On the convexity of the cost function for the (Q,R) inventory model

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Abstract
In this paper we investigate the convexity of the annual total cost function for the Hadley & Whitin (1963) continuous review (Q,R) inventory model with backorders and fixed lead-time, when the reorder point is non-negative and the cost per unit backordered is used to calculate the expected annual shortage cost. Efficient procedures for finding the order quantity and reorder point which ensure a global minimum are provided when the demand in the lead time follows the Normal and Log-Normal distributions. Convexity conditions are obtained for each distribution and numerical examples are given to explain how the values of the model cost parameters affect the optimal solution. The results indicate strong interaction between convexity and cost parameters as their values determine when the minimum cost obtained from solving the first order conditions is global and when the global minimum is attained setting the reorder point to zero.

Keywords: Logistics; Continuous review inventory model; Expected total cost function; Convexity conditions; Cost parameters.

1. Introduction

For optimizing continuous review (Q,R) inventory systems with backorders and fixed lead-time, the majority of works in the literature have used the annual total cost function proposed by Hadley and Whitin (1963). This cost function (hereafter "H-W") results from the sum of the expected annual ordering, inventory carrying and shortage costs. For evaluating the expected annual inventory carrying cost, the authors approximated the expected on-hand inventory at any time with the expression $\left(\mathrm{Q}/2+\text{safety stock}\right)$, while for the calculation of the expected annual shortage cost (hereafter "EASC") they used the cost per unit backordered. Further, they assumed that the lead-time demand follows the Normal distribution. Under these specifications, the authors claimed that this cost function is convex in both Q and R. So, setting the partial derivatives equal to zero, they developed an iterative procedure to determine the optimal sizes for the order quantity, Q, and the reorder point, R, and to obtain the minimum cost.

In the subsequent years, a number of papers has questioned the existence of the convexity of the H-W cost function. When the probability density function of the lead-time demand distribution takes on the value zero, Veinott (1964) showed that the expected size of backorders is not a convex function. But, according to the author, the H-W cost function is convex when the probability density function of the lead-time demand is non-increasing for any positive demand. Brooks and Lu (1969) proved that the expected size of backorders, although is a convex function in Q given R and convex in R given Q, this function is not in general convex in both Q and R. For Normal lead-time demand, the authors showed that convexity exists only when the reorder point is greater than the expected demand during lead-time, namely, when the service level is above 0.50.

Under the same principles, Minh (1975) supported that the H-W cost function with Normal lead-time demand is not convex, proving that the first partial derivatives with respect to Q and R vanish at two different points and concluding that there are two solutions from which the first is classified as a minimum and the second as a saddle point. Besides, the minimum solution can be obtained even when the reorder point is lower than the expected lead-time demand. Das (1988) showed that the global minimum of the H-W cost function can be identified by means of the first order conditions and the shape of the lead-time demand distribution. Particularly, when the lead-time demand is either unimodal or J-shaped, the author claimed that the local minimum is also the global minimum. Furthermore, for relatively low values of the shortage cost per unit backordered, Lau and Lau (2002) found that in some stages of applying the Hadley-Whitin iterative procedure, the service level takes on negative values and this has as a result the procedure to break down.

Apart from the aforementioned works, the convexity problem of the H-W cost function has been also investigated under alternative ways of evaluating the EASC. Using a specified fixed cost per stock-out occasion, Silver et al. (1998) found that the H-W cost function is convex for service levels greater than 0.50. Using Silver et al.'s expression of evaluating the EASC, Chung et al. (2009) concluded that the H-W cost function is not convex in general and proposed an alternative algorithm to locate the optimal values of Q and R. Lau et al. (2002a) examined the case of taking a negative service level when the shortage cost is computed using either a cost per unit short or a cost per stock-out occasion. The authors gave explanations why and when the Hadley-Whitin iterative procedure leads to nonsensical optimal solutions.

On the contrary, using a fractional charge per unit short per unit time to evaluate the EASC, Das (1983a) proved for Normal and Gamma lead-time demand distributions that the H-W cost function is strictly convex. Finally, when the items of an order do not arrive within a scheduled time, Dohi et al. (1999) studied the problem of finding the best timing to deliver the items after a shortage occurs. The authors presented a mathematical model to control a shortage period by an emergency order and concluded that controlling the order time limit, the inventory management system can allow shortages to a degree.

Under Normal and Log-Normal demand in a fixed lead-time, our paper re-examines the convexity problem of the H-W cost function when the cost per unit backordered is used for evaluating the EASC, and for Normal lead-time demand the reorder point is non-negative. Taking first and second partial derivatives with respect to Q and R, we obtain the general form of the Hessian determinant in terms only of the reorder point R. Then, transforming R into standard Normal values, specifications for the Hessian determinant are obtained for both Normal and Log-Normal lead-time demand. Further, for both distributions, following an analogous approach to that of Das (1988) and Chung et al. (2009), we rewrite the H-W cost function in terms only of the values of the standard Normal transferring in this way the study of convexity from the three dimensional to the two dimensional space.

This approach allow us for the first time to relate the conditions for convexity of the H-W cost function to the values of the following three cost parameters: the fixed ordering cost, the holding cost per unit per year, and the shortage cost per unit. In particular, in terms of values of these three cost parameters, conditions are derived in order to identify the following three cases which hold for both distributions: (a) the H-W cost function has a global minimum which is obtained from solving the first order conditions of minimizing the H-W cost function, (b) the cost at zero reorder point is smaller than the "minimum" cost obtained from solving the first order conditions, and (c) the H-W cost function is either increasing or non-decreasing in the two-dimensional space, in which case the minimum cost occurs at zero reorder point. The latter case is referred to the literature as the degeneracy problem (Lau et al., 2002a).

Being able to express the H-W cost function in terms only of the standard Normal values when the demand in fixed lead-time is Normal or Log-Normal and to perform the analysis in the two dimensional space is the main reason which justifies the choice of these two distributions. Further, for the Log-Normal, the reorder point is always non-negative, while for the Normal, the reorder point can be negative. We show in this paper that this fact justifies why the conditions of identifying the aforementioned three cases regarding convexity are different between Normal and Log-Normal. This is another reason of including these two distributions in our analysis. To mention here that for Normal lead-time demand, when the reorder point is negative the form of the H-W cost function changes and is given in Lau et al. (2002b). This latter case is not examined in the current work.

Apart from the reasons stated above, the choice of the Normal distribution to model continuous demand has been also made since its use in both research and practice offers tractable results and good approximations, especially when demand has a relatively low coefficient of variation, preferably below 0.3 (e.g. Lau, 1997; Janssen et al., 2009; Kevork, 2010). Parallel to the Normal, the choice of the Log-Normal has been made as it is positively skewed, the areas under its probability density function can be calculated through the standard Normal distribution, and Log-Normal demand is positive for any coefficient of variation (e.g. Tadikamalla, 1979, 1984; Das, 1983b). It is worth mentioning here that the majority of papers in the inventory literature examines convexity under continuous demand. For integer-valued demand processes, Ang et al. (2013) considered the single-item continuous review and periodic review systems and showed that convexity properties of the policy parameters in continuous space did not hold in discrete space. Finally, it is worthwhile to point out that the parameters of each demand distribution are considered as known. For the case of unknown demand parameters, several papers suggest methods to estimate them aiming mainly to study the behavior of service level (e.g. Syntetos & Boylan, 2008).

Based on the aforementioned discussion and remarks the rest of the paper is organized as follows. In Section 2 we revise first and second order conditions for the H-W cost function minimization problem and derive the general form of the Hessian determinant as function only of the reorder point. In sections 3 and 4, we give for Log-Normal and Normal lead-time demand respectively the following: (a) the cost function and the Hessian determinant in terms only of standard Normal values, (b) the conditions to have global minimum from solving the first order conditions of minimizing the H-W cost function or a global minimum at zero reorder point, and (c) a new algorithm leading to values for the order quantity and the reorder point which ensure global minimum cost. Then in Section 5, through a numerical experimentation, we investigate the managerial implications of changing the values of cost parameters on the optimal sizes of order quantities and reorder points, as well as, on the global minimum cost. To perform the analysis, we use "logical" values for the cost parameters which are suggested in the inventory literature. The last section concludes the paper summarizing the most important findings.

