')$.

[4]Assume that assets are nominal, i.e., $R(p) \equiv N$, and that trading constraints are given by exogenous portfolio restrictions, i.e., for every agent $i$ there is a set $Z^i \subseteq \mathbb{R}^{\mathcal{J}}$ such that $\Phi^i(p,q) = \mathbb{R}_+^{\mathcal{L}\times\mathcal{S}} \times Z^i$, $\forall (p,q) \in \mathbb{P}$. Then, $(\mathrm{NR}_1(\mathbb{P}'))$ holds if and only if the following non-redundancy condition imposed by Siconolfi (1987) holds,

$$\bigcup_{i\in\mathcal{I}} \left\{z^i \in \mathbb{R}^{\mathcal{J}} \setminus \{0\} :\ Nz^i = 0 \ \wedge\ \delta z^i \in Z^i,\, \forall \delta > 0\right\} = \emptyset.$$

Proposition 1. *Under Assumptions (A1)-(A4), for every $\mathbb{P}' \subseteq \mathbb{P}$ non-empty and compact,*

$$\mathrm{NR}_0(\mathbb{P}') \quad \Longrightarrow \quad \bigcup_{(p,q)\in\mathbb{P}'} \Omega(p,q) \;\; \textit{is bounded} \quad \Longrightarrow \quad 0 \notin \bigcup_{(p,q)\in\mathbb{P}'} \; \bigcup_{\mathcal{I}'\subseteq\mathcal{I}} \; \sum_{i\in\mathcal{I}'} \mathcal{A}_0^i(p,q).$$

Thus, each non-redundancy condition, $\mathrm{NR}_1(\mathbb{P}')$ or $\mathrm{NR}_0(\mathbb{P}')$, guarantees that Assumption (A5)(i) holds. Furthermore, as the following example illustrates, when assets are nominal and trading constraints are exogenous, Assumption (A5)(i) is weaker than the traditional non-redundancy hypothesis imposed by Siconolfi (1987).

Example 3. Consider an economy with exogenous trading constraints and nominal assets. There are three agents $\mathcal{I} = \{1,2,3\}$ and two assets $\mathcal{J} = \{1,2\}$, which have identical payments satisfying $N_{1,1} = N_{1,2} = 1$ and $(N_{s,j})_{s\neq 1} = 0$, $\forall j \in \mathcal{J}$. Also, there is $m > 0$ such that, $Z^1 = [-m,+\infty) \times \{0\}$, $Z^2 = \{0\} \times [-m,+\infty)$, and $Z^3 = [-m,+\infty) \times (-\infty,0]$. Then, Assumption (A5)(ii) holds, although $\left\{z \in \mathbb{R}^{\mathcal{J}} \setminus \{0\} : \; Nz = 0 \;\; \wedge \;\; \delta z \in Z^3, \forall \delta > 0\right\} \neq \emptyset$. □

## 5. Upper Bounds on Asset Prices

One of the main steps in any proof of equilibrium existence is to ensure that endogenous variables can be bounded without inducing frictions over individual demand correspondences. Under Assumption (A5) we can obtain natural upper bounds for individual allocations. However, it is also necessary to ensure that prices can be bounded. With this objective, some authors impose *financial survival conditions*, assuming that every agent has access to resources by short-selling any financial contract (see Angeloni and Cornet (2006), Hahn and Won (2007), and Aouani and Cornet (2009, 2011)). Notwithstanding, as we want to include financial market segmentation, we need to follow alternative approaches to establish bounds for asset prices.

Before discussing these alternatives, we introduce some concepts.

Definition 2. *A financial contract $j \in \mathcal{J}$ is an ultimate source of liquidity when, given $(p,q) \in \mathbb{P}$ with $p_0 = 0$, there exists $(\theta(p,q), \zeta(p,q)) \in (0,1) \times (0,1)$ such that, each agent $i$ can short-sell $\zeta(p,q)$ units of asset $j$ in order to demand the bundle $((1+\theta(p,q))W_0^i, ((1-\theta(p,q))W_s^i)_{s\in S})$.*

Thus, agents have access to liquidity even when they cannot obtain resources by selling physical endowments. It follows that, under Assumptions (A3)-(A4), an ultimate source of liquidity is a contract that any agent can short-sale in order to make trading-feasible a small increment on current consumption in exchange of a reduction on future demand. For notation convenience, let $\mathcal{J}_u$ be the (possibly empty) maximal subset of $\mathcal{J}$ composed by contracts that are ultimate sources of liquidity.

Assumption (A6)

*(i) Given $j \in \mathcal{J}_u$, for every $i \in \mathcal{I}$ we have that* $\Phi^i(p,q) + \widehat{e}_j \subseteq \Phi^i(p,q)$, $\forall (p,q) \in \mathbb{P}$.

*(ii) Given $j \notin \mathcal{J}_u$, for every $i \in \mathcal{I}$ and $(x^i, z^i) \in \Phi^i(p,q)$,*

$$(x^i, z^i) - \delta \widehat{e}_j \in \Phi^i(p,q),\ \forall \delta \in [0, \max\{z^i_j, 0\}], \quad \forall (p,q) \in \mathbb{P}.$$

Assumption (A6)(i) requires that all agents have access to invest in each asset belonging to $\mathcal{J}_u$, while (A6)(ii) holds if and only if long positions for assets in $\mathcal{J} \setminus \mathcal{J}_u$ can be reduced without compromising the trading feasibility of allocations.

We affirm that, under Assumptions (A1)-(A6), if $\mathcal{J}_u$ satisfies the super-replication property defined below, then there are endogenous bounds for asset prices.

Definition 3. *Agents can super-replicate financial payments investing in contracts $\mathcal{J}_u$ and buying commodities when for any compact set $\mathbb{P}_1 \subset (\mathbb{R}^{\mathcal{L}}_+ \setminus \{0\})^S$ there exists $(\widehat{x}, (\widehat{z}_k)_{k \in \mathcal{J}_u}) \geq 0$ such that,*

$$\sum_{j \notin \mathcal{J}_u} R_{s,j}(p_s) > 0 \quad \Longrightarrow \quad \sum_{j \notin \mathcal{J}_u} R_{s,j}(p_s) < p_s Y_s \widehat{x} + \sum_{k \in \mathcal{J}_u} R_{s,k}(p_s) \widehat{z}_k, \qquad \forall s \in S,\ \forall (p_s)_{s \in S} \in \mathbb{P}_1.$$

Intuitively, if agents can super-replicate financial payments investing in contracts $\mathcal{J}_u$ and buying commodities, then the price of any traded contract $j \notin \mathcal{J}_u$ can be bounded from above in terms of $(p_0, (q_k)_{k \in \mathcal{J}_u})$. In addition, since all agents have access to some amount of credit through any $k \in \mathcal{J}_u$, it is possible to normalize prices $(p_0, (q_k)_{k \in \mathcal{J}_u})$ without inducing frictions on individual demand correspondences (see Lemma 3 for detailed arguments).

Notice that, the continuity of assets payments (Assumption (A2)) ensures that any contract $j$ satisfying $(R_{s,j}(p_s))_{s \in S} \gg 0$, $\forall (p_s)_{s \in S} \in (\mathbb{R}^{\mathcal{L}}_+ \setminus \{0\})^S$ super-replicates the payments of the remaining assets. For instance, it holds when $j$ is a risk-free nominal asset, i.e., $R_{s,j} \equiv 1$, $\forall s \in S$.

