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Strong equilibrium in games with common and complementary local utilities

Nikolai S. Kukushkin*

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Abstract

A rather general class of strategic games is described where the coalition improvements are acyclic and hence strong equilibria exist: The players derive their utilities from the use of certain "facilities"; all players using a facility extract the same amount of "local utility" therefrom, which amount depends both on the set of users and on their actions, and is decreasing in the set of users; the "ultimate" utility of each player is the minimum of the local utilities at all relevant facilities. Two important subclasses are "games with structured utilities," basic properties of which were discovered in 1970s and 1980s, and "bottleneck congestion games," which attracted researchers' attention quite recently. The former games are representative in the sense that every game from the whole class is isomorphic to one of them. The necessity of the minimum aggregation for the "persistent" existence of strong equilibria, actually, just Pareto optimal Nash equilibria, is established. *MSC2010 Classification*: 91A10; *JEL Classification*: C 72.

Key words: Strong equilibrium; Weakest-link aggregation; Coalition improvement path; Congestion game; Game with structured utilities

1 Introduction

Both motivation for and the structure of this paper closely resemble those of Kukushkin (2007). Moreover, the models considered in either paper, when described in very general terms, sound quite similarly.

The players derive their utilities from the use of certain objects. Rosenthal (1973) called them "factors"; following Monderer and Shapley (1996) as well as Holzman and Law-Yone (1997), we call them "facilities" here. The players are free to choose facilities within certain limits. All the players using a facility extract the same amount of "local utility" therefrom, which amount may depend both on the set of users and on their actions. The "ultimate" utility of each player is an aggregate of the local utilities obtained from all relevant facilities.

Four crucial differences should be listed at the start. First, in Kukushkin (2007), following Rosenthal (1973), each player summed up relevant local utilities (strictly speaking, monotone transformations were allowed); here, each player takes into account only the worst local utility (again, monotone transformations may be allowed).

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Second, the main results of Kukushkin (2007) were about the acyclicity of *individual* improvements and, accordingly, the existence of Nash equilibria. Here, it is about the acyclicity of *coalitional* improvements and, accordingly, the existence of strong equilibria.

Thirdly, the games for which the main results of Kukushkin (2007) hold are naturally partitioned into two classes: "generalized congestion games" and "games with structured utilities." In the former class, the players choose *which* facilities to use and do not choose anything else; in the latter, each player chooses *how* to use facilities from a fixed list. Here, both those classes are present too, but we also allow games combining both types of choice, i.e., "which" and "how." It should be mentioned that, both here and in Kukushkin (2007), games with structured utilities form a representative subclass.

Finally, we have to assume here that whenever a new player starts to use a facility, those already there are not better off (even if are not hurt); the only exception is Proposition 4.5, which, characteristically, is about individual improvements only. In Kukushkin (2007), there was no need for such an assumption.

The idea of games with structured utilities and the minimum aggregation originated in Germeier and Vatel' (1974) although in a much less general form. Their approach was developed further in a series of papers, see Kukushkin et al. (1985) and references therein.

In Moulin (1982, Chapter 5), pirates were going to a treasure island; each pirate could choose between two ships, and the more pirates on board of either ship, the slower it went. The game was a particular case of Rosenthal's model, but the existence of a strong equilibrium, as in Germeier–Vatel's model, was established. It is remarkable that, since each player could only use a single facility (ship), we may assume that the minimum aggregation was applied and, therefore, the existence of a strong equilibrium (and even the acyclicity of coalition improvements) was to be expected.

The fact that the minimum aggregation and negative impacts in congestion games are conducive to coalition stability was gradually noticed quite recently (Fotakis et al., 2008; Harks et al., 2013).

Theorem 4.1 of this paper unifies and strengthens all those results. As long as each player uses the minimum aggregation and there are negative impacts at each facility, it does not matter which subsets of facilities and what methods of using them are available to each player: all coalition improvements are acyclic (to be more precise, there exists a "strong ω-potential") and hence strong equilibria exist and, in a sense, attract all adaptive dynamics.

Theorem 4.6 shows that every game satisfying the assumptions of Theorem 4.1 is isomorphic to a game with structured utilities and the minimum aggregation. In other words, the main findings of Kukushkin et al. (1985) remain relevant to every model of this type that has been considered since then. That paper, however, was silent on some important issues, e.g., algorithmic and computational aspects.

Perhaps the most interesting results of this paper are Theorems 6.1 and 6.3 which establish the necessity of the minimum aggregation for the "persistent" existence of Pareto optimal Nash equilibria, to say nothing of strong equilibria, and hence for the acyclicity of coalition improvements as well. The first result of this kind was in Kukushkin (1992); however, it was designed for a particular class of games, so rather peculiar combinations of the minimum and maximum were allowed, which are not good in a more general case.

The minimum operator is not at all unusual in the theory of production functions. Galbraith

(1958, Chapter XVIII) explicitly invoked Leontief's model to justify an attitude to public and private consumption ("social balance") that sounds indistinguishable from the minimum aggregation. Our Theorem 4.1 shows that players who have internalized this attitude do not need any taxes to provide for an efficient level of public consumption; it is difficult to say whether Galbraith himself expected such a conclusion.

The "weakest-link" aggregation has recently become rather popular in models of communication networks, "bottleneck congestion games," see Harks et al. (2013) and references therein.

Section 2 introduces principal improvement relations associated with a strategic game. Section 3 provides a formal description of our basic model as well as its main structural properties. Throughout Section 4, the players use the minimum aggregation. The main results there are Theorems 4.1 and 4.6.

In Section 5, we consider the maximum aggregation rule, which has the same implications in games with positive impacts (Theorem 5.1); moreover, infinite sets of facilities can be treated in this case (Theorem 5.4). The leximin/leximax aggregation of local utilities is also considered there; it ensures the acyclicity of individual improvements, but not of coalitional ones.

Section 6 contains the characterization results, Theorems 6.1 and 6.3, which establish the necessity of the minimum aggregation for the "persistent" existence of Pareto optimal Nash equilibria. In Section 7, some related questions of secondary importance are discussed.

More complicated proofs, concerning the necessity of the minimum aggregation for the acyclicity of strong coalition improvements (actually, just for the existence of a Pareto optimal Nash equilibrium) regardless of all other characteristics of the game, are deferred to the Appendix. The two sections differ in their context: Section A is about generalized congestion games; Section B, games with structured utilities.

2 Improvement dynamics in strategic games

A *strategic game* Γ is defined by a finite set of players N (we denote $n = \#N$), and strategy sets X_i and utility functions u_i on $X_N = \prod_{i \in N} X_i$ for all $i \in N$. We denote $\mathcal{N} = 2^N \setminus \{\emptyset\}$ (the set of potential coalitions) and $X_I = \prod_{i \in I} X_i$ for each $I \in \mathcal{N}$; instead of $X_{N \setminus \{i\}}$ and $X_{N \setminus I}$, we write X_{-i} and X_{-I}, respectively. It is sometimes convenient to consider utility functions u_i as components of a "joint" mapping $u_N \colon X_N \to \mathbb{R}^N$.

With every strategic game, a few improvement relations on X_N are associated ($I \in \mathcal{N}$, $y_N, x_N \in X_N$):

$$y_N \rhd_I x_N \rightleftharpoons \big[y_{-I} = x_{-I} \;\&\; \forall i \in I\, [u_i(y_N) > u_i(x_N)]\big]; \tag{1a}$$

$$y_N \rhd^{\mathrm{Ind}} x_N \rightleftharpoons \exists i \in N\, [y_N \rhd_{\{i\}} x_N] \tag{1b}$$

(*individual improvement* relation);

$$y_N \rhd^{\mathrm{Coa}} x_N \rightleftharpoons \exists I \in \mathcal{N}\, [y_N \rhd_I x_N] \tag{1c}$$

(*strong coalition improvement* relation).

A *maximizer* of an improvement relation $\rhd$, i.e., a strategy profile $x_N \in X_N$ such that $y_N \rhd x_N$ holds for no $y_N \in X_N$, is an equilibrium: a Nash equilibrium if $\rhd$ is $\rhd^{\mathrm{Ind}}$; a strong equilibrium if $\rhd$ is $\rhd^{\mathrm{Coa}}$.

An *individual improvement path* is a (finite or infinite) sequence $\{x_N^k\}_{k=0,1,\ldots}$ such that $x_N^{k+1} \rhd^{\mathrm{Ind}} x_N^k$ whenever x_N^{k+1} is defined; an *individual improvement cycle* is an individual improvement path such that $x_N^m = x_N^0$ for $m > 0$. A strategic game has the *finite individual improvement property* (*FIP*) if there exists no infinite individual improvement path; then every individual improvement path, if continued whenever possible, reaches a Nash equilibrium in a finite number of steps.

Replacing $\rhd^{\mathrm{Ind}}$ with $\rhd^{\mathrm{Coa}}$, we obtain the definitions of a *coalition improvement path*, a *coalition improvement cycle*, and the *finite coalition improvement property* (*FCP*). The latter implies that every coalition improvement path reaches a strong equilibrium in a finite number of steps.

For a finite game, the FIP (FCP) is equivalent to the acyclicity of the relation $\rhd^{\mathrm{Ind}}$ ($\rhd^{\mathrm{Coa}}$) and is equivalent to the existence of a "potential" in the following sense. An *order potential* of Γ is an irreflexive and transitive relation $\succ$ on X_N satisfying

$$\forall x_N, y_N \in X_N\, [y_N \rhd^{\mathrm{Ind}} x_N \Rightarrow y_N \succ x_N]. \tag{2}$$

A *strong order potential* of Γ is an irreflexive and transitive relation $\succ$ on X_N satisfying

$$\forall x_N, y_N \in X_N\, [y_N \rhd^{\mathrm{Coa}} x_N \Rightarrow y_N \succ x_N]. \tag{3}$$

Generally, the absence of finite cycles does not mean very much, so we employ a more demanding notion of a potential.

A binary relation $\succ$ on a metric space X_N is *ω-transitive* if it is transitive and the conditions $x_N^\omega = \lim_{k\to\infty} x_N^k$ and $x_N^{k+1} \succ x_N^k$ for all $k = 0, 1, \ldots$ always imply $x_N^\omega \succ x_N^0$.

Remark. Gillis (1959) and Smith (1974) considered this condition for orderings.

A *strong ω-potential* of Γ is an irreflexive and ω-transitive relation $\succ$ on X_N satisfying (3). By Theorem 1 of Kukushkin (2008), $\succ$ admits a maximizer on X_N if the latter is compact; as follows immediately from (3), every maximizer of $\succ$ is a strong equilibrium.

3 Common local utilities

A *game with common local utilities* may have an arbitrary (finite) set of players N and arbitrary sets of strategies X_i whereas the utility functions satisfy certain structural requirements. There is a finite set A of *facilities*. For every $i \in N$, there is a mapping $\mathrm{B}_i : X_i \to 2^{\mathrm{A}} \setminus \{\emptyset\}$; $\mathrm{B}_i(x_i)$ is interpreted as the set of facilities which player i uses under the strategy x_i. With every $\alpha \in \mathrm{A}$, a list of functions $\varphi_\alpha(I, \cdot)\colon X_I \to \mathbb{R}$ ($I \in \mathcal{N}$) is associated. For every $i \in N$ and $x_i \in X_i$, there is a mapping $U_i^{x_i} \colon \mathbb{R}^{\mathrm{B}_i(x_i)} \to \mathbb{R}$, an *aggregation rule*.

Given a strategy profile $x_N \in X_N$, we denote $N(\alpha, x_N) = \{i \in N \mid \alpha \in \mathrm{B}_i(x_i)\}$ for each $\alpha \in \mathrm{A}$: the set of players using α at x_N. The "ultimate" utility functions of the players are built of the local utilities:

$$u_i(x_N) = U_i^{x_i}\big(\langle \varphi_\alpha(N(\alpha, x_N), x_{N(\alpha,x_N)})\rangle_{\alpha\in \mathrm{B}_i(x_i)}\big), \tag{4}$$

for all $i \in N$ and $x_N \in X_N$.

