CVA, FVA (and DVA?) with stochastic spreads. A feasible replication approach under realistic assumptions.

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Abstract

In this paper we explore the different components that should be incorporated in the price of uncollateralized [1] derivatives. We do so by putting special focus on the hedge-ability of every term. In order to reflect the most realistic situation, we assume stochastic credit spreads for both counterparties. In such a framework, the counterparty acting as the hedger will be concerned about market risk (movements in the price of the underlying asset), both sources of the credit risk of the investor (spread changes and default event) and also his own credit risk.

Regarding his own credit risk, we assume that the derivatives hedger has no incentive to hedge the change in value of the derivative upon his own default, since the hedger will not be exposed to this change in value. Nevertheless, we assume that the hedger has a strong incentive to hedge the changes in the derivative's price due to changes in his credit spread curve, which is a source of risk that the derivatives hedger will be exposed to during the replication process. We also suggest a hedging strategy for this risk factor (spread changes of the investor) as we do for the other (market risk, spread changes and default event of the issuer).

We conclude that under these assumptions CVA (a unilateral version of it that does not depend on the hedger's funding curve) and FVA (a funding adjustment that does only depend on the investor's default indicator and not on the hedger's) are the only components to be incorporated in the price of financial derivatives.

*The opinions of this article are those of the author and do not reflect in any way the views or business of his employer.

[1] Although the results can be easily generalized to partially collateralized transactions.

Contents

1 Introduction

Over the last years, the finance community has come up with what could be considered one of the most controversial concepts of all time: that is DVA (debit value adjustment). While conceived by some as economically meaningful, since it contemplates counterparty credit risk in a symmetrical way, others argue that it is meaningless due to the fact that it cannot be hedged.

Apart from that, derivatives hedgers have also become concerned about funding costs (FCA) or benefits (FBA) incurred in the dynamic replication process. This has produced a big debate lately about which of this four components (CVA, DVA, FCA, FBA) should be incorporated into the pricing of financial derivatives.

Although the CVA/DVA/FVA debate has been previously analyzed from a replication perspective (see for example [2], [3]), [4]), the analysis is done under the assumption of non stochastic spreads. Under this framework, the only source of risk regarding the derivative's hedger credit worthiness is the default event, therefore only attention is paid on the hedging of the cashflow experienced by the hedger upon his time of default. In trying to hedge this default event, sometimes it is necessary for the hedger to get additional funding to purchase debt issued by himself in order to neutralize the positive cashflow produced by the default event. Nevertheless this situation is a little bit cumbersome, since this additional funding also represents debt issued by the hedger and also has a jump to default component that is ignored and that leaves the jump to default unhedged. Another possibility mentioned is for the hedger to trade on senior and junior debt issued by himself, although this assumption seems unrealistic. Apart from that, in this framework, no attention is paid on how the hedger can neutralize changes in the derivative's value due to changes in the hedger's credit spread which is something that the hedger will be exposed to on a continuous basis no matter whether the default event happens or not.

Regarding the hedging of the hedger's default event, let's first analyze from a pure practical perspective the real incentive to hedge it. The hedging of this component implies that the hedger could have to sell protection on himself. The CDS counterparty used to hedge this component will require some sort of over collateralization. This overcollateralization can accelerate the default and will for sure reduce the recovery left for the bondholders since the CDS counterparty will have priority on the assets posted as collateral, so that there is no incentive to hedge it for either equity holders or bondholders of the hedging firm.

Nevertheless it seems meaningful to include a term in the price of uncollateralized derivatives that reflects the hedger creditworthiness. In order to come up with the expression of this term, we will put special emphasis on it to be hedgeable to changes of the hedger's credit spread, although no attemp will be made in order to hedge the jump in the derivative's price upon the default of the hedger. The approach followed will imply that, contrary to what is suggested by DVA supporters, the price of an uncollateralized derivative will depend on which of the two counterparties acts as the hedger and which as the investor [2].

