Problems of utility and prospect theories. A ”certain-uncertain” inconsistency of the random-lottery incentive system

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Extended Abstract

Aczél and Luce (2007) emphasized a fundamental question: whether the Prelec weighting function $W(p)$ equals 1 at $p=1$ (or whether $W(1)=1$).

This paper has obtained three main groups of results:

1) The values $W_{Certain}$ and $W(1)$ are additionally defined or specified. The Aczél-Luce question whether $W(1)=1$ is modified to the question whether $W(1)=W_{Certain}$ and the question is emphasized whether the probability weighting function $W(p)$ is continuous.

If $W(p)$ reveals a discontinuity at $p=1$, then this is a topological feature. This can qualitatively change (at least) the mathematical aspects of the utility and prospect theories.

This is supported by a number of the evidences of the qualitative difference between subjects' treatments of the probabilities of probable and certain outcomes.

2) Purely mathematical theorems prove (under several conditions) if the dispersion of data (the noise) is non-zero, then the non-zero discontinuity take place at the probability $p=1$.

3) In the prevailing random-lottery incentive system of the experiments of the utility and prospect theories, the choices of certain outcomes are stimulated by uncertain lotteries.

Because of this evident "certain-uncertain" inconsistency, the deductions from the random-lottery incentive experiments, those include the certain outcomes, cannot be unquestionably correct.

The experiment of Starmer and Sugden (1991) evidently supports this consideration.

Contents

Introduction

A man is the main subject of economics. Decisions of a man are the fundamental operations of economics. Utility theory, as a branch of the economic theory, is specially devoted to the research of decisions of a man.

Bernoulli (1738) had given rise to researches of problems of the utility theory. Von Neumann and Morgenstern (1947) had provided promises of feasibility of correct and, naturally, rational fundamentals of the economic theory. But these promises were broken by the Nobel laureate Allais (1953). Other later works of various authors had shown that real man's decisions are undoubtedly inconsistent with rational models and, moreover, with the probability theory.

Nobel laureate Kahneman and Thaler (2006) pointed out that the problems of the utility and prospect theories, including Allais (Allais, 1953) and Ellsberg (Ellsberg, 1961) paradoxes, have still not been adequately overcome.

This paper discusses a new possible way to solve these problems and an inconsistency of a prevailing incentive system of experiments. The paper is an initial design of future articles. It develops early reports (see, e.g., Harin, 2009a-2010b), articles Harin (2012a, 2012b) and the recent report Harin (2014).

The short main chapters of the paper present the idea of the inconsistency of the system of experiments. The longer Appendices show the evidence of the new way of solution of the problems.

1. The Aczél-Luce question whether *W(1)=1*

1.1. The problems. A probability weighting function

An essential part of the abovementioned problems of the utility and prospect theories consists in the problems, those are connected with a probability weighting (see, e.g., Tversky and Wakker, 1995). The probability weighting means that subjects treat the probability p by a function $W(p)$ which is not equal to p (or $W(p)\neq p$). The function is defined both for probable and certain outcomes.

Prelec weighting function (Prelec, 1998) is one of the most popular probability weighting functions.

1.2. The question

One possible way to solve the above problems is to consider the vicinities of the boundaries of the probability scale, e.g. at $p\sim 1$ (see Aczél and Luce, 2007).

Aczél and Luce (2007) emphasized a fundamental question: whether $W(1)=1$ (whether the Prelec weighting function (see Prelec, 1998) $W(1)$ is equal to 1 at $p=1$). In this paper, we refer to this question as the Aczél-Luce question (or Luce question).

1.3. Additional definitions

There are a number of the evidences of the qualitative difference between subjects' treatment of the probabilities of probable and certain outcomes (see, e.g., Kahneman and Tversky, 1979, McCord and de Neufville, 1986, Halevy, 2008).

So, in the general case, we should distinguish between values of the probability weighting function of the certain outcome and of the limit of the probability weighting function of the uncertain outcome when the probability of the uncertain outcome tends to 1.

Let us additionally define or specify a value $W_{Certain}$ of the probability weighting function $W(p)$ for the certain outcome. We may assume $W_{Certain}$ to be equal to 1 or normalize other values of $W(p)$ by $W_{Certain}$.

Let us here additionally specify a value $W(1)$ as the limit of the probability weighting function $W(p)$ for the probable outcome when p tends to 1

$$W(1) \equiv \lim_{p \to 1} W(p) \ .$$

If $W(1)=W_{Certain}$, then $W(p)$ is continuous. In the general case, this has not been proven. So (if we define also $W_{Impossible}$ for the impossible case),

$$W(p) = \begin{cases} W_{\mathrm{Im}\,possible} & p = 0 \\ W(p) & p \in]0,1[\\ W_{Certain} & p = 1 \end{cases}$$

(see also Aczél and Luce, 2007) and $W(p)$ can be continuous or discontinuous.

1.4. A modification of the question

Let us reformulate, modify the Aczél-Luce question whether $W(1)=1$ into the question whether $W(1)=W_{Certain}$ or whether $W(p)$ is continuous.

To answer to the question and to prove or disprove the continuity of $W(p)$ we should determine and measure the difference

$$W_{Certain} - W(1) = ?$$

Note, that this does not put a question whether $W_{Certain}$ is equal to 1.

The answer $W(1) \neq W_{Certain}$ to the modified Aczél-Luce question means that the function has a discontinuity near $p=1$. This is not a quantitative but a qualitative, moreover, a topological feature. So, the answer to the question can qualitatively change the situation in the utility and prospect theories, at least in their mathematical aspects.

2. Purely mathematical restrictions

2.1. A synthesis of the two ways

The second possible way to solve the problems of utility and prospect theories has been widely discussed, e.g., in Schoemaker and Hershey (1992), Hey and Orme (1994), Chay et al (2005), Butler and Loomes (2007). The essence of this way consists in a proper attention to noise, imprecision, and other reasons that might cause dispersion, scattering, spread of the data.

A purely mathematical research (see, e.g., Harin, 2010a, 2012b) combines, synthesizes the two above-mentioned ways. That is, it considers the dispersion of data near the boundaries (or the influence of the dispersion of the data near the boundaries) of the probability scale.

2.2. Existence theorems. The proof of $W(1)<W_{Certain}$

Purely mathematical theorems (see, e.g., Harin, 2012b and the Appendix A3) proves a probability weighting function $W(p)$ cannot reach $W_{Certain}$, at a non-zero dispersion of data, for $W(p)\leq p$ at $p>1/2$ and for $W_{Certain}=1$. The theorem is based on a sequence of lemmas and theorems (more detailed see the Appendix A3):

For a finite non-negative function on an interval $[0, 1]$, the analog of the dispersion is proved to tend to 0, when the mean M of the function tends to a boundary of the interval. Hence, if the analog of the dispersion is not less than a non-zero value, then the non-zero restrictions exist on M. Namely, M cannot be closer to a boundary of the interval, e.g., to 1, than by another non-zero value. This signifies, that, at a non-zero analog of the dispersion, M cannot reach 1.