Remark. Clearly, only the values of $\varphi_\alpha(I, x_I)$ where $\alpha \in \mathrm{B}_i(x_i)$ for all $i \in I$ matter for (4); hence there is no need to define φ_α for pairs (I, x_I) not satisfying this condition. "Dually," if $\alpha \in \mathrm{A}$ and $i \in I$ are such that $\alpha \in \mathrm{B}_i(x_i)$ for all $x_i \in X_i$, then functions $\varphi_\alpha(I, \cdot)$ with $i \notin I$ are superfluous and hence could be deleted from the description of the game. We disregard such details here.

When considering infinite strategy sets, we impose appropriate topological assumptions. Each X_i is always a compact metric space and each function B_i is continuous, i.e., constant on each connected component of X_i. Every function $\varphi_\alpha(I, \cdot)$ is, at least, upper semicontinuous. Every function $U_i^{x_i}$ is increasing and continuous.

We say that player i has a *negative impact* on facility α if for each $I \in \mathcal{N}$ such that $i \notin I$, every $x_i \in X_i$ such that $\alpha \in \mathrm{B}(x_i)$, and every $x_I^\alpha \in X_I$ such that $\alpha \in \mathrm{B}(x_j^\alpha)$ for all $j \in I$,

$$\varphi_\alpha(I, x_I^\alpha) \geq \varphi_\alpha(I \cup \{i\}, \langle x_I^\alpha, x_i \rangle). \tag{5}$$

We say that player i has a *strictly negative impact* on facility α if the inequality in (5) is strict. We call Γ a *game with* (*strictly*) *negative impacts* if the appropriate condition holds for all $i \in N$ and $\alpha \in \mathrm{A}$. A definition of (*strictly*) *positive impacts* is obtained by reversing the inequality sign in (5).

The class of games with common local utilities includes both classes of games considered in Kukushkin (2007): "generalized congestion games" and "games with structured utilities." In a *generalized congestion game*, $X_i \subseteq 2^{\mathrm{A}} \setminus \emptyset$, $\mathrm{B}_i(x_i) = x_i$ (i.e., each player chooses just a set of facilities), and φ_α only depends on $\#I$ (so we use the notation $\varphi_\alpha(k)$ rather then $\varphi_\alpha(I, x_I)$ in this case). Rosenthal's (1973) congestion games proper are distinguished by additive aggregation of local utilities.

If, conversely, each B_i is a constant on the whole X_i, the game is a *game with structured utilities* as defined in Kukushkin (2007); in such games, each $\Upsilon_i = \mathrm{B}_i(x_i)$ is treated as a parameter of the model. We use the notation $N(\alpha) = \{i \in N \mid \alpha \in \Upsilon_i\}$ for such games; the local utility functions then are just $\varphi_\alpha \colon X_{N(\alpha)} \to \mathbb{R}$.

Henceforth, "a game" always means "a game with common local utilities."

4 Games with the minimum aggregation

Throughout this section, we assume that each player uses the minimum ("weakest-link") aggregation:

$$u_i(x_N) = \min_{\alpha \in \mathrm{B}_i(x_i)} \varphi_\alpha(N(\alpha, x_N), x_{N(\alpha, x_N)}) \tag{6}$$

for all $i \in N$ and $x_N \in X_N$. In economic terms, (6) means that all local utilities are absolute complements.

An important role in the study of such games is played by the leximin ordering on a (finite) Cartesian power of $\mathbb{R}$. Let us recall the standard definition.

Given a finite set M, $\#M = m$, and $v_M \in \mathbb{R}^M$, we denote $\pi(v_M) = \langle \pi_1(v_M), \ldots, \pi_m(v_M) \rangle$ the vector of the same values v_h for $h \in M$ in the increasing order: $\pi_1(v_M) \leq \cdots \leq \pi_m(v_M)$, and there is a one-to-one mapping $\sigma \colon \{1, \ldots, m\} \to M$ such that $\pi_h(v_M) = v_{\sigma(h)}$ for all h. Now we can define the ordering itself:

$$v'_M >_{\mathrm{Lmin}} v_M \rightleftharpoons \exists h \left[\pi_h(v'_M) > \pi_h(v_M) \;\&\; \forall h' < h \left[\pi_{h'}(v'_M) = \pi_{h'}(v_M)\right]\right]. \tag{7}$$

Obviously, $>_{\mathrm{Lmin}}$ is irreflexive and transitive. Two vectors $v_M, v'_M \in \mathbb{R}^M$ are incomparable if and only if $\pi(v_M) = \pi(v'_M)$; therefore, incomparability is an equivalence relation.

For further references, we also define the leximax ordering. The only difference is that we start with the greatest components when comparing two vectors.

$$v'_M >_{\mathrm{Lmax}} v_M \rightleftharpoons \exists h \left[\pi_h(v'_M) > \pi_h(v_M) \;\&\; \forall h' > h \left[\pi_{h'}(v'_M) = \pi_{h'}(v_M)\right]\right]. \tag{8}$$

Theorem 4.1. *Let Γ be a game with negative impacts where each player uses the minimum aggregation, i.e., conditions (5) and (6) hold everywhere. Let each X_i be a compact metric space, each B_i be continuous, and all functions $\varphi_\alpha(I, \cdot)$ be upper semicontinuous. Then Γ admits a strong ω-potential, and hence possesses a strong equilibrium.*

Proof. Considering utility functions u_i as components of a mapping $u_N \colon X_N \to \mathbb{R}^N$, we define $\succ$ on X_N by

$$y_N \succ x_N \rightleftharpoons u_N(y_N) >_{\mathrm{Lmin}} u_N(x_N),$$

where $>_{\mathrm{Lmin}}$ is the leximin ordering on $\mathbb{R}^N$ defined by (7). Obviously, $\succ$ is irreflexive and transitive.

Lemma 4.1.1. *$\succ$ is ω-transitive on X_N.*

Proof. For every $x_N \in X_N$, we denote $\vartheta(x_N) = \langle \vartheta_1(x_N), \dots, \vartheta_n(x_N)\rangle$ the vector of values $u_i(x_N)$ for $i \in N$ in the increasing order; in the above notation, $\vartheta_h(x_N) = \pi_h(u_N(x_N))$. Since each function u_i is upper semicontinuous in x_N, so is each ϑ_h.

Now let $x_N^{k+1} \succ x_N^k$ for all $k = 0, 1, \dots$ and $x_N^k \to x_N^\omega$; we have to show $x_N^\omega \succ x_N^0$. For each $k \in \mathbb{N}$, we denote $h(k)$ the h from (7) for $u_N(x_N^{k+1}) >_{\mathrm{Lmin}} u_N(x_N^k)$, i.e., $\vartheta_{h'}(x_N^{k+1}) = \vartheta_{h'}(x_N^k)$ for $h' < h(k)$ and $\vartheta_{h(k)}(x_N^{k+1}) > \vartheta_{h(k)}(x_N^k)$. Since N is finite, we may, replacing $\langle x_N^k \rangle_k$ with a subsequence if needed, assume that $h(k) = h$ does not depend on k. Now we have $\vartheta_{h'}(x_N^\omega) \geq \vartheta_{h'}(x_N^0)$ for $h' < h$ and $\vartheta_h(x_N^\omega) > \vartheta_h(x_N^0)$ by the upper semicontinuity; therefore, $u_N(x_N^\omega) >_{\mathrm{Lmin}} u_N(x_N^0)$. □

Lemma 4.1.2. *Given $y_N, x_N \in X_N$, we denote $N_+ = \{i \in N \mid u_i(y_N) > u_i(x_N)\}$ and $N_- = \{i \in N \mid u_i(y_N) < u_i(x_N)\}$. Let $\min_{i \in N_-} u_i(y_N) > \min_{i \in N_+} u_i(x_N)$, assuming $\min \emptyset = +\infty$. Then $y_N \succ x_N$.*

A straightforward proof is omitted.

Lemma 4.1.3. *$\succ$ satisfies (3).*

Proof. Supposing $y_N \rhd_I x_N$, we have to show $y_N \succ x_N$. If y_N Pareto dominates x_N, then we are home immediately. Let

$$u_j(y_N) < u_j(x_N); \tag{9}$$

then $j \notin I$, so $y_j = x_j$. By (6), there is $\alpha \in \mathrm{B}_j(y_j) = \mathrm{B}_j(x_j)$ such that $u_j(y_N) = \varphi_\alpha(N(\alpha, y_N), y_{N(\alpha, y_N)})$. Suppose $I \cap N(\alpha, y_N) = \emptyset$; then $N(\alpha, y_N) \subseteq N(\alpha, x_N)$ and $x_{N(\alpha, y_N)} = y_{N(\alpha, y_N)}$; hence $\varphi_\alpha(N(\alpha, x_N), x_{N(\alpha, x_N)}) \leq \varphi_\alpha(N(\alpha, y_N), y_{N(\alpha, y_N)})$ by (5); hence $u_j(x_N) \leq u_j(y_N)$, contradicting (9). Therefore, there must be $i \in I \cap N(\alpha, y_N)$ and hence $u_j(y_N) = \varphi_\alpha(N(\alpha, y_N), y_{N(\alpha, y_N)}) \geq u_i(y_N) > u_i(x_N)$. Since j satisfying (9) was arbitrary, Lemma 4.1.2 is applicable, implying $y_N \succ x_N$. □

Now Theorem 4.1 immediately follows from Lemmas 4.1.1 and 4.1.3. □

Without the negative impacts assumption, the existence of a strong equilibrium in generalized congestion games with the minimum aggregation cannot be ensured although a Nash equilibrium always exists. Without anonymity, there may be no equilibrium at all.

Example 4.2. Let us consider a two person generalized congestion game with the minimum aggregation (6): $N = \{1, 2\}$; $\mathrm{A} = \{a, b, c\}$; $X_1 = \{\mathrm{A}, \{a\}\}$, $X_2 = \{\mathrm{A}, \{b\}\}$; $\varphi_a(1) = \varphi_b(1) = 1$, $\varphi_a(2) = \varphi_b(2) = 3$, $\varphi_c(1) = 0$ $\varphi_c(2) = 2$ (i.e., every facility exhibits positive impacts). The matrix of the game looks as follows:

$$\begin{array}{ccc} & \text{abc} & \text{b} \\ \text{abc} & (2,2) & (0,3) \\ \text{a} & (3,0) & (1,1). \end{array}$$

We have a prisoner's dilemma.

Example 4.3. Let us consider a two person generalized congestion game: $N = \{1, 2\}$, $\mathrm{A} = \{a, b\}$, $X_1 = X_2 = \{\{a\}, \{b\}\}$, $\varphi_a(1) = 0$, $\varphi_a(2) = 2$, $\varphi_b(1) = 3$, $\varphi_b(2) = 1$ (i.e., a exhibits positive impacts; b, negative). The matrix of the game looks as follows:

$$\begin{array}{ccc} & \text{a} & \text{b} \\ \text{a} & (2,2) & (0,3) \\ \text{b} & (3,0) & (1,1). \end{array}$$

We have a prisoner's dilemma again.

Example 4.4. Let us consider a two person game where each player chooses a single facility, but there is no anonymity: $N = \{1, 2\}$, $\mathrm{A} = \{a, b\}$, $X_1 = X_2 = \{\{a\}, \{b\}\}$, $\varphi_a(\{2\}) = 0$, $\varphi_a(\{1\}) = 2$, $\varphi_a(N) = 4$, $\varphi_b(N) = 1$, $\varphi_b(\{2\}) = 3$, $\varphi_b(\{1\}) = 5$ (i.e., both facilities exhibit positive impacts). The matrix of the game looks as follows:

$$\begin{array}{ccc} & \text{a} & \text{b} \\ \text{a} & (4,4) & (2,3) \\ \text{b} & (5,0) & (1,1). \end{array}$$

There is no Nash equilibrium.

