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Non-Associativity of Lorentz Transformation and Associative Reciprocal Symmetric Transformation

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ABSTRACT

Lorentz transformation is not associative. The non-associativity makes it frame dependent; and it does not fulfill relativistic requirements including reciprocity principle. The non associativity also leads to ambiguities when three or more velocities are involved. We have proposed an associative Reciprocal Symmetric Transformation (RST) to replace Lorentz transformation. RST is complex and is compatible with Pauli and Dirac algebra.

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Keywords: Lorentz boost; Non-associativity; Reciprocity Principle; Reciprocal Symmetric Transformation; Thomas Precession; Quaternionic transformation; Pauli quaternion; Spin; Clifford algebra

1 Introduction

We have shown[1] that Lorentz-Einstein law of addition of velocities is not associative and that this law gives a frame dependent relative velocity. Frame dependence contradicts[2] the principle of relativity. Another implication of

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non-associativity is that it fails[3] to fulfill reciprocity principle; when 2 or more velocities are involved velocity **BC** (Fig.1) is not[4] the negative of velocity **CB** (Fig.2). Ungar[5] recognizes that Einstein's law is "neither commutative nor associative"[4], but he attributes the failure of reciprocity principle to "the non-commutativity of the relativistic composition of non-collinear velocities"[4].

Wigner[6] and Moller[7] have tried to justify this failure by in invoking Thomas Precession. Much later Oziewicz wrote, "There have been attempts[8] [Ungar 1988] to explain the non-associativity, and also Mocanu paradox, as the Thomas rotation, i.e. as non-transitivity of the parallelism of the spatial frames. We consider this attempt not satisfactory."[9]

Ungar[10] has introduced gyrovectors to make Einstein's law of addition both commutative and associative. The attempt is not satisfactory because:

(i) Non-commutativity is not a physical requirement. To try to make it commutative is based on wrong diagnosis.

(ii) Inclusion of gyrovectors involves a series of arbitrary prescriptions.

As Ungar[11] has observed, "The non-associativity of Einstein's velocity addition is not widely known". The popular belief remains that Einstein's law of addition of velocities is associative, although it is an exercise in elementary algebra to prove non-associativity (11)

Lorentz transformation may be put in a matrix form and matrix algebra is associative. This has misled some people to think that Lorentz transformation is associative. (This is clarified in Appendix A in section 10).

In this paper we intend to reiterate the non-associativity of Einstein's law of addition of velocities and to show that (space-time) Lorentz transformation is also not associative.

We want to present Reciprocal Symmetric Transformation (RST), in place of LT as the solution. RST is complex. This is explained by fact that it obeys Pauli quaternion algebra and is conform to Dirac electron theory which also obeys Pauli quaternion algebra. The complex nature of RST makes it compatible with quantum mechanics.

2. Failure of Reciprocity

Consider three moving bodies **A**, **B** and **C** in relative motion with velocities **u**,

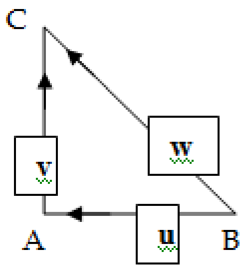

Fig.1

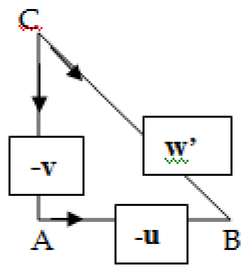

Fig.2

v and **w** as shown in Fig.1 We add (obeying Lorentz-Einstein law) velocities **u** and **v** to get $\mathbf{w}_L$ where

$$\mathbf{w}_L = \mathbf{u} \oplus_L \mathbf{v} = \frac{\mathbf{u}+\mathbf{v}}{1+\mathbf{u}\cdot\mathbf{v}/c^2} + \frac{1}{c^2}\frac{\gamma_u}{\gamma_u+1}\frac{\mathbf{u}\times(\mathbf{u}\times\mathbf{v})}{1+\mathbf{u}\cdot\mathbf{v}/c^2} \tag{1}$$