2. Theoretical Background

For the continuous review (Q,R) inventory system with fixed order quantity-reorder point and the demand to be backordered when the system is out of stock, we provide below the list of symbols, which are used throughout this paper and the required assumptions to develop the convexity conditions of the H-W cost function:

Notation

Q	:	order quantity.
R	:	reorder point.
A	:	fixed ordering cost (€).
h	:	holding cost per unit per year (€/unit/yr).
s	:	shortage cost per unit[1] (€/unit).
X	:	random variable representing total demand during lead-time.
$f(\mathrm{x})$	:	probability density function of X.
$F(\mathrm{x})$	:	cumulative distribution function of X.
μ	:	expected demand during lead-time.
σ	:	standard deviation of lead-time demand.
D	:	expected annual demand.

[1]Silver et al. (1998, p. 263) define s as the product of a fractional charge (B_2) per unit short times a unit variable cost.

Assumptions

(a) Lead-time demand distribution has the same mean and the same standard deviation at any inventory cycle.

(b) The reorder point is nonnegative $(R \geq 0)$ and kept constant at any inventory cycle.

(c) Lead-time is fixed and remains the same at any inventory cycle.

(d) When the order quantity is received, the inventory level is always raised above the reorder point[2].

With these assumptions and under the notation stated above, the H-W cost function for the (Q,R) inventory model is

$$C(Q,R) = \frac{A \cdot D}{Q} + h\left(\frac{Q}{2} + R - \mu\right) + \frac{s \cdot D}{Q} S(R), \tag{1}$$

where, for given R, $S(R) = \int_R^{\infty} (x - R) f(x) dx$ is the expected size of backorders in each inventory cycle. Differentiating (1) with respect to Q and R, the first order conditions are stated as

$$\frac{\partial C}{\partial Q} = -\frac{A \cdot D}{Q^2} + \frac{h}{2} - \frac{s \cdot D}{Q^2} \cdot S(R) = 0,$$

$$\frac{\partial C}{\partial R} = h - \frac{s \cdot D}{Q}[1 - F(R)] = 0,$$

from which we take

$$Q = \sqrt{2\frac{A}{h}D + 2\frac{s}{h}D \cdot S(R)}, \tag{2}$$

$$F(R) = 1 - \frac{h \cdot Q}{s \cdot D}. \tag{3}$$

Taking the second order conditions

[2] This assumption implies that at each inventory cycle the lead-time demand never exceeds the order quantity, ensuring that there is never more than one order outstanding at any point in time.

$$C_{QQ} = \frac{\partial^2 C}{\partial Q^2} = \frac{2 \cdot s \cdot D}{Q^3}\left\{\frac{A}{s} + S(R)\right\} > 0, \qquad C_{RR} = \frac{\partial^2 C}{\partial R^2} = \frac{s \cdot D}{Q} \cdot f(R) > 0,$$

and $$C_{QR} = \frac{\partial^2 C}{\partial Q \partial R} = \frac{s \cdot D}{Q^2}[1 - F(R)],$$

the Hessian determinant has the form $|H| = \left(s^2 D^2 / Q^4\right) g(R)$, where

$$g(R) = 2\left[\frac{A}{s} + S(R)\right] f(R) - \left[1 - F(R)\right]^2. \tag{4}$$

To attain a solution to the minimization of (1), the expression on the right hand side of (3) should be between zero and one. Further, to find the derivatives in the first and second order conditions we use the result (e.g. Hadley and Whitin, 1963)

$$\partial S(R)/\partial R = \frac{\partial}{\partial R}\left(\int_R^\infty x f(x)\,dx\right) - \frac{\partial}{\partial R}\left(R\left[1 - F(R)\right]\right) = -\left[1 - F(R)\right].$$

It follows from (4) that the range of the function $g(R)$ defines the convexity of the H-W cost function given in (1). This range depends on the forms of both the probability density and the cumulative distribution function of the lead-time demand. When the Log-Normal and the Normal distributions are used to model the lead-time demand, in the next two sections we shall derive the conditions which define the sign of $g(R)$ and determine the convexity of the H-W cost function.

3. Log-Normal Lead-Time Demand

Given that X has the Log-Normal distribution with mean μ and standard deviation σ, the following hold (e.g. Tadikamalla, 1979; Gallego et al., 2007):

(a) $\ln X \sim N(\lambda, \theta^2)$ with $\theta > 0$, $\mu = e^{\lambda + \theta^2/2}$, $\sigma^2 = e^{2\lambda + \theta^2}\left(e^{\theta^2} - 1\right)$, and thus the parameters λ and θ are determined respectively from $\theta = \sqrt{\ln\left(1 + cv^2\right)}$ and $\lambda = \ln\mu - \theta^2/2$, where $cv = \sigma/\mu$ is the coefficient of variation,

(b) $f(x)=(\theta\cdot x)^{-1}\varphi(\xi)$ and $F(x)=\Phi(\xi)$ where $\varphi(\xi)$ is the probability density function and $\Phi(\xi)$ is the cumulative distribution function of the standard Normal distribution evaluated at $\xi=(\ln(x)-\lambda)/\theta$ and

(c) $S(R)=\mu\left\{1-\Phi\left(\frac{\ln R-\lambda}{\theta}-\theta\right)\right\}-R\left\{1-\Phi\left(\frac{\ln R-\lambda}{\theta}\right)\right\}$ (e.g. Silver, 1980).

Setting $r=(\ln(R)-\lambda)/\theta$, the function $S(R)$ for the Log-Normal lead-time demand takes the form

$$S(R)=S_{LN}(r)=\mu\Phi(\theta-r)-e^{\lambda+r\cdot\theta}\Phi(-r). \tag{5}$$

Substituting (5) into (4), the function $g(R)$ is expressed in terms of r as

$$g(R)=g_{LN}(r)=2\left[\frac{A}{s}+S_{LN}(r)\right]\frac{\varphi(r)}{\theta\cdot e^{\lambda+r\theta}}-\left[\Phi(-r)\right]^{2}. \tag{6}$$

It follows from (6) that the sign of $g_{LN}(r)$, and by extension the sign of the Hessian in (4), is formed independently of the order quantity Q. Recall also that Q is determined from the expression on the right hand-side of (2). So, incorporating this expression into (1) and using (5), the cost function C(Q,R) is written in terms of r as

$$C_{LN}(r)=h\left(\sqrt{2\frac{A}{h}D+2\frac{s}{h}D\cdot S_{LN}(r)}+e^{\lambda+r\theta}-\mu\right), \tag{7}$$

whose first derivative is

$$C'_{LN}(r)=\frac{dC_{LN}(r)}{dr}=-h\theta e^{\lambda+r\theta}V_{LN}(r), \tag{8}$$

Where

$$V_{LN}(r)=\frac{\frac{s}{h}D\Phi(-r)}{\sqrt{2\frac{A}{h}D+2\frac{s}{h}D\cdot S_{LN}(r)}}-1 \tag{9}$$

With

$$V'_{LN}(r) = -\theta e^{\lambda + r\theta} \frac{\frac{s^2}{h^2} D^2}{\left[2\frac{A}{h}D + 2\frac{s}{h}D \cdot S_{LN}(r)\right]^{3/2}} g_{LN}(r). \tag{10}$$

To determine the sign of $g_{LN}(r)$, we take its derivative

$$g'_{LN}(r) = -2(r+\theta)\left[\frac{A}{s} + S_{LN}(r)\right]\frac{\varphi(r)}{\theta \cdot e^{\lambda + r\theta}}. \tag{11}$$

Since $S_{LN}(r)$ is always positive we find that $g'_{LN}(r) = 0$ when $r = -\theta$ and $g'_{LN}(r) > 0$ [or alternatively $g'_{LN}(r) < 0$] when $r < -\theta$ [or alternatively when $r > -\theta$]. Further, $\lim_{r \to -\infty} S_{LN}(r) = \mu$ and $\lim_{r \to +\infty} S_{LN}(r) = 0$ as

$$\lim_{r \to +\infty} e^{\lambda + r\theta}\Phi(-r) = e^{\lambda} \lim_{r \to +\infty} \frac{\frac{d}{dr}\Phi(-r)}{\frac{d}{dr}e^{-r\theta}} = \frac{e^{\lambda}}{\theta\sqrt{2\pi}} \lim_{r \to +\infty} e^{-0.5r^2 + r\theta} = \frac{e^{\lambda}}{\theta\sqrt{2\pi}} \frac{1}{e^{+\infty}} = 0.$$

So, using the limits of the function $S_{LN}(r)$ in (6) it holds that $\lim_{r \to -\infty} g_{LN}(r) = -1$ and $\lim_{r \to +\infty} g_{LN}(r) = 0$.