Proposition 4.5. *Every generalized congestion game where each player uses the minimum aggregation* (6) *admits an order potential* (2)*, and hence has the FIP and possesses a Nash equilibrium.*

Proof. We start with an observation that the leximin ordering $>_{\mathrm{Lmin}}$ is symmetric and separable. The first property is obvious; the second only needs a formal definition to become so:

$$\langle v_1, \ldots, v_m, v_{m+1}, \ldots, v_{m'} \rangle >_{\mathrm{Lmin}} \langle v'_1, \ldots, v'_m, v_{m+1}, \ldots, v_{m'} \rangle \iff \langle v_1, \ldots, v_m \rangle >_{\mathrm{Lmin}} \langle v'_1, \ldots, v'_m \rangle.$$

As the next step, we make the leximin ordering $>_{\mathrm{Lmin}}$ applicable to corteges of different lengths by adding $+\infty$'s to the shorter one. To be more precise, whenever $m' > m$ and the least m co-ordinates of $\langle v'_1, \ldots, v_{m'} \rangle$ coincide with respective (after re-arrangement) co-ordinates of $\langle v_1, \ldots, v_m \rangle$, we declare that $\langle v_1, \ldots, v_m \rangle >_{\mathrm{Lmin}} \langle v'_1, \ldots, v_{m'} \rangle$. The extended ordering remains symmetric and separable.

With every $x_N \in X_N$, we associate an unordered cortege:

$$\varkappa(x_N) = \Big\langle \varphi_\alpha(k) \Big\rangle_{\alpha \in \mathrm{A}\colon n(\alpha,x_N)>0,\ k=1,\dots,n(\alpha,x_N)}.$$

Now we define $\succ$ on X_N by $y_N \succ x_N \rightleftharpoons \varkappa(y_N) >_{\mathrm{Lmin}} \varkappa(x_N)$. To complete the proof, we have to check (2).

Let $y_N \rhd_{\{i\}} x_N$, i.e., $u_i(y_N) > u_i(x_N)$ and $y_{-i} = x_{-i}$. A is partitioned into four disjoint subsets: $\mathrm{A}^0 = x_i \cap y_i$, $\mathrm{A}^+ = y_i \setminus x_i$, $\mathrm{A}^- = x_i \setminus y_i$, $\mathrm{A}^* = \mathrm{A} \setminus (x_i \cup y_i)$; thus, $x_i = \mathrm{A}^0 \cup \mathrm{A}^-$ and $y_i = \mathrm{A}^0 \cup \mathrm{A}^+$. We define

$$\begin{aligned}\varkappa_{-i} = \Big\langle &\langle \varphi_\alpha(k) \rangle_{\alpha \in \mathrm{A}^0\colon n(\alpha,x_N)>1;\ k=1,\dots,n(\alpha,x_N)-1=n(\alpha,y_N)-1},\\ &\langle \varphi_\alpha(k) \rangle_{\alpha \in \mathrm{A}^+\colon n(\alpha,x_N)>0;\ k=1,\dots,n(\alpha,x_N)=n(\alpha,y_N)-1}, \langle \varphi_\alpha(k) \rangle_{\alpha \in \mathrm{A}^-\colon n(\alpha,y_N)>0;\ k=1,\dots,n(\alpha,y_N)=n(\alpha,x_N)-1},\\ &\langle \varphi_\alpha(k) \rangle_{\alpha \in \mathrm{A}^*\colon n(\alpha,x_N)>0;\ k=1,\dots,n(\alpha,x_N)=n(\alpha,y_N)} \Big\rangle;\end{aligned}$$

$$\varkappa_i(x_N) = \Big\langle \varphi_\alpha(n(\alpha,x_N)) \Big\rangle_{\alpha \in \mathrm{A}^0 \cup \mathrm{A}^-} \ \Big[= \Big\langle \varphi_\alpha(n(\alpha,x_N)) \Big\rangle_{\alpha \in x_i}\Big];$$

$$\varkappa_i(y_N) = \Big\langle \varphi_\alpha(n(\alpha,y_N)) \Big\rangle_{\alpha \in \mathrm{A}^0 \cup \mathrm{A}^+} \ \Big[= \Big\langle \varphi_\alpha(n(\alpha,y_N)) \Big\rangle_{\alpha \in y_i}\Big].$$

Since $u_i(y_N) > u_i(x_N)$, we have $\varkappa_i(y_N) >_{\mathrm{Lmin}} \varkappa_i(x_N)$. Since $\varkappa(x_N) = \langle \varkappa_{-i}, \varkappa_i(x_N) \rangle$ and $\varkappa(y_N) = \langle \varkappa_{-i}, \varkappa_i(y_N) \rangle$, we have $\varkappa(y_N) >_{\mathrm{Lmin}} \varkappa(x_N)$ by separability. $\square$

Remark. Essentially, the potential defined in the proof of Proposition 4.5 is the same as in Rosenthal (1973).

Among games with the minimum aggregation and negative impacts, games with structured utilities form a representative subclass. We call two strategic games Γ^* and Γ *indistinguishable* (from one another) if the sets N and X_i are the same in both, and $u_i^*(x_N) = u_i(x_N)$ for every $x_N \in X_N$ and $i \in N$.

Theorem 4.6. *For every game Γ with the minimum aggregation and negative impacts, there exists a game Γ^* with structured utilities and also with the minimum aggregation, which is indistinguishable from Γ.*

Proof. We define $\mathrm{A}^* = \mathrm{A} \times \mathcal{N}$, $\Upsilon_i^* = \{(\alpha, I) \in \mathrm{A}^* \mid i \in I\}$, so $N((\alpha, I)) = I$, and

$$\varphi^*_{(\alpha,I)}(x_I) = \begin{cases} \varphi_\alpha(I, x_I), & \text{if } \forall i \in I\,[\alpha \in \mathrm{B}_i(x_i)],\\ +\infty, & \text{else.} \end{cases}$$

Then we consider Γ^* with the same sets N and X_i, and utility functions $u_i^*(x_N) = \min_{(\alpha,I)\in\Upsilon_i^*} \varphi^*_{(\alpha,I)}(x_I)$.

Remark. The $+\infty$ in the definition of φ^* need not be understood literally: anything large enough would do.

Let us show that $u_i(x_N) = u_i^*(x_N)$ for every $i \in N$ and $x_N \in X_N$. Let $u_i(x_N) = \varphi_\alpha(N(\alpha, x_N), x_{N(\alpha,x_N)})$ with $i \in N(\alpha, x_N) = M$. We have $(\alpha, M) \in \Upsilon_i^*$ and $\varphi^*_{(\alpha,M)}(x_M) = \varphi_\alpha(M, x_M) = u_i(x_N)$; therefore, $u_i^*(x_N) \leq u_i(x_N)$.

Now let $(\alpha, I) \in \Upsilon_i^*$ and $\varphi^*_{(\alpha,I)}(x_I) < +\infty$; then $i \in I \subseteq N(\alpha, x_N)$. If $I \subset N(\alpha, x_N)$, then $\varphi^*_{(\alpha,I)}(x_I) = \varphi_\alpha(I, x_I) \geq \varphi_\alpha(N(\alpha, x_N), x_{N(\alpha,x_N)})$ by (5). If $I = N(\alpha, x_N)$, then $\varphi^*_{(\alpha,I)}(x_I) = \varphi_\alpha(N(\alpha, x_N), x_{N(\alpha,x_N)})$. In either case, $\varphi^*_{(\alpha,I)}(x_I) \geq u_i(x_N)$ by (6). Since $(\alpha, I) \in \Upsilon_i^*$ was arbitrary, $u_i^*(x_N) \geq u_i(x_N)$. □

5 Related aggregation rules

The maximum ("best-shot") aggregation is defined "dually" to (6):

$$u_i(x_N) = \max_{\alpha \in \mathrm{B}_i(x_i)} \varphi_\alpha(N(\alpha, x_N), x_{N(\alpha,x_N)}) \tag{10}$$

for all $i \in N$ and $x_N \in X_N$.

From the viewpoint of economics applications, there is a big difference between (6) and (10): the former satisfies the "decreasing marginal utility" condition, while the latter does not.

Theorem 5.1. *Let Γ be a game with positive impacts where each player uses the maximum aggregation (10). Let each X_i be compact, and all functions B_i be continuous, and all functions $\varphi_\alpha(I, \cdot)$ be upper semicontinuous. Then Γ admits a strong ω-potential and hence possesses a strong equilibrium.*

Proof. Similarly to Theorem 4.1, we define a strong ω-potential by the leximax ordering (8) rather than leximin (7): $y_N \succ x_N \rightleftharpoons u_N(y_N) >_{\mathrm{Lmax}} u_N(x_N)$. Then condition (3) is proven just dually. □

Proposition 5.2. *Every generalized congestion game where each player uses the maximum aggregation (10) admits an order potential (2), and hence has the FIP and possesses a Nash equilibrium.*

Theorem 5.3. *For every game Γ with the maximum aggregation and positive impacts, there exists a game Γ^* with structured utilities and also with the maximum aggregation, which is indistinguishable from Γ.*

Both proofs, dual to those of Proposition 4.5 and Theorem 4.6, are omitted.

A technical advantage of the maximum over minimum emerges if we consider games with infinite sets of facilities.

Theorem 5.4. *Let Γ be a strategic game where A is a metric space, every $\mathrm{B}_i(x_i)$ is compact, each mapping $\mathrm{B}_i : X_i \to 2^{\mathrm{A}} \setminus \{\emptyset\}$ is continuous in the Hausdorff metric on its target, and each $\varphi_{(\cdot)}(I, \cdot)$ (for $I \in \mathcal{N}$) is upper semicontinuous on $\mathrm{A} \times X_I$. Let each X_i be a compact metric space, each player use the maximum aggregation (10), and all impacts be positive. Then Γ admits a strong ω-potential and hence possesses a strong equilibrium.*

Proof. The same argument as in Theorem 4.1, or Theorem 5.1, is insufficient for an infinite A: it is not at all obvious that the maximum in (10) is attained for all $i \in N$ and $x_N \in X_N$, to say nothing about the upper semicontinuity of u_i.

Lemma 5.4.1. *Let $i \in N$, $x_N^k \to x_N$ and $\alpha^k \in \mathrm{B}_i(x_i^k)$ for all $k = 0, 1, \ldots$ Then there is $\alpha \in \mathrm{B}_i(x_i)$ such that*

$$\varphi_\alpha(N(\alpha, x_N), x_{N(\alpha,x_N)}) \geq \overline{\lim}_{k\to\infty} \varphi_{\alpha^k}(N(\alpha^k, x_N^k), x^k_{N(\alpha^k,x_N^k)}). \tag{11}$$

Proof. First, replacing $\langle\alpha^k\rangle_k$ with a subsequence if needed, we may assume that the upper limit in the right-hand side of (11) is just the limit. Since N is finite, we may (again replacing $\langle\alpha^k\rangle_k$ with a subsequence if needed) assume that $N(\alpha^k, x_N^k) = I$ is the same for all k. The condition $x_N^k \to x_N$ implies $\mathrm{B}_i(x_i^k) \to \mathrm{B}_i(x_i)$ in the Hausdorff metric. Let $\varepsilon_k \to 0$ (e.g., $\varepsilon_k = 1/k$); for each $k = 0, 1, \ldots,$ there is $\beta^k \in \mathrm{B}_i(x_i)$ and $h(k)$ for which the distance between β^k and $\alpha^{h(k)}$ is less than ε_k. Since $\mathrm{B}_i(x_i)$ is compact, we may assume $\beta^k \to \alpha \in \mathrm{B}_i(x_i)$ and hence $\alpha^{h(k)} \to \alpha$; therefore, we may assume $\alpha^k \to \alpha$ too.

