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## Generating Functions for $P(n, p, *)$ and $P(n, *, p)$

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### Abstract

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*This paper shows how to prove the Theorem  $P(n, p, *) = P(n, *, p)$ , i.e., the number of partitions of  $n$  into  $p$ -parts is equal to the number of partitions of  $n$  having largest part  $p$ .*

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**Key Words:** Irrelevant, decreasing order,  $p$ -parts

### 1. Introduction

We give some definitions of a partition,  $P(n, p, \leq q)$ ,  $P(n, \leq q, p)$ ,  $P(n, p, *)$  and  $P(n, *, p)$ . We generate the generating functions for  $P(n, p, \leq q)$ ,  $P(n, \leq q, p)$ ,  $P(n, p, *)$  and  $P(n, *, p)$  and prove the theorem  $P(n, p, *) = P(n, *, p)$  by graphically. Finally we give a numerical example when  $n = 8$ .

### 2. Definitions

**Partition:** A partition of a number is a representation of  $n$  as the sum of any number of positive integral parts. Thus,  $5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1$ . The order of the parts is irrelevant, so that parts to be arranged in decreasing order of magnitude, we denote by  $P(n)$ , the number of partitions of  $n$ . Thus,  $P(5) = 7$ .

$P(n, p, \leq q)$ : The number of partitions of  $n$  into  $p$ -parts, none of which exceeds  $q$ .

$P(n, \leq q, p)$ : The number of partitions of  $n$  into  $p$  or any smaller number of parts, the greatest of which is  $p$ .

$P(n, p, *)$ : The number of partitions of  $n$  into  $p$ -parts.

$P(n, *, p)$ : The number of partitions of  $n$  having largest part  $p$ .

### 3. Generating Functions for $P(n, p, \leq q)$

The Generating functions for  $P(n, p, \leq q)$  is of the form [1]:

$$\frac{1}{(1-zx)(1-zx^2)\dots(1-zx^q)}$$

$$= 1 + \sum_{n=1}^{\infty} x^n \left\{ \sum_{p=1}^n z^p P(n, p, \leq q) \right\}. \tag{1}$$

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It is convenient to define  $P(n, p, \leq q) = 0$  if  $n < p$ . The coefficient  $P(n, p, \leq q)$  is the number of partitions of  $n$  into  $p$ -parts, none of which exceeds  $q$ .

Again the generating function for  $P(n, \leq q, p)$  is of the form;

$$\begin{aligned} & \frac{1}{(1-zx)(1-zx^2)\dots(1-zx^q)} \\ &= 1 + xz + x^2(z+z^2) + x^3(z+z^2+z^3) + x^4(z+2z^2+z^3+z^4) + \dots\infty \\ &= 1 + \sum_{n=1}^{\infty} x^n \left\{ \sum_{p=1}^n z^p P(n, \leq q, p) \right\}. \end{aligned} \quad (2)$$

The proof of the Theorem  $P(n, p, \leq q) = P(n, \leq q, p)$  is given in Hardy and Wright [2]. If  $q \rightarrow \infty$ , in (1), such as  $\lim_{q \rightarrow \infty} x^q = 0$  when  $|x| < 1$ , then (1) becomes;

$$\begin{aligned} & \frac{1}{(1-zx)(1-zx^2)(1-zx^3)\dots\infty} \\ &= 1 + xz + x^2(z+z^2) + x^3(z+z^2+z^3) + x^4(z+2z^2+z^3+z^4) + \dots\infty \\ &= 1 + \sum_{n=1}^{\infty} x^n \left\{ \sum_{p=1}^n z^p P(n, p, *) \right\} \end{aligned} \quad (3)$$

where the coefficient  $P(n, p, *)$  is the number of partitions of  $n$  into  $p$ -parts. Again (2) becomes;