An alternative to obtain upper bounds for asset prices is to have individuals whose preferences satisfy a kind of impatience condition.

Assumption (A7)

*There is a non-empty subset of agents $\mathcal{I}^* \subseteq \mathcal{I}$ with strictly monotonic preferences such that:*

*(i) Given $i \in \mathcal{I}^*$ and $(\rho, x^i) \in (0,1) \times \mathbb{R}^{\mathcal{L} \times \mathcal{S}}_{++}$, there exists $\tau^i(\rho, x^i) \in \mathbb{R}^{\mathcal{L}}_+$ such that,*

$$V^i\left(x^i_0 + \tau^i(\rho, x^i), (\rho\, x^i_s)_{s \in S}\right) > V^i(x^i).$$

*(ii) Let $j \notin \mathcal{J}_u$, there exists $i \in \mathcal{I}^*$ and $z^i \in \mathbb{R}^{\mathcal{J}}_+$ with $z^i_j > 0$ and $-(0, z^i) \in \Phi^i(p,q)$, $\forall (p,q) \in \mathbb{P}$.*

Assumption (A7)(i) holds independently of the representation of preferences, and was introduced by Seghir and Torres-Martínez (2011) to analyze equilibrium existence in a model with borrowing constraints depending on first-period consumption. Intuitively, it requires the existence of agents that, in terms of preferences, can compensate any loss in utility associated with a reduction in future demand with an increment of present consumption. In particular, Assumption (A7) is satisfied when there is an agent $h$ such that $V^h$ is unbounded on first period consumption and, independent of prices $(p,q) \in \mathbb{P}$, the zero vector belongs to the interior of $\Phi^h(p,q)$.

In this context, the main idea behind the existence of upper bounds for asset prices is as follows: consider an agent $i \in \mathcal{I}^*$ such that, at prices $(p,q) \in \mathbb{P}$, her optimal consumption allocation is market feasible. Suppose that, as an alternative to her optimal strategy, she decides to make a promise on an asset $j \notin \mathcal{J}_u$ using the borrowed resources to increase first period consumption. Also, assume that this promise can be paid with her future endowments. As a consequence of (A7), if the new strategy generates a high enough liquidity, then she will ensure a utility level greater than the one associated to aggregated endowments. Thus, $q_j$ needs to be bounded (see Lemma 5 for detailed arguments).

Since one of our results of equilibrium existence is related with Seghir and Torres-Martínez (2011), it is interesting to discuss our assumptions when we restrict the attention to that framework.

Example 4 (Consumption-Dependent Borrowing Constraints)

Suppose that trading constrains are independent of prices and determine restrictions on borrowing and first-period consumption. Thus, given $(p,q) \in \mathbb{P}$ and $i \in \mathcal{I}$, we assume that

$$\Phi^i(p,q) = \{(x^i,z^i) \in \mathbb{E} : \ \exists (\theta^i,\varphi^i) \in \mathbb{R}^{\mathcal{J}}_+ \times \mathbb{R}^{\mathcal{J}}_+, \quad \varphi^i \in \Psi^i(x^i_0) \ \ \wedge \ \ z^i = \theta^i - \varphi^i \ \},$$

where $\Psi^i = (\Psi^i_j) : \mathbb{R}^{\mathcal{L}}_+ \twoheadrightarrow \mathbb{R}^{\mathcal{J}}_+$. In this context, Assumption (A3) holds if and only if $\{\Psi^i\}_{i\in\mathcal{I}}$ have a closed and convex graph. Assumption (A4)(i) holds if and only if, for every agent $i$, $0 \in \Psi^i(x^i_0)$ and $\Psi^i(x^i_0) \subseteq \Psi^i(y^i_0)$, $\forall y^i_0 \geq x^i_0$. This last property implies that, to ensure Assumption (A5), it is sufficient to require that $\{\Psi^i\}_{i\in\mathcal{I}}$ have bounded values. Since trading constraints only affect short-sales, Assumptions (A4)(ii) and Assumption (A6) always holds. Assumption (A7)(ii) holds if and only if there exists $\delta > 0$ such that $\delta(1,\ldots,1) \in \sum_{i\in\mathcal{I}^*} (\Psi^i_j(0))_{j\notin\mathcal{J}_u}$.

Therefore the hypotheses of the main result in Seghir and Torres-Martínez (2011) imply that Assumptions (A1)-(A7) hold. □

## 6. Examples of Trading Constraints

In this section we present some examples of trading constraints allowing: wealth-dependent financial access, investment-dependent credit access, debt constraints precluding bankruptcy, security exchanges, commodity options with deposit requirements, and assets that are backed by physical or financial collateral.

Example 5 (Income-Based Financial Access)

Given $(p,q) \in \mathbb{P}$ and $i \in \mathcal{I}$, assume there exists an asset $j$ such that,

$$(x^i, z^i) \in \Phi^i(p,q) \qquad \Longrightarrow \qquad z^i_j \in [\min\{p_0 \cdot (\tau_1 - w^i_0), 0\}, \max\{p_0 \cdot (w^i_0 - \tau_2), 0\}],$$

where $\tau_1, \tau_2 \in \mathbb{R}^{\mathcal{L}}_+$. Then, agent $i$ can short-sale asset $j$ if and only if the value of her first period endowment is greater than $p_0\tau_1$. Analogously, she can invest in asset $j$ if and only if her first period endowment is greater than $p_0\tau_2$. That is, $j$ can only be traded by high income agents.

If we suppose that for some $k \in \mathcal{J}$, $(x^i, z^i) \in \Phi^i(p,q) \implies z^i_k \in [\min\{p_0 \cdot (w^i_0 - \tau_1), 0\}, +\infty]$, then all agents can invest on $k$, but only low-income agents can short-sale it. □

Example 6 (Exclusive Credit Lines)

We can consider the case where the access to credit depends on the amount of investment in some financial contracts. That is, there exists $j \in \mathcal{J}$ and $\mathcal{J}' \subset \mathcal{J} \setminus \{j\}$ such that, for every $(p,q) \in \mathbb{P}$ and $i \in \mathcal{I}$, we have that

$$(x^i, z^i) \in \Phi^i(p,q) \qquad \Longrightarrow \qquad z^i_j \in \left[\min\left\{K - \sum_{k \in \mathcal{J}'} q_k z^i_k, 0\right\}, +\infty\right).$$

Hence, only investors that expend an amount greater than $K$ in assets belonging to $\mathcal{J}'$ have access to short-sale the financial contract $j$. □

Example 7 (Debt Constraints)

If there is $\kappa \in (0,1)$ such that, for any $(p,q) \in \mathbb{P}$ and for some $i \in \mathcal{I}$,

$$(x^i, z^i) \in \Phi^i(p,q) \qquad \Longrightarrow \qquad \kappa p_s \cdot (w^i_s + Y_s x^i_0) + \sum_{j \in \mathcal{J}} R_{s,j}(p_s) \min\{z^i_j, 0\} \geq 0,\ \forall s \in S,$$

then agent $i$'s trading constraints ensure that her debt is not greater than an exogenously-fixed portion of physical-resources' value. Notice that, if a portion $\rho > \kappa$ of physical resources can be garnished in case of bankruptcy, the above restriction ensures that $i$ honors her commitments. □

Example 8 (Security Exchanges)

Suppose that we split the sets of agents and financial contracts such that,

$$\mathcal{I} = \bigcup_{r=1}^{a} \mathcal{I}_r, \qquad \mathcal{J} = \bigcup_{r=1}^{b} \mathcal{J}_r,$$

and assume that for every $(p,q) \in \mathbb{P}$ and $i \in \mathcal{I}_r$,

$$(x^i, z^i) \in \Phi^i(p,q) \qquad \Longrightarrow \qquad \begin{cases} z^i_j \geq 0, & \forall j \in G_+(\mathcal{I}_r); \\ z^i_j \leq 0, & \forall j \in G_-(\mathcal{I}_r); \\ z^i_j = 0, & \forall j \notin G_+(\mathcal{I}_r) \cup G_-(\mathcal{I}_r), \end{cases}$$

where $G_+, G_- : \{\mathcal{I}_1, \ldots, \mathcal{I}_a\} \twoheadrightarrow \{\mathcal{J}_1, \ldots, \mathcal{J}_b\}$ are non-empty valued correspondences.

