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Mock Theta Conjectures

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Abstract

This paper shows how to prove the two Theorems first and second mock theta conjectures respectively.

Keywords: Mock theta, rank of partition.

1. Introduction

We give the definitions of π , rank of partition, $N(m, n)$, $N(m, t, n)$, $\rho_0(n)$, $\rho_1(n)$, z , $(x)_\infty$, $(zx)_\infty$, $(x^n)_m$, $(x^k; x^5)_m$ which are collected from Partitions Yesterday and Today [4], Generalizations of Dyson's Rank [3], Ramanujan's Lost Notebook [2]. We generate the generating functions for $\rho_0(n)$, and $\rho_1(n)$ [2] and prove the two Theorems first and second mock theta conjectures respectively. Finally we give two numerical examples which are related to first and second mock theta conjectures respectively when $n = 1$.

2. Definitions

π : A partition.

Rank of partition: The largest part of a partition π minus the number of parts of π .

$N(m, n)$: The number of partitions of n with rank m .

$N(m, t, n)$: The number of partition of n with rank congruent to m modulo t .

$\rho_0(n)$: The number of partitions of n with unique smallest part and all other parts \leq the double of the smallest part.

$\rho_1(n)$: The number of partitions of n with unique smallest part and all other parts \leq one plus the double of the smallest part.

z : The set of complex numbers.

$(x)_\infty$: The product of infinite factors is defined as follows:

$$(x)_\infty = (1-x)(1-x^2)(1-x^3)\dots\infty.$$

$(zx)_\infty$: The product of infinite factors is defined as follows:

$$(zx)_\infty = (1-zx)(1-zx^2)(1-zx^3)\dots\infty.$$

$(x^n)_m$: The product of m factors is defined as follows:

$$(x^n)_m = (1-x^n)(1-x^{n+1})(1-x^{n+2})\dots(1-x^{n+m-1}).$$

$(x^k; x^5)_m$: The product of m factors is defined as follows:

$$(x^k; x^5)_m = (1-x^k)(1-x^{k+5})(1-x^{k+10})\dots(1-x^{k+(m-1)5}).$$

3. Mock Theta Functions (2)

We quote the relations below [1, 2]:

$$F(x) = \frac{(1-x)(1-x^2)(1-x^3)\dots\infty}{(1-2x \cos \frac{2n\pi}{5} + x^2)(1-2x^2 \cos \frac{2n\pi}{5} + x^4)\dots\infty}.$$

$$f'(x) = 1 + \frac{x}{1-2x \cos \frac{2n\pi}{5} + x^2} + \frac{x^4}{(1-2x \cos \frac{2n\pi}{5} + x^2)(1-2x^2 \cos \frac{2n\pi}{5} + x^4)} + \dots\infty,$$

$n = 1$ or 2 .

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$$F(x^{\frac{1}{5}}) = A(x) - 4x^{\frac{1}{5}} \cos \frac{2n\pi}{5} B(x) + 2x^{\frac{2}{5}} \cos \frac{4n\pi}{5} C(x) - 2x^{\frac{3}{5}} \cos \frac{2n\pi}{5} D(x). \tag{1}$$

$$f'(x^{\frac{1}{5}}) = \left\{ A(x) - 4 \sin^2 \frac{2n\pi}{5} \Phi(x) \right\} + x^{\frac{1}{5}} B(x) + 2x^{\frac{2}{5}} \cos \frac{2n\pi}{5} C(x) - 2x^{\frac{3}{5}} \cos \frac{2n\pi}{5} \left\{ D(x) + 4 \sin^2 \frac{2n\pi}{5} \frac{\Psi(x)}{x} \right\}. \tag{2}$$

$$A(x) = \frac{1 - x^2 - x^3 + x^9 + \dots}{(1-x)^2 (1-x^4)^2 (1-x^6)^2 \dots}$$

$$B(x) = \frac{(1-x^5)(1-x^{10})(1-x^{15}) \dots}{(1-x)(1-x^4)(1-x^6) \dots}$$

$$C(x) = \frac{(1-x^5)(1-x^{10})(1-x^{15}) \dots}{(1-x^2)(1-x^3)(1-x^7) \dots}$$

$$D(x) = \frac{1 - x - x^4 + x^7 + \dots}{(1-x^2)^2 (1-x^3)^2 (1-x^7)^2 \dots}$$

$$\phi(x) = -1 + \left\{ \frac{1}{1-x} + \frac{x^5}{(1-x)(1-x^4)(1-x^6)} + \frac{x^{20}}{(1-x)(1-x^4)(1-x^6)(1-x^9)(1-x^{11})} + \dots \right\},$$