As far as the probability estimation corresponds to such a function and a non-zero dispersion of data takes place, then the non-zero restrictions exist on the probability estimation. This signifies, that at a non-zero dispersion of data, the probability estimation cannot reach 1.

As far as the probability is the limit of the probability estimation and a non-zero minimal dispersion of data takes place, then the non-zero restrictions exist on the probability. Namely, the probability cannot be closer, than by the non-zero value, to a boundary of the probability scale. This signifies, that, at a non-zero dispersion of the data, the probability p cannot reach 1, or $p<1$.

From this conclusion, it follows that, for $W_{Certain}=1$ and for $W(p)\leq p$ at $p>1/2$, if a non-zero dispersion of data takes place, then $W(p)<W_{Certain}$ for any reachable p. (more detailed proofs, e.g. the perception by the subjects A3.7, and clear graphical evidence of them see below in the Appendices).

As a matter of fact, the non-zero minimal dispersion of data can be caused, e.g., by non-zero noises those are practically unavoidable in economics.

2.3. Experiments: the evidences and the question

The existence theorem is supported by the experiments in various fields (see, e.g., Harin 2009a-2014). The risk aversion, the risk premium, the underweighting of high and the overweighting of low probabilities etc. support the theorem.

Nevertheless, at present, the experiments at $p{\sim}1$ seem to not support the theorem. So, the Aczél-Luce question is also the question about the theorem.

3. An analysis of a detail of the experiments

Let us analyze a fine detail of the experiments. Let us consider some typical descriptions of the experiments. We can see in the literature:

Loewenstein and Thaler (1989), page 188: "The students … were told that the experimenter would select and implement one of their choices at random."

Tversky and Thaler (1990), page 206: "The subjects are told that one of these pairs will be selected at random at the end of the session, and that they will play one of these bets."

Kahneman et al (1991), page 195: "One of the four market trials would subsequently be selected at random and only the trades made on this trial would be executed". Page 197: "One of the accepted offers (including the original endowment) was selected at random at the end of the experiment to determine the subject's payment."

Harbaugh et al (2001), page 1543: "… each observation consists of one choice from each of the 11 different budget sets."

Vossler et al (2012), page 158: "Participants are instructed that one of the 12 choice sets will be randomly chosen at the end of the experiment, each with equal probability."

Cappelen et al (2013), page 1402: "At the beginning of the experiment, stakeholders were told that the computer would randomly choose one of the situations and one of the choices in this situation to determine their final outcome."

Such a procedure can be seen not only in the field of the utility theory but also in other fields of the economics, see, e.g., Larkin and Leider (2012), page 193: "Subjects made fifteen choices between a lottery and a fixed payment. … Subjects were paid for one randomly selected decision".

We see that subjects are stimulated and paid by the choice of one from a number of situations. This is a well-known feature of the experiments in the field of the utility and prospect theories. But let us consider this feature more closely. We can see a fine detail in the literature (the highlighting and underlining is mine):

Andreoni and Sprenger (2012), page 3365: "One choice for each subject was selected for payment by drawing a numbered card at random. Subjects were told to treat each decision as if it were to determine their payments." and page 3366: "Section I provided a testable hypothesis for behavior across **<u>certain</u>** and uncertain intertemporal settings."

Von Gaudecker et al (2011), page 669: "Additionally, for one in every ten participants in these two treatments, one lottery was randomly selected and played out, and the payoff of that lottery was paid out." and page 667: "the probabilities of the high payoff in each option vary from 25 percent to **<u>100 percent</u>**".

Harrison et al (2005), page 898: "We undertook a new series of experiments that build closely on the basic design features of HL," ("HL" means here Holt and Laury, 2002) "but allow an identification of the extent to which the apparent scale effects on risk aversion are actually order effects." and "TABLE 1. …

Prob.	Payoff	Prob.	Payoff	Prob.	Payoff	Prob.	Payoff
<u>1</u>	$2	0	$1.60	**<u>1</u>**	$3.85	0	$0.10"

("Prob." means here probability(ies))

Holt and Laury (2002), page 1646: "… subjects began by indicating a preference, Option A or Option B, for each of the ten paired lottery choices in Table 1, with the understanding that one of these choices would be selected at random ex post and played to determine the earnings for the option selected." and "Even the most risk-averse person should switch over by decision 10 in the bottom row, since Option B yields a **sure** payoff of $3.85 in that case" and page 1645: TABLE 1 "Option A: **10/10** of $2.00, 0/10 of $1.60. Option B: **10/10** of $3.85, 0/10 of $0.10"*

Starmer and Sugden (1991), page 974: "… subjects in groups B and C knew that they were taking part in a random-lottery experiment in which questions 21 and 22 had equal chances of being for real." and "One problem, which we shall call P', required a choice between two lotteries R' (for "riskier") and S' (for "safer"). R' gave a 0.2 chance of winning £10.00 and a 0.75 chance of winning £7.00 (with the residual 0.05 chance of winning nothing); S' gave £7.00 **for sure**."

We see that the random incentive procedure is used not only in the probable but in the certain situations also. Let us consider this detail more closely.

3.1. An inconsistency between the certain outcomes and uncertain incentives

First, let us note that the stimulation by the payment for the choice of one from a number of situations may be named as an uncertain stimulation. We may name it also as a stimulation by uncertain incentives.

Further, suppose, that the subjects choose an uncertain choice, that is the choice, which probability is strictly less than *1* (and strictly more than *0*). In this case, the choice and the incentive are of the same type.

Suppose, that the subjects choose the certain choice, that is the choice, which probability is strictly equal to *1* (or strictly equal to *0*). In this case, the choice and the incentive are of the essentially different types. Moreover, this uncertain incentive can call the certain outcome into question.

So, there is an evident inconsistency between the certain type of the choice and the uncertain type of the incentive.

So, the correctness of the use of uncertain incentives for certain outcomes cannot be unquestionable. We may name this problem as a "certain-uncertain" inconsistency.

3.2. The role of the incentives

Incentives are widely discussed in economics (see, e.g., Starmer and Sugden, 1991, Holt and Laury, 2002, Gneezy et al 2006, Bordalo et al 2012, Larkin and Leider, 2012). Do incentives influence the choice of the subjects in the utility and prospect theories?

The correct answer to this question needs a special research.

However, we may be sure, that if incentives would not have any influence on the choice of the subjects, then there would be no reason to use such incentives.

So, we may not exclude that an incentive can influence a choice of a subject, at least partially.

3.3. The random-lottery incentive system

The above discussed random incentive procedure is usually called as the random-lottery incentive system (or the random lottery incentive system). The random-lottery incentive system is the well-known mechanism in experiments in the utility and prospect theories and, more widely, in experimental economics.

Starmer and Sugden (1991), page 971: "One common experimental design is to ask subjects to perform a number of tasks, each of which requires a choice among lotteries with monetary payoffs. At the end of the experiment, one of these tasks is selected by a random device, and the subject plays out the lottery that he or she chose in that task. This is the random lottery procedure.