$$\gamma_u = \frac{1}{\sqrt{1-(u/c)^2}} \tag{2}$$

$\oplus_L$ stand for Lorentz-Einstein addition. In Fig.2 we add velocities **–v** and **–u** to get $\mathbf{w}'_L$

$$\mathbf{w}'_L = (-\mathbf{v}) \oplus_L (-\mathbf{u}) \tag{3}$$

Lorentz-Einstein law of addition (1) gives

$$\mathbf{w}_L \neq \mathbf{w}'_L \tag{4}$$

(4) contradicts the principle of reciprocity required by the principle of relativity. Some[12] authors have called the inequality a paradox and have attributed it to "The non-commutativity and the non-associativity of the composition law of the non-collinear velocities"[13]. Some[4] others have

attributed it to non-commutativity only. We shall show below that non commutativity does not contribute to it; the difficulty is due to non-associativity only.

3. Frame Dependence

Let $\mathbf{u}$ be the velocity of a moving body. An observer moving with velocity $\mathbf{v}$ observers the relative velocity

$$\mathbf{z}_L = (-\mathbf{v}) \oplus_L \mathbf{u} \tag{5}$$

With respect to a second observer moving with velocity $\mathbf{y}$, $\mathbf{u}$ and $\mathbf{v}$ become

$$\mathbf{u}' = (-\mathbf{y}) \oplus_L \mathbf{u} \text{ and } \mathbf{v}' = (-\mathbf{y}) \oplus_L \mathbf{v} \tag{6}$$

The relative velocity $\mathbf{z}_L$ becomes

$$\mathbf{z}'_L = (-\mathbf{v}') \oplus_L \mathbf{u}' \tag{7}$$

Using (1) we find[5]

$$\mathbf{z}'_L \neq \mathbf{z}_L \tag{8}$$

Eq. (8) shows that relative motion is frame dependent and that (1) is not relativistic.

4. Ambiguity

Consider a body moving with velocity $\mathbf{u}$. An observer with velocity $\mathbf{v}$ observers the velocity $\mathbf{u}' = (-\mathbf{v}) \oplus_L \mathbf{u}$. A second observer is traveling with velocity $\mathbf{y}$ and

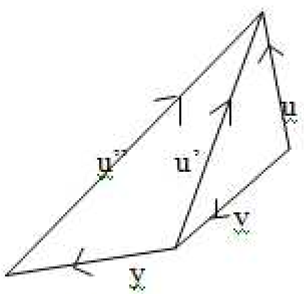

Fig.3

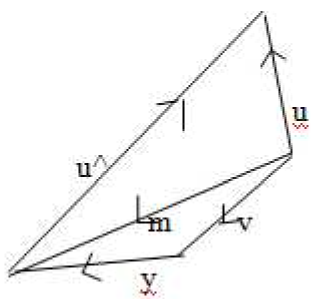

Fig.4

observers the velocity $\mathbf{u}'' = (-\mathbf{y}) \oplus_L \mathbf{u}'$. This situation is represented by Fig.3 and eq. (9). Fig.4 and eq. (10) represents the situation in which velocities **v** and **y** are added first to give **m;** $\mathbf{m} = \mathbf{v} \oplus_L \mathbf{y}$. The observer moving with velocity **m** observers $\mathbf{u}\hat{}$

$$\mathbf{u}'' = (-\mathbf{y}) \oplus_L \mathbf{u}' = (-\mathbf{y}) \oplus_L \{(-\mathbf{v}) \oplus_L \mathbf{u}\} \tag{9}$$

$$\mathbf{u}\hat{} = (-\mathbf{m}) \oplus_L \mathbf{u} = \{-(\mathbf{v} \oplus_L \mathbf{y})\} \oplus_L \mathbf{u} \tag{10}$$

The inequality

$$\mathbf{u}'' \neq \mathbf{u}\hat{} \tag{11}$$

shows an ambiguity in the computation of the resultant velocity when 3 or more velocities are added.