But we get;

$$\begin{aligned} & A(x^5) - 4x \cos \frac{2\pi}{5} B(x^5) + 2x^2 \cos \frac{4\pi}{5} C(x^5) - 2x^3 \cos \frac{2\pi}{5} D(x^5) \\ &= 1 - 4x \cos^2 \frac{2\pi}{5} + 2x^2 \cos \frac{4\pi}{5} - 2x^3 \cos \frac{2\pi}{5} + 2x^5 - 4x^6 \cos^2 \frac{2\pi}{5} + 2x^8 \cos \frac{2\pi}{5} - x^{10} + \dots \end{aligned}$$

$$\Psi(x) = -1 + \left\{ \frac{1}{1-x^2} + \frac{x^5}{(1-x^2)(1-x^3)(1-x^7)} + \frac{x^{20}}{(1-x^2)(1-x^3)(1-x^7)(1-x^8)(1-x^{12})} + \dots \right\}.$$

Now,

$$\begin{aligned} & \frac{x}{1-x} + \frac{x^3}{(1-x^2)(1-x^3)} + \frac{x^5}{(1-x^3)(1-x^4)(1-x^5)} + \dots \\ &= 3\phi(x) + 1 - A(x). \end{aligned}$$

And,

$$\begin{aligned} & \frac{x}{1-x} + \frac{x^2}{(1-x^2)(1-x^3)} + \frac{x^3}{(1-x^3)(1-x^4)(1-x^5)} + \dots \\ &= 3\Psi(x) + xD(x). \end{aligned}$$

We assume without loss of generality that $n = 1$. Let $\zeta = \exp \frac{2\pi i}{5}$, then we may write the definitions of $F(x)$ and $f'(x)$ as;

$$F(x) = \frac{(x)_{\infty}}{(\zeta x)_{\infty} (\zeta^{-1} x)_{\infty}}$$

and

$$\begin{aligned} f'(x) &= 1 + \sum_{n=1}^{\infty} \frac{x^{n^2}}{(1-\zeta x)(1-\zeta^{-1} x) \dots (1-\zeta^n x)(1-\zeta^{-1} x^n)} \\ &= \sum_{n=1}^{\infty} \frac{x^{n^2}}{(\zeta x)_n (\zeta^{-1} x)_n}, \end{aligned}$$

where we have used the relations;

$$(a)_0 = 1, (a)_n = (1-a)(1-ax) \dots (1-ax^{n-1}), \text{ for } n \geq 1$$

and

$$(a)_{\infty} = \lim_{n \rightarrow \infty} (a)_n = \prod_{n=1}^{\infty} (1-ax^{n-1}).$$

After replacing x by $x^{\frac{1}{5}}$ we see that (1) and (2) are identities for $F(x)$ and $f'(x)$. We note that the numerators in the definitions of $A(x)$ and $D(x)$ are theta series in x and hence may be written as infinite products using Jacobi's triple product identity;

$$\begin{aligned} & \prod_{n=1}^{\infty} (1 - zx^n)(1 - z^{-1}x^{n-1})(1 - x^n) \\ &= \prod_{n=-\infty}^{\infty} (-1)^n z^n x^{\frac{n(n+1)}{2}} \end{aligned} \tag{3}$$

$$= \dots + z^{-2}x - z^{-1} + 1 - zx + z^2x^3 - \dots \infty .$$

where $z \neq 0$ and $|x| < 1$.