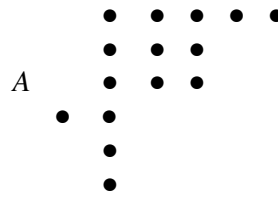
$$\begin{aligned} & \frac{1}{(1-zx)(1-zx^2)(1-zx^3)\dots\infty} \\ &= 1 + xz + x^2(z+z^2) + x^3(z+z^2+z^3) + x^4(z+2z^2+z^3+z^4) + \dots\infty \\ &= 1 + \sum_{n=1}^{\infty} x^n \left\{ \sum_{p=1}^n z^p P(n, *, p) \right\} \end{aligned} \quad (4)$$

where the coefficient  $P(n, *, p)$  is the number of partitions of  $n$  having largest part  $p$ .

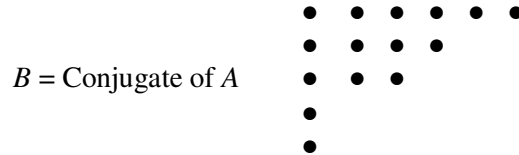
Now we can consider a Theorem as follows:

**Theorem:**  $P(n, p, *) = P(n, *, p)$  i.e., the number of partitions of  $n$  into  $p$ -parts is equal to the number of partitions of  $n$  having largest part  $p$ .

**Proof:** We establish a one-to-one correspondence between the partitions enumerated by  $P(n, p, *)$  and those enumerated by  $P(n, *, p)$ . Let  $n = a_1 + a_2 + \dots + a_p$  be a partition of  $n$  into  $p$ -parts. We transfer this into a partition of  $n$  having largest part  $p$  and can represent a partition of 15 graphically by an array of dots or nodes such as,

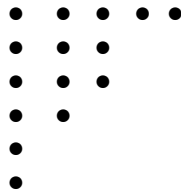


The dots in a column correspond to a part. Thus  $A$  represents the partition  $6+4+3+1+1$  of 15. We can also represent  $A$  by transposing rows and columns in which case it would represent the partition graphically as conjugate of  $A$ .



The dots in a column correspond to a part, so that it represents the partition  $5+3+3+2+1+1$  of 15. Such pair of partitions are said to be conjugate. The number of parts at 1<sup>st</sup> one portion is equal to the largest part of 2<sup>nd</sup> one partition, so that our corresponding is one-to-one.

Conversely, we can represent the partition  $B =$  conjugate of  $A$ , by transposing rows and columns, in which case it would represent the same partition like  $A$ , so we can say that the largest part of the partition is equal to the number of parts of the partition, then our corresponding is onto, i.e., the number of partitions of  $n$  into  $p$ -parts is equal to the number of partitions of  $n$  having largest part  $p$ . Consequently,



$$P(n, p, *) = P(n, *, p).$$

Hence the Theorem.

#### 4. A Numerical Example When $n = 8$

The list of partitions of 8 into 4 parts is given as follows:

$5+1+1+1 = 4+2+1+1 = 3+3+1+1 = 3+2+2+1 = 2+2+2+2$ . The number of such partitions is 5 i.e.,  $P(8, 4, *) = 5$ .

Again the list of partitions of 8 having largest part 4 is given by;

$4+4 = 4+3+1 = 4+2+1+1 = 4+1+1+1+1 = 4+2+2$ .

So the number of such partitions is 5, i.e.,  $P(8, *, 4) = 5$ . Here  $4+4$ ,  $4+3+1$ ,  $4+2+1+1$ ,  $4+1+1+1+1$  and  $4+2+2$  are the conjugate partitions of  $2+2+2+2$ ,  $3+2+2+1$ ,  $4+2+1+1$ ,  $5+1+1+1$  and  $3+3+1+1$  respectively. Thus the number of partitions of 8 into parts, the largest of which is 4 i.e.,  $P(8, 4, *) = P(8, *, 4)$ .

### **5. Conclusion**

For any positive integer of  $n$ , we can verify the Theorem  $P(n, p, *) = P(n, *, p)$ . We have already satisfied the Theorem when  $n = 8$ .

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