Generating Functions for $X(n)$ and $Y(n)$

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Generating Functions for $X(n)$ and $Y(n)$

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Abstract

This paper shows how to prove the Theorem $X(n) = Y(n)$, i.e., the number of partitions of $n$ with no part repeated more than twice is equal to the number of partitions of $n$ with no part is divisible by 3.

Key Words: Infinite factors, enumerated by $X(n)$

1. Introduction

We give some definitions of $X(n)$ and $Y(n)$ [1]. We generate the generating functions for $X(n)$ and $Y(n)$, and prove the Theorem $X(n) = Y(n)$. Finally we give a numerical example when $n = 8$.

2. Definitions

$X(n)$: The number of partitions of $n$ with no part repeated more than twice.

$Y(n)$: The number of partitions of $n$ with no part is divisible by 3.

3. Generating Functions

We consider a function, which is the product of infinite factors, one of which is $(1 + x^n + x^{2n})$ and it can be written as;

$$(1 + x + x^2)(1 + x^2 + x^4)(1 + x^3 + x^6)...\infty$$

$= 1 + x + 2x^2 + 2x^3 + 4x^4 + 5x^5 + 7x^6 + ...\infty$

$= 1 + \sum_{n=1}^{\infty} X(n) x^n \quad (1)$

Each element of the product comes from multiplying together one term from each bracket either $x^0$ or $x^n$ or $x^{n+n}$ from $(1 + x^n + x^{2n})$. So in the corresponding partitions no part occurs more than twice.

Therefore we can say that the coefficient $X(n)$ of $x^n$ in the above expansion is the number of partitions of $n$ with no part is repeated more than twice.

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The generating function for $Y(n)$ is of the form [2]:

$$
\frac{1}{(1-x)(1-x^2)(1-x^4)(1-x^5)\cdots(1-x^{3n-2})(1-x^{3n-1})}
= 1 + x + 2x^2 + 2x^3 + 4x^4 + 5x^5 + 7x^6 + \cdots \infty
$$

$$
= 1 + \sum_{n=1}^{\infty} Y(n) x^n
\tag{2}
$$

where the coefficient $Y(n)$ is the number of partitions of $n$ with no part is divisible by 3.

From equations (1) and (2) we get;

$$
= 1 + \sum_{n=1}^{\infty} X(n) x^n
= (1 + x + x^2)(1 + x^2 + x^4)(1 + x^3 + x^6)\cdots(1 + x^n + x^{2n})\cdots \infty
$$

$$
= \frac{1 - x^3 - 1 - x^6 - 1 - x^9 - 1 - x^{3n} \cdots}{1 - x - 1 - x^2 - 1 - x^3 \cdots 1 - x^n \cdots}
= \frac{1}{(1-x)(1-x^2)(1-x^4)(1-x^5)\cdots(1-x^{3n-2})(1-x^{3n-1})}
$$

$$
= 1 + \sum_{n=1}^{\infty} Y(n) x^n.
$$

Equating the coefficient of $x^n$ from both sides we get;

$$
X(n) = Y(n).
$$

**Theorem:** $X(n) = Y(n)$, i.e., the number of partitions of $n$ with no part is repeated more than twice is equal to the number of partitions of $n$ with no part is divisible by 3.

**Proof:** We develop a one-to-one correspondence between the partitions enumerated by $X(n)$ and those enumerated by $Y(n)$. Let $n = a_1 + a_2 + \ldots + a_r$ be a partition of $n$ with no part is repeated more than twice. We transfer this into a partition of $n$ with no part is divisible by 3. If a part $a_m$ of the partition, which is divisible by 3, enumerated by $X(n)$ can be expressed into three equal parts, such that: 6 = 2+2+2, 3 = 1+1+1. Rearranging the parts of the partition, we can say that the parts are not divisible by 3. Clearly, our correspondence is one-to-one.

Conversely, we start any partition of $n$ into with no part is divisible by 3, say $n = a_1 + a_2 + \ldots + a_r$, we consider the same part not less than thrice, it would be unique sum by same three parts by taking a group, such that, 5+1+1+1 = 5+3 and 2+2+2+1+1 = 6+1+1.

This gives $n$ as a partition with no part is repeated more than twice. Thus, we have the one-to-one correspondence. The corresponding is onto, so that $X(n) = Y(n)$. Hence the Theorem.
4. A Numerical Example

When \( n = 8 \), the listed partitions of 8 with no part repeated more than twice is given below:

\[
8 = 7+1 = 6+2 = 6+1+1 = 5+3 = 5+2+1 = 4+4 = 5+3+1 = 4+2+1+1 = 4+2+2 = 3+3+2 = 3+3+1+1 = 3+2+2+1.
\]

So, there are 13 partitions i.e., \( X(8) = 13 \). Again, the list of partitions of 8 with no part is divisible by 3 is given below:

\[
8 = 7+1 = 5+2+1 = 5+1+1+1+1 = 4+4 = 4+2+1+1 = 4+2+2 = 4+1+1+1+1 = 2+2+2+2 = 2+2+2+1+1 = 2+2+1+1+1+1+1 = 2+1+1+1+1+1+1+1 = 1+1+1+1+1+1+1+1+1.
\]

So, there are 13 partitions i.e., \( Y(8) = 13 \).

\[\therefore X(n) = Y(n)\]

5. Conclusion

For any positive integer of \( n \), we can verify the Theorem \( X(n) = Y(n) \). We have already satisfied the Theorem when \( n = 8 \).

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References


(1987)