Based on the above arguments, it follows that when r increases on the interval $(-\infty, -\theta)$ then $g_{LN}(r)$ is strictly increasing and takes values on $(-1, g_{LN}(-\theta))$. On the other hand, if r continues to increase on $(-\theta, +\infty)$ then $g_{LN}(r)$ becomes strictly decreasing with values on $(g_{LN}(-\theta), 0)$. Therefore, the maximum of $g_{LN}(r)$ attained at $r = -\theta$ is positive for any θ and, hence, there is a unique r_o on the interval $(-\infty, -\theta)$, for which $g_{LN}(r_o) = 0$. Using these results and the limiting values of $V_{LN}(r)$, which are

$$\lim_{r \to -\infty} V_{LN}(r) = \frac{\frac{s}{h}D}{\sqrt{2\frac{A}{h}D + 2\frac{s}{h}D \cdot \mu}} - 1 \tag{12}$$

and $\lim_{r\to+\infty} V_{LN}(r) = -1$, we state the two main findings determining the sign of (8) and by extension the monotony of (7):

(a) For $r < r_o$, $g_{LN}(r)$ is negative making $V_{LN}(r)$ to be strictly increasing and to take values on $\left(\lim_{r\to-\infty} V_{LN}(r), V_{LN}(r_o)\right)$ and

(b) For $r > r_o$, $g_{LN}(r)$ is positive making $V_{LN}(r)$ to be strictly decreasing with values on $\left(V_{LN}(r_o), -1\right)$.

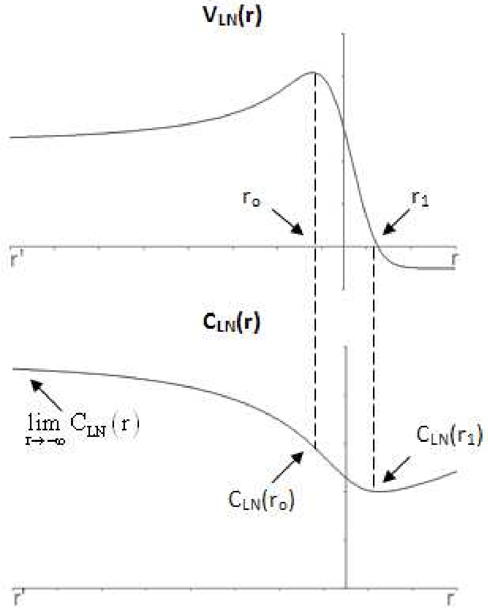
Based on the last two findings, we conclude that the monotony of $C_{LN}(r)$, and by extension the convexity of $C(Q,R)$, depend on whether $\lim_{r\to-\infty} V_{LN}(r)$ is positive or negative and $V_{LN}(r_o)$ is greater or less than zero, making us to distinguish the following three cases:

Case 1: $\lim_{r\to-\infty} V_{LN}(r) > 0$

In this case, $V_{LN}(r_o)$ is positive and the graph of $V_{LN}(r)$ intersects the horizontal axis at a unique point r_1, with $-\infty < r_o < r_1 < +\infty$. For $r > r_1$ [or alternatively $r < r_1$] it holds $V_{LN}(r) < 0$ [or alternatively $V_{LN}(r) > 0$]. So, it follows from (8) that when r increases up to r_1 $C_{LN}(r)$ is strictly decreasing with values on the interval $\left(\lim_{r\to-\infty} C_{LN}(r), C_{LN}(r_1)\right)$, while when r continues to increase taking values greater than r_1, then $C_{LN}(r)$ becomes strictly increasing taking values on $\left(C_{LN}(r_1), +\infty\right)$. Hence, $C_{LN}(r)$ has a global minimum attained at $r = r_1$.

Using the values $A = 70$, $s = 1.5$, $h = 0.6$, $D = 10000$ and $\mu = 300$ of Lau and Lau (2002) and setting $\sigma = 60$ to ensure negligible probability of taking negative values for demand (e.g. Kevork, 2010), we obtain $\lim_{r\to-\infty} V_{LN}(r) \approx 5.004806 > 0$, satisfying the condition of case 1. The value of r_1 is obtained from (9) solving the equation $V_{LN}(r_1)=0$. Using the Newton-Raphson method, this value is $r_1 \approx 1.533295$. With $\theta=\sqrt{\ln(1+cv^2)} \approx 0.198042$, $\lambda=\ln\mu-\theta^2/2 \approx 5.684172$, and setting the value of r_1 in (7), we take the global minimum $C_{LN}(r_1) \approx 998.1429$. For this specific numerical example, the graphs of $V_{LN}(r)$ and $C_{LN}(r)$ are illustrated in Figure 1.

Figure 1: The case of global minimum for the cost function when the lead-time demand is Log-Normal



Case 2: $\lim_{r\to-\infty} V_{LN}(r)<0$ **and** $V_{LN}(r_o)>0$

For this case, there are two values r_2 and r_3, with $-\infty<r_2<r_o<r_3<+\infty$, for which $V_{LN}(r)=0$. For any r smaller than r_2 or greater than r_3, $V_{LN}(r)$ is negative, while for $r_2<r<r_3$, $V_{LN}(r)$ is positive. Thus, it follows from (8) that, (a) $C_{LN}(r)$ is strictly increasing on $-\infty<r<r_2$, (b) $C_{LN}(r)$ becomes strictly decreasing on $r_2<r<r_3$ and (c) $C_{LN}(r)$ becomes

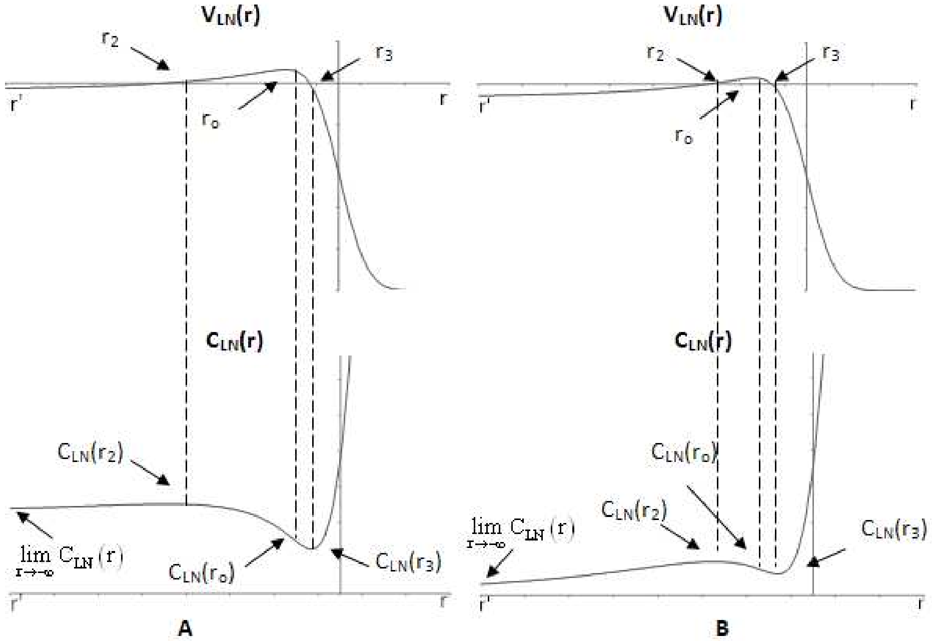
again strictly increasing on $r > r_3$. Hence, $C_{LN}(r)$ has a global maximum attained at $r = r_2$ but a local minimum at $r = r_3$. The latter happens because the cost at $R = 0$, which is given by

$$\lim_{r \to -\infty} C_{LN}(r) = h\left(\sqrt{2\frac{A}{h}D + 2\frac{s}{h}D \cdot \mu} - \mu\right),$$

might be smaller than the minimum cost $C_{LN}(r_3)$. Hence, the global minimum is the smallest between $\lim_{r \to -\infty} C_{LN}(r)$ and $C_{LN}(r_3)$.