This incentive system has several attractive features. It allows the experimenter to collect a considerable amount of data from each subject, thus economizing on the costs of recruiting subjects and allowing tests that compare a subject's responses to two or more tasks. At the same time, it avoids the problem of reference-point and wealth effects that would be created if subjects were paid according to their performances on each of a number of tasks. (A subject's response to one task might be affected by the amount he or she had won on a previous task.)"

Andreoni and Sprenger (2012), page 3365: "… random-lottery mechanism, which is widely used in experimental economics …"

Starmer (2000), page 371: "although most experiments involve real —usually monetary—incentives, the most common reward mechanism is the random lottery incentive system. In experiments with this design, subjects are rewarded according to their response to one task which is randomly selected at the end of the experiment."

Wakker (2007): "… the random-lottery incentive system has become the almost exclusively used incentive system for individual choice, and numerous studies have used and tested it. It is used by people well recognized in experimental economics … Remember that the random-lottery incentive system is the only real incentive system for individual choice known today that can avoid the income effect. Without it, real incentives for individual choice are no longer well possible."

Here, and in other works (see, e.g., Vossler and Rondeau, 2012, von Gaudecker et al, 2011) we find the elaborated researches of correctness but no mention of the "certain-uncertain" inconsistency.

So, we may conclude:

1) The random-lottery incentive system is widespread in the utility and prospect theories. Moreover there are no widespread mentions about differences between the results of the random-lottery incentive system and other systems.

2) The essence of the random-lottery incentive system corresponds to the random, uncertain name of the system.

3) The question of considering an isolated situation as a grand meta-lottery over many choice situations has been already brought up. Nevertheless, the specific "certain-uncertain" inconsistency of the random-lottery incentive system has not still been considered.

4. The "certain-uncertain" inconsistency of the random-lottery incentive system

4.1. Final considerations and future tasks

So, we see that the random-lottery incentive system is the prevailing experimental procedure in the utility and prospect theories. The tests of the Aczél-Luce question are often connected with this system.

The present tests of the modified Aczél-Luce question (those, maybe, are not carefully considered) can lead to the opinion that, at $p\sim 1$, the probable outcomes are qualitatively the same as the certain ones. Particularly, it can lead to the opinion that the probability weighting function $W(p)$ tends to $W_{Certain}$, when the probability tends to 1. But there are a number of the evidences of the qualitative difference between the probable and certain outcomes (see, e.g., Kahneman and Tversky, 1979, McCord and de Neufville, 1986, Halevy, 2008). The existence theorem supports this qualitative difference and the answer $W(1)<W_{Certain}$ to the modified Aczél-Luce question.

The random-lottery incentive system is concerned with the "certain-uncertain" inconsistency. This inconsistency means that the certain choice is stimulated by the uncertain incentive. Because of this evident "certain-uncertain" inconsistency, the deductions from the random-lottery incentive experiments, those include the certain outcomes, cannot be unquestionably correct.

So, these deductions need an additional proof, or an amendment, or a new approach. At present, the random-lottery incentive system seems to cannot determine the qualitative difference between certain and probable outcomes at $p\sim 1$. An additional explanation of the "certain-uncertain" inconsistency needs an additional full-length article or articles.

At that, it may be supposed that such a useful and prevailing tool as the random-lottery incentive system and the overwhelming majority of the data, which is already obtained by means of it, may and should be continued to use. The following may be supposed:

In the narrow middle of the probability scale (where the probability weighting function intercepts the line $W(p)=p$) and in the obvious cases, the data and deductions may be used "as it is".

In the wide middle of the probability scale, the deductions may be the same or slightly corrected. This may be true when the probability p is located sufficiently far from $p=1-r_{Restriction}$, where the restriction $r_{Restriction}$ is obtained from the theorem of existence of restrictions on the probability for the case of the random lottery (see Appendix A3).

When the probability tends to the restriction $p\rightarrow 1-r_{Restriction}$, the data should be used with non-linear corrections and the deductions should be recalculated by non-linear functions.

At the probabilities those are in the forbidden zone $p>1-r_{Restriction}$, a new approach may be needed to make the deductions correct.

4.2. An experimental support of the considerations of this paper

We can see in Starmer and Sugden (1991) the following:

Page 974:

"For groups A and D, this page began with an underlined text stating that question 22 would be played for real. For groups B and C, the corresponding text stated that one of the two questions would be played for real and that which question was to played out would be decided at the end of the experiment in the following way. The subject would roll a six-sided die. If the number on the die was 1, 2, or 3, then question 21 would be played; if the number was 4, 5, or 6, question 22 would be played."

"One problem, which we shall call P', required a choice between two lotteries R' (for "riskier") and S' (for "safer"). R' gave a 0.2 chance of winning £10.00 and a 0.75 chance of winning £7.00 (with the residual 0.05 chance of winning nothing); S' gave £7.00 for sure."

So, in the R'-S' problem, R' gives *£10.00*0.2+£7.00*0.75=£7.25*. S' gives *£7.00*1=£7.00*. Here *R'=£7.25>S'=£7.00*.

Here are the results of Starmer and Sugden (1991) of interest for this paper (the highlighting is mine):

Page 976,

TABLE 2-RESULTS:

		Problem P'	
Group	Incentive	R'	S'
B	Random lottery	19	21
C	Random lottery	22	18
D	**P'real**	**13**	**27**

We see the results for P'real incentive differ essentially from those for Random lottery incentive.

So, the experiment of Starmer and Sugden (1991) evidently supports the considerations of this paper. This holds both for the sign of the bias (which follows from the theorem) of the preferences of the subjects to the certain outcome and for the sign of the bias of the influence of the "certain-uncertain" inconsistency of the random-lottery incentive system. We see that the random lottery incentives can essentially modify subjects' choices in comparison with the real incentives, when these choices include the certain outcomes and the probability (*0.95=0.2+0.75*) of the probable choices is near the boundary of the probability scale.

As to the interfaces between no corrections, slight corrections and essential corrections, we evidently see that here, at the probabilities of *0.95*, the corrections should be quite essential.

A number of special additional independent researches should be done to consider, discuss and establish these corrections and revisions and interfaces between these corrections and revisions.

Conclusions

Aczél and Luce (2007) emphasized a fundamental question: whether $W(1)=1$ (whether the Prelec (probability) weighting function equals 1 at $p=1$).

The paper develops this work. There are a number of the evidences of the qualitative difference between subjects' treatment of the probabilities of probable and certain outcomes (see, e.g., Kahneman and Tversky, 1979, McCord and de Neufville, 1986, Halevy, 2008) and, in the general case, we should distinguish between the values of the probability weighting function of certain and uncertain outcomes. Two values of the probability weighting function $W(p)$ are specified: $W_{Certain}$ for the certain outcome and $W(1)$, as the limit of the probability weighting function $W(p)$, for the probable outcome when p tends to 1

$$W(1) \equiv \lim_{p \to 1} W(p) \ .$$

The Aczél-Luce question whether $W(1)=1$ is modified to the question whether $W(1)=W_{Certain}$. The answer $W(1) \neq W_{Certain}$ means a discontinuity of $W(p)$ at $p=1$. This is a topological feature. So, the answer can qualitatively change the utility and prospect theories, at least in their mathematical aspects.