Let $j \in N \setminus N(\alpha, x_N)$, i.e., $\alpha \notin \mathrm{B}_j(x_j)$; then the continuity of B_j immediately implies that $\alpha^k \notin \mathrm{B}_j(x_j^k)$ for all k large enough, i.e., $j \notin N(\alpha^k, x_N^k)$. Thus, $I \subseteq N(\alpha, x_N)$, and hence $\varphi_\alpha(N(\alpha, x_N), x_{N(\alpha,x_N)}) \geq \varphi_\alpha(I, x_I) \geq \lim_{k\to\infty} \varphi_{\alpha^k}(I, x_I^k)$, the first inequality following from the positive impacts assumption, the second from the upper semicontinuity assumption. Taking into account the first step of the proof and the definition of I, we have (11). □

To prove that the maximum in (10) is attained for any $i \in N$ and $x_N \in X_N$, we define $x_N^k = x_N$ for all k and pick a maximizing sequence for $\psi(\alpha) = \varphi_\alpha(N(\alpha, x_N), x_{N(\alpha,x_N)})$ as $\langle\alpha^k\rangle_k$; then the α from Lemma 5.4.1 obviously maximizes $\psi(\alpha)$.

Given $i \in N$ and $x_N^k \to x_N$, we pick $\alpha^k \in \mathrm{B}_i(x_i^k)$ such that $u_i(x_N^k) = \varphi_{\alpha^k}(N(\alpha^k, x_N^k), x^k_{N(\alpha^k,x_N^k)})$; then Lemma 5.4.1 means that u_i is upper semicontinuous at x_N. Therefore, each function ϑ_h defined as in the proof of Lemma 4.1.1 is upper semicontinuous too, and the ω-transitivity of $>_{\mathrm{Lmax}}$ follows in the same way. The condition (3) is proven exactly as in Theorem 5.1, or rather, dually to Theorem 4.1. Theorem 5.4 is proven. □

The maximum aggregation may seem exotic. It should be kept in mind, though, that Theorems 5.1 and 5.4 are quite meaningful, e.g., for congestion games with singleton strategies, where they show the acyclicity of strong coalition improvements and hence the existence of strong equilibria, under positive impacts. The same fact for games with negative impacts was noticed in Holzman and Law-Yone (1997). Actually, the local utilities in both cases may depend on the list of users rather than on their number only.

For games with negative impacts and the minimum aggregation, the dual of Lemma 5.4.1 is valid, but it implies the *lower* semicontinuity of u_i, which property is useless for a utility function.

Example 5.5. Let us consider a generalized congestion game with negative impacts: $N = \{1, 2\}$, $\mathrm{A} = [0, 1] = X_1 = X_2$, $\varphi_\alpha(1) = \alpha + 1$, $\varphi_\alpha(2) = \alpha$ for all $\alpha \in [0, 1]$. Suppose (x_1, x_2) to be a Nash equilibrium. If $1 \notin \{x_1, x_2\}$, then player 1 can switch to $y_1 = 1$ increasing his utility level. Let, say, $x_1 = 1$; then player 2 does not have a best response (the supremum is $\lim_{\alpha\to1} \varphi_\alpha(1) = 2$, but it is not attained). Thus, the game possesses no Nash equilibrium.

Generalizing our basic notions, we may assume that the preferences of the players may be described by orderings without numeric representations. Then we may consider games where the players use the leximin (or leximax) ordering to aggregate local utilities. We denote $M_i = \max_{x_i \in X_i} \# x_i$ and define L_i as $\bar{\mathbb{R}}^{M_i}$ with the leximin ordering (7) Now we define $u_i\colon X_N \to L_i$ by $u_i(x_N) = \Big\langle \langle \varphi_\alpha(n(\alpha, x_N)) \rangle_{\alpha \in x_i}, \langle +\infty, \dots, +\infty \rangle \Big\rangle$, the number of $+\infty$'s equaling $M_i - \# x_i$.

Proposition 5.6. *Every generalized congestion game where each player uses the leximin aggregation* (7) *admits an order potential* (2)*, and hence has the FIP and possesses a Nash equilibrium.*

The proof is essentially the same as in Proposition 4.5.

Leximin aggregation and minimum aggregation may seem very similar, but there is no analogue of Theorem 4.1 for the former case.

Example 5.7. Let us consider a generalized congestion game with negative impacts: $N = \{1, 2\}$, $\mathrm{A} = \{a, b, c, d, e, f, g\}$; $X_1 = \big\{\{a, b, c\}, \{d, e, f\}\big\}$; $X_2 = \big\{\{a, f, g\}, \{b, c, d\}\big\}$; $\varphi_a(2) = \varphi_b(2) = \varphi_d(2) = \varphi_e(1) = \varphi_g(1) = 0$; $\varphi_c(2) = 1$; $\varphi_a(1) = \varphi_d(1) = \varphi_f(2) = 2$; $\varphi_b(1) = \varphi_c(1) = \varphi_f(1) = 3$. Assuming that both players use the leximin aggregation, we obtain the 2×2 matrix of the game:

$$\begin{array}{ccc} & \text{afg} & \text{bcd} \\ \text{abc} & (\langle 0,3,3 \rangle, \langle 0,0,3 \rangle) & (\langle 0,1,2 \rangle, \langle 0,1,2 \rangle) \\ \text{def} & (\langle 0,2,2 \rangle, \langle 0,2,2 \rangle) & (\langle 0,0,3 \rangle, \langle 0,3,3 \rangle). \end{array}$$

We have a prisoner's dilemma: the northeastern corner is a unique Nash equilibrium, which is Pareto dominated by the southwestern corner.

Exact analogs of Proposition 5.6 and Example 5.7 for the leximax aggregation are easy to formulate.

6 Characterization results

A mapping $U\colon \mathbb{R}^{\Sigma(U)} \to \mathbb{R}$, where $\Sigma(U)$ is a finite set, is an *admissible aggregation function* if it is continuous and increasing in the sense of

$$\big[\forall s \in \Sigma(U) [v'_s > v_s]\big] \Rightarrow U(v'_{\Sigma(U)}) > U(v_{\Sigma(U)}). \tag{12}$$

The continuity of U and (12) imply

$$\big[\forall s \in \Sigma(U) [v'_s \geq v_s]\big] \Rightarrow U(v'_{\Sigma(U)}) \geq U(v_{\Sigma(U)}). \tag{13}$$

Remark. Exactly as in Kukushkin (2007), all results of this section remain valid if each U is assumed to be defined on a Cartesian power of an open interval in $\mathbb{R}$ and the attention is restricted to games where all values of local utilities belong to that interval. When the local utilities are, say, integer-valued, nothing is known about the necessity parts of the theorems; most likely, they are wrong.

Let $\mathfrak{U}$ be a set of admissible aggregation functions. We say that a game with common local utilities Γ *is consistent with the set* $\mathfrak{U}$ if for every $i \in N$ and $x_i \in X_i$, there are $U \in \mathfrak{U}$ and a bijection $\mu_i^{x_i}\colon \Sigma(U) \to \mathrm{B}(x_i)$ such that

$$U_i^{x_i}\big(\langle v_\alpha \rangle_{\alpha \in \mathrm{B}_i(x_i)}\big) = U\big(\langle v_{\mu_i^{x_i}(s)} \rangle_{s \in \Sigma(U^{x_i})}\big).$$

Theorem 6.1. *Let $\mathfrak{U}$ be a set of admissible aggregation functions such that $\#\Sigma(U) = 1$ for, at most, one $U \in \mathfrak{U}$. Then the following conditions are equivalent.*

1. *Every generalized congestion game with negative impacts which is consistent with $\mathfrak{U}$ has the FCP and hence possesses a strong equilibrium.*

2. *Every generalized congestion game with strictly negative impacts which is consistent with $\mathfrak{U}$ possesses a weakly Pareto optimal Nash equilibrium.*

3. *For every $U \in \mathfrak{U}$, there is a continuous and strictly increasing mapping $\lambda^U : \mathbb{R} \to \mathbb{R}$ such that:*

$$\forall U \in \mathfrak{U}\ \forall v_{\Sigma(U)} \in \mathbb{R}^{\Sigma(U)}\ \big[U(v_{\Sigma(U)}) = \lambda^U(\min\{v_s\}_{s\in\Sigma(U)})\big]; \tag{14a}$$

$$\forall U', U \in \mathfrak{U}\ \big[\lambda^{U'} = \lambda^U \ \text{ or }\ \lambda^{U'}(\mathbb{R}) \cap \lambda^U(\mathbb{R}) = \emptyset\big]. \tag{14b}$$

The implication [1 ⇒ 2] is trivial. The proofs of [2 ⇒ 3] and [3 ⇒ 1] are deferred to Sections A.1 and A.2, respectively.

The equivalence between Statements 1 and 2 of Theorem 6.1 does not hold without the uniqueness of $U \in \mathfrak{U}$ with $\#\Sigma(U) = 1$. For instance, if $\#\Sigma(U) = 1$ for all $U \in \mathfrak{U}$, then no restriction on λ^U is needed to ensure the existence of even a strong equilibrium (Konishi et al., 1997), but there may be no FIP (Milchtaich, 1996).

Proposition 6.2. *Let $\mathfrak{U}$ be a set of admissible aggregation functions such that every generalized congestion game with negative impacts which is consistent with $\mathfrak{U}$ possesses a weakly Pareto optimal Nash equilibrium. Then there is a continuous and strictly increasing mapping $\lambda^U : \mathbb{R} \to \mathbb{R}$, for every $U \in \mathfrak{U}$, such that:*

$$\forall U \in \mathfrak{U}\ \forall v_{\Sigma(U)} \in \mathbb{R}^{\Sigma(U)}\ \big[U(v_{\Sigma(U)}) = \lambda^U(\min\{v_s\}_{s\in\Sigma(U)})\big]; \tag{15a}$$

$$\forall U', U \in \mathfrak{U}\ \big[\lambda^{U'} = \lambda^U \ \text{ or }\ \lambda^{U'}(\mathbb{R}) \cap \lambda^U(\mathbb{R}) = \emptyset \ \text{ or }\ \#\Sigma(U) = \#\Sigma(U') = 1\big]. \tag{15b}$$

The proof, very similar to that of the implication [2 ⇒ 3] in Theorem 6.1, is deferred to Section A.3.

Theorem 6.3. *For every set $\mathfrak{U}$ of admissible aggregation functions, the following conditions are equivalent.*

1. *Every game with structured utilities which is consistent with $\mathfrak{U}$ and where the strategy sets are compact and utility functions are upper semicontinuous admits a strong ω-potential and hence possesses a strong equilibrium.*

2. *Every finite game with structured utilities which is consistent with $\mathfrak{U}$ possesses a weakly Pareto optimal Nash equilibrium.*

3. *For every $U \in \mathfrak{U}$, there is a continuous and strictly increasing mapping $\lambda^U : \mathbb{R} \to \mathbb{R}$ such that either*

$$\forall U \in \mathfrak{U}\ \forall v_{\Sigma(U)} \in \mathbb{R}^{\Sigma(U)}\ \big[U(v_{\Sigma(U)}) = \lambda^U(\min\{v_s\}_{s\in\Sigma(U)})\big], \tag{16a}$$

or

$$\forall U \in \mathfrak{U}\ \forall v_{\Sigma(U)} \in \mathbb{R}^{\Sigma(U)}\ \big[U(v_{\Sigma(U)}) = \lambda^U(\max\{v_s\}_{s\in\Sigma(U)})\big]; \tag{16b}$$

besides,

$$\forall U', U \in \mathfrak{U}\left[\lambda^{U'} = \lambda^{U} \text{ or } \#\Sigma(U) \neq \#\Sigma(U') \text{ or } \lambda^{U'}(\mathbb{R}) \cap \lambda^{U}(\mathbb{R}) = \emptyset\right]. \qquad (17)$$

The implication [1 ⇒ 2] is trivial. The proofs of [2 ⇒ 3] and [3 ⇒ 1] are deferred to Sections B.1 and B.2, respectively.

7 Concluding remarks

7.1. Everything in this paper is about games with ordinal preferences. Improvement relations (1) are invariant to strictly increasing transformations of utility functions u_i. Moreover, Theorems 4.1 and 4.6, as well as Theorems 5.1–5.4, would remain valid if we assumed that every $\varphi_\alpha(I, \cdot)$ maps X_I to an arbitrary chain rather than $\mathbb{R}$, cf. Proposition 5.6. On the other hand, the chain must be the same for all I and α; thus, the preferences are "co-ordinal" here.