We will make the following assumptions in order to obtain the different components to be included in the price of uncollateralized derivatives:

- The price of a derivative should reflect all of its hedging costs.
- Since nowadays a very high percentage (if not all) of uncollateralized transactions imply a counterparty acting as an investor (risk taker) and a hedger (risk hedger), the derivative's price should just reflect the hedging costs borne by the hedger.
- The hedger will only be willing to hedge the fluctuations in the derivative's price that he will experience while being alive, that is, while not having defaulted.
- There is neither CVA nore FVA to be made to fully collateralized derivatives (with continuous collateral margining in cash, symmetrical collateral mechanisms and no thresholds, minimum transfer amounts, ...).

Market assumptions:

- There is a liquid CDS (credit default swap) curve for the investor.
- There is a liquid curve of bonds issued by the hedger.
- Continuous hedging is possible, unlimited liquidity, no bid-offer spreads, no trading costs.
- Recovery rates are either deterministic or there are recovery locks available so that recovery risk is not a concern.

Model assumptions:

- Both the hedger and the investor are defaultable. Simultaneous default is not possible.
- The underlying asset follows a difusion process under the real world measure.
- The derivative's underlying asset is unaffected by the default event of any of the counterparties.

[2] Notice that the same happens with any manufactured product. That is, the price of a car reflects the manufacturing costs of the car manufacturer and has nothing to do with the manufacturing cost of the car buyer if he was to build his own car.

- Both the credit spreads of the investor and of the hedger are stochastic following correlated diffusion processes under the real world measure.

Since funding costs arise due to asymmetries between the collateral characteristics of derivatives traded with investors (that could be uncollateralized or partially collateralized) and those of the hedging instruments (usually traded in the interbank market, where deals are fully collateralized), in order to incorporate funding costs we will assume this situation. That is, an uncollateralized derivative (the one that the hedger trades with the investor) is hedged with a generic derivative collateralized in cash. In order to simplify the algebra we will assume that interest rates are not stochastic, although the results achieved are also valid under stochastic interest rates.

The structure of the paper is as follows:

- In section 4 we will explore the different risks that will impact the replication price of financial derivatives. We will discuss which of them can generally be hedged by the derivative's hedger and which will be experienced by the hedger while not having defaulted. We will see that the same risks that can be hedged are the only ones that will be experienced by the hedger while not having defaulted.
- In section 3 we will describe the replication strategy to be followed by the derivatives hedger in order to be immune to changes in the different risk factors. The replication strategy will lead to the partial differential equation (PDE) followed by the derivative's price, which will help us to identify the different components to be incorporated in it.
- In section 4 we compare DVA hedging vs FVA hedging in a simplified framework.
- In section 5 we summarize the main conclusions.
- In appendix A we review the PDE followed by any credit derivative. We do so by analyzing the hedging of both jump to default and spread risks. We distinguish between credit derivatives collateralized in cash and bonds that can be repoed.

2 The hedgeable risks

We will assume that under the real world measure $\mathbb{P}$, the evolution of the relevant market variables (price of the derivative's underlying asset and credit spreads of the investor and the hedger) are governed by the following stochastic differential equations:

$$\begin{aligned} dS_t &= \mu_t^S S_t dt + \sigma_t^S S_t dW_t^{S,\mathbb{P}} \\ dh_t^I &= \mu_t^I dt + \sigma_t^I dW_t^{I,\mathbb{P}} \\ dh_t^H &= \mu_t^H dt + \sigma_t^H dW_t^{H,\mathbb{P}} \end{aligned} \tag{1}$$

Where S_t represents the price of the derivative's underlying asset at time t, h_t^I the short term CDS spread of the investor, h_t^H the short term CDS spread of the derivative's hedger, $\mu_t^S,\ \mu_t^I,\ \mu_t^H$ the real world drifts of the 3 processes and $\sigma_t^S(t,S_t),\ \sigma_t^I(t,h_t^I),\ \sigma_t^H(t,h_t^H)$ their volatilities. $W_t^{S,\mathbb{P}},\ W_t^{I,\mathbb{P}},\ W_t^{H,\mathbb{P}}$ are brownian motions under the real world measure $\mathbb{P}$.

We will assume that the 3 processes are correlated with time dependent correlations:

$$\rho_t^{S,I}dt = dW_t^{S,\mathbb{P}}dW_t^{I,\mathbb{P}},\quad \rho_t^{H,I}dt = dW_t^{H,\mathbb{P}}dW_t^{I,\mathbb{P}},\quad \rho_t^{S,H}dt = dW_t^{S,\mathbb{P}}dW_t^{H,\mathbb{P}}$$

Notice that although we could have assumed a n-dimensional Heath Jarrow Morton model for credit spreads, we have assumed that the evolution of the credit curves is governed by one factor models in order to simplify the algebra.

