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On automatic derivation of first order conditions in dynamic stochastic optimisation problems*

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Abstract

This note presents an algorithm for deriving first order conditions applicable to the most common optimisation problems encountered in dynamic stochastic models automatically. Given a symbolic library or a computer algebra system one can efficiently derive first order conditions which can then be used for solving models numerically (steady state, linearisation).

Keywords: DSGE, stochastic optimisation, first order conditions, symbolic computations.

JEL classification: C61, C63, C68.

1 Introduction

Toolboxes aimed at solving dynamic stochastic general equilibrium models (e.g. Dynare [1]) require users to derive first order conditions for agents' optimisation problems manually. This is most probably caused by the lack of an algorithm for deriving them automatically, given objective function and constraints. This note presents such algorithm which is applicable to most common optimisation problems encountered in dynamic stochastic models.

Given a symbolic library or a computer algebra system satisfying some functional requirements, one can efficiently derive the first order conditions which can then be used for solving the model numerically (steady state, linearisation).

The approach presented here is fairly general and can be extended in order to handle more complicated optimisation problems.

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2 The problem

This is the standard setup presented in economic textbooks. A detailed exposition can be for example found in [3] or [2].

Time is discrete, infinite, and it begins at $t = 0$. In each period $t = 1, 2, \dots$ a realisation of the stochastic event ξ_t is observed. A history of events up to time t is denoted by s_t. More formally, let $(\Omega, \mathcal{F}, \mathrm{P})$ be a discrete probabilistic space with the filtration $\{\emptyset, \Omega\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \mathcal{F}_t \subset \mathcal{F}_{t+1} \cdots \subset \Omega$. Each event at date t (ξ_t) and every history up to time t (s_t) is $\mathcal{F}_t$-measurable. Let $\pi(s_t)$ denote the probability of history s_t up to time t. The conditional probability $\pi(s_{t+1}|s_t)$ is the probability of the event ξ_{t+1} such that $s_{t+1} = s_t \cap \xi_{t+1}$.

In what follows it is assumed that variable with the time index t is $\mathcal{F}_t$-measurable.

In the period $t = 0$ an agent determines vectors of control variables $x(s_t) = \left(x^1(s_t), \dots, x^N(s_t)\right)$ at all possible events s_t as a solution to her optimisation problem. The objective function U_0 (lifetime utility) is recursively given by the following equation:

$$U_t(s_t) = F\left(x_{t-1}(s_{t-1}), x_t(s_t), z_{t-1}(s_{t-1}), z_t(s_t), \mathrm{E_t} H^1(x_{t-1}, x_t, U_{t+1}, z_{t-1}, z_t, z_{t+1}), \dots, \mathrm{E_t} H^J(\dots)\right), \tag{1}$$

with constraints satisfying:

$$\begin{aligned} &G^i\left(x_{t-1}(s_{t-1}), x_t(s_t), z_{t-1}(s_{t-1}), z_t(s_t), \mathrm{E_t} H^1(x_{t-1}, x_t, U_{t+1}, z_{t-1}, z_t, z_{t+1}), \dots, \mathrm{E_t} H^J(\dots)\right) = 0, \\ &x_{-1} \text{ given.} \end{aligned} \tag{2}$$

where $x_t(s_t)$ are decision variables and $z_t(s_t)$ are exogenous variables and $i = 1, \dots, I$ indexes constraints. We shall denote the expression $\mathrm{E_t} H^j(x_{t-1}, x_t, U_{t+1}, z_{t-1}, z_t, z_{t+1})$ compactly as $\mathrm{E_t} H^j_{t+1}$ with $j = 1, \dots, J$. We have:

$$\mathrm{E_t} H^j_{t+1} = \sum_{s_{t+1} \subset s_t} \pi(s_{t+1}|s_t) H^j\left(x_{t-1}(s_{t-1}), x_t(s_t), U_{t+1}(s_{t+1}), z_{t-1}(s_{t-1}), z_t(s_t), z_{t+1}(s_{t+1})\right).$$

Let us now modify the problem by substituting $q^j_t(s_t)$ for $\mathrm{E_t} H^j_{t+1}$ and adding constraints of the form $q^j_t(s_t) = \mathrm{E_t} H^j_{t+1}$.