To illustrate this case, we use the values of A, h, μ and σ of the numerical example of case 1, reducing only the value of s from $s = 1.5$ to $s = 0.107$. Then, we compute $\lim_{r \to -\infty} V_{LN}(r) \approx -0.03333 < 0$. Setting the right hand-side of (6) equal to zero and solving the equation $g_{LN}(r_o) = 0$, we obtain $r_o \approx -2.38601$. Substituting this value into (9) we take $V_{LN}(r_o) \approx 0.066315377 > 0$. The negative value of $\lim_{r \to -\infty} V_{LN}(r)$ and the positive value of $V_{LN}(r_o)$ indicate that we are in case 2. Then, from (9), solving the equation $V_{LN}(r) = 0$, we take $r_2 \approx -7.81845$ and $r_3 \approx -1.30556$. Using these values of r_2 and r_3 in (7) we take the global maximum $C_{LN}(r_2) \approx 927.5234$ and the minimum $C_{LN}(r_3) \approx 923.7302$. This is the global minimum because $C_{LN}(r_3) < \lim_{r \to -\infty} C_{LN}(r) \approx 926.8875$ (see Figure 2A). On the contrary, if instead of $s = 0.107$, we set $s = 0.103$, and we follow the same procedure, we obtain $\lim_{r \to -\infty} C_{LN}(r) \approx 920.3636$ and $C_{LN}(r_3) \approx 921.7499$. Thus, in this case the global minimum occurs at $R = 0$ (see Figure 2B). To solve the equations $g_{LN}(r_o) = 0$ and $V_{LN}(r) = 0$ we implemented again the Newton-Raphson method.

Figure 2: The case of two local minima for the cost function when the lead-time demand is Log-Normal



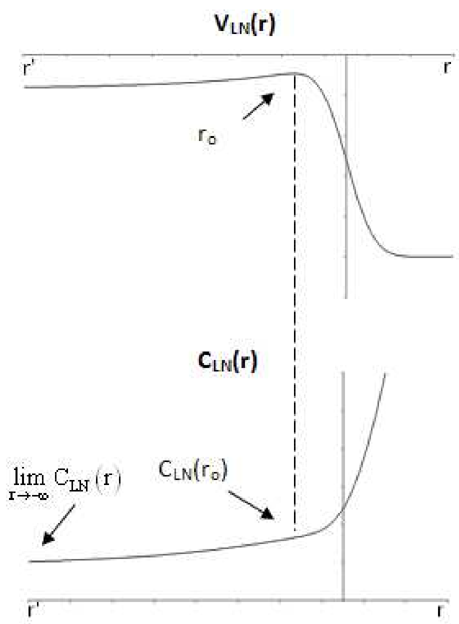
Case 3: $\lim_{r\to-\infty} \mathbf{V_{LN}}(\mathbf{r})<\mathbf{0}$ **and** $\mathbf{V_{LN}}(\mathbf{r_o})\le \mathbf{0}$

When $V_{LN}(r_o)<0$ then $C_{LN}(r)$ is an increasing function of r on $(-\infty,+\infty)$, while if $V_{LN}(r_o)=0$, $C_{LN}(r)$ is non-decreasing. Hence, in this case, $C_{LN}(r)$ has its unique minimum at $R=0$, namely, when $r\to-\infty$. Using again the values A, h, μ and σ of the numerical example of case 2, and reducing further the shortage cost by setting it at $s=0.09$, we obtain $r_o\approx-2.45284$, $\lim_{r\to-\infty} V_{LN}(r)\approx-0.16581<0$ and $V_{LN}(r_o)\approx-0.092019318<0$. These values indicate that we are in case 3. Then, using the values of λ and θ obtained in the numerical example of case 1 we take the minimum cost $\lim_{r\to-\infty} C_{LN}(r)\approx 898.8883$. This case under the parameter values which are used in the above numerical example is shown in Figure 3.

Summarizing, therefore, the algorithm for finding the minimum cost under Log-Normal lead-time demand distribution consists of the following steps:

Step 1:	Find r_o from (6) solving the eq. $g_{LN}(r_o)=0$.
Step 2:	Find $\lim_{r\to-\infty} V_{LN}(r)$ from eq. (12). If $\lim_{r\to-\infty} V_{LN}(r)>0$ then go to Step 3. Otherwise go to Step 4.
Step 3:	Find r_1 from (9) solving the eq. $V_{LN}(r_1)=0$. Set $r^*=r_1$ and go to Step 6.
Step 4:	If $V_{LN}(r_o)>0$ find $r_2<r_o$ and $r_3>r_o$ from (9) solving the eq. and go to Step 5. Otherwise go to Step 7.
Step 5:	If $C_{LN}(r_3)<\lim_{r\to-\infty} C_{LN}(r)$ then set $r^*=r_3$ and go to Step 6. Otherwise, go to Step 7.
Step 6:	Find $S_{LN}(r^*)$ from (5) and set the optimal order quantity $Q^*=\sqrt{2\frac{A}{h}D+2\frac{s}{h}\cdot D\cdot S_{LN}(r^*)}$, the optimal reorder point $R^*=e^{\lambda+r^*\theta}$ and the minimum total cost $C(Q^*,R^*)=C_{LN}(r^*)=h(Q^*+R^*-\mu)$. Go to Step 8.
Step 7:	Set the optimal reorder point $R^*=0$, the optimal order quantity $Q^*=\sqrt{2\frac{A}{h}D+2\frac{s}{h}D\cdot\mu}$ and the minimum total cost $C(Q^*,0)=h(Q^*-\mu)$.
Step 8:	End of algorithm.

Figure 3: The case of an increasing cost function when the lead-time demand is Log-Normal



4. Normal Lead-Time Demand

The following hold for the Normal variable X with mean μ and variance σ^2:

(a) $f(\mathrm{x}) = \sigma^{-1}\varphi(\xi)$ and $F(\mathrm{x}) = \Phi(\xi)$, where $\varphi(\xi)$ is the probability density function and $\Phi(\xi)$ is the cumulative distribution function of the standard Normal evaluated at $\xi = (\mathrm{x}-\mu)/\sigma$ and

(b) $\mathrm{S}(\mathrm{R}) = (\mu-\mathrm{R})\Phi\left(1-\frac{\mathrm{R}-\mu}{\sigma}\right)+\sigma\varphi\left(\frac{\mathrm{R}-\mu}{\sigma}\right)$ (e.g. Lau et al., 2002b).