The other two sources on uncertainty are the default indicator processes $N_t^{I,\mathbb{P}} = 1_{\{\tau_I \le t\}},\ N_t^{H,\mathbb{P}} = 1_{\{\tau_H \le t\}}$ with real world default intensities $\lambda_t^{I,\mathbb{P}},\ \lambda_t^{H,\mathbb{P}}$. Parameters associated with the investor will carry a superscript I whereas those of the hedger a superscript H. τ_I and τ_H will represent the default times of the investor and the hedger.

The cash flows that the derivative's hedger will face in the replication process will depend on each and every one of the sources of uncertainty $(S_t, h_t^I, h_t^H, N_t^{I,\mathbb{P}}, N_t^{H,\mathbb{P}})$. Therefore $V_t = V(t, S_t, h_t^I, h_t^H, N_t^{I,\mathbb{P}}, N_t^{H,\mathbb{P}})$ (V_t represents the derivative's value from the investor's perspective). Assuming that both the investor and the hedger have not defaulted by time t, the change in value from t to $t+dt$ experienced by V_t will be given by (applying Itô's Lemma for jump diffusion processes)

$$dV_t = \underbrace{\frac{\partial V_t}{\partial S_t}dS_t}_{\text{Delta risk}} + \underbrace{\frac{\partial V_t}{\partial h_t^I}dh_t^I}_{\text{Spread risk to I}} + \underbrace{\frac{\partial V_t}{\partial h_t^H}dh_t^H}_{\text{Spread risk to H}} + \underbrace{\Delta V_t^I dN_t^{I,\mathbb{P}}}_{\text{Default risk to I}} + \underbrace{\Delta V_t^H dN_t^{H,\mathbb{P}}}_{\text{Default risk to H}} + \underbrace{O(dt)}_{\text{Theta}} \tag{2}$$

ΔV_t^I represents the jump in the value of the derivative if default of the investor happened at time t and ΔV_t^H the jump if the hedger defaulted.

Of all the risk terms in (2) the hedger will only be exposed to $\frac{\partial V_t}{\partial S_t}dS_t$, $\frac{\partial V_t}{\partial h_t^I}dh_t^I$, $\frac{\partial V_t}{\partial h_t^H}dh_t^H$ and $\Delta V_t^I dN_t^{I,\mathbb{P}}$. Keep in mind that $\Delta V_t^H dN_t^{H,\mathbb{P}}$ is conditional on the hedger having defaulted. Since the hedger will not be there to experience the change in value of the derivative, there will be no incentive at all to hedge it.

Nevertheless we will analyze whether each one of the components of (2) can actually be hedged:

- $\frac{\partial V_t}{\partial S_t} dS_t$: This component can be hedged by trading in a fully collateralized derivative written on the underlying asset. Therefore the hedger will be able to hedge the market risk component without having a net cash flow [3].
- $\frac{\partial V_t}{\partial h^I_t} dh^I_t$ and $\Delta V^I_t dN^{I,\mathbb{P}}_t$: In order to be hedged to both spread and default risks of the investor, the hedger will have to trade in two credit default swaps written on the investor with different maturities. Notice that this is because we have assumed a one factor model for the evolution of the credit spread curve. Had we assumed an n factor model, then the hedger would have to trade in CDSs with $n+1$ different maturities. If we assume that the investor is not perceived by the market as correlated with the hedger, the hedger will be able to either buy or sell protection on the investor. Notice that this hedging component will imply a zero net cash flow, since CDSs are collateralized market instruments.
- $\frac{\partial V_t}{\partial h^H_t} dh^H_t$ and $\Delta V^H_t dN^{H,\mathbb{P}}_t$ altogether: The hedger will have to trade on two different credit instruments written on himself (or $n+1$ under a n factor model for the evolution of its credit curve). In general he will have to go long or short its own credit risk. Since the market will never be willing to buy protection written on the hedger from the hedger, the hedging of this two components will have to be done by trading on the hedger's own debt. Notice that the hedging could imply a net purchase of debt, so that it could never be done unless V_t was positive (the hedger has received funds from the investor) and enough to purchase the net debt, which will not happen in general. If it was not enough, then the hedging would not be possible. Notice that issuing debt to purchase the hedger own debt is not an option, since the issuance of debt will generate DVA with the funding provider, leaving the overall DVA unaffected.
- $\frac{\partial V_t}{\partial h^H_t} dh^H_t$: Notice that no matter whether the hedger makes the unrealistic assumption of being default free, in the process of replicating the derivative it will be exposed to its own funding spread (which will be related to the short term CDS spread h^H_t). This implies that the pricing equation would depend on its current funding curve and, unless the hedger unrealistically believes it to be non stochastic, the hedger should have an incentive to hedge this source of risk. As we will see, this source of risk can always be hedged in a one factor world by trading on two bonds with different maturities while forcing the net purchase to be equal to the derivative's NPV as seen from the investor (if the NPV as seen from the investor is negative the net