We shall also use $F_t(s_t)$ and $G^i_t(s_t)$ to denote expressions $F\left(x_{t-1}(s_{t-1}), x_t(s_t), z_{t-1}(s_{t-1}), z_t(s_t), q^1_t(s_t), \dots, q^j_t(s_t)\right)$ and $G^i\left(x_{t-1}(s_{t-1}), x_t(s_t), z_{t-1}(s_{t-1}), z_t(s_t), q^1_t(s_t), \dots, q^j_t(s_t)\right)$ respectively.

Then the agent's problem may be written as:

$$\begin{aligned} &\max_{(x_t)_{t=0}^{\infty}, (U_t)_{t=0}^{\infty}} U_0 \\ &\text{s.t. :} \\ &U_t(s_t) = F_t(s_t), \\ &G^i_t(s_t) = 0, \\ &q^j_t(s_t) = \mathrm{E_t} H^j_{t+1}, \\ &x_{-1} \text{ given.} \end{aligned} \tag{3}$$

3 The First Order Conditions

The Lagrangian for the problem (3) may be written as follows:

$$\begin{aligned}\mathcal{L} = U_0 &+ \sum_{t=0}^{\infty}\sum_{s_t}\pi(s_t)\lambda_t(s_t)\left[F_t(s_t) - U_t(s_t)\right] \\ &+ \sum_{t=0}^{\infty}\sum_{s_t}\pi(s_t)\lambda_t(s_t)\sum_{i=1}^{I}\mu_t^i(s_t)G_t^i(s_t) \\ &+ \sum_{t=0}^{\infty}\sum_{s_t}\pi(s_t)\lambda_t(s_t)\sum_{j=1}^{J}\eta_t^j(s_t)(\mathrm{E}_t H_{t+1}^j - q_t^j(s_t)).\end{aligned}$$

The first order condition for maximizing the Lagrangian with respect to $U_t(s_t)$ is:

$$0 = -\pi(s_t)\lambda_t(s_t) + \pi(s_{t-1})\lambda_{t-1}(s_{t-1})\pi(s_t|s_{t-1})\sum_{j=1}^{J}\eta_{t-1}^j(s_{t-1})H_{t,3}^j(s_t),$$

where 3 in $H_{t,3}^j(s_t)$ stands for a partial derivative of $H_t^j(s_t)$ with respect to its third argument, i.e. $U_t(s_t)$ (we shall adopt such notation throughout this note).

Using the property $\pi(s_{t-1})\pi(s_t|s_{t-1}) = \pi(s_t)$, aggregating the equation with respect to (s_t) and dividing it by $\pi(s_t)$ yields:

$$\lambda_t(s_t) = \lambda_{t-1}(s_{t-1})\sum_{j=1}^{J}\eta_{t-1}^j(s_{t-1})H_{t,3}^j(s_t),$$

which implies:

$$\lambda_{t+1}(s_{t+1}) = \lambda_t(s_t)\sum_{j=1}^{J}\eta_t^j(s_t)H_{t+1,3}^j(s_{t+1}).$$

In general, Lagrange multipliers on time aggregators, i.e. equation (1), are non-stationary. For instance, in case of exponential discounting, one will have $\lambda_{t+1} = \beta\lambda_t$. Dividing the equation by $\lambda_t(s_t)$ we obtain:

$$\frac{\lambda_{t+1}(s_{t+1})}{\lambda_t(s_t)} = \sum_{j=1}^{J}\eta_t^j(s_t)H_{t+1,3}^j(s_{t+1}).$$

Now let us set $\lambda_t(s_t) = 1$. This is equivalent to reinterpreting $\lambda_{t+1}t(s_{t+1})$ as $\frac{\lambda_{t+1}(s_{t+1})}{\lambda_t(s_t)}$ in all equations. We have:

$$\lambda_{t+1}(s_{t+1}) = \sum_{j=1}^{J}\eta_t^j(s_t)H_{t+1,3}^j(s_{t+1}). \tag{4}$$

The first order condition for maximizing the Lagrangian $\mathcal{L}$ with respect to $x_t(s_t)$ gives:

$$\begin{aligned}0 = \pi(s_t)\lambda_t(s_t)F_{t,2}(s_t) &+ \sum_{s_{t+1}\subset s_t}\pi(s_{t+1})\lambda_{t+1}(s_{t+1})F_{t+1,1}(s_{t+1}) \\ + \pi(s_t)\lambda_t(s_t)\sum_{i=1}^{I}\mu_t^i(s_t)G_{t,2}^i(s_t) &+ \sum_{s_{t+1}\subset s_t}\pi(s_{t+1})\lambda_{t+1}(s_{t+1})\sum_{i=1}^{I}\mu_{t+1}^i(s_{t+1})G_{t+1,1}^i(s_{t+1}) \\ + \pi(s_t)\lambda_t(s_t)\sum_{j=1}^{J}\eta_t^j(s_t)H_{t+1,2}^j(s_{t+1}) &+ \sum_{s_{t+1}\subset s_t}\pi(s_{t+1})\lambda_{t+1}(s_{t+1})\sum_{j=1}^{J}\eta_{t+1}^j(s_{t+1})H_{t+2,1}^j(s_{t+2}).\end{aligned}$$