Then, we obtain a structure of *exchanges*, $\{\mathcal{J}_1, \ldots, \mathcal{J}_b\}$, where an agent $i \in I_r$ can only short-sale assets that are available in the exchanges belonging to $G_-(\mathcal{I}_r)$, whereas she can only invest in assets traded in exchanges belonging to $G_+(\mathcal{I}_r)$. Notice that the markets of debt and investment are not necessarily segmented, as $G_+(\mathcal{I}_r)$ and $G_-(\mathcal{I}_r)$ are not required to be disjoint. Also, by Assumption (A6)(i), if $j \in \mathcal{J}_u$, then $j \in \bigcap_{r=1}^{a}(G_+(\mathcal{I}_r) \cap G_-(\mathcal{I}_r))$.

Since the same agent can participate in several exchanges—because $G_+$ and $G_-$ are not necessarily singled-valued—we obtain a model of exchanges with heterogeneous participation, multi-membership, and price-dependent trading constraints.[5] □

Example 9 (Commodity Options)

Let $j \in \mathcal{J}$ be a financial contract such that, for every $(p,q) \in \mathbb{P}$,

$$R_{s,j}(p_s) = \max\{Y_s y - K, 0\}, \qquad \forall s \in S,$$

$$(x^i, z^i) \in \Phi^i(p,q) \quad \Longrightarrow \quad \kappa\, p_0 \cdot y + \min\{z^i_j, 0\} \geq 0,$$

where $y \in \mathbb{R}^{\mathcal{L}}_+$, $K > 0$ and $\kappa \in [0,1)$. Then, $j$ is a commodity option that gives the right to buy in the second period, at a strike price $K$, the bundle obtained by the transformation of $y$ through time. To short-sell this option, agents are required to buy a portion $\kappa$ of $y$ as guarantee. □

Example 10 (Collateralized Assets)

We can include non-recourse collateralized assets.[6] Indeed, a collateralized contract $j$ can be characterized by a pair $(C_j, (D_{s,j}(p_s))_{s\in S})$, where $C_j = (C_{j,l})_{l\in\mathcal{L}} \in \mathbb{R}^{\mathcal{L}}_+ \setminus \{0\}$ is the collateral

[5] Faias and Luque (2013) address an equilibrium model with exchanges where individual preferences satisfy the kind of impatience condition imposed by Seghir and Torres-Martínez (2011). Different to the example above, they allow cross listing and transactions fees.

[6] In the absence of payment enforcement mechanisms over collateral repossession, the monotonicity of preferences guarantees that borrowers of a collateralized loan always deliver the minimum between promises and collateral values. Therefore, lenders that finance these loans perfectly foresight the payments that they will receive. Hence, as in

guarantee, and $(D_{s,j}(p_s))_{s\in S} \in \mathbb{R}^S_+$ are the state contingent promises, which determine payments $R_{s,j}(p_s) = \min\{D_{s,j}(p_s), p_s Y_s C_j\}$, $\forall s \in S$. Since borrowers are required to pledge the associated collateral, we assume that, given $(x^i, z^i) \in \Phi^i(p,q)$, the following properties hold

$$x^i_0 + C_j z^i_j \geq 0, \quad \text{and} \qquad ((x^i_0 - \alpha C_j, (x^i_s)_{s\in S}), z^i) + \alpha \widehat{e}_j \in \Phi^i(p,q), \ \forall \alpha \in [0, -\min\{z^i_j, 0\}].$$

Thus, individual consumption plans include the required collateral guarantees and any reduction in short positions reduces the requirements of collateral. That is, there is no *cross-collateralization* of payments, i.e., several loans backed by the same collateral. Notice that, payments associated to non-recourse collateralized loans can be super-replicated by the collateral bundle.

To include assets backed by financial collateral, we can assume that there are $j, k \in \mathcal{J}$ such that, given $(p,q) \in \mathbb{P}$ and $i \in \mathcal{I}$, for any $s \in S$ we have that $R_{s,j} = \min\{T_{s,j}(p_s), R_{s,k}(p_s)\}$ and

$$(x^i, z^i) \in \Phi^i(p,q) \qquad \Longrightarrow \quad \exists (\theta^i, \varphi^i) \in \mathbb{R}^{\mathcal{J}}_+ \times \mathbb{R}^{\mathcal{J}}_+ : \quad \theta^i_k \geq \varphi^i_j \ \wedge \ z^i = \theta^i - \varphi^i.$$

where $T_{s,j} : \mathbb{R}^{\mathcal{L}}_+ \to \mathbb{R}_+$. Then, each unit of asset $j$ promises to deliver an amount $T_{s,j}(p_s)$ at a state of nature $s$, and it is backed by one unit of financial contract $k$ in case of default.[7] □

## 7. Equilibrium Existence

Our first result ensures the compatibility between equilibrium and markets segmentation.

Theorem 1. *Under Assumptions (A1)-(A5)(i) and (A6), if agents can super-replicate financial payments investing in assets in $\mathcal{J}_u$ and buying commodities, then there exists a competitive equilibrium.*

The super-replication property trivially holds when there is an ultimate source of liquidity with strictly positive payments, when $\mathcal{J} = \mathcal{J}_u$ or when there is a bundle of commodities that super-replicate financial payments. Thus, departing from Theorem 1, we can obtain the following results.

Corollary 1. *Under Assumptions (A1)-(A5)(i) and (A6), if there exists $j \in \mathcal{J}_u$ such that $(R_{s,j}(p_s))_{s\in S} \gg 0$, $\forall (p_s)_{s\in S} \in (\mathbb{R}^{\mathcal{L}}_+ \setminus \{0\})^S$, then there exists a competitive equilibrium.*

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Geanakoplos and Zame (2013), we can capture with a same financial contract both the collateralized line of credit and the collateralized loan obligation (CLO) that passthrough the payments made by borrowers.

[7]Notice that, as $k$ is used as financial collateral, the investment in it may not be reduced without affecting the trading feasibility. Thus, under the conditions of Theorem 1, $k \in \mathcal{J}_u$.

Corollary 2. *Under Assumptions (A1)-(A5)(i), there is a competitive equilibrium if all assets are ultimate sources of liquidity (i.e., financial survival holds).*

Corollary 3. *Under Assumptions (A1)-(A5)(i) and (A6)(ii), there is a competitive equilibrium if one of the following conditions is satisfied:*

*(i) all assets are backed by physical collateral;*

*(ii) all assets are real and claims are measure in units of non-perishable commodities.*

In particular, we extend Geanakoplos and Zame (2013) to include financial market segmentation and price-dependent trading constraints. Notice that, the results above are compatible with the exclusion of some agents from credit markets only if $\mathcal{J}_u = \emptyset$. However, as the following result shows, even without require financial payments to be super-replicated by physical markets it is possible to guarantee equilibrium existence in a model that allows exclusion of credit markets.