Replacing x by x^5 and z by x^{-3} we get from (3);

$$\prod_{n=1}^{\infty} (1 - x^{5n-3})(1 - x^{5n-2})(1 - x^{5n})$$

$$= \dots + x^{11} + 1 - x^2 + x^9 - \dots \infty$$

$$= 1 - x^2 - x^3 + x^9 + x^{11} - \dots \infty .$$

Again replacing x by x^5 and z by x^{-3} (3) becomes;

$$\prod_{n=1}^{\infty} (1 - x^{5n-4})(1 - x^{5n-1})(1 - x^{5n})$$

$$= \dots + x^{13} - x^4 + 1 - x + x^7 - \dots \infty$$

$$= 1 - x - x^4 + x^7 + x^{13} - \dots \infty$$

In fact we have;

$$A(x) = \prod_{n=1}^{\infty} \frac{(1 - x^{5n-3})(1 - x^{5n-2})(1 - x^{5n})}{(1 - x^{5n-4})^2(1 - x^{5n-1})^2}$$

$$B(x) = \prod_{n=1}^{\infty} \frac{(1 - x^{5n})}{(1 - x^{5n-4})(1 - x^{5n-1})}$$

$$C(x) = \prod_{n=1}^{\infty} \frac{(1 - x^{5n})}{(1 - x^{5n-3})(1 - x^{5n-2})}$$

$$D(x) = \prod_{n=1}^{\infty} \frac{(1 - x^{5n-4})(1 - x^{5n-1})(1 - x^{5n})}{(1 - x^{5n-3})(1 - x^{5n-2})}$$

3.1 Rank of a Partition

The rank of a partition is defined as the largest part minus the number of parts. Thus the partition $6 + 5 + 2 + 1 + 1 + 1 + 1$ of 17 has rank, $6 - 7 = -1$ and the conjugated partition, $7 + 3 + 2 + 2 + 2 + 1$ has rank, $7 - 6 = 1$. i.e., the rank of a partition and that of the conjugate partition differ only in sign. The rank of a partition of 5 belongs to any one of the residues (mod 5) and we have exactly 5 residues. There is similar result for all partitions of 7 leading to (mod 7).

The generating function for the rank is of the form [3];

$$\sum_{n=1}^{\infty} (-1)^{n-1} x^{\frac{n}{2}(3n-1)+|m|n} (1 - x^n) \prod_{j=1}^{\infty} (1 - x^j)^{-1}$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \left\{ x^{\frac{n}{2}(3n+2+|m|-1)} - x^{\frac{n}{2}(3n+2+|m|+1)} \right\} \sum_{k=0}^{\infty} P(k) x^k$$

$$= (x^{|m|+1} + 0 \cdot x^{|m|+2} + x^{|m|+3} + \dots \infty) - (x^{2|m|+5} + x^{2|m|+6} + \dots \infty)$$

$$= \sum_{n=0}^{\infty} N(m, n) x^n .$$

The generating function for $N(m, t, n)$ is of the form;

$$\sum_{\substack{n=-\infty \\ n \neq 1}}^{\infty} (-1)^n x^{\frac{n}{2}(3n+1)} \frac{(x^{mn} + x^{n(t-m)})}{1 - x^m} \prod_{j=1}^{\infty} (1 - x^j)^{-1}$$

$$= \sum_{\substack{n=-\infty \\ n \neq 1}}^{\infty} (-1)^n x^{\frac{n}{2}(3n+1)} (x^{mn} + x^{n(t-m)}) \times$$

$$(1 + x^m + x^{2m} + \dots \infty) \sum_{k=0}^{\infty} P(k) x^k$$

$$= \sum_{n=0}^{\infty} N(m, t, n) x^n ;$$

which shows that all the coefficients of x^{-n} (where n is any positive integer) are zero.