7.2. Our assumption that all users receive the same intermediate utility from a facility should not be viewed as a simplifying technical condition. Making it, we concentrate on relationships between "fellow travellers," which can be considered as basic as, e.g., those between competitors for a scarce resource. At the moment, there is no evidence to suggest that similar results could hold in a broader context.

There is some literature on group formation games where each utility function only depends on the strategy chosen by the player and on the number of players who have chosen the same strategy, but different players may have different functions. Typically, there is just the existence of equilibria in such models, without acyclicity of improvements (Milchtaich, 1996; Konishi et al., 1997), so there is no ground to expect a close connection with this paper.

Conditions for the existence of strong equilibrium in congestion games with negative impacts were studied by Holzman and Law-Yone (1997). Their findings also seem unrelated to ours since only a very weak version of the acyclicity of coalition improvements was established.

7.3. We could define the *weak* coalition improvement relation similarly to (1):

$$y_N \rhd_I^{\mathrm{wCo}} x_N \rightleftharpoons \left[y_{-I} = x_{-I} \;\&\; \forall i \in I\,[u_i(y_N) \geq u_i(x_N)] \;\&\; \exists i \in I\,[u_i(y_N) > u_i(x_N)]\right];$$

$$y_N \rhd^{\mathrm{wCo}} x_N \rightleftharpoons \exists I \in \mathcal{N}\,[y_N \rhd_I^{\mathrm{wCo}} x_N].$$

A maximizer of $\rhd^{\mathrm{wCo}}$ can be called a "very" strong equilibrium. The notion may seem too strong, but there is nothing unusual in the existence of such equilibria in the games considered here (Kukushkin et al., 1985).

7.4. There is an obvious asymmetry in the implications of the existence of a strong (ω-)potential (3) as presented in Section 2: In a finite game, strong equilibria exist and all myopic adaptive dynamics converge to an equilibrium in a finite number of steps. In an infinite game, only the existence of a strong equilibrium was asserted. Actually, something can be said about adaptive dynamics in compact games too.

The simplest picture emerges if we consider improvement paths parameterized with countable ordinals. Then the existence of a strong ω-potential in a compact game implies that every coalition improvement path, if continued whenever possible, reaches a strong equilibrium at some stage (Kukushkin, 2010, Theorem 3.21). In other words, the only difference between finite and infinite games is that finite paths should be replaced with transfinite ones in the latter case.

For those who believe whatever happens after the first limit to be irrelevant, the situation is much more complicated and some questions remain open. A clear-cut theorem about the possibility to approximate an equilibrium with a finite improvement path in a continuous enough game with a potential is presented in Kukushkin (2011). Strictly speaking, it is about Nash equilibrium, but the same argument can be applied to coalition improvements and strong equilibria, cf. Kukushkin (2010, Theorem 4.3).

7.5. The leximin and leximax orderings are often met in the social choice theory, see, e.g., Moulin (1988). Economists naturally dislike the latter, but usually find it difficult to get rid of in their axiomatic characterizations (d'Aspremont and Gevers, 1977; Deschamps and Gevers, 1978).

7.6. Just as in the case of Kukushkin (2007), some forms of the main results of this paper can be found in Kukushkin (2004). The greatest advances over that paper are in Theorems 6.1 and 6.3 here: a much broader notion of a family of aggregation rules is employed. Under this notion, the special role of "aggregation rules" for the case of a single local utility in generalized congestion games was discerned. It should be stressed that the possibility to reverse the implication in Proposition 6.2 remains unclear.

7.7. In Kukushkin (2007), a similarity was noted between the necessity proofs there and the famous Debreu–Gorman Theorem (Debreu, 1960; Gorman, 1968) on additive representation of separable orderings. There seems to be no general theorem on abstract preference orderings that could display parallel similarities with Theorems 6.1 and 6.3 here.

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Appendix: Proofs

A Generalized congestion games

A.1 Proof of [2 ⇒ 3] in Theorem 6.1

As a first step, we show that every function $U \in \mathfrak{U}$ is symmetric.

Lemma A.1.1. *Let $U \in \mathfrak{U}$, $s', s'' \in \Sigma(U)$, $v^-, v^+ \in \mathbb{R}$, and $v'_{\Sigma(U)}, v''_{\Sigma(U)} \in \mathbb{R}^{\Sigma(U)}$ be such that $v^+ > v^-$, $v''_{s''} = v'_{s'} = v^+$, $v''_{s'} = v'_{s''} = v^-$, and $v''_s = v'_s$ for all $s \in \Sigma(U) \setminus \{s', s''\}$. Then $U(v''_{\Sigma(U)}) = U(v'_{\Sigma(U)})$.*

Proof. Supposing the contrary, we may, without restricting generality, assume $u^+ = U(v''_{\Sigma(U)}) > U(v'_{\Sigma(U)}) = u^-$. Now let us consider a generalized congestion game with strictly negative impacts which is consistent with $\mathfrak{U}$: $N = \{1, 2\}$; the facilities are $\mathrm{A} = \{a, b, c, d\} \cup E$, where $E = \{e_s\}_{s \in \Sigma(U) \setminus \{s', s''\}}$;

$X_1 = \big\{\{a,c\} \cup E, \{b,d\} \cup E\big\}$; $X_2 = \big\{\{d,a\} \cup E, \{c,b\} \cup E\big\}$; $\varphi_t(1) = v^+$ and $\varphi_t(2) = v^-$ for each $t \in \{a,b,c,d\}$, while $\varphi_{e_s}(2) = v_s$ for all $s \in \Sigma(U) \setminus \{s', s''\}$; $U_i^{x_i}$ is U for both $i \in N$ and all $x_i \in X_i$; $\mu_1^{x_1}(a) = \mu_1^{x_1}(b) = \mu_2^{x_2}(c) = \mu_2^{x_2}(d) = s''$ and $\mu_2^{x_2}(a) = \mu_2^{x_2}(b) = \mu_1^{x_1}(c) = \mu_1^{x_1}(d) = s'$ whenever appropriate, whereas $\mu_i^{x_i}(e_s) = s$ for both $i \in N$ and all $s \in \Sigma(U) \setminus \{s', s''\}$. The 2×2 matrix of the game looks as follows:

$$\begin{array}{ccc} & \text{daE} & \text{cbE} \\ \text{acE} & (u^-, u^+) & (u^+, u^-) \\ \text{bdE} & (u^+, u^-) & (u^-, u^+). \end{array}$$

There is no Nash equilibrium in the game. □

Remark. Neither continuity, nor monotonicity of U were needed in the proof.

Lemma A.1.1 shows that the mappings $\mu_i^{x_i}$ do not matter and hence may be ignored in the following. Moreover, we may assume that $\Sigma(U) = \{1, \ldots, m\}$ (with m depending on U, naturally). As a next step, we show that the impossibility of a prisoner's dilemma implies that each indifference curve in each two-dimensional section must exhibit a similarity with either minimum or maximum.

Lemma A.1.2. *Let $U \in \mathfrak{U}$, $v_1 > v_2$, and*

$$U(v_1, v_2, v_3, \ldots, v_m) > U(v_2, v_2, v_3, \ldots, v_m); \tag{18}$$

then $U(v_1, \bar{v}_2, v_3, \ldots, v_m) = U(v_1, v_2, v_3, \ldots, v_m)$ for all $\bar{v}_2 \leq v_2$.

Proof. A non-strict inequality immediately follows from the monotonicity of U. Let us suppose that $U(v_1, \bar{v}_2, \ldots, v_m) = u' < u = U(v_1, v_2, \ldots, v_m)$ for some $\bar{v}_2 < v_2$. Taking into account (18) and the continuity of U, we may, increasing $\bar{v}_2$ if needed, assume $U(v_2, v_2, v_3, \ldots, v_m) < u'$. By the continuity of U, there is $\delta > 0$ such that $v_2 + \delta < v_1$ and $U(v_2 + \delta, v_2 + \delta, v_3 + \delta, \ldots, v_m + \delta) = u'' < u'$; we denote $U(v_1 + \delta, v_2 + \delta, v_3 + \delta, \ldots, v_m + \delta) = u^+ > u$. Thus,

$$u'' < u' < u < u^+. \tag{19}$$

Now let us consider a generalized congestion game with strictly negative impacts which is consistent with $\mathfrak{U}$: $N = \{1, 2\}$; the facilities are $\mathrm{A} = \{a,b,c,d\} \cup E \cup F$, where $E = \{e_s\}_{3 \leq s \leq m}$ and $F = \{f_s\}_{3 \leq s \leq m}$; $X_1 = \big\{\{a,c\} \cup E, \{b,d\} \cup F\big\}$; $X_2 = \big\{\{a,d\} \cup E, \{b,c\} \cup F\big\}$; $\varphi_a(1) = v_1 + \delta$, $\varphi_a(2) = \bar{v}_2$, $\varphi_b(1) = v_2 + \delta$, $\varphi_b(2) = v_2$, $\varphi_c(1) = \varphi_d(1) = v_1$, $\varphi_c(2) = \varphi_d(2) = v_2 + \delta$, $\varphi_{e_s}(1) = \varphi_{f_s}(1) = v_s + \delta$ and $\varphi_{e_s}(2) = \varphi_{f_s}(2) = v_s$ $(s = 3, \ldots, m)$; $U_i^{x_i} = U$ for both $i \in N$ and all $x_i \in X_i$. The 2×2 matrix of the game looks as follows:

$$\begin{array}{ccc} & \text{adE} & \text{bcF} \\ \text{acE} & (u', u') & (u^+, u'') \\ \text{bdF} & (u'', u^+) & (u, u). \end{array}$$

Taking into account (19), we see that the northwestern corner is a unique Nash equilibrium, which is strongly Pareto dominated by the southeastern corner. □

Lemma A.1.3. *Let $U \in \mathfrak{U}$, $v_1 > v_2$, and*

$$U(v_1, v_1, v_3, \ldots, v_m) > U(v_1, v_2, v_3, \ldots, v_m); \tag{20}$$

then $U(\bar{v}_1, v_2, v_3, \ldots, v_m) = U(v_1, v_2, v_3, \ldots, v_m)$ for all $\bar{v}_1 \geq v_1$.

Proof. A non-strict inequality immediately follows from the monotonicity of U. Let us suppose

$$U(\bar{v}_1, v_2, \ldots, v_m) = u^+ > u = U(v_1, v_2, \ldots, v_m) \tag{21}$$

for some $\bar{v}_1 > v_1$. By the continuity of U, (20) and (21) imply the existence of $v'_1 \in]v_2, v_1[$ such that $u < U(v'_1, v_1, v_3, \ldots, v_m) < u^+$. By the same continuity, we may pick $\delta > 0$ such that $v'_1 + \delta < v_1$, $U(v_2 + \delta, v_1 + \delta, v_3 + \delta, \ldots, v_m + \delta) = u' < U(v'_1, v_1, v_3, \ldots, v_m)$ and $U(v'_1 + \delta, v_1 + \delta, v_3 + \delta, \ldots, v_m + \delta) = u'' < u^+$; by monotonicity,

$$u < u' < u'' < u^+. \tag{22}$$

Now let us consider a generalized congestion game with strictly negative impacts which is consistent with $\mathfrak{U}$: $N = \{1, 2\}$; the facilities are $\mathrm{A} = \{a, b, c, d\} \cup E \cup F$, where $E = \{e_s\}_{3 \le s \le m}$ and $F = \{f_s\}_{3 \le s \le m}$; $X_1 = \big\{\{a, c\} \cup E, \{b, d\} \cup F\big\}$; $X_2 = \big\{\{a, d\} \cup E, \{b, c\} \cup F\big\}$; $\varphi_a(1) = \bar{v}_1$, $\varphi_a(2) = v_2 + \delta$, $\varphi_b(1) = v_1$, $\varphi_b(2) = v'_1 + \delta$, $\varphi_c(1) = \varphi_d(1) = v_1 + \delta$, $\varphi_c(2) = \varphi_d(2) = v_2$, $\varphi_{e_s}(1) = \varphi_{f_s}(1) = v_s + \delta$ and $\varphi_{e_s}(2) = \varphi_{f_s}(2) = v_s$ $(s = 3, \ldots, m)$; $U_i^{x_i} = U$ for both $i \in N$ and all $x_i \in X_i$. The 2×2 matrix of the game looks as follows:

$$\begin{array}{ccc} & \text{adE} & \text{bcF} \\ \text{acE} & (u', u') & (u^+, u) \\ \text{bdF} & (u, u^+) & (u'', u''). \end{array}$$

Taking into account (22), we see that the northwestern corner is a unique Nash equilibrium, which is strongly Pareto dominated by the southeastern corner. □

As a next step, we establish a restriction on mutual location of "minimum-like" angles (23) and "maximum-like" angles (24).