Purely mathematical theorems prove $p<1$, at a non-zero dispersion of data. The perception of the probability by the subjects and, so, the probability weighting function $W(p)$ can be biased by the restriction $r_{Restriction}$ (see A.3.7).

For $W(p) \leq p$ at $p>1/2$ and $W_{Certain}=1$, an assumption may be proposed

$$W_{Certain} - W(1) \geq r_{Restriction} > 0 \ .$$

If the dispersion of data (the noise) is non-zero, then $r_{Restriction}$ is non-zero also.

The Appendices A.1.1-A.3.8 of the paper clearly show the truth and evidence of the theorem. The data support the theorem in wide diapasons, but at $p \sim 1$ the experiments may be treated as they do not.

However, in the prevailing random-lottery incentive system of these experiments, the choices of certain outcomes are stimulated by uncertain lotteries.

Because of this evident "certain-uncertain" inconsistency, the deductions from the random-lottery incentive experiments, those include the certain outcomes, cannot be unquestionably correct.

Starmer and Sugden (1991) experiment evidently supports this conclusion.

It may be supposed that such a useful tool as the random-lottery incentive system and the majority of its data may and should be continued to use:

In the middle of the probability scale, the deductions may be the same or slightly corrected.

When the probability tends to the restriction $p \to 1-r_{Restriction}$, the deductions should be recalculated by non-linear functions. Due to Starmer and Sugden (1991), at the probabilities about 0.95, the corrections should be quite essential.

At the probabilities in the forbidden zone $p>1-r_{Restriction}$, a new approach may be needed to make the deductions correct and to measure $W_{Certain}-W(1)$.

The further consideration of the idea of the "certain-uncertain" inconsistency may be developed only after its independent confirmations will take place.

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Appendices

Contents of the Appendices

A1. An illustrative example of restrictions on the mean

Let us consider briefly an illustrative example of restrictions on the mean (see, e.g., Harin, 2012a).

A1.1. Two points

Let us suppose given an interval $[A, B]$ (see Figure 1). Let us suppose that two points are determined on this interval: a left point x_{Left} and a right point x_{Right} : $x_{Left}<x_{Right}$. The coordinates of the middle, mean point may be calculated as $M=(x_{Left}+x_{Right})/2$.

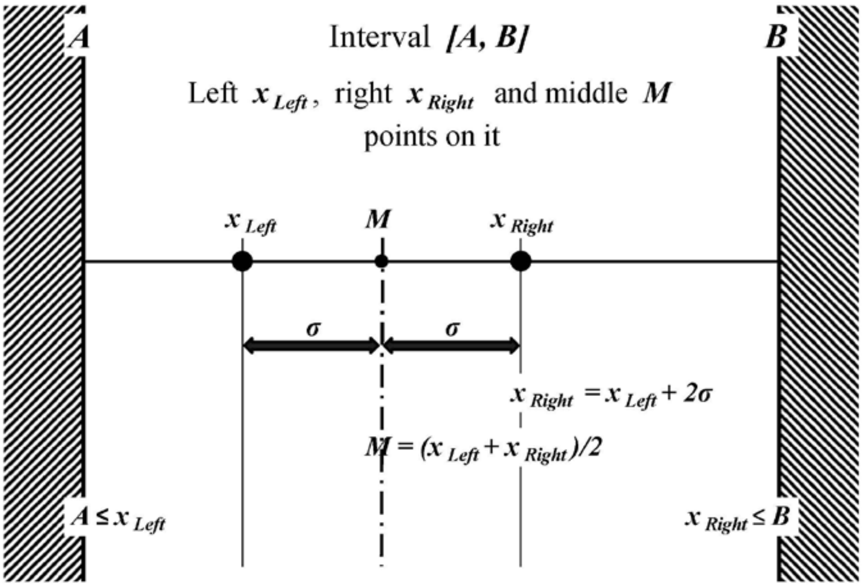

Figure 1. An interval $[A, B]$. Left x_{Left}, right x_{Right} and middle, mean M points on it

Let us suppose that $x_{Right}-x_{Left}\geq 2\sigma=2Const_{\sigma}>0$. So, of course, $x_{Right}\geq x_{Left}+2\sigma$ and $x_{Left}\leq x_{Right}-2\sigma$. For the sake of simplicity, Figures 1-3 represent the case of the equality $x_{Right}-x_{Left}=2\sigma$ and also, of course, $x_{Right}=x_{Left}+2\sigma$, $x_{Left}=x_{Right}-2\sigma$ and $M-x_{Left}=x_{Right}-M=\sigma=Const_{\sigma}>0$.

So, $M=x_{Left}+\sigma>x_{Left}$ and $M=x_{Right}-\sigma<x_{Right}$.

Suppose further that $x_{Left}\geq A$ and $x_{Right}\leq B$.

One can easily see that two types of zones for M can exist in the interval:

1) The mean point M can be located only in the zone which will be referred to as "allowed" (see Figure 2).

2) The mean point M cannot be located in the zones which will be referred to as "forbidden" (see Figure 3).

A1.2. Allowed zone

Due to the conditions of the example, the left point x_{Left} may not be located further left than the left border of the interval $x_{Left} \geq A$ and the right point x_{Right} may not be located further right than the right border of the interval $x_{Right} \leq B$.

For M, we have $M = x_{Left} + \sigma \geq A + \sigma > A$ and $M = x_{Right} - \sigma \leq B - \sigma < B$ (see Figure 2).

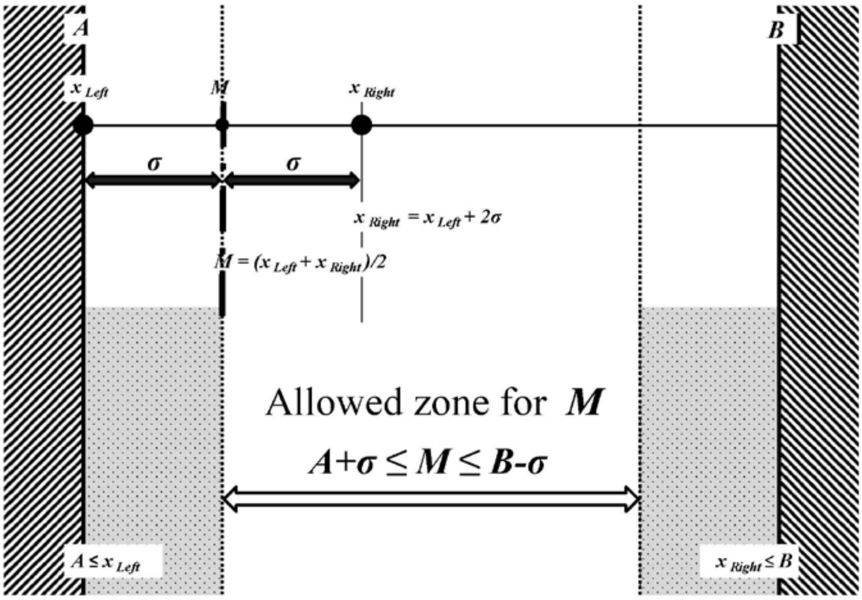

Figure 2. The allowed zone for M

The width of the allowed zone for M is equal to

$$B - \sigma - (A + \sigma) = (B - A) - 2\sigma .$$

It is less than the width $(B-A)$ of the total interval $[A, B]$ by 2σ. Also, the allowed zone is a proper subset of the total interval.