5. Diagnosis and the Solution of the Problem

1. *Intuitive approach*

In the first triangle **u** comes first then **v** and they add up to **w**. In the second triangle **-v** comes first then **-u** and they add up to $\mathbf{w}'$. Instead of (4) we need

$\mathbf{w} = -\mathbf{w}'$

$$\mathbf{w} = -\mathbf{w}' \quad \text{or} \quad (-\mathbf{v}) \oplus (-\mathbf{u}) = -(\mathbf{u} \oplus \mathbf{v}) \tag{12}$$

In (12) we have used the notation $\oplus$ (without suffix *L*) to represent an associative addition.

2. *Matrix Approach*

Let $\mathbf{M}(\mathbf{u})$ be the matrix which takes Σ to Σ' i.e. $\mathbf{M}(\mathbf{u})\Sigma = \Sigma'$ and $\mathbf{M}(-\mathbf{u})\Sigma' = \Sigma$. Then

$$\mathbf{M}(-\mathbf{u})\mathbf{M}(\mathbf{u}) = \mathbf{M}^{-1}(\mathbf{u})\mathbf{M}(\mathbf{u}) = \mathbf{M}(0) = \mathbf{1} \text{ with } \mathbf{M}(-\mathbf{u}) = \mathbf{M}^{-1}(\mathbf{u}) \tag{13}$$

$\mathbf{M}(0) = \mathbf{1}$ is the identity matrix $\mathbf{M}(0)\Sigma = \Sigma$. By repeated application of the matrices we have

$$\mathbf{M}(\mathbf{v})\mathbf{M}(\mathbf{u})\Sigma = \mathbf{M}(\mathbf{v} \oplus \mathbf{u})\Sigma = \Sigma'' \tag{14}$$

Combining the matrices and their inverses we have

$$\mathbf{M}(-\mathbf{u})\mathbf{M}(-\mathbf{v})\mathbf{M}(\mathbf{v})\mathbf{M}(\mathbf{u})\Sigma = \mathbf{M}(-(\mathbf{v} \oplus \mathbf{u}))\mathbf{M}(\mathbf{v} \oplus \mathbf{u})\Sigma = \mathbf{M}(0)\Sigma = \Sigma \tag{14}$$

Or

$$\mathbf{M}(-\mathbf{u})\mathbf{M}(-\mathbf{v}) = \mathbf{M}((-\mathbf{u}) \oplus (-v)) = \mathbf{M}(-(\mathbf{v} \oplus \mathbf{u})) \tag{15}$$

(15) gives (12) corresponding to the familiar matrix rule $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

3. *Algebraic Approach*

Theorem:

If $\oplus$ stands for an associative addition

$$-(\mathbf{u} \oplus \mathbf{v}) = (-\mathbf{v}) \oplus (-\mathbf{u}) \tag{16}$$

Proof:

Using associativity of $\oplus$ and $\mathbf{v} \oplus (-\mathbf{v}) = 0$, we have

$$(\mathbf{u} \oplus \mathbf{v}) \oplus \{(-\mathbf{v}) \oplus (-\mathbf{u})\} = \mathbf{u} \oplus \{\mathbf{v} \oplus (-\mathbf{v})\} \oplus (-\mathbf{u}) = \mathbf{u} \oplus (-\mathbf{u}) = 0 \tag{17}$$

Therefore,

$$(\mathbf{u} \oplus \mathbf{v}) \oplus \{(-\mathbf{v}) \oplus (-\mathbf{u})\} = 0 \tag{18}$$

Also we have

$$\{\mathbf{u} \oplus \mathbf{v}\} \oplus \{-(\mathbf{u} \oplus \mathbf{v})\} = 0 \tag{19}$$

Comparison between (14) and (15) shows

$$-(\mathbf{u} \oplus \mathbf{v}) = (-\mathbf{v}) \oplus (-\mathbf{u}) \tag{20}$$

6. Pauli Quaternion

From 3-vectors **u**, **v** and **w** we construct 4-vectors

$$\mathbf{u} \Rightarrow (1.\sigma_0 + \mathbf{u}\cdot\boldsymbol{\sigma}) = (1.\sigma_0 + u_x\sigma_x + u_y\sigma_y + u_z\sigma_z) \tag{21}$$

Where σ s have the following properties[14] [Quaternion and Pauli Quaternion differ by a factor of i].