Simplification yields:

$$
\begin{aligned}
0 = \pi(s_t)\lambda_t(s_t) & \left[F_{t,2}(s_t) + \sum_{i=1}^{I} \mu_t^i(s_t) G_{t,2}^i(s_t) + \sum_{j=1}^{J} \eta_t^j(s_t) H_{t+1,2}^j(s_{t+1}) \right] \\
& + \sum_{s_{t+1} \subset s_t} \pi(s_{t+1})\lambda_{t+1}(s_{t+1}) \left[F_{t+1,1}(s_{t+1}) + \sum_{i=1}^{I} \mu_{t+1}^i(s_{t+1}) G_{t+1,1}^i(s_{t+1}) + \sum_{j=1}^{J} \eta_{t+1}^j(s_{t+1}) H_{t+2,1}^j(s_{t+2}) \right].
\end{aligned}
$$

Setting $\lambda_t(s_t) = 1$ as before, dividing the equation by $\pi(s_t)$, and making use of the property $\pi(s_{t+1}) = \pi(s_{t+1}|s_t)\pi(s_t)$ we obtain:

$$
\begin{aligned}
0 = & F_{t,2}(s_t) + \sum_{i=1}^{I} \mu_t^i(s_t) G_{t,2}^i(s_t) + \sum_{j=1}^{J} \eta_t^j(s_t) H_{t+1,2}^j(s_{t+1}) \\
& + \sum_{s_{t+1} \subset s_t} \pi(s_{t+1}|s_t)\lambda_{t+1}(s_{t+1}) \left[F_{t+1,1}(s_{t+1}) + \sum_{i=1}^{I} \mu_{t+1}^i(s_{t+1}) G_{t+1,1}^i(s_{t+1}) + \sum_{j=1}^{J} \eta_{t+1}^j(s_{t+1}) H_{t+2,1}^j(s_{t+2}) \right].
\end{aligned}
$$

After rearrangement we arrive at stochastic Euler equations:

$$
\begin{aligned}
0 = & F_{t,2}(s_t) + \sum_{i=1}^{I} \mu_t^i(s_t) G_{t,2}^i(s_t) + \sum_{j=1}^{J} \eta_t^j(s_t) H_{t+1,2}^j(s_{t+1}) \\
& + \mathrm{E}_t \left[\lambda_{t+1} \left(F_{t+1,1} + \sum_{i=1}^{I} \mu_{t+1}^i(s_{t+1}) G_{t+1,1}^i + \sum_{j=1}^{J} \eta_{t+1}^j(s_{t+1}) H_{t+2,1}^j(s_{t+2}) \right) \right].
\end{aligned} \tag{5}
$$

Finally, differentiating the Lagrangean $\mathcal{L}$ with respect to $q_t^j(s_t)$ gives:

$$
0 = \pi(s_t)\lambda_t(s_t) F_{t,4+j}(s_t) + \pi(s_t)\lambda_t(s_t) \sum_{i=1}^{I} \mu_t^i(s_t) G_{t,4+j}^i(s_t) - \pi(s_t)\lambda_t(s_t)\eta_t^j(s_t).
$$

Setting $\lambda_t(s_t) = 1$ and dividing the equation by $\pi(s_t)$ yields:

$$
0 = F_{t,4+j}(s_t) + \sum_{i=1}^{I} \mu_t^i(s_t) G_{t,4+j}^i(s_t) - \eta_t^j(s_t). \tag{6}
$$

There are $N + 1 + J$ first order conditions: one w.r.t. to U_t (4), N w.r.t. x_t^n (5) and J w.r.t. q_t^j (6). There are also I conditions $G_t^i = 0$, the equation $F\left(x_{t-1}, x_t, z_{t-1}, z_t, q_t^1, \dots, q_t^j\right) = U_t$ and J equations defining q_t^j. The overall number of equations $(N + I + 2J + 2)$ equals the number of variables: N decision variables x_t^n, the variable U_t, J variables q_t^j, the Lagrange multiplier λ_t, I Lagrange multipliers μ_t^i and J Lagrange multipliers η_t^j (which gives $N + I + 2J + 2$ variables).

