Generating Functions for X(n) and Y(n)

Das, Sabuj and Mohajan, Haradhan

Department of Mathematics. Raozan University College, Chittagong, Bangladesh., Premier University, Chittagong, Bangladesh.

10 January 2014

Generating Functions for $X(n)$ and $Y(n)$

Sabuj Das[1]
Haradhan Kumar Mohajan[2]

Abstract

This paper shows how to prove the Theorem $X(n)=Y(n)$, i.e., the number of partitions of n with no part repeated more than twice is equal to the number of partitions of n with no part is divisible by 3.

Key Words: Infinite factors, enumerated by $X(n)$

1. Introduction

We give some definitions of $X(n)$ and $Y(n)$ [1]. We generate the generating functions for $X(n)$ and $Y(n)$, and prove the Theorem $X(n)=Y(n)$. Finally we give a numerical example when $n = 8$.

2. Definitions

$X(n)$: The number of partitions of n with no part repeated more than twice.
$Y(n)$: The number of partitions of n with no part is divisible by 3.

3. Generating Functions

We consider a function, which is the product of infinite factors, one of which is $(1+x^n+x^{2n})$ and it can be written as;

$$=(1+x+x^2)(1+x^2+x^4)(1+x^3+x^6)\ldots\infty$$
$$=1+x+2x^2+2x^3+4x^4+5x^5+7x^6+\ldots\infty$$
$$=1+\sum_{n=1}^{\infty}X(n)x^n \quad (1)$$

Each element of the product comes from multiplying together one term from each bracket either x^0 or x^n or x^{n+n} from $(1+x^n+x^{2n})$. So in the corresponding partitions no part occurs more than twice.

Therefore we can say that the coefficient $X(n)$ of x^n in the above expansion is the number of partitions of n with no part is repeated more than twice.

[1] Senior Lecturer, Department of Mathematics. Raozan University College, Bangladesh.
[2] Premier University, Chittagong, Bangladesh.

The generating function for $Y(n)$ is of the form [2];

$$\frac{1}{(1-x)(1-x^2)(1-x^4)(1-x^5)\ldots(1-x^{3n-2})(1-x^{3n-1})\ldots}$$
$$=1+x+2x^2+2x^3+4x^4+5x^5+7x^6+\ldots\infty$$
$$=1+\sum_{n=1}^{\infty}Y(n)x^n \tag{2}$$

where the coefficient $Y(n)$ is the number of partitions of *n* with no part is divisible by 3.

From equations (1) and (2) we get;

$$=1+\sum_{n=1}^{\infty}X(n)x^n$$
$$=(1+x+x^2)(1+x^2+x^4)(1+x^3+x^6)\ldots(1+x^n+x^{2n})\ldots\infty$$
$$=\frac{1-x^3}{1-x}.\frac{1-x^6}{1-x^2}.\frac{1-x^9}{1-x^3}\cdots\frac{1-x^{3n}}{1-x^n}\cdots$$
$$=\frac{1}{(1-x)(1-x^2)(1-x^4)(1-x^5)\ldots(1-x^{3n-2})(1-x^{3n-1})\ldots}$$
$$=1+\sum_{n=1}^{\infty}Y(n)x^n.$$

Equating the coefficient of x^n from both sides we get;

$X(n)=Y(n)$.

Theorem: $X(n)=Y(n)$, i.e., the number of partitions of *n* with no part is repeated more than twice is equal to the number of partitions of *n* with no part is divisible by 3.

Proof: We develop a one-to-one correspondence between the partitions enumerated by $X(n)$ and those enumerated by $Y(n)$. Let $n=a_1+a_2+\ldots+a_r$ be a partition of *n* with no part is repeated more than twice. We transfer this into a partition of *n* with no part is divisible by 3. If a part a_m of the partition, which is divisible by 3, enumerated by $X(n)$ can be expressed into three equal parts, such that: 6 = 2+2+2, 3 = 1+1+1. Rearranging the parts of the partition, we can say that the parts are not divisible by 3. Clearly, our correspondence is one-to-one.

Conversely, we start any partition of *n* into with no part is divisible by 3, say $n=a_1+a_2+\ldots+a_r$, we consider the same part not less than thrice, it would be unique sum by same three parts by taking a group, such that, 5+1+1+1 = 5+3 and 2+2+2+1+1 = 6+1+1.

This gives *n* as a partition with no part is repeated more than twice. Thus, we have the one-to-one correspondence. The corresponding is onto, so that $X(n)=Y(n)$. Hence the Theorem.

4. A Numerical Example

When $n = 8$, the listed partitions of 8 with no part repeated more than twice is given below:

8 = 7+1 = 6+2 = 6+1+1 = 5+3 = 5+2+1 = 4+4 = 5+3+1 = 4+2+1+1 = 4+2+2 = 3+3+2 = 3+3+1+1= 3+2+2+1.

So, there are 13 partitions i.e., $X(8) = 13$. Again, the list of partitions of 8 with no part is divisible by 3 is given below:

8 = 7+1 = 5+2+1 = 5+1+1+1+1 = 4+4 = 4+2+1+1 = 4+2+2= 4+1+1+1+1 = 2+2+2+2 =2+2+2+1+1= 2+2+1+1+1+1 = 2+1+1+1+1+1+1 = 1+1+1+1+1+1+1+1.

So, there are 13 partitions i.e., $Y(8) = 13$.

$\therefore X(n) = Y(n)$

5. Conclusion

For any positive integer of n, we can verify the Theorem $X(n) = Y(n)$. We have already satisfied the Theorem when $n = 8$.

6. Acknowledgment

It is a great pleasure to express our sincerest gratitude to our respected Professor Md. Fazlee Hossain, Department of Mathematics, University of Chittagong, Bangladesh. We will remain ever grateful to our respected Late Professor Dr. Jamal Nazrul Islam, RCMPS, University of Chittagong, Bangladesh.

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