If the distance 2σ between the left x_{Left} and right x_{Right} points is non-zero, then the difference between the width of the allowed zone and the width of the interval is non-zero also. If the distance is greater than $2\sigma,$ then the difference is greater than 2σ also.

So, the mean point M can be located only in the allowed zone of the interval.

A1.3. Restrictions, forbidden zones

Let us define the term "restriction" for the purposes of this paper:

Definition. A **restriction** (more exactly, a **restriction on the mean**) signifies the impossibility for the mean to be located closer to a border of the interval than some fixed distance. In other words, a restriction implies here a forbidden zone for the mean near a border of the interval.

The value of a restriction or the width of a forbidden zone signifies the minimal possible distance between the mean and a border of the interval. For brevity, the term "the value of a restriction" may be shortened to "restriction".

If $A \le x_{Left}$, $x_{Right} \le B$ and $x_{Right}\text{-}x_{Left}=2\sigma$, then restrictions, forbidden zones with the width of one sigma σ exist between the mean point and the borders of the interval (see Figure 3). So there are two forbidden zones, located near the borders of the interval. The mean point M cannot be located in these forbidden zones.

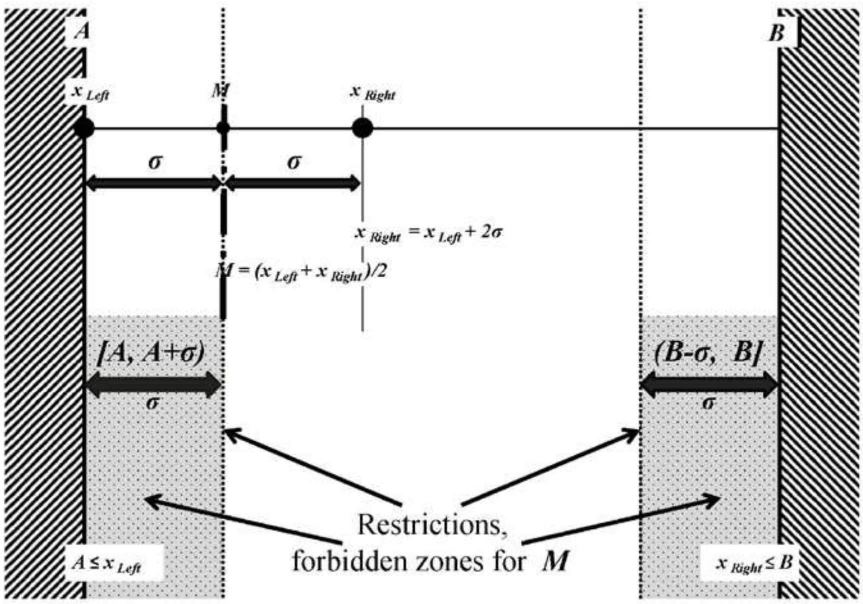

Figure 3. The forbidden zones, restrictions on M

The restrictions, the forbidden zones are shown by two dotted lines and by painting in the bottom part of Figure 3.

As we can easily see, restrictions on the mean or forbidden zones exist between the allowed zone of the mean M and the borders A and B of the interval $[A; B]$. The value of the restriction, or, equivalently, the width of the forbidden zone, is equal to σ.

So, the restrictions of the value σ on the mean point M exist near the borders of the interval

A2. An illustrative example of restrictions on the probability

Let us consider briefly an illustrative example of restrictions on the probability (see, e.g., Harin, 2012a).

A.2.1. A classical round target

Consider a classical example: an aiming firing at a target. Suppose a classical round target (Figure 4) of the diameter *2L*.

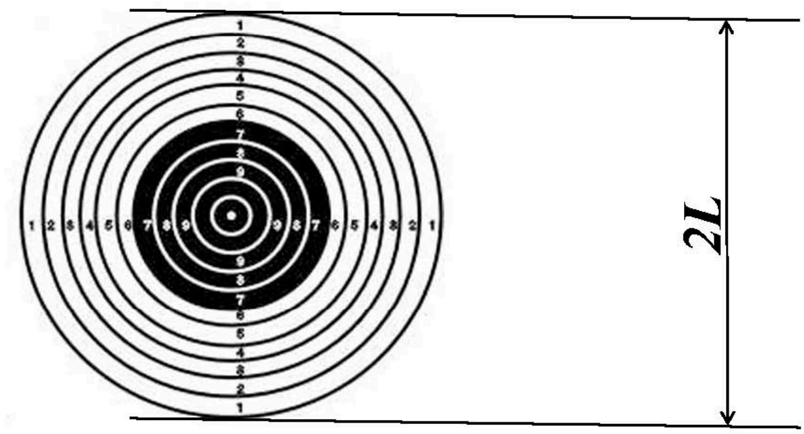

Figure 4. A target for firing

Suppose Mr. Somebody performs an aiming firing by batches of pellets, small shots at a target.

A2.2. Two types of dispersion

For the obviousness suppose (Figure 5) the dispersion of pellets hits is uniformly distributed in a zone of the diameter 2σ (See an example of the normal distribution below in A.2.6.).

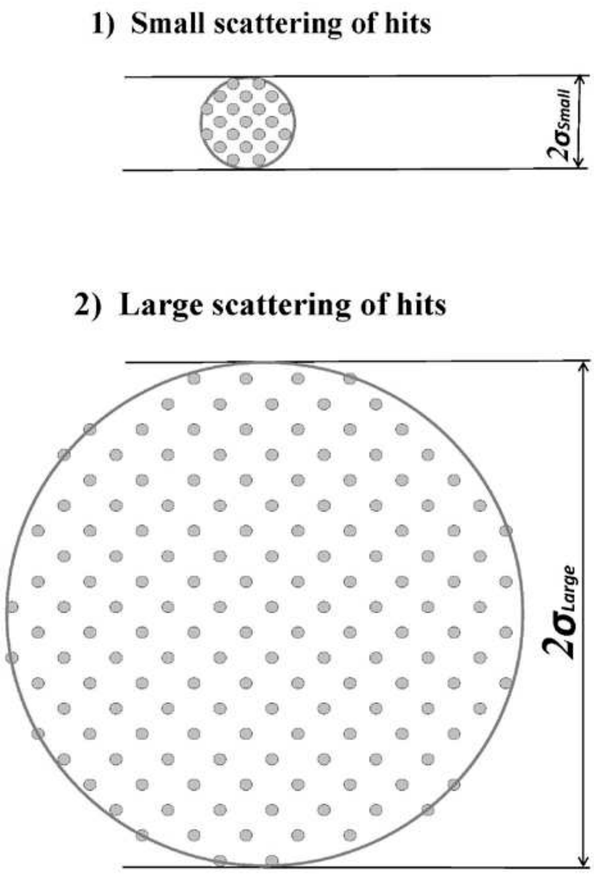

Figure 5. Dispersion of hits is uniformly distributed
in a zone of the diameter 2σ

Notes about this figure:

Note 1: This is only a simplified example (See an example of the normal distribution below in A.2.6.).