4 An example

The purpose of this section is to illustrate the FOC derivation procedure presented above with an example of a typical optimisation problem encountered in numerous RBC models. Assume a representative firm in a competitive setting maximises an objective (discounted profits) at time 0 (Π_0), given definition of profits earned each period (π_t), the technology available (Cobb-Douglas production function), and the low of motion for capital (K_t). The firm owns capital and employs labour L_t at wage W_t. It uses production factors to produce Y_t, which it sells at price P_t. All prices are treated as given. The firm discounts its next-period profits with the growth rate of the Lagrange multiplier (λ_t^c) in household's (firm's owner) problem corresponding to the budget constraint, i.e. λ_t^c is a shadow price of consumption. Therefore, the expected discounted profit is:

$$\mathrm{E}_0\left[\sum_{t=0}^{\infty}\frac{\beta^t\lambda_t^c}{\lambda_0^c}\pi_t\right].$$

The firm's optimisation problem can be written as follows:

$$\begin{aligned}
&\max_{K_t,L_t^d,Y_t,I_t,\pi_t}\Pi_t=\pi_t+\mathrm{E}_t\left[\beta\frac{\lambda_{t+1}^c}{\lambda_t^c}\Pi_{t+1}\right] \qquad (7)\\
&\text{s.t. :}\\
&\pi_t=P_tY_t-L_tW_t-I_t, \qquad (\lambda_t^\pi)\\
&Y_t={K_{t-1}}^{\alpha}\left(L_te^{Z_t}\right)^{1-\alpha}, \qquad (\lambda_t^Y)\\
&K_t=I_t+K_{t-1}\left(1-\delta\right), \qquad (\lambda_t^k)
\end{aligned}$$

where Z_t is an exogenous variable determining labour productivity and α, β, and δ are respectively: capital share, discount factor, and depreciation rate. λ_t^π, λ_t^Y, λ_t^k are Lagrange multipliers for the constraints.

In order to derive the FOCs for this maximisation problem using equations (4), (5), and (6), it is helpful to define the symbols used in the previous section for the problem (7):

$$\begin{aligned}
&x_t\equiv[K_t,L_t,Y_t,I_t,\pi_t],\\
&z_t\equiv[Z_t],\\
&U_t\equiv\Pi_t, \qquad \lambda_t\equiv\lambda_t^\Pi\\
&q_t^1=\mathrm{E_t}H_{t+1}^1\equiv\mathrm{E_t}\left[\beta\frac{\lambda_{t+1}^c}{\lambda_t^c}\Pi_{t+1}\right], \qquad (\eta_t^1)\\
&F_t\equiv x_t^5+q_t^1,\\
&G_t^1\equiv P_tx_t^3-x_t^2W_t-x_t^4-x_t^5, \qquad (\mu_t^1\equiv\lambda_t^\pi)\\
&G_t^2\equiv {x_{t-1}^1}^{\alpha}\left(x_t^2e^{z_t^1}\right)^{1-\alpha}-x_t^3, \qquad (\mu_t^2\equiv\lambda_t^Y)\\
&G_t^3\equiv x_t^4+x_{t-1}^1\left(1-\delta\right)-x_t^1. \qquad (\mu_t^3\equiv\lambda_t^k)
\end{aligned}$$

Substituting these definitions in equations (4)-(6) one obtains the following set of first order conditions:

- from equation (4):

$$-\lambda_t^\Pi+\eta_{t-1}^1\beta{\lambda_{t-1}^c}^{-1}\lambda_t^c=0,$$

- from equation (5):

$$-\lambda_t^k+\mathrm{E}_t\left[\lambda_{t+1}^\Pi\left(\alpha\lambda_{t+1}^Y{K_t}^{-1+\alpha}\left(L_{t+1}^de^{Z_{t+1}}\right)^{1-\alpha}+\lambda_{t+1}^k\left(1-\delta\right)\right)\right]=0,$$

$$-W_t\lambda_t^\pi+\lambda_t^Ye^{Z_t}\left(1-\alpha\right){K_{t-1}}^{\alpha}\left(L_t^de^{Z_t}\right)^{-\alpha}=0,$$

$$P_t\lambda_t^\pi-\lambda_t^Y=0,$$

$$-\lambda_t^\pi+\lambda_t^k=0,$$

$$1-\lambda_t^\pi=0,$$

- from equation (6):