Theorem 2. *Under Assumptions (A1)-(A7) there exists a competitive equilibrium for the economy with endogenous market segmentation.*

This result extends Seghir and Torres-Martínez (2011) in order to include price-dependent trading constraints and investment restrictions. It also guarantees that their main result holds under weaker assumptions. In fact, we only impose the impatience condition on a subset of agents. More importantly, they assume that sets of trading admissible short-sales are compact, an hypothesis that is stronger than Assumption (A5)(ii).[8]

Recently, Pérez-Fernández (2013) also extends the results of Seghir and Torres-Martínez (2011) including price-dependent trading constraints in an environment with non-ordered preferences. In his model, the relationship between investment and debt is more general than ours, because Assumption (A7)(ii) does not necessarily hold. However, as in Seghir and Torres-Martínez (2011), it is assumed that correspondences of trading admissible allocations have compact values.

## 8. Concluding Remarks

In this paper we extend the theory of general equilibrium with incomplete financial markets to include price-dependent trading constraints that restrict both consumption alternatives and admissible portfolios. Our approach is general enough to incorporate several types of dependencies

[8]See Example 4 for a detailed comparison between assumptions in the two models.

between prices, consumption, and financial access. For instance, the access to liquidity may depend on individuals income, the short-sale of derivatives may require the deposit of margins, and borrowers could be required to pledge physical and/or financial collateral.

As we want to include financial segmentation and credit-access exclusion, our results of equilibrium existence do not rely on financial survival conditions. Hence, we propose two ways to ensure the existence of a competitive equilibrium, based on either the super-replication of promises (Theorem 1) or a kind of agents' impatience (Theorem 2). The super-replication property holds when there is a risk-free nominal asset with unrestricted investment and such that all agents can sell it. The impatience condition holds when utility functions are unbounded in the first period consumption.

## Appendix A: Proof of Proposition 1

Let $\mathbb{P}' \subseteq \mathbb{P}$ be a non-empty and compact set.

Assume that there is an unbounded sequence $\{(x_n^i, z_n^i)_{i\in\mathcal{I}}\}_{n\in\mathbb{N}} \in \bigcup_{(p,q)\in\mathbb{P}'} \Omega(p,q)$. Then, there exists a sequence $\{(p_n, q_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{P}'$ such that, $(x_n^i, z_n^i) \in \Omega(p_n, q_n)$, $\forall n \in \mathbb{N}$. Also, Assumption (A4)(i) ensures that, for every $n$ and $i$, $(W, z_n^i) \in \Phi^i(p_n, q_n)$, where $W = (W_s)_{s\in\mathcal{S}} := \sum_{i\in\mathcal{I}} W^i$. Hence, for some agent $h$ there is an unbounded subsequence $\{z_{n_k}^h\}_{k\in\mathbb{N}} \subseteq \{z_n^h\}_{n\in\mathbb{N}}$ such that, for every $k \in \mathbb{N}$, $z_{n_k}^h \neq 0$, $\|z_{n_k}^h\|_\Sigma \leq \|z_{n_{k+1}}^h\|_\Sigma$, and $(W, z_{n_k}^h) \in \Phi^h(p_{n_k}, q_{n_k})$. Let $((\tilde{p}, \tilde{q}), \tilde{z}^h)$ be a cluster point of $\{(p_{n_k}, q_{n_k}), z_{n_k}^h/\|z_{n_k}^h\|_\Sigma\}_{k\in\mathbb{N}}$.

We affirm that, $R(\tilde{p})\tilde{z}^h = 0$ and $\tilde{q}\cdot\tilde{z}^h = 0$. First, if there is an state of nature $s \in S$ such that $\sum_{j\in\mathcal{J}} R_{s,j}(\tilde{p}_s)\tilde{z}_j^h < 0$, then $\delta_0 \sum_{j\in\mathcal{J}} R_{s,j}(\tilde{p}_s)\tilde{z}_j^h < -2\tilde{p}_s \cdot W_s$, for some $\delta_0 > 0$. This implies that, for $k \in \mathbb{N}$ large enough, $\delta_0 \sum_{j\in\mathcal{J}} R_{s,j}(p_{n_k,s}) z_{n_k,j}^h/\|z_{n_k}^h\|_\Sigma < -2p_{n_k,s}\cdot W_s$. Since $\lim_k \|z_{n_k}^h\|_\Sigma = +\infty$, it follows that for $k$ large enough $\sum_{j\in\mathcal{J}} R_{s,j}(p_{n_k,s}) z_{n_k,j}^h < -2p_{n_k,s}\cdot W_s$, a contradiction with $(x_{n_k}^i, z_{n_k}^i)_{i\in\mathcal{I}} \in \Omega(p_{n_k}, q_{n_k})$. Second, if there is $s \in S$ such that $\sum_{j\in\mathcal{J}} R_{s,j}(\tilde{p}_s)\tilde{z}_j^h > 0$, then $\delta_1 \sum_{j\in\mathcal{J}} R_{s,j}(\tilde{p}_s)\tilde{z}_j^h > 2(\#\mathcal{I}-1)\tilde{p}_s\cdot W_s$, for some $\delta_1 > 0$. Hence, for $k \in \mathbb{N}$ large enough, we have that $\sum_{j\in\mathcal{J}} R_{s,j}(p_{n_k,s}) z_{n_k,j}^h > 2(\#\mathcal{I}-1)\, p_{n_k,s}\cdot W_s$. Due to $\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}} R_{s,j}(p_{n_k,s}) z_{n_k,j}^i = 0$, there exists $h' \neq h$ such that $\sum_{j\in\mathcal{J}} R_{s,j}(\tilde{p}_{n_k,s}) z_{n_k,j}^{h'} < -2\,\tilde{p}_{n_k,s}\cdot W_s$, a contradiction with $(x_{n_k}^i, z_{n_k}^i)_{i\in\mathcal{I}} \in \Omega(p_{n_k}, q_{n_k})$. The property $\tilde{q}\cdot\tilde{z}^h = 0$ follows by analogous arguments.

In addition, $(W^h, \delta\tilde{z}^h) \in \Phi^h(\tilde{p}, \tilde{q})$ for every $\delta > 0$. Indeed, given $\delta > 0$ there exists $k(\delta) \in \mathbb{N}$ such that $\|z_{n_k}^h\|_\Sigma \geq \delta, \forall k \geq k(\delta)$. Hence, as $\Phi^h$ has convex values and $(W, 0) \in \Phi^h(p_{n_k}, q_{n_k})$ for every $k \in \mathbb{N}$, it follows that $(W, \delta z_{n_k}^h/\|z_{n_k}^h\|_\Sigma) \in \Phi^h(p_{n_k}, q_{n_k})$ for any $k \geq k(\delta)$, which in turn implies that $(W, \delta\tilde{z}^h) \in \Phi^h(\tilde{p}, \tilde{q})$. Furthermore, Assumption (A1) guarantees that there is $\sigma \in (0,1)$ such that $\sigma W \ll W^h$. As for every $\delta > 0$ we have that $(1-\sigma)(0,0) + \sigma(W, \delta\tilde{z}^h/\sigma) \in \Phi^h(\tilde{p}, \tilde{q})$, the property follows from Assumption (A4)(i). Therefore, $\tilde{z}^h \in \mathcal{A}_0^h(\tilde{p}, \tilde{q})$, which implies that $\bigcup_{(p,q)\in\mathbb{P}'} \bigcup_{i\in\mathcal{I}} \mathcal{A}_0^i(p,q) \neq \emptyset$. This concludes the proof of the first implication.

