Aumann and Serrano's Economic Index of Risk for Sums of Gambles

Li, Minqiang

Bloomberg LP

May 2014

Aumann and Serrano's Economic Index of Risk for Sums of Gambles

Minqiang Li

E-mail: minqiang.li@gmail.com

May 3, 2014

We study Aumann and Serrano's (2008) risk index for sums of gambles that are not necessarily independent. We show that if the dependent parts of two gambles are similarly ordered, or more generally positively quadrant dependent, then the risk index of the sum of two gambles is always larger than the minimum of the risk indices of the two gambles. For negative dependence, the risk index of the sum is always smaller than the maximum of the two risk indices. The above results agree with our intuitions well. For example, the result for negative dependence agrees with our intuition of risk diversification. Thus this result can be considered another attractive property of Aumann and Serrano's risk index.

Keywords: Risk index; Additive gambles; Subadditivity; Positive quadrant dependence

JEL classifications: A10; C00; D81

I Introduction

Aumann and Serrano (2008) proposed an index of riskiness that assigns to each gamble a single fixed number. We study sums of gambles that are not necessarily independent. We show that if the dependent parts of two gambles are similarly ordered, or more generally positively quadrant dependent, then the risk index of the sum of two gambles is always larger than the minimum of the risk indices of the two gambles. For negative dependence, the risk index of the sum is always smaller than the maximum of the two risk indices. The above results agree with our intuitions well. For example, the result for negative dependence agrees with our intuition of risk diversification. Thus this result can be considered another attractive property of Aumann and Serrano's risk index.

II Sums of Additive gambles

A gamble g is a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E}g$ is finite and positive and $\mathbb{P}(g < 0) > 0$. Unlike Aumann and Serrano (2008), we do not assume that g takes only finitely many values. Furthermore, we do not assume that g is bounded or has a continuous density function. The risk index $R(g)$ is the unique positive solution (if exists) of

$$\mathbb{E}e^{-g/R(g)} = 1. \tag{1}$$

Considering sums of gambles are useful in practice. For example, an investor's portfolio might consist of different positions, each considered a different gamble. It might be useful to be able to get some quick idea of the riskiness of the whole portfolio given the riskiness of the components and their dependence structure. From a financial engineering point of view, many new financial products can be thought of as the result of adding gambles (such as sector index funds) or splitting a gamble into many others (such as collateralized mortgage obligations).

In the following, we will always assume that $g + h$ is a well-defined gamble for two gambles g and h. Aumann and Serrano (2008) show that the riskiness (and thus the attractiveness) of $g + h$ always lies between those of g and h. We examine sums of additive gambles more closely. We will generalize (5.8.1) of Aumann and Serrano to situations where we do not necessarily have independence. It turns out that in line with our intuition, if two gambles g and h are positively dependent (in senses presented rigorously in Propositions 2 and 3), then the risk index of the gamble $g + h$ cannot be smaller than the minimum of the risk indices of g and h. On the other hand, the risk index of the gamble $g + h$ cannot be larger than the maximum of the risk indices of g and h if we have negative dependence.

The following proposition is a straight-forward generalization of the results in Aumann and Serrano (2008) to arbitrary number of additive gambles.

Proposition 1. *Let g_i where $i = 1, \cdots, N$ be N additive gambles.*

1. *(Subadditivity) Let $\lambda_i > 0$ for $i = 1, \cdots, N$, then*

$$R\Big(\sum_{i=1}^{N} \lambda_i g_i\Big) \leq \sum_{i=1}^{N} \lambda_i R(g_i); \tag{2}$$