Lemma A.1.4. *Let $U, U' \in \mathfrak{U}$, $v_1 > v_2$,*

$$U(v_1, v_1, v_3, \ldots, v_m) > U(v_1, v_2, v_3, \ldots, v_m), \tag{23}$$

$v'_1 > v'_2$, and

$$U'(v'_1, v'_2, v'_3, \ldots, v'_{m'}) > U'(v'_2, v'_2, v'_3, \ldots, v'_{m'}). \tag{24}$$

Then $v_1 > v'_2$.

Proof. Supposing the contrary, $v'_2 \ge v_1$, we denote $u_1^- = U(v'_1, v_2, v_3, \ldots, v_m)$, $u_1^+ = U(v_1, v_1, v_3, \ldots, v_m)$, $u_2^- = U'(v_1, v_1, v'_3, \ldots, v'_{m'})$, and $u_2^+ = U'(v'_1, v_2, v'_3, \ldots, v'_{m'})$. We have $u_1^+ > u_1^-$ by Lemma A.1.3 since $v'_1 > v'_2 \ge v_1$, and $u_2^+ > u_2^-$ by Lemma A.1.2 since $v'_2 \ge v_1 > v_2$.

Now we consider a generalized congestion game with strictly negative impacts which is consistent with $\mathfrak{U}$: $N = \{1, 2\}$; the facilities are $\mathrm{A} = \{a, b, c, d\} \cup E \cup F$, where $E = \{e_s\}_{s \in \{3, \ldots, m\}}$ and $F = \{f_s\}_{s \in \{3, \ldots, m'\}}$; $X_1 = \big\{\{a, b\} \cup E, \{c, d\} \cup E\big\}$; $X_2 = \big\{\{a, c\} \cup F, \{b, d\} \cup F\big\}$; $\varphi_a(2) = \varphi_d(2) = v_2$, $\varphi_a(1) = \varphi_d(1) = \varphi_b(2) = \varphi_c(2) = v_1$, $\varphi_b(1) = \varphi_c(1) = v'_1$, $\varphi_{e_s}(1) = v_s$ for each $s \in \{3, \ldots, m\}$, and $\varphi_{f_s}(1) = v'_s$ for each $s \in \{3, \ldots, m'\}$; $U_1^{x_1}$ is U for each $x_1 \in X_1$ and $U_2^{x_2}$ is U' for each $x_2 \in X_2$. The 2×2 matrix of the game looks as follows:

$$\begin{array}{ccc} & \text{acF} & \text{bdF} \\ \text{abE} & (u_1^-, u_2^+) & (u_1^+, u_2^-) \\ \text{cdE} & (u_1^+, u_2^-) & (u_1^-, u_2^+). \end{array}$$

There is no Nash equilibrium in the game. □

Given $U \in \mathfrak{U}$, we denote:

$$V_U^{\min} = \{v_1 \in \mathbb{R} \mid \exists v_2, \ldots, v_m \in \mathbb{R}\, [v_1 > v_2 \;\&\; U(v_1, v_1, v_3, \ldots, v_m) > U(v_1, v_2, v_3, \ldots, v_m)]\};$$

$$V_U^{\max} = \{v_2 \in \mathbb{R} \mid \exists v_1, v_3, \ldots, v_m \in \mathbb{R}\, [v_1 > v_2 \;\&\; U(v_1, v_2, v_3, \ldots, v_m) > U(v_2, v_2, v_3, \ldots, v_m)]\};$$

$$v_U^{\min} = \inf V_U^{\min};\; v_U^{\max} = \sup V_U^{\max}.$$

(If $V_U^{\min} = \emptyset$, then we assume $v_U^{\min} = +\infty$; if $V_U^{\max} = \emptyset$, then $v_U^{\max} = -\infty$.) By Lemma A.1.4, $v_U^{\min} \geq v_U^{\max}$. For $v \in \mathbb{R}$, we define

$$\lambda^U(v) = U(v, v, \ldots, v).$$

Clearly, λ^U is continuous and strictly increasing.

Lemma A.1.5. *For every $U \in \mathfrak{U}$ and $v_1, v_2, v_3, \ldots, v_m \in \mathbb{R}$, there hold*

$$U(v_1, v_2, v_3, \ldots, v_m) = \lambda^U(\min_m v_m) \tag{25}$$

whenever $\min_m v_m \geq v_U^{\max}$, and

$$U(v_1, v_2, v_3, \ldots, v_m) = \lambda^U(\max_m v_m) \tag{26}$$

whenever $\max_m v_m \leq v_U^{\min}$.

Proof. Let $\min_m v_m > v_U^{\max}$. Without restricting generality, we may assume $v_1 \geq v_2 \geq \cdots \geq v_m$. By the definition of $v_U^{\max}$ and symmetry of U, we have $U(v_1, v_2, \ldots, v_{m-1}, v_m) = U(v_1, v_2, \ldots, v_{m-2}, v_m, v_m) = \cdots = U(v_1, v_m, \ldots, v_m, v_m) = U(v_m, \ldots, v_m) = \lambda^U(\min\{v_1, v_2, \ldots, v_m\})$.

If $\min_m v_m = v_U^{\max}$, we obtain the same equality by continuity. If $\max_m v_m \leq v_U^{\min}$, we argue dually. $\square$

Lemma A.1.6. *For every $U \in \mathfrak{U}$, either $v_U^{\min} = v_U^{\max} = +\infty$ or $v_U^{\min} = v_U^{\max} = -\infty$.*

Proof. Supposing that $v_U^{\max} < v' < v'' < v_U^{\min}$, we would have $U(v', v'', \ldots, v'') = \lambda^U(v')$ by (25) and $U(v', v'', \ldots, v'') = \lambda^U(v'')$ by (26), which is impossible since λ^U is strictly increasing.

Supposing that $v' < v_U^{\max} = v_U^{\min} < v''$, we would have $U(v_U^{\max}, v'', \ldots, v'') = \lambda^U(v_U^{\max})$ by (25) and $U(v', \ldots, v', v_U^{\max}) = \lambda^U(v_U^{\max})$ by (26), which contradicts monotonicity (12). $\square$

Lemma A.1.7. *Either $U(v_1, v_2, v_3, \ldots, v_m) = \lambda^U(\min_m v_m)$ for every $U \in \mathfrak{U}$ and all $v_1, v_2, v_3, \ldots, v_m \in \mathbb{R}$, or $U(v_1, v_2, v_3, \ldots, v_m) = \lambda^U(\max_m v_m)$ for every $U \in \mathfrak{U}$ and all $v_1, v_2, v_3, \ldots, v_m \in \mathbb{R}$.*

Immediately follows from Lemmas A.1.5, A.1.6, and A.1.4.

Lemma A.1.8. *For every $U \in \mathfrak{U}$ and all $v_1, v_2, v_3, \ldots, v_m \in \mathbb{R}$, there holds $U(v_1, v_2, v_3, \ldots, v_m) = \lambda^U(\min_m v_m)$.*

Proof. In light of Lemma A.1.7, it is enough to show that the maximum aggregation is not "good" in the case of negative impacts. If $\mathfrak{U}$ contains functions of $m \geq 3$ arguments, the dual to Example 4.2 will do. Otherwise, we need an example more.

Let us consider a generalized congestion game with strictly negative impacts and the maximum aggregation: $N = \{1,2,3\}$; the facilities are $\mathrm{A} = \{a,b,c,d,e\}$; $X_1 = \big\{\{a,e\},\{b,d\}\big\}$; $X_2 = \big\{\{a,c\},\{d,e\}\big\}$; $X_3 = \big\{\{a,b\},\{c,e\}\big\}$; $\varphi_a(3) = \varphi_e(3) = 0$, $\varphi_c(2) = \varphi_e(2) = 1$, $\varphi_a(2) = \varphi_d(2) = \varphi_e(1) = 2$, $\varphi_b(2) = 3$, $\varphi_c(1) = 4$, $\varphi_d(1) = 5$, $\varphi_a(1) = 6$, and $\varphi_b(1) = 7$; every $U_i^{x_i}$ is the same U defined by (26). Denoting $u^k = \lambda^U(k)$ for each $k \in \{1,2,\ldots,7\}$, we obtain the following $2 \times 2 \times 2$ matrix of the game (player 1 chooses rows, player 2 columns, and player 3 matrices):

$$\begin{array}{c} \\ \\ \text{ae} \\ \text{bd} \end{array} \begin{array}{c} \text{ab} \\ \begin{array}{cc} \text{ac} & \text{de} \end{array} \\ \begin{bmatrix} (u^2,u^4,u^7) & (u^2,u^5,u^7) \\ (u^5,u^4,u^3) & (u^3,u^2,u^6) \end{bmatrix} \end{array} \begin{array}{c} \text{ce} \\ \begin{array}{cc} \text{ac} & \text{de} \end{array} \\ \begin{bmatrix} (u^2,u^2,u^1) & (u^6,u^5,u^4) \\ (u^7,u^6,u^2) & (u^7,u^2,u^4) \end{bmatrix} \end{array}.$$

The individual improvement relation is acyclic (as it should be according to Proposition 4.5) and the southwestern corner of the left matrix is a unique Nash equilibrium. However, this equilibrium is strongly Pareto dominated by the northeastern corner of the right matrix. $\square$

Thus, (14a) is proven. Let us turn to (14b).

Lemma A.1.9. *Let $U, U' \in \mathfrak{U}$ and $\lambda^U \neq \lambda^{U'}$. Then $\lambda^U(\mathbb{R}) \cap \lambda^{U'}(\mathbb{R}) = \emptyset$.*

Proof. Let us suppose the contrary, $\lambda^U(\mathbb{R}) \cap \lambda^{U'}(\mathbb{R}) \neq \emptyset$. Since both λ^U and $\lambda^{U'}$ are homeomorphisms, $\lambda^U(\mathbb{R}) \cap \lambda^{U'}(\mathbb{R})$ is open and $\{v \in \mathbb{R} \mid \lambda^{U'}(v) = \lambda^U(v)\}$ is closed in $\mathbb{R}$, there must be $v' \neq v$ such that $\lambda^{U'}(v') = \lambda^U(v)$. Let $\#\Sigma(U') > 1$.

1. Supposing first that $v > v'$, we denote $u^1 = \lambda^{U'}(v')$. Then we pick $\underline{v} \in]v', v[$, denote $u^0 = \lambda^U(\underline{v})$ and $u^3 = \lambda^{U'}(\underline{v})$ (so $u^0 < \lambda^U(v) = u^1 = \lambda^{U'}(v') < u^3$), and pick $\bar{v} > v$ so that $u^2 = \lambda^U(\bar{v}) < u^3$; $u^2 > u^1$ is satisfied automatically.