Setting $\mathrm{z} = (\mathrm{R}-\mu)/\sigma$, we obtain

$$\mathrm{S}(\mathrm{R}) = \mathrm{S}_{\mathrm{NM}}(\mathrm{z}) = \sigma\left[\varphi(\mathrm{z}) - \mathrm{z}\Phi(-\mathrm{z})\right], \tag{13}$$

and substituting (13) into (4), the function $g(\mathrm{R})$ for the Normal lead-time demand is specified as

$$g(\mathrm{R}) = g_{\mathrm{NM}}(\mathrm{z}) = 2\left[\frac{\mathrm{A}}{\mathrm{s}} + \mathrm{S}_{\mathrm{NM}}(\mathrm{z})\right]\frac{\varphi(\mathrm{z})}{\sigma} - \left[\Phi(-\mathrm{z})\right]^2. \tag{14}$$

with

$$g'_{\mathrm{NM}}(\mathrm{z}) = -2\mathrm{z}\left[\frac{\mathrm{A}}{\mathrm{s}} + \mathrm{S}_{\mathrm{NM}}(\mathrm{z})\right]\frac{\varphi(\mathrm{z})}{\sigma}. \tag{15}$$

As $\mathrm{S}_{\mathrm{NM}}(\mathrm{z})$ is always positive, it holds that $g'_{\mathrm{NM}}(\mathrm{z}) = 0$ when $\mathrm{z} = 0$ and $g'_{\mathrm{NM}}(\mathrm{z}) > 0$ [or alternatively $g'_{\mathrm{NM}}(\mathrm{z}) < 0$] if $\mathrm{z} < 0$ [or alternatively when $\mathrm{z} > 0$]. Further, $\lim_{\mathrm{z}\to-\infty} \mathrm{S}_{\mathrm{NM}}(\mathrm{z}) = +\infty$ and $\lim_{\mathrm{z}\to+\infty} \mathrm{S}_{\mathrm{NM}}(\mathrm{z}) = 0$ as

$$\lim_{\mathrm{z}\to+\infty} \mathrm{z}\Phi(-\mathrm{z}) = \lim_{\mathrm{z}\to+\infty} \frac{\frac{\mathrm{d}}{\mathrm{dz}}\mathrm{z}}{\frac{\mathrm{d}}{\mathrm{dz}}\left[\Phi(-\mathrm{z})\right]^{-1}} = \lim_{\mathrm{z}\to+\infty} \frac{\frac{\mathrm{d}}{\mathrm{dz}}\left[\Phi(-\mathrm{z})\right]^{-2}}{\frac{\mathrm{d}}{\mathrm{dz}}\varphi(\mathrm{z})} = \lim_{\mathrm{z}\to+\infty} \frac{2\frac{\mathrm{d}}{\mathrm{dz}}\left[\Phi(-\mathrm{z})\right]}{\frac{\mathrm{d}}{\mathrm{dz}}\mathrm{z}} = -2\lim_{\mathrm{z}\to+\infty}\varphi(\mathrm{z}) = 0.$$

Using the limits of the function $\mathrm{S}_{\mathrm{NM}}(\mathrm{z})$ in (14) we find that $\mathrm{g}_{\mathrm{NM}}(\mathrm{z}) \to -1$ if $\mathrm{z} \to -\infty$ and $\mathrm{g}_{\mathrm{NM}}(\mathrm{z}) \to 0$ when $\mathrm{z} \to +\infty$. It follows, therefore, that $g_{\mathrm{NM}}(\mathrm{z})$ is a strictly increasing function taking values on $\left(-1, g_{NM}(0)\right)$ when z increases on the interval $(-\infty, 0)$, while if z continues to increase on $(0, +\infty)$ then $g_{NM}(z)$ becomes strictly decreasing with values on $\left(g_{NM}(0), 0\right)$. Thus, $g_{NM}(z)$ has a positive maximum at $z = 0$, and since $\lim_{z \to -\infty} g_{NM}(z) = -1$, $\lim_{z \to +\infty} g_{NM}(z) = 0$, there is a unique z_o on the interval $-\infty < z_o < 0$ for which $g_{NM}(z_o) = 0$.

In Eq. (1) of section 2 we give the general form of the H-W cost function, while the expression on the right hand-side of (2) describes how to determine the order quantity Q. Incorporating (2) into (1) and using (13), the H-W cost function C(Q,R) is written as

$$C_{NM}(z) = h\left(\sqrt{2\frac{A}{h}D + 2\frac{s}{h}D \cdot S_{NM}(z)} + z \cdot \sigma\right) \tag{16}$$

with

$$C'_{NM}(z) = \frac{dC_{NM}(z)}{dz} = -h\sigma V_{NM}(z) \tag{17}$$

where

$$V_{NM}(z) = \frac{\frac{s}{h}D\Phi(-z)}{\sqrt{2\frac{A}{h}D + 2\frac{s}{h}D \cdot S_{NM}(z)}} - 1 \tag{18}$$

with

$$V'_{NM}(z) = -\frac{\sigma \cdot s^2 \cdot D^2}{h^2\left[2\frac{A}{h}D + 2\frac{s}{h}D \cdot S_{NM}(z)\right]^{3/2}} g_{NM}(z) \tag{19}$$

and $\lim_{z \to \pm\infty} V_{NM}(z) = -1$.

The arguments stated above lead to the following two findings:

(a) For $z < z_o$, $g_{NM}(z)$ is negative making $V_{NM}(z)$ to be strictly increasing with values on the interval $\left(-1, V_{NM}(z_o)\right)$, and

(b) For $z > z_o$, $g_{NM}(z)$ is positive making $V_{NM}(z)$ to be strictly decreasing taking values on $(V_{NM}(z_o), -1)$.

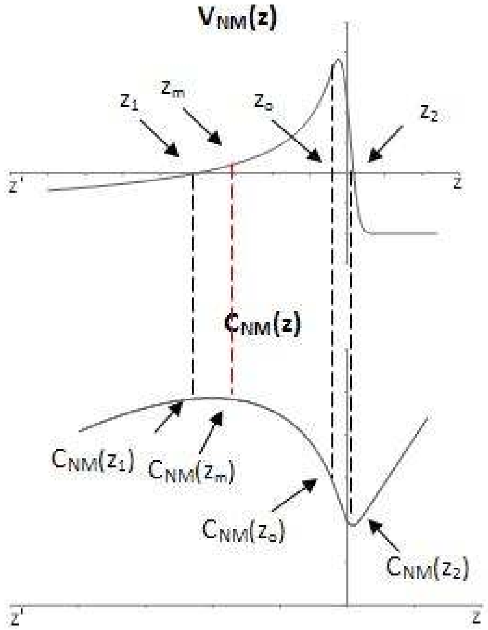
When $V_{NM}(z_o) > 0$ the graph of $V_{NM}(z)$ intersects the horizontal axis at two points z_1 and z_2, with $-\infty < z_1 < z_0 < z_2 < +\infty$. For any $z < z_1$ or $z > z_2$, $V_{NM}(z)$ is negative, while for $z_1 < z < z_2$, $V_{NM}(z)$ is positive. So, $C_{NM}(z)$ is strictly increasing on $-\infty < z < z_1$, $C_{NM}(z)$ becomes strictly decreasing on $z_1 < z < z_2$ and $C_{NM}(z)$ becomes again strictly increasing on $z_2 < z < +\infty$. It is concluded, therefore, that $C_{NM}(z)$ has a global maximum attained at $z = z_1$ and a minimum at $z = z_2$.

Let $z_m = -\mu/\sigma$ be the value at which $R = 0$. If the minimum at $z = z_2$ is global or local, this depends on whether z_m is smaller or larger than z_1. We distinguish, therefore, the following three cases, with the last one to be referred to $V_{NM}(z_o) \le 0$.

Case 1: $V_{NM}(z_o) > 0$ and $z_m > z_1$

In this case z_m is located between z_1 and z_2 and hence $C_{NM}(z_2) < C_{NM}(z_m) < C_{NM}(z_1)$. Thus, the minimum at $z = z_2$ is global. To illustrate numerically this case we consider again the values $A = 70$, $s = 1.5$, $h = 0.6$, $D = 10000$, $\mu = 300$ and $\sigma = 60$. Solving the equation $g_{NM}(z_o) = 0$ we take $z_o \approx -1.20717$, which being substituted into (18) gives $V_{NM}(z_o) \approx 7.956302 > 0$. Further, solving $V_{NM}(z) = 0$ we obtain $z_1 \approx -207.55493$ and $z_2 \approx 1.536974$. So, we find that $z_1 < z_m = -5$. Replacing the values of z_1, z_m and z_2 into (16) we take $C_{NM}(z_1) \approx 7528$, $C_{NM}(z_m) \approx 2317.9992$ and the global minimum $C_{NM}(z_2) \approx 987.5743$. Case 1 under the above parameter values is illustrated in Figure 4.