Notice that, if there is $(p,q) \in \mathbb{P}'$ and $\mathcal{I}' \subseteq \mathcal{I}$ such that $0 \in \sum_{i\in\mathcal{I}'} \mathcal{A}_0^i(p,q)$, then for every $i \in \mathcal{I}'$ there is $z^i \in \mathbb{R}^{\mathcal{J}} \setminus \{0\}$ such that $q\cdot z^i = 0$, $R(p)z^i = 0$, and $(W^i, \delta z^i) \in \Phi^i(p,q)$, $\forall\delta > 0$, with $\sum_{i\in\mathcal{I}'} z^i = 0$. We conclude that, $\sum_{i\in\mathcal{I}'} \delta z^i = 0$ and $(W^i, \delta z^i) \in C^i(p,q)$, $\forall i \in \mathcal{I}'$, $\forall\delta > 0$. Hence, $\Omega(p,q)$ is unbounded. □

## Appendix B: Proof of Theorem 1

Given $M \in \mathbb{N}$, consider the space $\mathbb{P}(M)$ of normalized prices, which is defined by

$$(p,q) \equiv ((p_0,(q_k)_{k\in\mathcal{J}_u}),(q_j)_{j\in\mathcal{J}\setminus\mathcal{J}_u},(p_s)_{s\in S}) \in \mathbb{P}(M) := \mathcal{P}_0 \times [0,M]^{\mathcal{J}\setminus\mathcal{J}_u} \times \mathcal{P}_1^S,$$

where $\mathcal{P}_0 := \{y \in \mathbb{R}_+^{\mathcal{L}\cup\mathcal{J}_u} : \|y\|_\Sigma = 1\}$, and $\mathcal{P}_1 := \{y \in \mathbb{R}_+^{\mathcal{L}} : \|y\|_\Sigma = 1\}$.[9]

Lemma 1. *Under Assumptions (A2), (A3) and (A4)(i), for every agent $i \in \mathcal{I}$ the choice set correspondence $C^i : \mathbb{P}(M) \twoheadrightarrow \mathbb{E}$ is lower hemicontinuous with closed graph and non-empty and convex values.*

Proof. Fix $i \in \mathcal{I}$. Assumption (A4)(i) ensures that for every $(p,q) \in \mathbb{P}$ the allocation $(W^i,0) \in C^i(p,q)$, which implies that $C^i$ is non-empty valued. Assumptions (A2) and (A3) imply that $C^i$ has convex values and closed graph. To prove that $C^i$ is lower hemicontinuous, let $\mathring{C}^i : \mathbb{P}(M) \twoheadrightarrow \mathbb{E}$ be the correspondence that associates to each $(p,q) \in \mathbb{P}(M)$ the set of allocations $(x^i,z^i) \in C^i(p,q)$ satisfying budget constraints with strict inequalities. We affirm that $\mathring{C}^i$ is lower hemicontinuous and has non-empty values. Since $C^i$ is the closure of $\mathring{C}^i$, these properties imply that $C^i$ is lower hemicontinuous (see Border (1985, 11.19(c))).

Thus, we close the proof ensuring the claimed properties for $\mathring{C}^i$.

To prove that $\mathring{C}^i$ has non-empty values, fix $(\mu_0,\mu_1) \in (0,1)\times(0,1)$ such that $\mu_0 > \mu_1$. It follows from Assumption (A4)(i) that $((\mu_0 W_0^i,(\mu_1 W_s^i)_{s\in S}),0) \in \Phi^i(p,q)$ for all $(p,q) \in \mathbb{P}(M)$.

Notice that, for any $(p,q) \in \mathbb{P}(M)$ with $p_0 \neq 0$ we have that $((\mu_0 W_0^i,(\mu_1 W_s^i)_{s\in S}),0) \in \mathring{C}^i(p,q)$. Thus, fix $(p,q) \in \mathbb{P}(M)$ such that $p_0 = 0$. Since $\Phi^i$ has convex values, it follows from Assumption (A2) that there exists $\lambda \in (0,1)$, high enough, such that

$$\begin{aligned}(\widetilde{x}^i,\widetilde{z}^i) := {} & \lambda\,((\mu_0 W_0^i,(\mu_1 W_s^i)_{s\in S}),0) \\ & + \frac{(1-\lambda)}{\max\{\#\mathcal{J}_u,1\}} \sum_{k\in\mathcal{J}_u} \left[\Big(W^i + \theta_k(p,q)(W_0^i,-(W_s^i)_{s\in S})),0\Big) - \zeta_k(p,q)\,\widehat{e}_k\right] \in \Phi^i(p,q);\end{aligned}$$

$$\lambda\mu_0 + \sum_{k\in\mathcal{J}_u} \frac{(1-\lambda)}{\max\{\#\mathcal{J}_u,1\}}(1+\theta_k(p,q)) \quad < \quad 1;$$

$$\frac{(1-\lambda)}{\max\{\#\mathcal{J}_u,1\}} \sum_{k\in\mathcal{J}_u} \zeta_k(p,q) \max_{(\tilde{p},\tilde{q})\in\mathbb{P}(M)} \max_{s\in S} R_{s,k}(\tilde{p}_s) \quad < \quad \frac{\lambda(\mu_0-\mu_1)}{2} \min_{i\in\mathcal{I}} \min_{(s,l)\in S\times\mathcal{L}} W_{s,l}^i;$$

where $(\theta_k,\zeta_k)_{k\in\mathcal{J}_u}$ are the functions that guarantee that contracts in $\mathcal{J}_u$ are ultimate sources of liquidity (see Definition 2). Notice that, the first condition above ensures that $(\widetilde{x}^i,\widetilde{z}^i)$ is trading feasible at prices $(p,q)$, the second requirement implies that $\widetilde{x}_0^i \ll w_0^i$, and the last inequality guarantees that, at each state of nature $s \in S$, debts can be paid with the resources that became available after the consumption of $\widetilde{x}_s^i$. Thus, the definition of $\mathbb{P}(M)$ guarantees that $(\widetilde{x}^i,\widetilde{z}^i) \in \mathring{C}^i(p,q)$. Hence, $\mathring{C}^i$ has non-empty values.[10]

To prove that $\mathring{C}^i$ is lower hemicontinuous, fix $(p,q) \in \mathbb{P}(M)$ and $(x^i,z^i) \in \mathring{C}^i(p,q)$. Given a sequence $\{(p_n,q_n)\}_{n\in\mathbb{N}} \subset \mathbb{P}(M)$ that converges to $(p,q)$, the lower hemicontinuity of $\Phi^i$ (Assumption (A3)) ensures that there exists a sequence $\{(x^i(n),z^i(n))\}_{n\in\mathbb{N}} \subset \mathbb{E}$ converging to $(x^i,z^i)$ such that $(x^i(n),z^i(n)) \in$

[9] Trading constraints are not necessarily homogeneous of degree zero in prices. Consequently, the normalization of prices may induce a selection of equilibria.

[10] Dividing by $\max\{\#\mathcal{J}_u,1\}$ we ensure that the arguments above still hold when $\mathcal{J}_u$ is an empty set.