$$-\sigma_y\sigma_x = \sigma_x\sigma_y = i\sigma_z \text{ with cyclic permutations} \tag{22}$$

$$\sigma_0\sigma_x = \sigma_x\sigma_0 = \sigma_x \text{ and } \sigma_x\sigma_x = 1 \text{ and also for } y \text{ and } z \tag{23}$$

We define quantities [in sections 6 and 7 we shall set $c = 1$].

$$\mathbb{U} = \gamma_u(1.\sigma_0 + \mathbf{u}\cdot\boldsymbol{\sigma}) \tag{24}$$

Multiplying we get

$$\mathbb{U}\mathbb{V} = \gamma_u(1.\sigma_0 + \mathbf{u}\cdot\boldsymbol{\sigma}).\ \gamma_v(1.\sigma_0 + \mathbf{v}\cdot\boldsymbol{\sigma}) = \gamma_w(1.\sigma_0 + \mathbf{w}\cdot\boldsymbol{\sigma}) = \mathbb{W} \tag{24}$$

Where $$\mathbf{w} = \frac{\mathbf{u} + \mathbf{v} + i\mathbf{u}\times\mathbf{v}/c}{1 + \mathbf{u}\cdot\mathbf{v}/c^2} \text{ and } \gamma_u\gamma_v(1 + \mathbf{u}\cdot\mathbf{v}) = \gamma_w \tag{25}$$

w of (24) and (25) agrees with (12). Multiplication of σ s is associative. Therefore, $\oplus$ is associative. Also

$$\gamma_u(1.\sigma_0 - \mathbf{v}\cdot\boldsymbol{\sigma})\ .\gamma_u(1.\sigma_0 - \mathbf{u}\cdot\boldsymbol{\sigma}) = \gamma_w(1.\sigma_0 - \mathbf{w}\cdot\boldsymbol{\sigma}) \tag{26}$$

7. Pauli Quaternion in Matrix Form

Let σ s be matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{and } \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{27}$$

$1.\sigma_0 + \mathbf{u}\cdot\boldsymbol{\sigma} = 1.\sigma_0 + u_x\sigma_x + u_y\sigma_y + u_z\sigma_z$ then becomes a 2×2 matrix. $\mathbb{U}$ is now the matrix below

$$\mathbb{U} = \gamma_u(1.\sigma_0 + \mathbf{u}\cdot\boldsymbol{\sigma}) = \gamma_u\begin{pmatrix} 1 + u_z & u_x - iu_y \\ u_x + iu_y & 1 - u_z \end{pmatrix} \tag{28}$$

Corresponding to (24) we have

$$\mathbb{U}\mathbb{V} = \mathbb{W} \tag{29}$$

The inverse matrix corresponding to (28) is

$$U^{-1} = \gamma_u (1.\sigma_0 - \mathbf{u} \cdot \boldsymbol{\sigma}) \tag{28}$$

$$(W)^{-1} = (UV)^{-1} = V^{-1}U^{-1} \tag{30}$$

Note that the order is reversed as in (26). Lorentz transformation does not have this property.

8. Compatibility between RST and Dirac Theory

We write Dirac's relativistic equation as[15]

$$\left(E - c\boldsymbol{\alpha} \cdot \mathbf{p} - \beta mc^2\right)\psi = 0 \tag{31}$$

For the purpose of comparison with (21) we set $m = 0$ and we write (31) as

$$\left(E\sigma_0 - c\boldsymbol{\sigma} \cdot \mathbf{p}\right)\psi = 0 \tag{32}$$

We multiply on the left by $\left(E\sigma_0 + c\boldsymbol{\sigma} \cdot \mathbf{p}\right)$ to get

$$\left(E\sigma_0 + c\boldsymbol{\sigma} \cdot \mathbf{p}\right)\left(E\sigma_0 - c\boldsymbol{\sigma} \cdot \mathbf{p}\right)\psi = \left(E^2 - (c\mathbf{p})^2\right)\psi \tag{33}$$