Note 2: The case 1) represents the case of small diameter $2\sigma_{Small}$ of the zone of dispersion of pellets hits.

The case 2) represents the case of large diameter $2\sigma_{Large}$ of the zone of dispersion of pellets hits.

Suppose the point of aiming may be varied between the center of the target and a point which is outside the target.

A2.3. Small dispersion

The case, when the diameter $2\sigma_{Small}$ of the zone of dispersion of hits is considerably less than the diameter $2L$ of the target, is drawn on the figure 6.

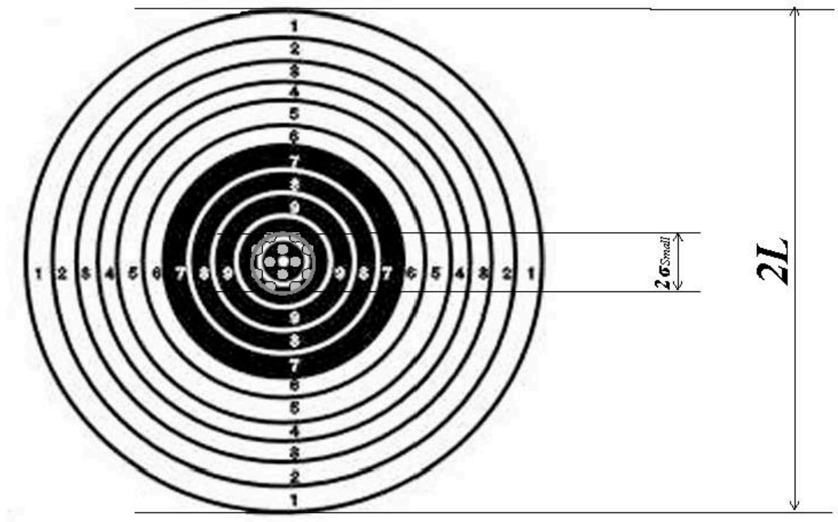

Figure 6. Firing for the small dispersion of hits
Note: The diameter $2\sigma_{Small}$ of the zone of dispersion of hits is considerably less than the diameter $2L$ of a target.

At the condition of the small dispersion of hits, the maximum possible probability of hit in the target can be equal to *1* (can reach the boundary of the probability scale).

When the point of aiming is varied between the center of the target and a point which is outside the target, the probability of hit in the target is varied from *1* to *0*. There are no restrictions in the probability scale.

A2.4. Large dispersion

The case, when the diameter $2\sigma_{Large}$ of the zone of dispersion of hits is considerably more than the diameter $2L$ of the target, is drawn on the figure 7.

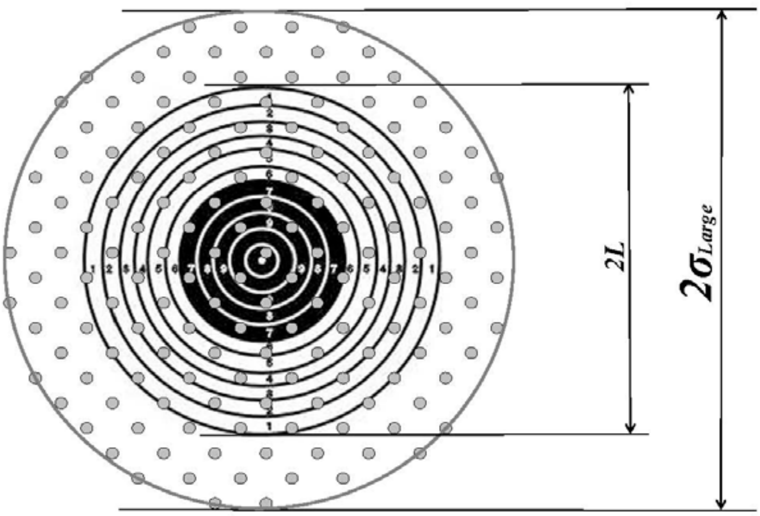

Figure 7. Firing for the large dispersion of hits

Note: The diameter $2\sigma_{Large}$ of the zone of dispersion of hits is considerably more than the diameter $2L$ of the target.

A2.5. Restriction on the probability

At the condition of the large dispersion of hits (exactly speaking at the condition the diameter $2\sigma_{Large}$ of the zone of dispersion of hits is more than the diameter $2L$ of a target), the maximum possible probability of hit in the target can not be equal to *1*.

So, the situation for the probability for this case is drawn on the figure 8.

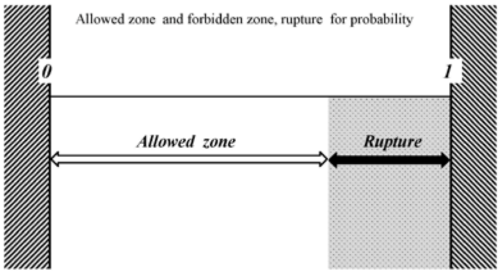

Figure 8. Restriction on the probability: Allowed zone and forbidden zone
Note: See the example of two restrictions for two boundaries below in A.2.6.

The value $P_{AllowedMax}$ of the maximal allowed probability of the allowed zone [0, $P_{AllowedMax}$] may be estimated as the ratio of the mean number of the hits in the target to the total number of the hits. In this particular case, when the distribution of hits is supposed to be uniform, this ratio equals to the ratio of the area of hits scattering to the area of the target

$$P_{AllowedMax} = S_{T\arg et} / S_{HitsL\arg e} = \pi L^2 / \pi\sigma^2{}_{\mathrm{L}\arg e} = L^2 / \sigma^2{}_{\mathrm{L}\arg e} \ .$$

If

$$L < \sigma_{\mathrm{L}\arg e} \ ,$$

then

$$P_{AllowedMax} < 1 \ .$$

In this particular case, the probabilities of hit in the target, that are larger than $P_{AllowedMax}$, are impossible. The allowed probabilities of hit in the target belong to the allowed zone [0, $P_{AllowedMax}$].

The value of the restriction $R_{Restriction}$ may be estimated as the difference between unit and the maximal allowed probability $P_{AllowedMax}$ of hit in the target

$$R_{\mathrm{Re}\,striction} = 1 - P_{AllowedMax} > 0 \ ,$$

and, if $L<\sigma_{Large}$, then $R_{Restriction}$ is a positive nonzero quantity. At the conditions of the figure 7, it is evident the probability $P_{AllowedMax}$ can not be more, then *0.5-0.7* (*50%-70%*) and the restriction $R_{Restriction}$ is as more as *0.3-0.5* (*30%-50%*).

A2.6. An example of the normal distribution and of two restrictions for two boundaries

Let us consider concisely an example of the normal distribution and of two restrictions for two boundaries.