2. *If all gambles are independent, the riskiness of $\sum_{i=1}^{N} g_i$ lies between the minimum riskiness and the maximum riskiness. That is,*

$$\min_i R(g_i) \leq R\Big(\sum_{i=1}^{N} g_i\Big) \leq \max_i R(g_i); \tag{3}$$

Proof: Statement 1 in the special two gambles case has been proven using the convexity of the exponential function in Aumann and Serrano (2008) by Sergiu Hart. The general statement follows from induction and the homogeneity of the risk index. Below we give another proof based on generalized Hölder's inequality (see, for example, Finner 1992, or Kuptsov 2001). For any $k = 1, \cdots, N$, let $p_k = \sum_{i=1}^{N} \lambda_i R(g_i)/(\lambda_k R(g_k)) > 1$. Then $\sum_{k=1}^{N} 1/p_k = 1$. We have

$$\begin{aligned}
\mathbb{E} \exp\left(-\frac{\sum_{k=1}^{N} \lambda_k g_k}{\sum_{i=1}^{N} \lambda_i R(g_i)}\right) &= \mathbb{E} \prod_{k=1}^{N} \exp\left(-\frac{\lambda_k g_k}{\sum_{i=1}^{N} \lambda_i R(g_i)}\right) \\
&= \left\|\prod_{k=1}^{N} \exp\left(-\frac{\lambda_k g_k}{\sum_{i=1}^{N} \lambda_i R(g_i)}\right)\right\|_1 \leq \prod_{k=1}^{N} \left\|\exp\left(-\frac{\lambda_k g_k}{\sum_{i=1}^{N} \lambda_i R(g_i)}\right)\right\|_{p_k} \\
&= \prod_{k=1}^{N} \left(\mathbb{E} e^{-g_k/R(g_k)}\right)^{1/p_k} = 1.
\end{aligned} \tag{4}$$

This proves the subadditivity. The equality obtains if and only if all the g_i's are multiples of each other. Statement 2 follows from (5.8.1) in Aumann and Serrano (2008) and induction. $\square$

The independence assumption in the second statement is quite strong for actual applications. For example, the profit/loss of a call option (viewed as a gamble) is positively correlated with that of a digital call option, and negatively correlated with that of a put option. The following proposition gives some partial results when we do not have independence.

Proposition 2. *We have the following statements for sums of additive gambles:*

1. *Suppose there exists a random variable Z such that g_1 and g_2 are both nonincreasing functions (or both nondecreasing) in Z, then $R(g_1 + g_2) \geq \min(R(g_1), R(g_2))$. More generally, suppose there exist $N + 1$ independent random variables $\widetilde{g}_i$ $(i = 1, \cdots, N)$ and Z, such that $g_i - \widetilde{g}_i$ are all nonincreasing functions (or all nondecreasing) in Z, then $R(\sum_{i=1}^{N} g_i) \geq \min_i R(g_i)$.*

2. *Suppose there exists a random variable Z such that g_1 is nonincreasing in Z and g_2 is nondecreasing in Z (or vice versa), then $R(g_1+g_2) \leq \max(R(g_1), R(g_2))$. More generally, suppose there exists three independent random variables $\widetilde{g}_1$, $\widetilde{g}_2$ and Z, such that $g_1 - \widetilde{g}_1$ is a nonincreasing function in Z while $g_2 - \widetilde{g}_2$ is nondecreasing in Z (or vice versa), then $R(g_1+g_2) \leq \max(R(g_1), R(g_2))$.*

Proof: The main ingredient for the proof is Čebyšev's algebraic inequality (see Mitrinović, Pečarić, and Fink 1993, or Theorem 236 in Hardy, Littlewood and Pólya 1934), which was used by Merton in his development of portfolio selection theory (p. 25, Merton 1990). It states that if f_1 and f_2 are two random variables both nonincreasing (or nondecreasing) functions in Z, then $\mathrm{cov}(f_1, f_2) \geq 0$, and $\mathrm{cov}(f_1, f_2) \leq 0$ if one is nonincreasing and the other nondecreasing.