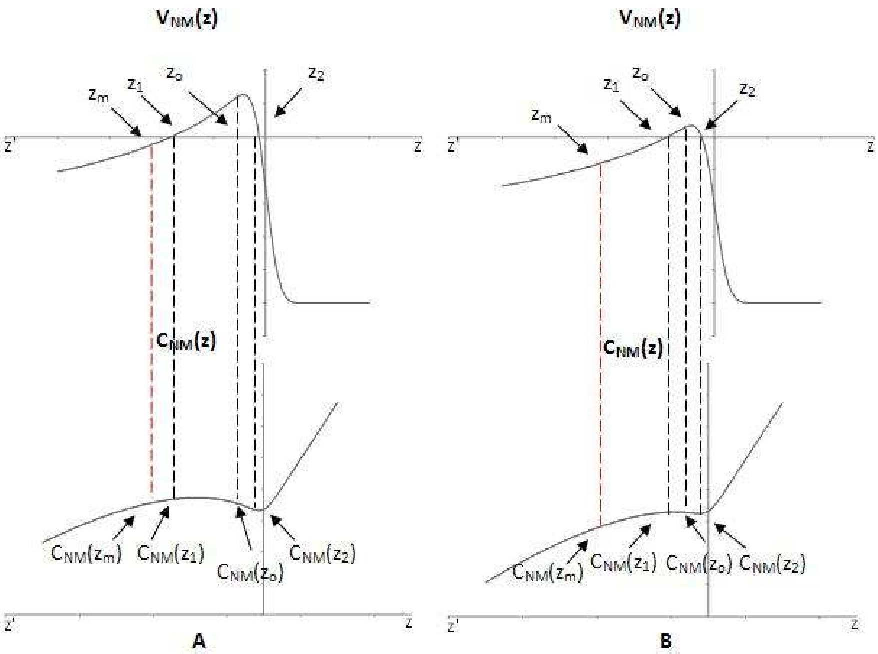
Figure 4: The case of global minimum for the cost function when the lead-time demand is Normal



Case 2: $V_{NM}(z_o)>0$ and $z_m<z_1$

When z_m is located to the left of z_1, it is not certain that $C_{NM}(z)$ has a global minimum at $z=z_2$ as $C_{NM}(z_m)$ might be smaller than $C_{NM}(z_2)$. So, the global minimum will be the smallest between $C_{NM}(z_m)$ and $C_{NM}(z_2)$. Using the values $A=70$, $s=0.11$, $h=0.6$, $D=10000$, $\mu=300$ and $\sigma=60$ we obtain $V_{NM}(z_o)\approx 0.076901>0$, $z_1\approx -4.67168808$ and $z_2\approx -1.17698526$ realizing that $z_1>z_m=-5$. Replacing into (16) we take the global maximum $C_{NM}(z_1)\approx 931.8182$ and the minimum $C_{NM}(z_2)\approx 926.0677$. The latter minimum value is global because $C_{NM}(z_2)<C_{NM}(z_m)\approx 931.7554$ (see Figure 5A). On the contrary, if we set $s=0.102$ then $C_{NM}(z_m)\approx 918.7265$ and $C_{NM}(z_2)\approx 921.725$. So, in this case the global minimum occurs at $z_m=-5$, namely, when $R=0$. (see Figure 5B).

Figure 5: The case of two local minima for the cost function when the lead-time demand is Normal

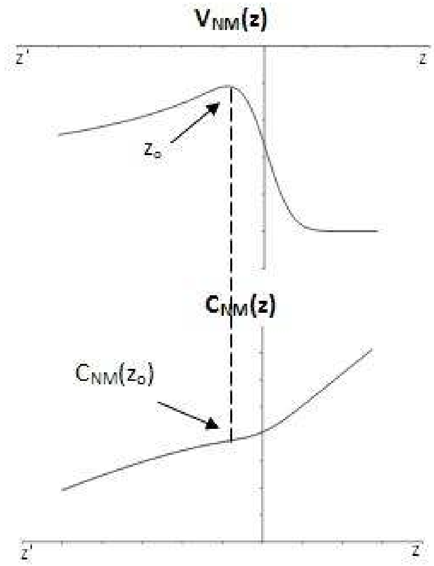


Case 3: $\mathbf{V_{NM}(z_o) \le 0}$

If $V_{NM}(z_o) < 0$ then $V_{NM}(z)$ is negative and $C'_{NM}(z)$ is positive for any z on $(-\infty, +\infty)$. So, $C_{NM}(z)$ is an increasing function, and in the special case where $V_{NM}(z_o) = 0$, $C_{NM}(z)$ is non-decreasing. Hence, $C_{NM}(z)$ has its unique minimum at $R = 0$. With $A = 70$, $s = 0.08$, $h = 0.6$, $D = 10000$, $\mu = 300$ and $\sigma = 60$ we take $z_o \approx -2.29011$ and $V_{NM}(z_o) \approx -0.19754 < 0$. Then replacing $z_m = -5$ into (16) we obtain the global minimum $C_{NM}(z_m) \approx 882.0734$ (see Figure 6).

It is worthwhile to mention at this point that, in the three cases of the Normal lead-time demand, whenever a nonlinear equation had to be solved, the Newton–Rapshon method was used again. Furthermore, in Figures 4,5 and 6, although we have extended the graph for values greater than z_m in order to demonstrate the three cases, the cost function C(Q,R) is defined only for $z_m \le z < +\infty$ where $R \ge 0$. The case of $R < 0$ is not included in the analysis as the mathematical form of C(Q,R) changes.

Figure 6: The case of an increasing cost function when the lead-time demand is Normal



Summarizing, therefore, the steps of algorithm for finding the minimum cost under Normal lead-time demand distribution are the following:

Step 1:	Find z_o from (14) solving the eq. $g_{NM}(z_o)=0$.
Step 2:	Find $V_{NM}(z_o)$ from eq. (18). If $V_{NM}(z_o)>0$ then go to Step 3. Otherwise, set $z^*=z_m$ and go to Step 5.
Step 3:	Find z, $z_1<z_o$ and $z_2>z_o$, from (18) solving the eq. $V_{NM}(z)=0$. If $z_m>z_1$ then set $z^*=z_2$ and go to Step 5. Otherwise, go to Step 4.
Step 4 :	If $C_{NM}(z_2)<C_{NM}(z_m)$ then set $z^*=z_2$ and go to Step 5. Otherwise, set $z^*=z_m$ and go to Step 5.
Step 5:	Find $S_{NM}(z^*)$ from (13) and set the optimal order quantity $Q^*=\sqrt{2\frac{A}{h}D+2\frac{s}{h}\cdot D\cdot S_{NM}(z^*)}$, the optimal reorder point $R^*=\mu+z^*\sigma$ and the minimum total cost $C^*(Q^*,R^*)=C_{NM}(z^*)=h(Q^*+R^*-\mu)$.
Step 6:	End of algorithm.

5. Relating convexity conditions to the cost parameters

From the numerical examples which have been worked out in the Log-Normal and Normal distributions, we realize that the convexity of the H-W cost function depends on the value of the shortage cost, s. Particularly, keeping the other two cost parameters fixed, when the value of s is relatively large and starts to decline, then we move for both distributions from case 1 (where the minimum cost obtained from the solution of the first order conditions is global) to case 3 (where the global minimum cost is attained at $R = 0$). In the current section, we demonstrate numerically that, apart from s, the convexity of the H-W cost function depends also on the values of the fixed ordering cost, A, and the holding cost per unit per year, h.

In Table 1, for Log-Normal lead-time demand, we present the minimum cost of the H-W cost function which is attained after (a) solving the first order conditions, and (b) setting the reorder point equal to zero. In the same Table, we also display the service level, the optimal order quantity, Q^*, and the optimal reorder point, R^*, which give (a) for case 1 the global minimum cost obtained after solving the first order conditions, (b) for case 2, the smallest between the minimum cost from the solution of the first order conditions and the cost at $R = 0$, and (c) for case 3 the minimum cost at $R = 0$. The computation of service levels, optimal order quantities, reorder points, and minimum costs was performed at different values of the cost parameters A, h and s, when each time we changed one of them and kept the other two fixed. The same information as above is given in Table 2, with the exception that the lead-time demand is Normal for which cases 1,2 and 3 have the same meaning as in Log-Normal.