Conditions

Let us consider firing at a target in the one-dimensional approach. Let the dimension of the target be equal to $2L>0$ and the scatter of hits, when aim is precise, obeys the normal law with the dispersion σ^2. Then (see, e.g., Abramowitz and Stegun, 1972) the maximal probability P_{in_Max} of hitting the target and the minimal probability $P_{out\_min}=1-P_{in\_Max}$ of missing it equal:

Results

For $\sigma=0$:
$P_{in\_Max}=1$ and $P_{out\_min}=0$.
That is, there are no ruptures in the probability scale for hits and misses, that is $r_{expect}=1-P_{in\_Max}=P_{out\_min}=0$.

For $L=3\sigma$:
$0\leq P_{in}\leq P_{in\_Max}=0{,}997<1$ and $0<0{,}003=P_{out\_min}\leq P_{out}\leq 1$.
For this case, the ruptures r_{expect} in the probability scale for hits and misses are equal to $r_{expect}=0{,}003>0$.

For $L=2\sigma$:
$0\leq P_{in}\leq P_{in\_Max}=0{,}95<1$ and $0<0{,}05=P_{out\_min}\leq P_{out}\leq 1$.
For this case, the ruptures r_{expect} in the probability scale for hits and misses are equal to $r_{expect}=0{,}05>0$.

For $L=\sigma$:
$0\leq P_{in}\leq P_{in\_Max}=0{,}68<1$ and $0<0{,}32=P_{out\_min}\leq P_{out}\leq 1$.
For this case, the ruptures r_{expect} in the probability scale for hits and misses are equal to $r_{expect}=0{,}32>0$.

Conclusion

For zero $\sigma=0$ there are no ruptures ($r_{expect}=0$).

For non-zero $\sigma>0$: The non-zero rupture $r_{expect}>0$ appears between the zone of possible values of the probability of hitting $0\leq P_{in}\leq P_{in\_Max}=1-r_{expect}<1$ and 1. The same non-zero rupture $r_{expect}>0$ appears between the zone of possible values of the probability of missing $0<r_{expect}=P_{out\_min}\leq P_{out}\leq 1$ and 0.

A3. The existence theorems of restrictions

Let us consider briefly (see, e.g., Harin, 2012b) existence theorems, from restrictions on the mean to restrictions on the probability, the bias of subjects' perception of the probability and the case of the random-lottery incentive system.

A3.1. Preliminary notes

Let us suppose given a finite interval, $X=[A, B] : 0<Const_{AB}\le(B-A)<\infty$, a set of points $\{x_k\} : k=1, 2, \dots K : 2\le K\le\infty$, and a finite non-negative function $f_K(x_k)$: at $x_k<A$ and $x_k>B$ the statement $f_K(x_k)\equiv 0$ is true; at $A\le x_k\le B$ the statement $0\le f_K(x_k)<\infty$ is true, and

$$\sum_{k=1}^{K} f_K(x_k) = W_K \ ,$$

where W_K (the total weight of $f_K(x_k)$) is a constant and

$$0 < W_K < \infty \ .$$

Without loss of generality, the function $f_K(x_k)$ may be normalized so that $W_K=1$.

Definition A3.1. Let us define an analog of the moment of n-th order of the function $f_K(x_k)$ relative to a point x_0:

$$E(X-X_0)^n = \frac{1}{W_K}\sum_{k=1}^{K}(x_k-x_0)^n f_K(x_k) = \sum_{k=1}^{K}(x_k-x_0)^n f_K(x_k) \ .$$

From now on, for brevity, we refer to this analog of the moment of n-th order as simply the moment of n-th order.

Let us suppose the mean $M\equiv E(X)$ of the function $f_K(x_k)$ exists

$$E(X) \equiv \frac{1}{W_K}\sum_{k=1}^{K} x_k f_K(x_k) = \sum_{k=1}^{K} x_k f_K(x_k) \equiv M \ .$$

Let us suppose at least one central moment $E(X-M)^n : 2\le n<\infty$, of the function $f_K(x_k)$ exists

$$E(X-M)^n = \frac{1}{W_K}\sum_{k=1}^{K}(x_k-M)^n f_K(x_k) = \sum_{k=1}^{K}(x_k-M)^n f_K(x_k) \ .$$

One may prove (see, e.g., Harin, 2013), that a function, which attains the maximal possible central moment, is concentrated at the borders of the interval. At that, the moduli of the central moments of such a function are not greater than the estimate

$$Max(|E(X-M)^n|) \le (M-A)^n \frac{B-M}{B-A} + (B-M)^n \frac{M-A}{B-A} \ .$$

A3.2. General lemma for the mean

Lemma A3.2. If, for the function $f_K(x_k)$ defined in Section A3.1, $M \equiv E(X)$ tends to A or to B, then, for $n : 2 \leq n < \infty$, $E(X-M)^n$ tends to zero.

Proof. For $M \to A$, the estimate gives

$$
\begin{aligned}
&| E(X-M)^n | \leq (M-A)^n \frac{B-M}{B-A} + (B-M)^n \frac{M-A}{B-A} = \\
&= [(M-A)^{n-1} + (B-M)^{n-1}] \frac{(M-A)(B-M)}{B-A} < \\
&< [(B-A)^{n-1} + (B-A)^{n-1}] \frac{(M-A)(B-M)}{B-A} \leq \\
&\leq 2(B-A)^{n-1}(M-A) \xrightarrow[M \to A]{} 0
\end{aligned}
.
$$

This rough estimate is already sufficient for the purpose of this paper. But a more precise estimate (see, e.g., Harin, 2013) may be obtained:

$$| E(X-M)^n | \leq (B-A)^{n-1}(M-A) \xrightarrow[M \to A]{} 0 \ .$$

For $M \to B$, the estimate is similar and gives

$$| E(X-M)^n | \leq (B-A)^{n-1}(B-M) \xrightarrow[M \to B]{} 0 \ .$$

So, if $(B-A)$ and n are finite and $M \to A$ or $M \to B$, then $E(X-M)^n \to 0$.

A3.3. General theorem for the mean

Let us define two terms for the purposes of this paper:

Definition A3.3.1. A **restriction on the mean** $\boldsymbol{r_{Mean}}$ (or, simply, a **restriction**) signifies the impossibility for the mean to be located closer to a border of the interval than some fixed distance. In other words, a restriction implies here a forbidden zone for the mean near a border of the interval.

The value of a restriction or the width of a forbidden zone signifies the minimal possible distance between the mean and a border of the interval. For brevity, the term "the value of a restriction" may be shortened to "restriction".

Definition A3.3.2. Let us define "**restriction on dispersion** of the n-th order" $r^n_{Dispersion.n} \equiv \boldsymbol{r^n_{Disp.n}} : r_{Disp.n} > 0$ (where dispersion is taken in the broad sense, as scattering, spread, variation, etc.) to be the minimal absolute value of the analog of the n-th order central moment $E(X-M)^n$: $|E(X-M)^n| \geq r^n_{Disp.n} > 0$.

For $n=2$ the restriction on the dispersion of second order is the minimal possible dispersion (in the particular sense) $r^2_{Disp.2} = \sigma^2_{Min}$.