$\Phi^i(p_n, q_n)$, $\forall n \in \mathbb{N}$. Thus, for $n \in \mathbb{N}$ large enough, $(x^i(n), z^i(n)) \in \mathring{C}^i(p_n, q_n)$. It follows from the sequential characterization of hemicontinuity that $\mathring{C}^i$ is lower hemicontinuous (see Border (1985, 11.11(b))). $\square$

For notation convenience, let $(\widehat{x}, (\widehat{z}_k)_{k\in\mathcal{J}_u})$ be an allocation that allows agents to super-replicate financial payments when second period commodity prices belong to $\mathcal{P}_1^S$. Also, define

$$\overline{Q} := \max\left\{1,\ \|\widehat{x}\|_\Sigma + \max_{k\in\mathcal{J}_u} \widehat{z}_k\right\};$$

$$\overline{\Omega} := 2 \sup_{(p,q)\in\mathbb{P}(\overline{Q}):\,(p,q)\gg 0}\ \sup_{(x^i,z^i)_{i\in\mathcal{I}}\in\Omega(p,q)} \sum_{i\in\mathcal{I}} \|z^i\|_\Sigma.$$

Notice that, Assumption (A5)(i) guarantees that $\overline{\Omega}$ is finite.

Given $(p,q) \in \mathbb{P}(M)$, for any $i \in \mathcal{I}$ we consider the truncated choice set $C^i(p,q) \cap \mathbb{K}$, where

$$\mathbb{K} := \left[0, 2\overline{W}\right]^{\mathcal{L}\times\mathcal{S}} \times \left[-\overline{\Omega},\ \#\mathcal{I}\ \overline{\Omega}\right]^{\mathcal{J}},$$

$$\overline{W} := \left(\#J\,\#\mathcal{I}\ \overline{\Omega} + \sum_{(s,l)\in\mathcal{S}\times\mathcal{L}}\ \sum_{i\in\mathcal{I}} W^i_{s,l}\right)\left(1 + \max_{s\in S}\ \max_{p_s\in\mathcal{P}_1}\ \sum_{j\in\mathcal{J}} R_{s,j}(p_s)\right).$$

Let $\Psi_M : \mathbb{P}(M)\times\mathbb{K}^{\mathcal{I}} \twoheadrightarrow \mathbb{P}(M)\times\mathbb{K}^{\mathcal{I}}$ be the correspondence given by

$$\Psi_M(p,q,(x^i,z^i)_{i\in\mathcal{I}}) = \phi_{0,M}((x^i_0,z^i)_{i\in\mathcal{I}}) \times \prod_{s\in S}\phi_s((x^i_s)_{i\in\mathcal{I}}) \times \prod_{i\in\mathcal{I}}\phi^i(p,q),$$

where

$$\begin{aligned}
\phi_{0,M}((x^i_0,z^i)_{i\in\mathcal{I}}) &:= \operatorname*{argmax}_{(p_0,q)\in\mathcal{P}_0\times[0,M]^{\mathcal{J}\setminus\mathcal{J}_u}} p_0\cdot\sum_{i\in\mathcal{I}}(x^i_0 - w^i_0) + q\cdot\sum_{i\in\mathcal{I}} z^i;\\
\phi_s((x^i_s)_{i\in\mathcal{I}}) &:= \operatorname*{argmax}_{p_s\in\mathcal{P}_1} \quad p_s\cdot\sum_{i\in\mathcal{I}}(x^i_s - W^i_s), \quad \forall s\in S;\\
\phi^i(p,q) &:= \operatorname*{argmax}_{(x^i,z^i)\in C^i(p,q)\cap\mathbb{K}} \quad V^i(x^i), \quad \forall i\in\mathcal{I}.
\end{aligned}$$

Lemma 2. *Under Assumptions (A1)-(A5)(i), $\Psi_M$ has a non-empty set of fixed points.*

Proof. By Kakutani's Fixed Point Theorem, it is sufficient to to prove that $\Psi_M$ has a closed graph with non-empty and convex values. Since $\mathbb{P}(M)$ is non-empty, convex and compact, Berge's Maximum Theorem establishes that $\{\phi_{0,M}, \{\phi_s\}_{s\in S}\}$ have a closed graph with non-empty and convex values.

It remains to prove that the same properties hold for $\{\phi^i\}_{i\in\mathcal{I}}$. Given $i \in \mathcal{I}$, Lemma 1 implies that $C^i$ has a closed graph with non-empty and convex values. Since $\mathbb{K}$ is compact and convex and $(W^i, 0) \in \mathbb{K}$, it follows that $(p,q) \in \mathbb{P}(M) \twoheadrightarrow C^i(p,q)\cap\mathbb{K}$ has a closed graph and non-empty, compact, and convex values. The proof of Lemma 1 also ensures that $C^i$ is lower hemicontinuous and $(W^i,0) \in C^i(p,q)\cap\operatorname{int}(\mathbb{K})$. As $(p,q)\in\mathbb{P}(M)\twoheadrightarrow\operatorname{int}(\mathbb{K})$ has open graph, it follows that $(p,q)\in\mathbb{P}(M)\twoheadrightarrow C^i(p,q)\cap\operatorname{int}(\mathbb{K})$ is lower hemicontinuous (see Border (1985, 11.21(c))). Therefore, $(p,q)\in\mathbb{P}(M)\twoheadrightarrow C^i(p,q)\cap\mathbb{K}$ is lower hemicontinuous too (see Border (1985, 11.19(c))). Berge's Maximum Theorem and the continuity and quasi-concavity of $V^i$ guarantees that $\phi^i$ satisfies the required properties. $\square$

Lemma 3. *Under Assumptions (A1)-(A5)(i) and (A6), assume that agents can super-replicate financial payments investing in assets $\mathcal{J}_u$ and buying commodities. Let $(\overline{p},\overline{q},(\overline{x}^i,\overline{z}^i)_{i\in\mathcal{I}})$ be a fixed point of $\Psi_M$ such that $\overline{p}\gg 0$ and*

$$\sum_{i\in\mathcal{I}}\overline{z}^i_k\leq 0,\ \forall k\in\mathcal{J}_u;\qquad\qquad \sum_{i\in\mathcal{I}}\overline{x}^i_{s,l}<2\overline{W},\ \forall(s,l)\in\mathcal{S}\times\mathcal{L}.$$

*Then, for any $j\notin\mathcal{J}_u$ we have that,*

$$\overline{q}_j>0\ \ \wedge\ \sum_{i\in\mathcal{I}}\overline{z}^i_j>0\ \Longrightarrow\ \overline{q}_j\leq\overline{Q}\,.$$

Proof. Let $j\notin\mathcal{J}_u$ such that $\overline{q}_j>0$ and $\sum_{i\in\mathcal{I}}\overline{z}^i_j>0$. Due to $\overline{p}\gg 0$, it follows from (A2) that $(R_{s,j}(\overline{p}_s))_{s\in S}\neq 0$. Hence, there exists $a\in S$ such that $\sum_{r\notin\mathcal{J}_u}R_{a,r}(\overline{p}_a)>0$. Since financial payments can be super-replicated by investments in $\mathcal{L}\cup\mathcal{J}_u$, it follows from Definition 3 that

$$\sum_{r\notin\mathcal{J}_u}R_{a,r}(\overline{p}_a)<\overline{p}_aY_a\widehat{x}+\sum_{k\in\mathcal{J}_u}R_{a,k}(\overline{p}_a)\widehat{z}_k\leq\overline{p}_aY_a\widehat{x}+\left(\max_{k\in\mathcal{J}_u}\ \widehat{z}_k\right)\sum_{k\in\mathcal{J}_u}R_{a,k}(\overline{p}_a).$$