Now we define the generating function;

$$r_a(d) \text{ for } N(a, t, tn + d)$$

where $r_a(d) = r_a(d, t) = \prod_{n=0}^{\infty} N(a, t, tn + d) x^n$, and

$$r_{a,b}(d) = r_{a,b}(d, t) = r_a(d) - r_b(d).$$

$$= \prod_{n=0}^{\infty} \{N(a, t, tn + d) - N(b, t, tn + d)\} x^n .$$

The generating function $\phi(x)$ is of the form;

$$\phi(x) = -1 + \left\{ \frac{1}{1-x} + \frac{x^5}{(1-x)(1-x^4)(1-x^6)} + \frac{x^{20}}{(1-x)(1-x^4)(1-x^6)(1-x^9)(1-x^{11})} + \dots \infty \right\} ,$$

$$= -1 + (1 + x + x^2 + \dots \infty) + x^5 (1 + x + x^2 + \dots \infty)$$

$$\begin{aligned} & (1+x^4+\dots\infty)(1+x^6+\dots\infty)+\dots\infty \\ & = x+x^2+x^3+x^4+2x^5+2x^6+2x^7+2x^8+\dots\infty \\ & = \sum_{n=0}^{\infty} \{N(1,5,5n)-N(2,5,5n)\}x^n \\ & = r_{1,2}(0). \end{aligned}$$

The generating function $A(x)$ is defined as;

$$\begin{aligned} A(x) &= \frac{1-x^2-x^3+x^9+\dots\infty}{(1-x)^2(1-x^4)^2(1-x^6)^2\dots\infty} \\ &= (1-x^2-x^3+x^9+\dots\infty)(1+2x+3x^2+\dots\infty) \\ & (1+2x^4+3x^8+\dots\infty)\dots\infty \\ &= 1+2x+2x^2+x^3+2x^4+\dots\infty \\ &= 1+\sum_{n=0}^{\infty} \{N(0,5,5n)-N(2,5,5n)+ \\ & N(1,5,5n)-2N(2,5,5n)\}x^2 \\ &= 1+\sum_{n=0}^{\infty} \{N(0,5,5n)-N(2,5,5n)\}x^n + \\ & 2\sum_{n=0}^{\infty} \{N(1,5,5n)-N(2,5,5n)\}x^n \\ &= 1+r_{0,2}(0)+2r_{1,2}(0). \end{aligned}$$

The generating function is of the form;

$$\begin{aligned} & \prod_{n=1}^{\infty} \frac{1-x^{5n}}{(1-x^{5n-4})(1-x^{5n-1})} \\ &= \prod_{n=1}^{\infty} (1-x^{5n})(1+x^{5n-4}+\dots\infty)(1+x^{5n-1}+\dots\infty) \\ &= (1-0)+(3-2)x+(12-11)x^2+x^3+2x^4+\dots\infty \\ &= \sum_{n=0}^{\infty} \{N(0,5,5n+1)-N(2,5,5n+1)\}x^n \end{aligned}$$

$$= r_{0,2}(1).$$

The generating function is of the form;

$$\begin{aligned} & \prod_{n=1}^{\infty} \frac{1-x^{5n}}{(1-x^{5n-3})(1-x^{5n-2})} \\ &= \prod_{n=1}^{\infty} (1-x^{5n})(1+x^{5n-3}+x^{10n-6}+\dots\infty) \\ &= (1-0)+(3-3)x+(16-15)x^2+\dots\infty \\ &= \sum_{n=0}^{\infty} \{N(0,5,5n+2)-N(2,5,5n+2)\}x^n \times \\ & (1+x^{5n-2}+x^{10n-4}+\dots\infty) \\ &= r_{1,2}(2). \end{aligned}$$

The generating function $\Psi(x)$ is of the form;

$$\begin{aligned} \Psi(x) &= -1 + \left\{ \frac{1}{1-x^2} + \frac{x^5}{(1-x^2)(1-x^3)(1-x^7)} + \right. \\ & \left. \frac{x^{20}}{(1-x^2)(1-x^3)(1-x^7)(1-x^8)(1-x^{12})} + \dots\infty \right\} \\ &= -1 + (1+x^2+x^4+\dots\infty) + x^5(1+x^2+\dots\infty) \times \\ & (1+x^3+x^6+\dots\infty)(1+x^7+\dots\infty) + \dots\infty \\ &= x^2+x^4+x^6+x^7+2x^8+x^9+2x^{10}+\dots\infty. \end{aligned}$$