(33) will be correct if σ s of (32) have the properties of (22) and (23). To see the correspondence to spin consider the product of type

$$\left(b\sigma_0 + \boldsymbol{\sigma} \cdot \mathbf{B}\right).\left(c\sigma_0 + \boldsymbol{\sigma} \cdot \mathbf{C}\right) = bc + \mathbf{B} \cdot \mathbf{C} + \boldsymbol{\sigma} \cdot \left(b\mathbf{C} + c\mathbf{B} + i\mathbf{B} \times \mathbf{C}\right) \tag{34}$$

In the presence of electromagnetic field[16], in Dirac theory, we find term like[14]

$$\left(\boldsymbol{\sigma} \cdot \mathbf{B}\right).\left(\boldsymbol{\sigma} \cdot \mathbf{C}\right) = \mathbf{B} \cdot \mathbf{C} + i\boldsymbol{\sigma} \cdot \left(\mathbf{B} \times \mathbf{C}\right) \tag{35}$$

We get (35) from (34) by setting $b = c = 0$. The last term of (34) or (35) gives spin[17]. Therefore, the imaginary cross term of (25) corresponds to spin[18].

9. Conclusion

We have seen that failure of reciprocity and ambiguity are consequence of the non-associativity of Einstein's law of addition of velocities.

Reciprocal Symmetric Transformation (RST) we are proposing is associative. RST is Clifford algebraic, and spin is innate in it as in Dirac equation.

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APPENDIX

Non-Associativity and Matrix Representation

Consider the non-associativity of Einstein's addition relation

$$\mathbf{y} \oplus_L (\mathbf{v} \oplus_L \mathbf{u}) \neq (\mathbf{y} \oplus_L \mathbf{v}) \oplus_L \mathbf{u} \tag{36}$$

We want to see the matrix representation corresponding to (36). Consider the quantity $\not{y}$ (y slash) defined by

$$\not{y} = \frac{1+\mathbf{y}/c}{\sqrt{1-(y/c)^2}} = \gamma_y(1+\mathbf{y}/c) \tag{37}$$

$$\not{y} \otimes_L \not{v} = \gamma_m(1+\mathbf{m}/c) = \not{m} \tag{38}$$

where $\mathbf{m}$ and γ_m are given by

$$\mathbf{m} = \mathbf{y} \oplus_L \mathbf{v} = \frac{\mathbf{y} + \mathbf{v}/\gamma_y + \{1-1/\gamma_y\}\frac{\mathbf{v.y}}{\mathbf{y}^2}\mathbf{y}}{1+\mathbf{y}\cdot\mathbf{v}/c^2} \quad \text{and} \quad \gamma_m = \gamma_y.\gamma_v(1+\mathbf{v}\cdot\mathbf{y}/c^2) \tag{39}$$

To go to matrix form, from the 4 vector $\not{u}$ we form the column vectors

$$\not{u} \to \gamma_u \begin{pmatrix} 1 \\ \mathbf{u}/c \end{pmatrix} = \not{u} \tag{40}$$

$\not{u}$ may, therefore, stand for the row vector as in (37) as well as for the column as in (40). We hope this ambiguity will not be a problem. With $\not{y}$ and $\not{v}$ columns, to execute (38), we define, from $\not{y} \otimes_L$, the matrix

$$\not{y} \otimes_L = \gamma_y \begin{pmatrix} 1 \\ \mathbf{y}/c \end{pmatrix} \otimes_L \equiv \gamma_y \begin{pmatrix} 1 & \mathbf{y}^{\mathrm{T}}/c \\ \mathbf{y}/c & 1/\gamma_y + \{1-1/\gamma_y\}\frac{\mathbf{y}}{y^2}\mathbf{y}^{\mathrm{T}} \end{pmatrix} \tag{41}$$

Then (38) gives

$$\not{y} \otimes_L \not{v} = \gamma_y\gamma_v \begin{pmatrix} 1 & \mathbf{y}^{\mathrm{T}}/c \\ \mathbf{y}/c & 1/\gamma_y + \{1-1/\gamma_y\}\frac{\mathbf{y}}{y^2}\mathbf{y}^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} 1 \\ \mathbf{v}/c \end{pmatrix} = \not{m} = \gamma_m \begin{pmatrix} 1 \\ \mathbf{m}/c \end{pmatrix} \tag{42}$$