For statement 1, we prove the more general conclusion. Let $\beta > 0$, then by independence,

$$\mathbb{E}e^{-\beta \sum_{i=1}^N g_i} = \mathbb{E}\prod_{i=1}^N e^{-\beta\widetilde{g}_i} e^{-\beta(g_i - \widetilde{g}_i)} = \prod_{i=1}^N \mathbb{E}e^{-\beta\widetilde{g}_i} \cdot \mathbb{E}\prod_{k=1}^N e^{-\beta(g_k - \widetilde{g}_k)}. \tag{5}$$

The product of two positive nonincreasing functions is still nonincreasing. The same is true for nondecreasing functions. Thus, by repeated use of Čebyšev's algebraic inequality, we have

$$\mathbb{E}\prod_{k=1}^N e^{-\beta(g_k - \widetilde{g}_k)} \geq \mathbb{E}e^{-\beta(g_N - \widetilde{g}_N)} \cdot \mathbb{E}\prod_{k=1}^{N-1} e^{-\beta(g_k - \widetilde{g}_k)} \geq \cdots \geq \prod_{k=1}^N \mathbb{E}e^{-\beta(g_k - \widetilde{g}_k)}. \tag{6}$$

Putting the above two equations together, we have by independence again

$$\mathbb{E}e^{-\beta \sum_{i=1}^N g_i} \geq \prod_{i=1}^N \mathbb{E}e^{-\beta\widetilde{g}_i} \cdot \prod_{k=1}^N \mathbb{E}e^{-\beta(g_k - \widetilde{g}_k)} = \prod_{i=1}^N \mathbb{E}e^{-\beta g_i}. \tag{7}$$

Now let $\beta = \max_i \alpha(g_i)$, then $\mathbb{E}e^{-\beta \sum_{i=1}^N g_i} \geq 1$ since $\mathbb{E}e^{-\beta g_i} \geq 1$ for all $i = 1, \cdots, N$. Thus,

$$\beta \geq \alpha\Big(\sum_{i=1}^N g_i\Big), \tag{8}$$

or equivalently, $R(\sum_{i=1}^N g_i) \geq \min_i R(g_i)$.

The proof for statement 2 is similar and thus omitted. □

An interesting application of the above proposition is the following. Let g_1 and g_2 be multivariate normally distributed gambles with positive means μ_1 and μ_2, variances σ_1^2 and σ_2^2 and correlation coefficient ϱ. We already know that when $\varrho = 0$, $\min(R(g_1), R(g_2)) \leq R(g_1+g_2) \leq \max(R(g_1), R(g_2))$ by Proposition 1. Proposition 2 above allows us to draw conclusions when $\varrho \neq 0$. Through a Gram-Schmidt orthogonalization, we see that when $\varrho > 0$, we have $\min(R(g_1), R(g_2)) \leq R(g_1+g_2) \leq R(g_1) + R(g_2)$. When $\varrho < 0$, we have

$R(g_1+g_2) \le \max(R(g_1), R(g_2))$. With some elementary but interesting algebra, these statements can be verified explicitly since we have $R(g_i) = \sigma_i^2/\mu_i$ for $i = 1, 2$, and

$$R(g_1+g_2) = \frac{\sigma_1^2 + \sigma_2^2 + 2\varrho\sigma_1\sigma_2}{\mu_1+\mu_2}. \tag{9}$$

When dealing with the sum of two gambles, Proposition 2 can be further generalized. Two random variables X and Y are said to be *positively quadrant dependent* (Lehmann 1966) if for any x and y we have

$$\mathbb{P}(X \le x, Y \le y) \ge \mathbb{P}(X \le x)\ \mathbb{P}(Y \le y). \tag{10}$$

We say X and Y are negatively quadrant dependent if the above equation reverses sign. Intuitively, X and Y are positively quadrant dependent if the probability that they are simultaneously small (or simultaneously large) is at least as great as it would be were they independent. In Proposition 2, the dependent parts of g_1 and g_2, namely $g_1 - \widetilde{g}_1$ and $g_2 - \widetilde{g}_2$, are assumed to be concordant (or in another terminology, similarly ordered). The positive quadrant dependence is a weaker notion than concordance. Concordance implies positive quadrant dependence but the reverse is not true. The proposition below shows that we can replace the concordance with positive quadrant dependence.