From both Tables it is observed that when the value of s reduces or the values of A or h increase then we move gradually from case 1 to case 3. Further, the solution of the first order conditions gives a global minimum even when the service levels are below 0.50. For example, for Log-Normal lead-time demand with $s = 0.12$ convexity exists even when the service level reaches the small size of 0.20, while for the Normal with $s = 0.13$ the existence of a convex solution is met at service level equal to 0.26. If the values of the cost parameters lead to case 2, then the minimum cost obtained from solving the first order conditions is global for relatively higher values of s, and relatively lower values for A or h. Besides, for case 2 when we change the value of s or A, we end up in small up to negligible differences between the cost at $R = 0$ and the minimum cost obtained from the first order conditions.

On the contrary, these differences become significant when we change h. For example, in the case of the Log-Normal lead-time demand, when $h = 50$, the cost at $R = 0$ is 21% smaller than the minimum cost obtained from the first order conditions. For the Normal lead-time demand, at $h = 48$, the corresponding reduction is at 19%.

The two Tables also demonstrate the implications of changes of the cost parameters on the optimal values of the order quantity and the reorder point. When s declines, to attain a global minimum cost from the solution of first order conditions, the optimal inventory policy aims to larger order quantities and smaller reorder points. In this way, the size of backorders increases and this is justified from the reduction of the unit shortage cost. Increasing Q^* and reducing R^* is also the optimal inventory policy when the fixed ordering cost rises. In this way the firm manages to reduce the number of orders in the year. Finally, as the holding cost increases, it is less costly for the firm to keep small amounts of inventories. In this case the optimal policy imposes the simultaneous reduction of Q^* and R^*.

Observe also that if h increases then Q^* decreases faster than R^*. This means that at some value of h, the optimal positive reorder point might be larger than the corresponding optimal order quantity. This violates the assumption that at each inventory cycle the order quantity should exceed the lead-time demand. For Normal lead-time demand, this assumption is translated to the equation $\Phi(Q)=1$ which holds only when $Q>R$. We can also find values for the cost parameters for which the minimization of the H-W cost function does not lead in acceptable solutions with reference to either the verification of model assumptions or the sign of the minimum cost. For example, under Log-Normal lead-time demand, for $A=1$, $s=1$, $D=10000$, $\mu=300$ and $\sigma=1$, if $h=10$ then we are in case 1 and although the global minimum cost is positive, this is attained when $R^{*}=301.6975$ is greater than $Q^{*}=45.1368$. If h increases further and reaches the size of 2000, then we are in case 3 with $Q^{*}=54.8635$, but the minimum cost will be equal to -490273.0662. Of course, the question in the latter numerical examples is how logical from the practice point of view such values for the cost parameters are.

Based on these arguments and closing this section, we should notify that the algorithms which are given in the previous two sections ensure the existence only of a convex mathematical solution. The case of finding a convex solution which is meaningless from the practice point of view is a problem of how logical values for the cost parameter have been used in the algorithms. Such a problem, however, is beyond the scope of this paper and a subject for future research.

6. Conclusions

In this paper, we re-examine the convexity problem of the Hadley-Whitin cost function under Normal and Log-Normal demand when the lead-time is fixed and the cost per unit backordered is used for determining the expected annual shortage cost. To investigate for both distributions, when the minimum cost obtained from solving the first order conditions is global, we express the cost function in terms of the standard Normal values. In this way, it is feasible to transfer the study from the three dimensional to the two dimensional space and to relate the convexity conditions to the three cost parameters of the model; the fixed ordering cost, the unit shortage cost, and the unit holding cost.

Table 1: Optimal solutions for different values of ordering cost, holding cost and shortage cost when the demand distribution is Log-Normal; Q^* is the optimal order quantity; R^* is the optimal reorder point; $C_{LN}(r_1)$ and $C_{LN}(r_3)$ are the minimum costs for the cases 1 and 2 respectively from solving the first order conditions; $\lim_{r\to-\infty} C_{LN}(r)$ is the minimum cost at $R=0$.

A=70, h=0.6, D=10000, μ=300 and σ=60							
	s	**Service-Level**	Q^*	R^*	$C_{LN}(r_1)$	$C_{LN}(r_3)$	$\lim_{r\to-\infty} C_{LN}(r)$
Case 1	**10**	0.9906	1561.1518	468.5901	1037.8452	-	-
	1.5	0.9374	1565.0223	398.5491	998.1429	-	-
	0.5	0.8116	1570.0620	350.4430	972.3030	-	-
	0.18	0.4725	1582.5324	290.1809	943.6280	-	-
	0.14	0.3184	1590.3604	267.9152	934.9654	-	-
	0.12	0.2004	1599.1094	249.0890	928.9190	-	-
Case 2	**0.107**	0.0959	1612.3989	227.1513	-	923.7302	926.8875
	0.103	0.0553	1833.9393	0.0000	-	921.7499	920.3636
	0.1	0.0146	1825.7419	0.0000	-	920.0035	915.4451
Case 3	**0.09**	0.0000	1798.1472	0.0000	-	-	898.8883
	0.05	0.0000	1683.2508	0.0000	-	-	829.9505
s=1.5, h=0.6, D=10000, μ=300 and σ=60							
	A	**Service-Level**	Q^*	R^*	$C_{LN}(r_1)$	$C_{LN}(r_3)$	$\lim_{r\to-\infty} C_{LN}(r)$
Case 1	**1**	0.9913	218.7441	470.9393	233.8101	-	-
	70	0.9374	1565.0223	398.5491	998.1429	-	-
	700	0.8051	4872.8265	348.7919	2952.9711	-	-
	4600	0.5026	12435.7154	294.5500	7458.1592	-	-
	15000	0.1023	22442.3847	228.8231	13422.7247	-	-
	18300	0.0073	24818.0940	181.3250	14819.6514	-	-
Case 2	**18400**	0.0043	24891.5428	174.9329	-	14859.8854	14859.9468
	18500	0.0011	25132.9797	0.0000	-	14899.9967	14899.7878
	18510	0.0006	25139.6102	0.0000	-	14904.0001	14903.7661
Case 3	**18600**	0.0000	25199.2063	0.0000	-	-	14939.5238
	19000	0.0000	25462.3906	0.0000	-	-	15097.4343
A=70, s=1.5, D=10000, μ=300 and σ=60							
	h	**Service-Level**	Q^*	R^*	$C_{LN}(r_1)$	$C_{LN}(r_3)$	$\lim_{r\to-\infty} C_{LN}(r)$
Case 1	**0.01**	0.9921	11865.2647	474.4008	120.3967	-	-
	0.6	0.9374	1565.0223	398.5491	998.1429	-	-
	6	0.7889	527.6649	344.8627	3435.1656	-	-
	19	0.5877	325.5349	307.3669	6325.1339	-	-
	20	0.5742	319.3564	305.2762	6492.6520	-	-
	21	0.5608	313.6802	303.2314	6655.1416	-	-
Case 2	**24**	0.5214	299.1502	297.3115	-	7115.0812	8598.7341
	40	0.3085	509.9020	0.0000	-	9030.3019	8396.0781
	50	0.1214	456.0702	0.0000	-	9850.2898	7803.5085
Case 3	**60**	0.0000	416.3332	0.0000	-	-	6979.9920
	70	0.0000	385.4496	0.0000	-	-	5981.4751

Table 2: Optimal solutions for different values of ordering cost, holding cost and shortage cost when the demand distribution is Normal; Q^* is the optimal order quantity; R^* is the optimal reorder point; $C_{NM}(z_2)$ is the minimum cost from solving the first order conditions; $C_{NM}(z_m)$ is the minimum cost at $R=0$.