$$1 - \eta_t^1 = 0.$$

Substituting for Lagrange multipliers one gets familiar results:

$$-1 + \mathrm{E}_t \left[\beta {\lambda_t^c}^{-1} \lambda_{t+1}^c \left(1 - \delta + \alpha P_{t+1} {K_t}^{-1+\alpha} \left(L_{t+1}^d e^{Z_{t+1}} \right)^{1-\alpha} \right) \right] = 0,$$

$$-W_t + P_t e^{Z_t} \left(1-\alpha\right) {K_{t-1}}^{\alpha} \left(L_t^d e^{Z_t} \right)^{-\alpha} = 0.$$

5 Algorithm

The equations derived in section 3 can be collected to yield an automatic method for deriving the first order conditions manually. This method is presented as Algorithm 1.

$\sharp$ FOC w.r.t. U
for $j \leftarrow 1, \ldots, J$ **do**
    $Z^j \leftarrow \eta_t^j \frac{\partial H_{t+1}^j}{\partial U_{t+1}}$
end for
$Z \leftarrow \sum_j Z^j$
$C^0 \leftarrow Z - \lambda_{t+1}$
$\sharp$ FOCs w.r.t. x
for $n \leftarrow 1, \ldots, N$ **do**
    for $i \leftarrow 1, \ldots, I$ **do**
        $L_n^i \leftarrow \mu_t^i \frac{\partial G_t^i}{\partial x_t^n}$
        $P_n^i \leftarrow \mu_{t+1}^i \frac{\partial G_{t+1}^i}{\partial x_t^n}$
    end for
    for $j \leftarrow 1, \ldots, J$ **do**
        $M_n^j \leftarrow \eta_t^j \frac{\partial H_{t+1}^j}{\partial x_t^n}$
        $Q_n^j \leftarrow \eta_{t+1}^j \frac{\partial H_{t+2}^j}{\partial x_t^n}$
    end for
    $L_n \leftarrow \sum_i L_n^i$
    $M_n \leftarrow \sum_j M_n^j$
    $P_n \leftarrow \sum_i P_n^i$
    $Q_n \leftarrow \sum_j Q_n^j$
    $R_n \leftarrow \frac{\partial F_{t+1}}{\partial x_t^n} + P_n + Q_n$
    $C^n \leftarrow \frac{\partial F_t}{\partial x_t^n} + L_n + M_n + \mathrm{E}_t[\lambda_{t+1} R_n]$
end for
$\sharp$ FOCs w.r.t. q
for $j \leftarrow 1, \ldots, J$ **do**
    for $i \leftarrow 1, \ldots, I$ **do**
        $S_j^i \leftarrow \mu_t^i \frac{\partial G_t^i}{\partial q_t^j}$
    end for
    $S_j \leftarrow \sum_i S_j^i$
    $C^{N+j} \leftarrow \frac{\partial F_t}{\partial q_t^j} + S_j - \eta_t^j$
end for

Algorithm 1: Derivation of First Order Conditions for (3)

Note, that the last set of first order conditions can always be solved for η_t and used to eliminate it from all remaining equations.

In order to implement the above algorithm, one needs a symbolic library or a computer algebra system with the following functionality:

- variables and time-indexed variables,
- basic arithmetical operations (+, −, /, ^) and elementary functions,
- lag and conditional expected value operators,
- substitution and differentiation operations.

6 Special case — the deterministic model

In the case of deterministic model the objective function U_0 (lifetime utility) is recursively given by the following equation:

$$U_t = F\left(x_{t-1}, x_t, z_{t-1}, z_t, z_{t+1}, U_{t+1}\right), \tag{8}$$

with constraints satisfying:

$$G^i\left(x_{t-1}, x_t, z_{t-1}, z_t, z_{t+1}, U_{t+1}\right), \tag{9}$$

where x_t are decision variables and z_t are exogenous variables and $i = 1, \ldots, I$ indexes constraints.

As earlier, we shall also use F_t and G^i_t to denote the expressions $F\left(x_{t-1}, x_t, z_{t-1}, z_t, z_{t+1}, U_{t+1}\right)$ and $G^i\left(x_{t-1}, x_t, z_{t-1}, z_t, z_{t+1}, U_{t+1}\right)$, respectively.