Theorem A3.3. If, for the finite non-negative discrete function $f_K(x_k)$ defined in Section A3.1, with the mean $M \equiv E(X)$ and the analog of an n-th $(2 \leq n < \infty)$ order central moment $E(X-M)^n$ of the function, a non-zero restriction on dispersion of the n-th order $r^n_{Disp.n} = Const_{Disp.n} > 0 : |E(X-M)^n| \geq r^n_{Disp.n}$, exists, then the non-zero restriction $r_{Mean} > 0$ on the mean $E(X)$ exists and $A < (A + r_{Mean}) \leq M \equiv E(X) \leq (B - r_{Mean}) < B$.

Proof. From the conditions of the theorem and from the lemma A3.2 for $M \to A$, we have

$$0 < r^n{}_{Disp.n} \leq | E(X-M)^n | \leq (B-A)^{n-1}(M-A)$$

and

$$0 < \frac{r^n{}_{Disp.n}}{(B-A)^{n-1}} \leq (M-A) \ .$$

So,

$$(M-A) \geq r_{Mean} = \frac{r^n{}_{Disp.n}}{(B-A)^{n-1}} > 0 \ .$$

For $M \to B$, the proof is similar and gives

$$(B-M) \geq r_{Mean} = \frac{r^n{}_{Disp.n}}{(B-A)^{n-1}} > 0 \ .$$

So, as long as $(B\text{-}A)$ and n are finite and $r^n{}_{Disp.n}=Const_{Disp.n}>0$, then $r_{Mean}=Const_M>0$ and $A<(A+r_{Mean})\leq M\leq(B\text{-}r_{Mean})<B$.

Note

This estimate is an ultra-reliable one. It is, in a sense, as ultra-reliable as the Chebyshev inequality. Preliminary calculations (see, e.g., Harin, 2009) which were performed for real cases, such as the normal, uniform and exponential distributions with the minimal values $\sigma^2{}_{Min}$ of the analog of the dispersion (in the particular sense), gave the restrictions r_{Mean} on the mean of the function, which are not worse than

$$r_{Mean} \geq \frac{\sigma_{Min}}{3} \ .$$

A3.4. Lemma for the probability estimation

Lemma A3.4. If $f_K(x_k)$ is defined as in section A3.1, and either $E[X]\to 0$ or $E[X]\to 1$, then, for $1<n<\infty$,

$$| E(X-M)^n | \to 0 \ .$$

Proof. As long as the conditions of this lemma satisfy the conditions of the lemma A3.2, then the statement of this lemma is as true as the statement of the lemma A3.2.

A3.5. Theorem for probability estimation

Theorem A3.5. If $\{x_k\}$ and a probability estimation, frequency F_K, are defined as in section A3.1 and $M\equiv E[X]\equiv F_K$, there are n : $1<n<\infty$, and $r_{Disp.n}>0$: $E[(X\text{-}M)^n]\geq r^n{}_{Disp.n}>0$, then, for the probability estimation, frequency $F_K\equiv M\equiv E[X]$, a restriction r_{mean} exists for which $0<r_{Mean}\leq F_K\leq(1\text{-}r_{Mean})<1$.

Proof. As long as the conditions of this theorem satisfy the conditions of the theorem A3.3, then the statement of this theorem is as true as the statement of the theorem A3.3.

A3.6. Theorem for probability

Theorem A3.6. If, for the probability scale [*0; 1*], a probability *P* and the probability estimation, frequency F_K, for a series of tests of number $K : K>>1$, are determined and, when the number *K* of tests tends to infinity, the frequency F_K tends at that to the probability *P*, that is

$$P = \lim_{K\to\infty} F_K \ ,$$

non-zero restrictions r_{mean} : $0<r_{Mean}\leq F_K\leq(1-r_{Mean})<1$ exist between the zone of the possible values of the frequency and every boundary of the probability scale, then the same non-zero restrictions r_{Mean} : $0<r_{Mean}\leq P\leq(1-r_{Mean})<1$ exist between the zone of the possible values of the probability *P* and every boundary of the probability scale.

Proof. Consider the left boundary *0* of the probability scale [*0; 1*]. The frequency F_K is not less than r_{Mean}:

$$F_K \geq r_{mean} \ .$$

Hence, we obtain for *P*:

$$P = \lim_{K\to\infty} F_K \geq \lim_{K\to\infty} r_{mean} = r_{mean} \ .$$

So, $P\geq r_{Mean}$. Note that this is true for both monotonous and dominated convergence. The reason is the fixation of the minimal value of all the F_K by the conditions of the theorem. For the right boundary *1* of the probability scale the proof is similar to that above.

A3.7. The bias of the perception of the probability

Let us make a note about the bias (more detailed see Harin, 2012b).

In almost any real case, a finite non-zero degree of uncertainty is inherent in real measurements of probability. The total magnitude of this uncertainty can be both negligible and high, relative to a useful signal, but it does not tend to zero.

Subjects are experienced and intuitively feel the restrictions.

In the ideal case, the probability is the same as it is claimed by the experimenters. In the real case (and from the point of view of the experienced subjects), the probability near every boundary is restricted and cannot be closer to the boundary than the restriction enables. So, near a boundary it is biased (in comparison with the ideal case) from a boundary to the middle of the probability scale.

Note that the bias may be supposed to exist not only in the zones of the restrictions but also beyond them and to vanish at the middle of the scale.

So, the restrictions near the boundaries can bias subjects' perception of the probability from the boundaries to the middle of the probability scale. The bias is directed to the middle and is maximal just near every boundary.

So, subjects' perception of probability can be biased from the boundaries to the middle of the probability scale due to the data dispersion (noise) restrictions. The probability weighting function *W(p)* should represent this bias.

A3.8. An additional statement for the case of the random-lottery incentive system

Let us additionally define $r_{Restriction} \equiv r_{Mean}$ and $p \equiv P$.

At that, $r_{Restriction} \geq r_{Random\text{-}Lottery}$, where $r_{Random\text{-}Lottery}$ is the restriction caused specifically by the random-lottery incentive system.

Statement A3.8. If the probability p satisfies the conditions of the theorem for probability A3.6, the probability weighting function $W(p)$ is defined for the certain and probable outcomes, $W(p) \leq p$ at $p > 1/2$ and $W_{Certain} = 1$, then

$$W_{Certain} - W(1) \geq r_{\mathrm{Re}striction} > 0 \ .$$

Proof. Owing to the theorem for probability A3.6. and the additional definitions of this statement,

$$p \leq 1 - r_{\mathrm{Re}striction} \ .$$

Owing to $W(p) \leq p$ at $p > 1/2$

$$W(p) \leq p \leq 1 - r_{\mathrm{Re}striction} \ .$$

Owing to $W_{Certain} = 1$

$$W(p) \leq 1 - r_{\mathrm{Re}striction} = W_{Certain} - r_{\mathrm{Re}striction} \ .$$

So,

$$W_{Certain} - W(p) \leq r_{\mathrm{Re}striction} \ .$$