			A=70, h=0.6, D=10000, μ=300 and σ=60			
	s	**Service-Level**	Q^*	R^*	$C_{NM}(z_2)$	$C_{NM}(z_m)$
	10	0.9907	1547.8307	441.2386	1013.4416	-
	2	0.9534	1552.5175	400.7404	991.9548	-
	1.5	0.9379	1553.7387	392.2185	987.5743	-
Case 1	**0.5**	0.8127	1560.7715	353.2751	968.4280	-
	0.3	0.6867	1566.6996	329.1843	957.5303	-
	0.18	0.4741	1577.6374	296.1051	944.2455	-
	0.13	0.2646	1593.3402	262.2485	933.3533	-
	0.11	0.1196	1614.0653	229.3809	926.0677	931.7554
Case 2	**0.104**	0.0582	1836.6636	0.0000	922.9844	921.9982
	0.102	0.0299	1831.2109	0.0000	921.7250	918.7265
	0.08	0.000	1770.1224	0.0000	-	882.0734
Case 3	**0.05**	0.000	1683.2508	0.0000	-	829.9505
	0.03	0.000	1622.7549	0.0000	-	793.6529
			s=1.5, h=0.6, D=10000, μ=300 and σ=60			
	A	**Service-Level**	Q^*	R^*	$C_{NM}(z_2)$	$C_{NM}(z_m)$
	1	0.9919	203.5533	444.1487	208.6212	-
	40	0.9528	1179.8099	400.3628	768.1036	-
	70	0.9379	1553.7383	392.2189	987.5743	-
Case 1	**700**	0.8054	4863.8085	351.6747	2949.2899	-
	4600	0.5028	12430.5974	300.4175	7458.6089	-
	15000	0.1020	22448.9538	223.8001	13423.6523	-
	17000	0.0435	23913.5023	197.2874	14286.4738	-
	18300	0.0060	24850.4626	149.2056	14819.8009	14820.0000
Case 2	**18400**	0.0026	25066.5780	0.0000	14859.9761	14859.9468
	18435	0.0009	25089.8386	0.0000	14874.0027	14873.9032
	18500	0.000	25132.9797	0.0000	-	14899.7878
Case 3	**19000**	0.000	25462.3906	0.0000	-	15097.4343
	20000	0.000	26108.7469	0.0000	-	15485.2482
			A=70, s=1.5, D=10000, μ=300 and σ=60			
	h	**Service-Level**	Q^*	R^*	$C_{NM}(z_2)$	$C_{NM}(z_m)$
	0.03	0.9863	6852.3875	432.3385	209.5418	-
	0.05	0.9823	5313.2495	426.2098	271.9730	-
	0.4	0.9494	1896.1141	398.3649	797.7916	-
Case 1	**0.6**	0.9378	1553.7389	392.2182	987.5743	-
	1.6	0.8971	964.4213	375.9217	1664.5488	-
	6	0.7928	518.0576	348.9658	3402.1403	-
	14	0.6655	358.3616	325.6560	5376.2468	-
	19	0.5976	317.6788	314.8294	6317.6566	8357.0268
Case 2	**40**	0.3164	509.9020	0.0000	9107.1762	8396.0781
	48	0.1339	465.4747	0.0000	9799.9859	7942.7842
	49	0.000	460.7004	0.0000	-	7874.3218
Case 3	**50**	0.000	456.0702	0.0000	-	7803.5086
	60	0.000	416.3332	0.0000	-	6979.9921

The analysis demonstrates that as the unit shortage cost declines we move for both distributions from the situation where the minimum cost obtained from solving the first order conditions is global to a situation where the global minimum cost occurs at a zero reorder point. The same happens when the fixed ordering cost or the unit holding cost increases. We also observe that a global minimum after solving the first order conditions is attained for cases where the fixed ordering cost and the unit holding cost are kept at relatively small values and at the same time, the unit shortage cost is not negligible. Under such circumstances, the convexity of the cost function exists even when the service levels drop below 0.50.

Through a numerical experimentation, we also study the managerial impacts of changing the values of the cost parameters on inventory policies such that the minimum cost obtained from the solution of first order conditions is global. For both distributions we find out that, when the unit shortage cost reduces or the fixed ordering cost increases inventory policy making aims to larger order quantities and to lower reorder points. In this way, a larger amount of orders are backordered, or less orders take place in the year. On the other hand, when the unit holding cost increases then the optimal inventory policy imposes smaller order quantities and lower reorder points. This results in lower expected annual inventory costs.

Finally, for each distribution this paper offers an algorithm to determine the optimal order quantity and reorder point in order to attain a global minimum cost. At this point, we notify that the two algorithms offer a solution without taking into account whether this solution is meaningful from the practice point of view. We illustrate numerically that there are values for the cost parameters for which either assumptions of the model are violated or minimum costs are negative. When this situation occurs, the only recommendation we can make at this stage is the check of how logical from the practice point of view the values of the cost parameters are, and to relegate this problem for future research.

References

Ang M., Song L.S., Wang M. and Zhang H. (2013). On properties of discrete (r,q) and (s,T) inventory systems. *European Journal of Operational Society, **229***, 95-105.

Brooks R.S. and Lu J.Y. (1969). On the convexity of the backorder function for an E.O.Q policy. *Management Science*, **15**, 453-454.

Chung K.J, Ting P.S. and Hou K.L. (2009). A simple cost minimization procedure for the (Q,r) inventory system with a specified fixed cost per stockout occasion. *Applied Mathematical Modelling*, **33**, 2538-2543.

Das C. (1983a). Q,r inventory models with time-weighted backorders. *Journal of the Operational Research Society*, **34**, 401-412.

Das C. (1983b). Inventory control for lognormal demand. *Computers and Operations Research*, **10**, 267-276.

Das C. (1988). On the minimum of a nonconvex inventory function. *Management Science*, **34**, 1023-1026.

Dohi T., Okamura H. and Osaki S. (1999). Optimal order-limit policies for an (r,Q) inventory system. *IMA Journal of Mathematics in Business and Industry*, **10**, 127-145.

Gallego, G., Katircioglu, K., and Ramachandran, B., (2007). Inventory management under highly uncertain demand. *Operations Research Letters*, **35**, 281-289

Hadley G. and Whitin T.M. (1963). *Analysis of Inventory Systems*. Prentice-Hall, Englewood Cliffs, New Jersey, USA.

Janssen E., Strijbosch L. and Brekelmans R. (2009). Assessing the effects of using demand parameters estimates in inventory control and improving the performance using a correction function. *International Journal of Production Economics*, **118**, 34-42.

Kevork I.S. (2010). Estimating the optimal order quantity and the maximum expected profit for single-period inventory decisions. *Omega*, **38**, 218-227.

Lau A.H.L. and Lau H.S. (2002). A comparison of different methods for estimating the average inventory level in a (Q,R) system with backorders. *International Journal of Production Economics*, **79**, 303-316.

Lau A.H.L., Lau H.S. and Pyke D.F. (2002a). Degeneracy in inventory models. *Naval Research Logistics*, **49**, 686-705.

Lau A.H.L., Lau H.S. and Robinson L.W. (2002b). Convenient Expressions for Computing the Exact Annual Cost of a Continuous-Review (Q, R) System with Backordering. *Journal of the Operational Research Society*, **53**, 655-663.

Lau H.S. (1997). Simple formulas for the expected costs in the newsboy problem: An educational note. *European Journal of Operational Research*, **3**, 557-561.

Minh T.V. (1975). *Convexity in a Hadley-Whitin model*. MSc thesis, Naval Postgraduate School, Monterey, CA.

Silver E.A. (1980). Comments on the lognormal approximation of lead time demand. *Omega*, **8**, 701-702.

Silver E.A., Pyke D.F. and Peterson R. (1998). *Inventory Management and Production Planning and Scheduling*, 3rd ed., John Wiley & Sons, New York, NY.

Syntetos A.A. and Boylan J.E. (2008). Demand forecasting adjustments for service-level achievement. *IMA Journal of Management Mathematics*, **19**, 175-192.

Tadikamalla P.R. (1979). The lognormal approximation to the lead time demand, *Omega*, **7**, 553-556.

Tadikamalla P.R. (1984). A comparison of several approximations to the lead time demand distribution. *Omega*, **12**, 575-581.

Veinott A.F. (1964). Review of Hadley and Whitin in *Journal of the American Statistical Association*, **59**, 283-285.