Then the agent's problem may be written as:

$$\begin{aligned}
&\max_{(x_t)_{t=0}^{\infty}, (U_t)_{t=0}^{\infty}} U_0\\
&\text{s.t. :}\\
&U_t = F_t,\\
&G^i_t = 0,\\
&x_{-1} \text{ given.}
\end{aligned} \tag{10}$$

The Lagrangian for problem (10) may be written as follows:

$$\mathcal{L} = U_0 + \sum_{t=0}^{\infty} \lambda_t \left(F_t - U_t\right) + \sum_{t=0}^{\infty} \lambda_t \sum_{i=1}^{I} \mu^i_t G^i_t.$$

The first order condition for maximizing the Lagrangian with respect to $U_t(s_t)$ is:

$$0 = -\lambda_t + \lambda_{t-1} F_{t-1,6} + \lambda_{t-1} \sum_{i=1}^{I} \mu^i_{t-1} G^i_{t-1,6}.$$

Rearranging and aggregating the formula yields:

$$\lambda_t = \lambda_{t-1} \left(F_{t-1,6} + \sum_{i=1}^{I} \mu^i_{t-1} G^i_{t-1,6}\right),$$

which implies:

$$\lambda_{t+1} = \lambda_t \left(F_{t,6} + \sum_{i=1}^{I} \mu_t^i G_{t,6}^i \right).$$

Dividing the equation by λ_t we obtain:

$$\frac{\lambda_{t+1}}{\lambda_t} = F_{t,6} + \sum_{i=1}^{I} \mu_t^i G_{t,6}^i.$$

Now let us set $\lambda_t = 1$. This is equivalent to reinterpreting λ_{t+1} as $\frac{\lambda_{t+1}}{\lambda_t}$ in all equations. We have:

$$\lambda_{t+1} = F_{t,6} + \sum_{i=1}^{I} \mu_t^i G_{t,6}^i. \tag{11}$$

The first order condition for maximizing the Lagrangian $\mathcal{L}$ with respect to x_t gives:

$$0 = \lambda_t F_{t,2} + \lambda_{t+1} F_{t+1,1} + \lambda_t \sum_{i=1}^{I} \mu_t^i G_{t,2}^i + \lambda_{t+1} \sum_{i=1}^{I} \mu_{t+1}^i G_{t+1,1}^i.$$

After setting $\lambda_t = 1$ as before and rearrangement we arrive at:

$$0 = F_{t,2} + \sum_{i=1}^{I} \mu_t^i G_{t,2}^i + \lambda_{t+1} \left(F_{t+1,1} + \sum_{i=1}^{I} \mu_{t+1}^i G_{t+1,1}^i \right). \tag{12}$$

Equations (11) and (12) can be collected to yield an automatic method for deriving first order conditions of a deterministic model. This method is presented as Algorithm 2.

$\sharp$ FOC w.r.t. U
$A \leftarrow \frac{\partial F_t}{\partial U_{t+1}}$
for $i \leftarrow 1, \ldots, I$ **do**
    $B^i \leftarrow \mu_t^i \frac{\partial G_t^i}{\partial U_{t+1}}$
end for
$B \leftarrow \sum_i B^i$
$C^0 \leftarrow A + B - \lambda_{t+1}$
$\sharp$ FOCs w.r.t. x
for $n \leftarrow 1, \ldots, N$ **do**
    $K_n \leftarrow \frac{\partial F_t}{\partial x_t^n}$
    $M_n \leftarrow \frac{\partial F_{t+1}}{\partial x_t^n}$
    for $i \leftarrow 1, \ldots, I$ **do**
        $L_n^i \leftarrow \mu_t^i \frac{\partial G_t^i}{\partial x_t^n}$
        $P_n^i \leftarrow \mu_{t+1}^i \frac{\partial G_{t+1}^i}{\partial x_t^n}$
    end for
    $L_n \leftarrow \sum_i L_n^i$
    $P_n \leftarrow \sum_i P_n^i$
    $C^n \leftarrow K_n + L_n + \lambda_{t+1} (M_n + P_n)$
end for

Algorithm 2: Derivation of First Order Conditions — deterministic model

7 Summary

We have derived the first order conditions for a general dynamic stochastic optimisation problem with objective function given by a recursive forward-looking equation. We have also constructed an algorithm for deriving first order conditions on a computer automatically. This algorithm has already been implemented in the DSGE solution framework called `gEcon`[1] which is being developed at the Chancellery of the Prime Minister of the Republic of Poland.

It is hoped that the algorithm presented here will reduce the burden and risk associated with pen & paper derivations of first order conditions in DSGE models and allow researchers in the field to focus on economic aspects of the models.

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