Hence

$$\begin{aligned} \frac{\Psi(x)}{x} &= x+x^3+x^4+x^5+x^6+2x^7+x^8+2x^9+\dots\infty \\ &= \sum_{n=0}^{\infty} \{N(2,5,5n+3)-N(0,5,5n+3)\}x^n \\ &= r_{2,0}(3) \end{aligned}$$

and

$$r_{0,2}(3) = -\frac{\Psi(x)}{x}.$$

The generating function $D(x)$ is of the form;

$$\begin{aligned} D(x) &= \frac{1-x-x^4+x^7+\dots}{(1-x^2)^2(1-x^3)^2(1-x^7)^2\dots} \\ &= (1-x-x^4+x^7+\dots)(1+2x^2+3x^4+\dots) \\ &\quad (1+2x^3+\dots)(1+2x^7+\dots)\dots \\ &= 1-x+2x^2+0.x^3+\dots \\ &= \sum_{n=0}^{\infty} \{N(0,5,5n+3) - N(1,5,5n+3) + \\ &\quad N(0,5,5n+3) - N(2,5,5n+3)\} x^n \\ &= \sum_{n=0}^{\infty} \{N(0,5,5n+3) - N(1,5,5n+3)\} x^n + \\ &\quad \sum_{n=0}^{\infty} \{N(0,5,5n+3) - N(2,5,5n+3)\} x^n \\ &= r_{0,1}(3) + r_{0,2}(3). \end{aligned}$$

3.2 Mock Theta Conjectures

The generating function for $\rho_0(n)$ is of the form [5];

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(x^{n+1})_{n+1}} \\ &= \frac{x}{1-x} + \frac{x^3}{(1-x^2)(1-x^3)} + \frac{x^5}{(1-x^3)(1-x^4)(1-x^5)} + \dots \\ &= x(1+x+x^2+\dots) + x^3(1+x^2+x^4+\dots) \times \\ &\quad (1+x^3+x^6+\dots) + \dots \\ &= x+x^2+2x^3+x^4+3x^5+2x^7+\dots \\ &= \sum_{n=0}^{\infty} \rho_0(n) x^n, \tag{4} \end{aligned}$$

which is convenient to define $\rho_0(0) = 0$

Now we prove the Theorem, which is known as First Mock Theta Conjecture.

Theorem 1: $N(1,5,5n) = N(0,5,5n) + \rho_0(n)$, where $\rho_0(n)$ is the number of partitions of n with unique smallest part and all other parts \leq the double of the smallest part.

Proof: From (4) we have;

$$\begin{aligned} &\frac{x}{1-x} + \frac{x^3}{(1-x^2)(1-x^3)} + \frac{x^5}{(1-x^3)(1-x^4)(1-x^5)} + \dots \\ &= \sum_{n=0}^{\infty} \rho_0(n) x^n \\ &\Rightarrow x+x^2+2x^3+x^4+3x^5+2x^6+\dots \\ &= \sum_{n=0}^{\infty} \rho_0(n) x^n \\ &\Rightarrow 3\phi(x) + 1 - A(x) = \sum_{n=0}^{\infty} \rho_0(n) x^n \text{ (by above)} \\ &3r_{1,2}(0) + 1 - (1+r_{0,2}(0)+r_{1,2}(0)) = \sum_{n=0}^{\infty} \rho_0(n) x^n \text{ (by above)} \\ &\Rightarrow r_{1,2}(0) - r_{0,2}(0) = \sum_{n=0}^{\infty} \rho_0(n) x^n \text{ (by above)} \\ &\Rightarrow \sum_{n=0}^{\infty} \{N(1,5,5n) - N(2,5,5n) + \\ &\quad N(0,5,5n) - N(2,5,5n)\} x^n = \sum_{n=0}^{\infty} \rho_0(n) x^n \\ &\Rightarrow \sum_{n=0}^{\infty} \{N(1,5,5n) - N(0,5,5n)\} x^n = \sum_{n=0}^{\infty} \rho_0(n) x^n. \end{aligned}$$

Equating the coefficient of x^n on both sides, we get

$$N(1,5,5n) = N(0,5,5n) + \rho_0(n). \text{ Hence the Theorem.}$$

The generating function for $\rho_1(n)$ is defined as;

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