Let us consider a generalized congestion game with strictly negative impacts, which is consistent with $\mathfrak{U}$: $N = \{1,2\}$; the facilities are $\mathrm{A} = \{a,b,c\} \cup D \cup E$, where $D = \{d_s\}_{2 \leq s \leq m}$ and $E = \{e_s\}_{3 \leq s \leq m'}$; $X_1 = \big\{\{a\} \cup D, \{b,c\} \cup E\big\}$; $X_2 = \big\{\{a,b\} \cup E, \{c\} \cup D\big\}$; $\varphi_a(2) = \varphi_b(1) = \varphi_c(2) = \underline{v}$, $\varphi_a(1) = \varphi_c(1) = \bar{v}$, $\varphi_b(2) = v'$, $\varphi_{d_s}(2) = \varphi_{e_{s'}}(2) = \bar{v}$ and $\varphi_{d_s}(1) = \varphi_{e_{s'}}(1) > \bar{v}$ for all appropriate s and s'; $U_i^{x_i}$ is U if x_i contains D and U' otherwise. The 2×2 matrix of the game looks as follows:

$$\begin{array}{ccc} & \text{abE} & \text{cD} \\ \text{aD} & (u^0,u^3) & (u^2,u^2) \\ \text{bcE} & (u^1,u^1) & (u^3,u^0). \end{array}$$

We have a prisoner's dilemma: strategies with the "U' aggregation" are dominant, but the northeastern corner strongly Pareto dominates the southwestern one.

2. Supposing $v' > v$, we denote $u^0 = \lambda^{U'}(v)$ and $u^4 = \lambda^U(v) > u^0$; then we pick $\underline{v} \in]v, v'[$ and $v^+ > \bar{v} > v'$, and denote $u^3 = \lambda^{U'}(\underline{v}) < u^4 < \lambda^{U'}(\bar{v}) = u^6 < \lambda^{U'}(v^+) = u^7$. Then we pick $v'' \in]v, \underline{v}[$ so that $u^5 = \lambda^U(v'') < u^6$; $u^5 > u^4$ is satisfied automatically. Finally, we pick $v''' \in]v, v''[$, and denote $u^1 = \lambda^{U'}(v''')$ and $u^2 = \lambda^{U'}(v'')$; we have $u^0 < u^1 < \cdots < u^7$.

Let us consider a generalized congestion game with strictly negative impacts, which is consistent with $\mathfrak{U}$: $N = \{1,2,3\}$; the facilities are $\mathrm{A} = \{a,b,c,d\} \cup E \cup F$, where $E = \{e_s\}_{2\le s\le m}$ and $F = \{f_s\}_{2\le s\le m'}$; $X_1 = \bigl\{\{a\} \cup E, \{d\} \cup F\bigr\}$; $X_2 = \bigl\{\{a,b\} \cup F \setminus \{f_2\}, \{c\} \cup F\bigr\}$; $X_3 = \bigl\{\{d\} \cup F, \{b\} \cup F\bigr\}$; $\varphi_a(2) = v$, $\varphi_a(1) = \varphi_b(2) = v''$, $\varphi_b(1) = v^+$, $\varphi_c(1) = v'''$, $\varphi_d(2) = \underline{v}$, $\varphi_d(1) = \varphi_{e_{s'}}(1) = \varphi_{f_s}(3) = \bar{v}$, $\varphi_{f_s}(2) = v^+$ and $\varphi_{f_s}(1) > v^+$ for all appropriate s and s'; $U_i^{x_i}$ is U if x_i contains E and U' otherwise. The $2\times 2\times 2$ matrix of the game looks as follows (again, player 1 chooses rows, player 2 columns, and player 3 matrices):

$$\begin{array}{c} \\ \\ \mathrm{aE} \\ \mathrm{dF} \end{array}
\begin{array}{c} \mathrm{dF} \\ \begin{array}{cc} \mathrm{abF} & \mathrm{cF} \end{array} \\ \begin{bmatrix} (u^4,u^0,u^6) & (u^5,u^1,u^6) \\ (u^3,u^2,u^3) & (u^3,u^1,u^3) \end{bmatrix} \end{array}
\begin{array}{c} \mathrm{bF} \\ \begin{array}{cc} \mathrm{abF} & \mathrm{cF} \end{array} \\ \begin{bmatrix} (u^4,u^0,u^2) & (u^5,u^1,u^7) \\ (u^6,u^2,u^2) & (u^6,u^1,u^7) \end{bmatrix} \end{array}.$$

There is no Nash equilibrium in the game. $\square$

A.2 Proof of [3 ⇒ 1] in Theorem 6.1

Let $\mathfrak{U}$ be a set of admissible aggregation functions satisfying both conditions (14) from Theorem 6.1 and such that $\#\Sigma(U) = 1$ for, at most, one $U \in \mathfrak{U}$. The condition (14b) obviously implies that $\mathfrak{U}$ is partitioned into a (finite or infinite) number of subsets W^α ($\alpha \in \mathrm{A}$) such that $\lambda^U = \lambda^{U'}$ whenever U and U' belong to the same W^α, and $\lambda^U(\mathbb{R}) \cap \lambda^{U'}(\mathbb{R}) = \emptyset$ whenever they do not. The latter condition, in turn, means that the set A is linearly ordered in the sense that $\alpha > \alpha' \iff \bigl[\lambda^U(u) > \lambda^{U'}(u')$ whenever $U \in W^\alpha$, $U' \in W^{\alpha'}$, and $u, u' \in \mathbb{R}\bigr]$.

Let Γ be a generalized congestion game with negative impacts which is consistent with $\mathfrak{U}$. For each player $i \in N$, the order on A generates an ordering on X_i: $x_i \succeq y_i \iff \bigl[U_i^{x_i} \in W^\alpha \;\&\; U_i^{y_i} \in W^{\alpha'} \;\&\; \alpha \ge \alpha'\bigr]$. Obviously, $u_i(x_i, z_{-i}) > u_i(y_i, z'_{-i})$ for all $z_{-i}, z'_{-i} \in X_{-i}$ whenever $x_i \succ y_i$. Assuming the possibility of a coalition improvement cycle in Γ, we immediately see that all strategies of each player involved in the cycle must be equivalent in that ordering. Denoting Γ^* the game with the same players, facilities, and strategies, but with the minimum aggregation (6), we see that the same cycle is a coalition improvement cycle in Γ^* as well; however, this contradicts Theorem 4.1.

A.3 Proof of Proposition 6.2

The condition here is the same as Statement 2 of Theorem 6.1. Therefore, we can argue exactly as in Section A.1 until we reach Lemma A.1.9, where the uniqueness of $U \in \mathfrak{U}$ for which $\#\Sigma(U) = 1$ was relied upon indeed. However, if $\#\Sigma(U') = \#\Sigma(U) = 1$, then (15b), unlike (14b), does not require anything of such U' and U, so the lemma is not needed.

B Structured utilities

B.1 Proof of [2 ⇒ 3] in Theorem 6.3

There is a considerable similarity with the proof of Theorem 6.1. Again, we start with the symmetry of every function $U \in \mathfrak{U}$.

Lemma B.1.1. *Let $U \in \mathfrak{U}$, $s', s'' \in \Sigma(U)$, $v^-, v^+ \in \mathbb{R}$, and $v'_{\Sigma(U)}, v''_{\Sigma(U)} \in \mathbb{R}^{\Sigma(U)}$ be such that $v^+ > v^-$, $v''_{s''} = v'_{s'} = v^+$, $v''_{s'} = v'_{s''} = v^-$, and $v''_s = v'_s$ for all $s \in \Sigma(U) \setminus \{s', s''\}$. Then $U(v''_{\Sigma(U)}) = U(v'_{\Sigma(U)})$.*

Proof. Supposing the contrary, we may, without restricting generality, assume $u^+ = U(v''_{\Sigma(U)}) > U(v'_{\Sigma(U)}) = u^-$. Now let us consider a finite game with structured utilities which is consistent with $\mathfrak{U}$: $N = \{1,2\}$; the facilities are $\mathrm{A} = \{a, b\} \cup C$, where $C = \{c_s\}_{s \in \Sigma(U) \setminus \{s', s''\}}$; $\Upsilon_1 = \Upsilon_2 = \mathrm{A}$; $X_1 = X_2 = \{1,2\}$; $\varphi_a(x_1, x_2) = v^-$ if $x_1 = x_2$ and $\varphi_a(x_1, x_2) = v^+$ otherwise; $\varphi_b(x_1, x_2) = v^-$ if $x_1 \neq x_2$ and $\varphi_a(x_1, x_2) = v^+$ otherwise; $\varphi_{c_s}(x_1, x_2) = v_s$ for all $s \in \Sigma(U) \setminus \{s', s''\}$ and $(x_1, x_2) \in X_N$; $U_i^{x_i}$ is U for both $i \in N$ and all $x_i \in X_i$; $\mu_1^{x_1}(a) = \mu_2^{x_2}(b) = s''$ and $\mu_1^{x_1}(b) = \mu_2^{x_2}(a) = s'$ for all $x_i \in X_i$, whereas $\mu_i^{x_i}(c_s) = s$ for both $i \in N$ and all $x_i \in X_i$ and $s \in \Sigma(U) \setminus \{s', s''\}$. The 2×2 matrix of the game (as usual, player 1 chooses rows, numbered from top to bottom, while player 2 chooses columns, numbered from left to right) looks as follows:

$$\begin{array}{cc} (u^-, u^+) & (u^+, u^-) \\ (u^+, u^-) & (u^-, u^+). \end{array}$$

There is no Nash equilibrium in the game. ∎

Remark. Exactly as in the proof of Theorem 6.1, neither continuity, nor monotonicity of U were needed.

Lemma B.1.1 shows that the mappings $\mu_i^{x_i}$ do not matter and hence may be ignored in the following. Moreover, we may assume that $\Sigma(U) = \{1, \ldots, m\}$ (with m dependent on U, naturally). As a next step, we show that the impossibility of a prisoner's dilemma implies that each indifference curve in each two-dimensional section must exhibit a similarity with either minimum or maximum.

Lemma B.1.2. *Let $U \in \mathfrak{U}$, $v_1 > v_2$, and*

$$U(v_1, v_2, v_3, \ldots, v_m) > U(v_2, v_2, v_3, \ldots, v_m);$$

then $U(v_1, \bar{v}_2, v_3, \ldots, v_m) = U(v_1, v_2, v_3, \ldots, v_m)$ for all $\bar{v}_2 \leq v_2$.

Proof. A non-strict inequality immediately follows from the monotonicity of U. Let us suppose that $U(v_1, \bar{v}_2, \ldots, v_m) = u' < u = U(v_1, v_2, \ldots, v_m)$ for some $\bar{v}_2 < v_2$. As in Lemma A.1.2, we may assume that $u^- = U(v_2, v_2, v_3, \ldots, v_m) < u'$. By the continuity of U, there is $\bar{v}_1 \in]v_2, v_1[$ such that $u' < U(\bar{v}_1, v_2, v_3, \ldots, v_m) = u'' < u$. Thus,

$$u^- < u' < u'' < u. \tag{27}$$

Now let us consider a finite game with structured utilities which is consistent with $\mathfrak{U}$: $N = \{1,2\}$; the facilities are $\mathrm{A} = \{a_1, a_2, b\} \cup C$, where $C = \{c_s\}_{s \in \{3, \ldots, m\}}$; $\Upsilon_i = \{a_i, b\} \cup C$ for both i; $X_1 = X_2 = \{1,2\}$; $U_i^{x_i}$ is U for both $i \in N$ and all $x_i \in X_i$; $\varphi_{a_i}(1) = v_2$, $\varphi_{a_i}(2) = v_1$; $\varphi_b(1,1) = \bar{v}_1$, $\varphi_b(1,2) = \varphi_b(2,1) = v_2$, $\varphi_b(2,2) = \bar{v}_2$; $\varphi_{c_s}(x_1, x_2) = v_s$ $(s = 3, \ldots, m)$. The 2×2 matrix of the game looks as follows:

$$\begin{array}{cc} (u'', u'') & (u^-, u) \\ (u, u^-) & (u', u'). \end{array}$$

Taking into account (27), we see that the southeastern corner ($x_1 = x_2 = 2$) is a unique Nash equilibrium, which is strongly Pareto dominated by the northwestern corner ($x_1 = x_2 = 1$). ∎

Lemma B.1.3. *Let $U \in \mathfrak{U}$, $v_1 > v_2$, and*

$$U(v_1, v_1, v_3, \ldots, v_m) > U(v_1, v_2, v_3, \ldots, v_m); \tag{28}$$

then $U(\bar{v}_1, v_2, v_3, \ldots, v_m) = U(v_1, v_2, v_3, \ldots, v_m)$ for all $\bar{v}_1 \geq v_1$.