We affirm that,

$$\overline{q}_j\quad\leq\quad\overline{p}_0\widehat{x}+\left(\max_{k\in\mathcal{J}_u}\ \widehat{z}_k\right)\sum_{k\in\mathcal{J}_u}\overline{q}_k.$$

Let $i$ be an agent that invests in asset $j$. If the inequality above does not hold, then there is $\varepsilon>0$ such that, $i$ can reduce her long position on asset $j$ in $\varepsilon\overline{z}^i_j$ units, change her first-period consumption to $\overline{x}^i_0+\varepsilon\overline{z}^i_j\widehat{x}$, and increase in $(\max_{r\in\mathcal{J}_u}\ \widehat{z}_r)\varepsilon\overline{z}^i_j$ units the investment in each $k\in\mathcal{J}_u$.[11] With this strategy, $i$ changes her wealth at state of nature $s\in S$ by

$$\left(\overline{p}_sY_s\widehat{x}+\left(\max_{k\in\mathcal{J}_u}\ \widehat{z}_k\right)\sum_{k\in\mathcal{J}_u}R_{s,k}(\overline{p}_s)-R_{s,j}(\overline{p}_s)\right)\varepsilon\overline{z}^i_j\geq 0,$$

where the last inequality follows from Definition 3 and holds as strict inequality for $s=a$. This contradicts the optimality of $(\overline{x}^i,\overline{z}^i)$ on $C^i(\overline{p},\overline{q})\cap\mathbb{K}$. We conclude that $\overline{q}_j\leq\overline{Q}$. □

Lemma 4. *Under Assumptions (A1)-(A5)(i) and (A6), assume that agents can super-replicate financial payments investing in assets $\mathcal{J}_u$ and buying commodities. Then, for any $M>\overline{Q}$ the fixed points of $\Psi_M$ are competitive equilibria.*

Proof. Given $M>\overline{Q}$, let $(\overline{p},\overline{q},(\overline{x}^i,\overline{z}^i)_{i\in\mathcal{I}})$ be a fixed point of $\Psi_M$. Adding first period budget constraints across agents, the definition of $\phi_{0,M}$ guarantees that,

$$p_0\cdot\sum_{i\in\mathcal{I}}(\overline{x}^i_0-w^i_0)+q\cdot\sum_{i\in\mathcal{I}}\overline{z}^i\leq\overline{p}_0\cdot\sum_{i\in\mathcal{I}}(\overline{x}^i_0-w^i_0)+\overline{q}\cdot\sum_{i\in\mathcal{I}}\overline{z}^i\leq 0,\quad\ \forall\,(p_0,q)\in\mathcal{P}_0\times[0,M]^{\mathcal{J}\setminus\mathcal{J}_u}.$$

Hence,

$$\sum_{i\in\mathcal{I}}(\overline{x}^i_0-w^i_0)\leq 0,\qquad\qquad \sum_{i\in\mathcal{I}}\overline{z}^i_k\leq 0,\ \forall k\in\mathcal{J}_u,$$

[11]As the new strategy needs to be on $\mathbb{K}$, the value of $\varepsilon$ may depend on $(\overline{q}_j,\overline{x}^i_0,(\overline{z}^i_k)_{k\in\mathcal{J}_u},\overline{z}^i_j)$.

and $\overline{q}_j = M$ for every $j \notin \mathcal{J}_u$ such that $\sum_{i\in\mathcal{I}} \overline{z}^i_j > 0$. Furthermore, adding individual budget constraints at any state of nature in the second period, the definition of $\mathbb{K}$ guarantees that,

$$p_s \cdot \sum_{i\in\mathcal{I}} (\overline{x}^i_s - W^i_s) \leq \overline{p}_s \cdot \sum_{i\in\mathcal{I}} (\overline{x}^i_s - W^i_s) \leq \overline{W}, \quad \forall p_s \in \mathcal{P}_1,\, \forall s \in S.$$

We obtain that $\sum_{i\in\mathcal{I}} \overline{x}^i_{s,l} < 2\overline{W}$, $\forall (s,l) \in \mathcal{S}\times\mathcal{L}$, which implies that $\overline{p} \gg 0$. In another case, Assumptions (A1) and (A4)(i) guarantee that at least one agent can improve her utility by increasing her consumption without additional costs. A contradiction to the optimality of plans $(\overline{x}^i, \overline{z}^i)_{i\in\mathcal{I}}$.

The strict positivity of commodity prices has several consequences. First, by Assumption (A2) asset promises are non-trivial and Assumptions (A1) and (A4)(ii) ensure that asset prices are strictly positive. Second, as $(\overline{p}, \overline{q}, (\overline{x}^i, \overline{z}^i)_{i\in\mathcal{I}})$ satisfies the hypotheses of Lemma 3 and $M > \overline{Q}$, we obtain that $\sum_{i\in\mathcal{I}} \overline{z}^i \leq 0$. Third, Assumption (A1) guarantees that budget constraints are satisfied by equality.

We conclude that,

$$(\overline{p}, \overline{q}) \in \mathbb{P}(\overline{Q}), \qquad (\overline{p}, \overline{q}) \gg 0, \qquad \sum_{i\in\mathcal{I}} (\overline{x}^i - W^i) = 0, \qquad \sum_{i\in\mathcal{I}} \overline{z}^i = 0,$$

and Assumption (A5)(i) implies that $(\overline{x}^i, \overline{z}^i)_{i\in\mathcal{I}} \in \Omega(\overline{p}, \overline{q}) \cap \operatorname{int}(\mathbb{K})$.[12]

As for any $i \in \mathcal{I}$ the allocation $(\overline{x}^i, \overline{z}^i)$ belongs to $C^i(\overline{p}, \overline{q}) \cap \operatorname{int}(\mathbb{K})$, given $(x^i, z^i) \in C^i(\overline{p}, \overline{q})$ with $x^i \neq \overline{x}^i$ there exists $\lambda \in (0,1)$ such that $\lambda(\overline{x}^i, \overline{z}^i) + (1-\lambda)(x^i, z^i) \in C^i(\overline{p}, \overline{q}) \cap \mathbb{K}$. The strongly quasi-concavity of utility functions (Assumption (A1)) implies that,

$$V^i(\lambda(\overline{x}^i, \overline{z}^i) + (1-\lambda)(x^i, z^i)) > \min\{V^i(\overline{x}^i), V^i(x^i)\}.$$

Since $(\overline{x}^i, \overline{z}^i) \in \phi^i(\overline{p}, \overline{q})$, we obtain that $V^i(x^i) < V^i(\overline{x}^i)$. Thus, $(\overline{x}^i, \overline{z}^i)$ is an optimal choice for agent $i$ in $C^i(\overline{p}, \overline{q})$, which concludes the proof. □

## Appendix C

Proof of Corollary 1. Notice that, given a compact set $\mathbb{P}_1 \subset (\mathbb{R}^{\mathcal{L}}_+ \setminus \{0\})^S$, the allocation

$$(\widehat{x}, (\widehat{z}_k)_{k\in\mathcal{J}_u}) := \max_{p\in\mathbb{P}_1} \sum_{s\in S} \left( \sum_{k\notin\mathcal{J}_u} R_{s,k}(p_s)/R_{s,j}(p_s) \right) \widehat{e}_j$$

satisfies the conditions of Definition 3. Thus, agents can super-replicate financial payments just by investing in asset $j$. □.