We now go to calculate

$$\not{y} \otimes_L (\not{v} \otimes_L \not{u}) = \not{u}'' \tag{43}$$

and

$$(\not{y} \otimes_L \not{v}) \otimes_L \not{u} = \not{m} \otimes_L \not{u} = \not{u}\hat{} \tag{44}$$

Using (41) we have from (43) (dropping γ_y, γ_v and γ_u)

$$\begin{pmatrix} 1 & \mathbf{y}^{\mathrm{T}}/c \\ \mathbf{y}/c & 1/\gamma_y + \{1-1/\gamma_y\}\dfrac{\mathbf{y}}{y^2}\mathbf{y}^{\mathrm{T}} \end{pmatrix} \left\{ \begin{pmatrix} 1 & \mathbf{v}^{\mathrm{T}}/c \\ \mathbf{v}/c & 1/\gamma_v + \{1-1/\gamma_v\}\dfrac{\mathbf{v}}{v^2}\mathbf{v}^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} 1 \\ \mathbf{u}/c \end{pmatrix} \right\} = \begin{pmatrix} 1 \\ \mathbf{u}''/c \end{pmatrix} \tag{45}$$

Since matrix multiplication is associative, we may re-arrange and re-write (45) as below

$$\left\{ \begin{pmatrix} 1 & \mathbf{y}^{\mathrm{T}}/c \\ \mathbf{y}/c & 1/\gamma_y + \{1-1/\gamma_y\}\dfrac{\mathbf{y}}{y^2}\mathbf{y}^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} 1 & \mathbf{v}^{\mathrm{T}}/c \\ \mathbf{v}/c & 1/\gamma_v + \{1-1/\gamma_v\}\dfrac{\mathbf{v}}{v^2}\mathbf{v}^{\mathrm{T}} \end{pmatrix} \right\} \begin{pmatrix} 1 \\ \mathbf{u}/c \end{pmatrix} = \begin{pmatrix} 1 \\ \mathbf{u}''/c \end{pmatrix} \tag{46}$$

(45) and (46) both represent (43), since matrix multiplication is associative.

We now go to find (44). Using (42) and (41) we have

$$\not{m} \otimes_L \not{u} = \gamma_m \gamma_u \begin{pmatrix} 1 & \mathbf{m}^{\mathrm{T}}/c \\ \mathbf{m}/c & 1/\gamma_m + \{1-1/\gamma_m\}\dfrac{\mathbf{m}}{m^2}\mathbf{m}^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} 1 \\ \mathbf{u}/c \end{pmatrix} = \gamma_{u\hat{}} \begin{pmatrix} 1 \\ \mathbf{u}\hat{}/c \end{pmatrix} \tag{47}$$

where $\mathbf{u}\hat{} = \mathbf{m} \oplus_L \mathbf{u}$ corresponds to the right hand side of (36). $\not{u}'' \neq \not{u}\hat{}$ since

$$\gamma_m \begin{pmatrix} 1 & \mathbf{m}^{\mathrm{T}}/c \\ \mathbf{m}/c & 1/\gamma_m + \{1_m - 1/\gamma_m\}\dfrac{\mathbf{m}}{m^2}\mathbf{m}^{\mathrm{T}} \end{pmatrix} \neq \gamma_y \gamma_v \begin{pmatrix} 1 & \mathbf{y}^{\mathrm{T}}/c \\ \mathbf{y}/c & 1/\gamma_y + \{1-1/\gamma_y\}\dfrac{\mathbf{y}}{y^2}\mathbf{y}^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} 1 & \mathbf{v}^{\mathrm{T}}/c \\ \mathbf{v}/c & 1/\gamma_v + \{1-1/\gamma_v\}\dfrac{\mathbf{v}}{v^2}\mathbf{v}^{\mathrm{T}} \end{pmatrix} \tag{48}$$

Inequality (48) is responsible for non-associativity $\not{u}'' \neq \not{u}\hat{}$.

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