Proof. A non-strict inequality immediately follows from the monotonicity of U. Let us suppose that $U(\bar{v}_1, v_2, \ldots, v_m) = u'' > u = U(v_1, v_2, \ldots, v_m)$ for some $\bar{v}_1 > v_1$; we may assume, without restricting generality, that $u'' < u^+ = U(v_1, v_1, v_3, \ldots, v_m)$.

By the continuity of U, (28) implies the existence of $v_1' \in]v_2, v_1[$ such that $u < u' = U(v_1', v_1, v_3, \ldots, v_m) < u''$. Thus,

$$u < u' < u'' < u^+. \tag{29}$$

Now let us consider a finite game with structured utilities which is consistent with $\mathfrak{U}$: $N = \{1, 2\}$; the facilities are $\mathrm{A} = \{a_1, a_2, b\} \cup C$, where $C = \{c_s\}_{s \in \{3,\ldots,m\}}$; $\Upsilon_i = \{a_i, b\} \cup C$ for both i; $X_1 = X_2 = \{1, 2\}$; $U_i^{x_i}$ is U for both $i \in N$ and all $x_i \in X_i$; $\varphi_{a_i}(1) = v_2$, $\varphi_{a_i}(2) = v_1$; $\varphi_b(1, 1) = \bar{v}_1$, $\varphi_b(1, 2) = \varphi_b(2, 1) = v_1$, $\varphi_b(2, 2) = v_1'$; $\varphi_{c_s}(x_1, x_2) = v_s$ $(s = 3, \ldots, m)$. The 2×2 matrix of the game looks as follows:

$$\begin{matrix} (u'', u'') & (u, u^+) \\ (u^+, u) & (u', u'). \end{matrix}$$

Taking into account (29), we see that the southeastern corner ($x_1 = x_2 = 2$) is a unique Nash equilibrium, which is strongly Pareto dominated by the northwestern corner ($x_1 = x_2 = 1$). □

Lemma B.1.4. *Let $U, U' \in \mathfrak{U}$, $v_1 > v_2$,*

$$U(v_1, v_1, v_3, \ldots, v_m) > U(v_1, v_2, v_3, \ldots, v_m), \tag{30}$$

$v_1' > v_2'$, and

$$U'(v_1', v_2', v_3', \ldots, v_{m'}') > U'(v_2', v_2', v_3', \ldots, v_{m'}'). \tag{31}$$

Then $v_1 > v_2'$.

Proof. Supposing the contrary, $v_2' \geq v_1$, we denote $u_1^- = U(v_1', v_2, v_3, \ldots, v_m)$, $u_1^+ = U(v_1, v_1, v_3, \ldots, v_m)$, $u_2^- = U'(v_1, v_1, v_3', \ldots, v_{m'}')$, and $u_2^+ = U'(v_1', v_2, v_3', \ldots, v_{m'}')$. We have $u_1^+ > u_1^-$ by Lemma B.1.3 since $v_1' > v_2' \geq v_1$, and $u_2^+ > u_2^-$ by Lemma B.1.2 since $v_2' \geq v_1 > v_2$.

Now we consider a finite game with structured utilities which is consistent with $\mathfrak{U}$: $N = \{1, 2\}$; the facilities are $\mathrm{A} = \{a, b\} \cup C \cup D$, where $C = \{c_s\}_{s \in \{3,\ldots,m\}}$ and $D = \{d_s\}_{s \in \{3,\ldots,m'\}}$; $\Upsilon_1 = \{a, b\} \cup C$, $\Upsilon_2 = \{a, b\} \cup D$; $X_1 = X_2 = \{1, 2\}$; $U_1^{x_1}$ is U for both $x_1 \in X_1$ and $U_2^{x_2}$ is U' for both $x_2 \in X_2$; $\varphi_a(x_1, x_2) = v_1$ if $x_1 = x_2$, $\varphi_a(x_1, x_2) = v_1'$ otherwise; $\varphi_b(x_1, x_2) = v_1$ if $x_1 = x_2$, $\varphi_b(x_1, x_2) = v_2$ otherwise; $\varphi_{c_s}(x_1) = v_s$ for both $x_1 \in X_1$ and all $s = 3, \ldots, m$; $\varphi_{d_s}(x_2) = v_s$ for both $x_2 \in X_2$ and all $s = 3, \ldots, m'$. The 2×2 matrix of the game looks as follows:

$$\begin{matrix} (u_1^-, u_2^+) & (u_1^+, u_2^-) \\ (u_1^+, u_2^-) & (u_1^-, u_2^+). \end{matrix}$$

There is no Nash equilibrium in the game. □

Lemma B.1.5. *Either $U(v_1, v_2, v_3, \ldots, v_m) = \lambda^U(\min_m v_m)$ for every $U \in \mathfrak{U}$ and all $v_1, v_2, v_3, \ldots, v_m \in \mathbb{R}$, or $U(v_1, v_2, v_3, \ldots, v_m) = \lambda^U(\max_m v_m)$ for every $U \in \mathfrak{U}$ and all $v_1, v_2, v_3, \ldots, v_m \in \mathbb{R}$.*

The statement follows from Lemma B.1.4 in the same way as Lemma A.1.7 followed from Lemma A.1.4.

Finally, let us turn to (17).

Lemma B.1.6. *Let $U, U' \in \mathfrak{U}$, $\#\Sigma(U) = \#\Sigma(U')$, and $\lambda^U \neq \lambda^{U'}$. Then $\lambda^U(\mathbb{R}) \cap \lambda^{U'}(\mathbb{R}) = \emptyset$.*

Proof. Supposing the contrary, we, exactly as in the proof of Lemma A.1.9, obtain the existence of $v' > v$ such that $\lambda^{U'}(v') = \lambda^U(v)$. We denote $u^+ = \lambda^U(v)$ and $u^- = \lambda^{U'}(v)$; obviously, $u^- < u^+$. Then we pick $v'' < v$ such that $\lambda^U(v'') = u \in]u^-, u^+[$, and pick $v^0 < v'$ such that $\lambda^U(v^0) = u^0 \in]u, u^+[$. Thus, $u^- < u < u^0 < u^+$.

Now we consider a finite game with structured utilities which is consistent with $\mathfrak{U}$: $N = \{1, 2\}$; the facilities are $\mathrm{A} = \{a_s\}_{s \in \{1,\ldots,m\}}$, where $m = \#\Sigma(U) = \#\Sigma(U')$; $\Upsilon_i = \mathrm{A}$ for both i; $X_1 = X_2 = \{1, 2\}$; U_i^1 is U and U_i^2 is U' for both i; for each $s \in \{1, \ldots, m\}$, $\varphi_{a_s}(1, 1) = v''$, $\varphi_{a_s}(2, 1) = \varphi_{a_s}(1, 2) = v$, and $\varphi_{a_s}(2, 2) = v^0$. Since $\varphi_{a_s}(x_N)$ does not depend on s, the 2×2 matrix of the game is the same whether (16a) or (16b) holds:

$$\begin{array}{cc} (u, u) & (u^+, u^-) \\ (u^-, u^+) & (u^0, u^0). \end{array}$$

We have a prisoner's dilemma: strategies with the "U aggregation" ($x_i = 1$) are dominant, but the southeastern corner strongly Pareto dominates the northwestern one. $\square$

Remark. Unlike Lemma A.1.9, there is nothing special about the case of $\#\Sigma(U) = 1$ here.

B.2 Proof of [3 ⇒ 1] in Theorem 6.3

Let $\mathfrak{U}$ be a set of admissible aggregation functions satisfying Condition 3 from Theorem 6.3. Denoting $\mathfrak{U}^m = \{U \in \mathfrak{U} \mid \#\Sigma(U) = m\}$ for every $m \in \mathbb{N}$, we may argue in the same way as in Section A.2 and obtain the partitioning of each (nonempty) $\mathfrak{U}^m$ into subsets W^α ($\alpha \in \mathrm{A}(m)$) such that $\lambda^U = \lambda^{U'}$ whenever U and U' belong to the same W^α and the set $\mathrm{A}(m)$ is linearly ordered in the sense that $\alpha > \alpha' \iff \left[\lambda^U(u) > \lambda^{U'}(u') \text{ whenever } U \in W^\alpha,\, U' \in W^{\alpha'}, \text{ and } u, u' \in \mathbb{R}\right]$.

Let Γ be a game with structured utilities which is consistent with $\mathfrak{U}$ and where the strategy sets are compact and utility functions upper semicontinuous. We have to prove that Γ admits an ω-potential.

For each $i \in N$, we have $\#\Sigma(U_i^{x_i}) = \#\Upsilon_i$ for all $x_i \in X_i$. Therefore, the order on $\mathrm{A}(\#\Upsilon_i)$ generates an ordering on X_i (exactly as in Section A.2): $y_i \trianglerighteq_i x_i \iff \left[U_i^{y_i} \in W^\alpha \;\&\; U_i^{x_i} \in W^{\alpha'} \;\&\; \alpha \geq \alpha'\right]$. Obviously, $u_i(y_i, z_{-i}) > u_i(x_i, z'_{-i})$ for all $z_{-i}, z'_{-i} \in X_{-i}$ whenever $y_i \succ_i x_i$. It follows immediately that $y_i \trianglerighteq_i x_i$ whenever $y_N \triangleright_I x_N$ and $i \in I$.

Now we define a preorder on X_N by

$$y_N \succeq_N x_N \rightleftharpoons \forall i \in N\, [y_i \trianglerighteq_i x_i],$$

and denote $\succ_N$ and $\sim_N$ its asymmetric and symmetric components. The upper semicontinuity of u_i implies that each $\trianglerighteq_i$ is ω-transitive nad hence $\succeq_N$ is ω-transitive as well.

Apart from "genuine" utilities u_i, we introduce, for each $i \in N$, "neutral" utility functions u_i^0 by (6), i.e., "without λ's."

Let (16a) hold. We denote $>_{\mathrm{Lmin}}$ the leximin ordering on X_N defined by utility functions u_i^0 as in the proof of Theorem 4.1. Now we define our potential as a lexicography:

$$y_N \ggcurly x_N \iff \big[y_N \succ_N x_N \ \text{ or } \ [y_N \sim_N x_N \ \& \ y_N >_{\mathrm{Lmin}} x_N]\big]. \tag{32}$$

Obviously, $\ggcurly$ is irreflexive and transitive. To show its ω-transitivity, we assume that $x_N^k \to x_N^\omega$ and $x_N^{k+1} \ggcurly x_N^k$ for all $k \in \mathbb{N}$. Then, by definition, $x_N^{k+1} \succeq_N x_N^k$ for all k, and hence $x_N^\omega \succeq_N x_N^0$ since that relation is ω-transitive. If $x_N^\omega \succ_N x_N^0$, we are home by the first component in (32). Otherwise, we have $x_N^{k+1} \sim_N x_N^k$ for all k, and hence are home by the second component in (32) since $>_{\mathrm{Lmin}}$ is ω-transitive.

Finally, let $y_N \rhd^{\mathrm{Coa}} x_N$; we have to show that $y_N \ggcurly x_N$. First, $y_N \succeq_N x_N$. If $y_N \succ_N x_N$, then we are home immediately. Otherwise, the same λ's are applied to each u_i^0 in both cases; hence $y_N >_{\mathrm{Lmin}} x_N$ exactly as in the proof of Theorem 4.1.

If (16b) holds, we argue dually, replacing $>_{\mathrm{Lmin}}$ with $>_{\mathrm{Lmax}}$.

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