Proof of Corollary 2. Since all assets are ultimate sources of liquidity, the results of Lemma 3 are not necessary to ensure equilibrium existence. Thus, Assumption (A6) can be dispensed. □.

[12] Assumption (A5)(i) is only required to ensure that $\left[(\overline{p}, \overline{q}) \gg 0 \;\wedge\; (\overline{x}^i, \overline{z}^i)_{i\in\mathcal{I}} \in \Omega(\overline{p}, \overline{q})\right] \Longrightarrow (\overline{x}^i, \overline{z}^i)_{i\in\mathcal{I}} \in \operatorname{int}(\mathbb{K})$. Notice that, if we change the upper bound $\overline{\Omega}$ for an arbitrary positive number in the definition of $\mathbb{K}$, then all the other arguments in the proof of Theorem 1 still hold.

Proof of Corollary 3. Since assets are either backed by physical collateral or have payments measured in units of a non-perishable commodity (i.e., a commodity $l \in \mathcal{L}$ such that $Y_s(l,l) > 0$, $\forall s \in S$), it follows that agents can super-replicate the financial payments by buying commodities. Thus, the result of Lemma 3 can be obtained without assuming that assets in $\mathcal{J}_u$ have unrestricted investment (Assumption (A6)(i)).□

## Appendix D: Proof of Theorem 2

Let $\mathcal{P} = \{(p_s)_{s\in\mathcal{S}} \in \mathbb{R}_+^{\mathcal{L}\times\mathcal{S}} : \|p_s\|_\Sigma \leq 1,\ \forall s \in \mathcal{S}\}$.

Given $N > 0$, define

$$\widehat{\mathbb{K}}(N) := \left[0, 2\widehat{W} + N\right]^{\mathcal{L}\times\mathcal{S}} \times \left[-\widehat{\Omega},\ \#\mathcal{I}\,\widehat{\Omega}\right]^{\mathcal{J}},$$

where

$$\widehat{W} := \left(\#J\,\#\mathcal{I}\,\widehat{\Omega} + \sum_{(s,l)\in\mathcal{S}\times\mathcal{L}}\ \sum_{i\in\mathcal{I}} W^i_{s,l}\right)\left(1 + \max_{s\in S}\ \max_{p_s\in\mathcal{P}_1}\ \sum_{j\in\mathcal{J}} R_{s,j}(p_s)\right),$$

$$\widehat{\Omega} := 2 \sup_{(p,q)\in\mathcal{P}\times\mathbb{R}^{\mathcal{J}}_+ :\, (p,q)\gg 0}\ \ \sup_{(x^i,z^i)_{i\in\mathcal{I}}\in\Omega(p,q)}\ \sum_{i\in\mathcal{I}} \|z^i\|_\Sigma.$$

Notice that, Assumption (A5)(ii) guarantees that $\widehat{\Omega}$ is finite.

Let $\Psi_{(M,N)}$ be the correspondence obtained by replacing $\mathbb{K}$ by $\widehat{\mathbb{K}}(N)$ in the definition of $\Psi_M$. Hence, identical arguments to those given in the proof of Theorem 1 ensure that, under Assumptions (A1)-(A5)(ii), and even with $\mathcal{J}_u = \emptyset$, the results of Lemmata 1 and 2 still hold. Thus, we focus on the determination of upper bounds for prices $(\overline{q}_j)_{j\notin\mathcal{J}_u}$.

Lemma 5. *Under Assumptions (A1)-(A7), let $(\overline{p},\overline{q},(\overline{x}^i,\overline{z}^i)_{i\in\mathcal{I}})$ be a fixed point of $\Psi_{(M,N)}$ satisfying $\overline{x}^i_{s,l} < 2\widehat{W}$, $\forall (s,l)\in\mathcal{S}\times\mathcal{L}$. Then, there is $\widehat{Q} > 0$ such that, for $N$ large enough, we have that $\overline{q}_j \leq \widehat{Q}$, $\forall j \notin \mathcal{J}_u$.*

Proof. For any $i \in \mathcal{I}^*$, let $\rho^i \in (0,1)$ such that $2\widehat{W}\rho^i = 0.25 \min_{l\in\mathcal{L}} W^i_{s,l}$. Hence, Assumption (A7)(i) and the strict monotonicity of preferences imply that,

$$V^i(\overline{x}^i) \leq V^i(2\widehat{W}(1,\ldots,1)) < V^i\left(2\widehat{W}(1,\ldots,1) + \tau^i(\rho^i, 2\widehat{W}), \left(\frac{W^i_s}{2}\right)_{s\in S}\right).$$

Fix $j \notin \mathcal{J}_u$ and $i = i(j) \in \mathcal{I}^*$ satisfying Assumption (A7)(ii). Then, there is $z^i \geq 0$ such that $z^i_j > 0$ and $-(0,z^i) \in \Phi^i(p,q)$, $\forall (p,q) \in \mathbb{P}$. Since $\Phi^i$ has convex values and $(0,0) \in \Phi^i(\overline{p},\overline{q})$, it follows that $-(0,\varepsilon z^i) \in \Phi^i(\overline{p},\overline{q})$, $\forall \varepsilon \in [0,1]$. Also, Assumption (A2) ensures that there is $\varepsilon^i \in (0,1)$ such that, for any state of nature $s \in S$, $\varepsilon^i \max_{p_s\in\mathcal{P}_1} \sum_{k\in\mathcal{J}} R_{s,k}(p_s)\, z^i_k < \left(\min_{l\in\mathcal{L}} W^i_{s,l}\right)/2$.

Then, for each $N > \widehat{N} := \max_{i\in\mathcal{I}^*} \|\tau^i(\rho^i, 2\widehat{W})\|_\Sigma$ we have that

$$\left(\left(2\widehat{W}(1,\ldots,1) + \tau^i(\rho^i, 2\widehat{W}), \left(\frac{W^i_s}{2}\right)_{s\in\mathcal{S}}\right), -\varepsilon^i z^i\right) \ \in\ \Phi^i(\overline{p},\overline{q}) \cap \widehat{\mathbb{K}}(N).$$

Consequently, as $(\overline{x}^i,\overline{z}^i)$ is an optimal choice for agent $i$ in $C^i(\overline{p},\overline{q}) \cap \widehat{\mathbb{K}}(N)$, it follows that

$$2\widehat{W}\|\overline{p}_0\|_\Sigma + \overline{p}_0 \cdot (\tau^i(\rho^i, 2\widehat{W}) - w^i_0) > \varepsilon^i\, \overline{q}\cdot z^i \geq \varepsilon^i \overline{q}_j z^i_j,$$

which implies that $\overline{q}_j \leq (2\widehat{W} + \widehat{N})/(\varepsilon^i z^i_j)$. Since $i = i(j)$ was fixed, we can consider

$$\widehat{Q} := \max_{j \notin \mathcal{J}_u} \frac{2\widehat{W} + \widehat{N}}{\varepsilon^{i(j)} z_j^{i(j)}}. \qquad \square$$

Lemma 6. *Under Assumptions (A1)-(A7), fix $(M, N) \gg (\widehat{Q}, \widehat{N})$. Then, each fixed point of $\Psi_{(M,N)}$ is a competitive equilibria.*

This result follows from analogous arguments to those made in the proof of Lemma 4.

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