Proposition 3. *We have the following statements for sums of additive gambles:*

1. *Suppose g_1 and g_2 are positively quadrant dependent, then $R(g_1+g_2) \ge \min(R(g_1), R(g_2))$. More generally, suppose there exist independent random variables $\widetilde{g}_1$, $\widetilde{g}_2$, such that $\widetilde{g}_1$ and $\widetilde{g}_2$ are both independent with $g_1 - \widetilde{g}_1 + g_2 - \widetilde{g}_2$. If $g_1 - \widetilde{g}_1$ and $g_2 - \widetilde{g}_2$ are positively quadrant dependent, then $R(g_1+g_2) \ge \min(R(g_1), R(g_2))$.*

2. *Suppose g_1 and g_2 are negatively quadrant dependent, then $R(g_1+g_2) \le \max(R(g_1), R(g_2))$. More generally, suppose there exist independent random variables $\widetilde{g}_1$, $\widetilde{g}_2$, such that $\widetilde{g}_1$ and $\widetilde{g}_2$ are both independent with $g_1 - \widetilde{g}_1 + g_2 - \widetilde{g}_2$. If $g_1 - \widetilde{g}_1$ and $g_2 - \widetilde{g}_2$ are negatively quadrant dependent, then $R(g_1+g_2) \le \max(R(g_1), R(g_2))$.*

Proof: A very useful characterization states that two random variables X and Y are positively quadrant dependent if and only if $\mathrm{cov}(s(X), t(Y)) \ge 0$ for all nondecreasing functions of s and t such that the integrals in the covariance are well-defined. Notice that $g_1 - \widetilde{g}_1$ and $g_2 - \widetilde{g}_2$ are positively quadrant dependent, then so are $e^{-\beta(g_1-\widetilde{g}_1)}$ and $e^{-\beta(g_2-\widetilde{g}_2)}$. The proof is almost exactly the same as that of Proposition 2 for $N = 2$. Instead of relying on Čebyšev's algebraic inequality, we use the characterization result for positive quadrant dependence. □

As one example of applying the above proposition, let S_1, S_2 and S_3 be three independent random variables standing for three future financial quantities. Let $g_1 \equiv \max(S_1+S_2-K_1, 0) - p_1$

be the profit or loss of a spread option with strike price K_1 and price p_1. Similarly for $g_2 \equiv \max(S_1 + S_3 - K_2, 0) - p_2$. Assume that the strike prices and option prices are such that g_1 and g_2 are gambles. Then by Example 1.(iv) in Lehmann (1966), g and h are positively quadrant dependent. Proposition 3 then tells us that $R(g_1 + g_2) \geq \min(R(g_1), R(g_2))$.

III Conclusion

We study in more detail sums of gambles that are not necessarily independent. In particular, we show that if the dependent parts of two gambles are positively quadrant dependent, then the risk index of the sum of two gambles is always larger than the minimum of the risk indices of the two gambles. For negative dependence, the risk index of the sum is always smaller than the maximum of the two risk indices.

References

Aumann, R.J., and R. Serrano, 2008. An economic index of riskiness. *Journal of Political Economy* 116(5), 810–836.

Finner, H., 1992. A generalization of Hölder's inequality and some probability inequalities. *Annals of Probability* 20(4), 1893–1901.

Hardy, G.H., J.E. Littlewood, and G. Pólya, 1934. *Inequalities*, Cambridge University Press.

Kuptsov, L.P., 2001. *Hölder Inequality*, in Encyclopedia of Mathematics (edited by Hazewinkel M.), Kluwer Academic Publishers.

Lehmann, E.L., 1966. Some concepts of dependence. *Annals of Mathematical Statistics* 37(5), 1137–1153.

Merton R.C., 1990. *Continuous–time Finance*, Blackwell Publishing.

Mitrinović, D.S., J. Pečarić, and A.M. Fink, 1992. *Classical and New Inequalities in Analysis (Mathematics and its Applications)*, Kluwer Academic Publishers.