[3] When a market participant enters into a collateralized transaction with a positive value (respectively negative) pays (receives) the value of the deal to (from) the counterparty, but receives (posts) the value as collateral. This produces a net cash flow of zero.

purchase will be negative, representing an issuance). Notice also that when a hedger enters into a non collateralized derivative, the hedger will modify the sensitivity of his overall debt with respect to changes in his funding spread. Hedging this component will leave the sensitivity unchanged.

It is important to stress that the same sources of risk that the hedger will not be able to hedge are the same sources of risk whose cash flows will never be paid or received by the issuer (since it will already be defaulted). Therefore it is convenient to get rid of these sources of risk if we define price as the value of the replicating portfolio.

3 The replication strategy

As already mentioned, we will consider the most general situation in which both the issuer and the investor are defaultable and the realistic assumption that spreads are stochastic. Therefore, the hedger will hedge the risk factors that he is exposed to on every path under which he finds himself not defaulted (that are in fact the only ones that are hedgeable). These risk factors are:

- Market risk due to changes in S_t.
- Investor's spread risk due to changes in h_t^I.
- Investor's default event.
- Hedger's spread risk due to changes in h_t^H.

In order to hedge the exposure to the first three factors, the hedger will have to trade in the following instruments:

- Market risk: We will assume that market risk is hedged with a fully collateralized derivative on the same underlying asset. H_t will represent its NPV from the hedger's perspective.
- Spread risk and default risk of the investor: In a world where the dynamics of the credit curve is governed by a one factor model, the hedger will have to trade on two CDSs with different maturities written on the investor. CDS(t,t+dt) will represent the value of an overnight credit default swap (with unit notional) under which the protection buyer pays a premium at time $t + dt$ equal to $h_t^I dt$. If the default time of the investor $t < \tau^I \leq t + dt$, then the protection buyer receives $(1 - R_I)$ (R_I represents the investor's recovery rate) at time $t{+}dt$. We will assume that $h_t^I dt$ is such that $CDS(t, t+dt) = 0$. CDS(t,T) is a cash collateralized credit default swap maturing on a later date $T > t$. In general $CDS(t, T) \neq 0$.

Before analyzing how to become immune to changes in the hedger's credit spread, we will analyze the hedger's funding situation. If $V_t > 0$, the hedger will have excess cash with which he will be able to buy back his own debt. If $V_t < 0$, the hedger will have to issue new debt.

In either case the hedger will have to decide the spread duration (sentitivity to spread changes) of the debt issued/bought back. In a world with deterministic spreads this decision will be irrelevant, but under the realistic assumption of stochastic spreads this is no longer true. Notice that an uncollateralized derivative can either become an asset or a liability during the replication process. Therefore when a new uncollateralized derivative is replicated, the hedger will see that the spread duration of his debt is altered unless he imposes that the spread duration of the incoming uncollateralized derivative is perfectly matched with the spread duration of the bonds issued/bought back.

Notice that the hedger can match the spread duration of the uncollateralized derivative by trading on bonds with two different maturities while imposing that the net buyback is equal to V_t (issuance if V_t is negative). This is what we will refer to as the self financing condition of the replication strategy.

We will assume that the hedger trades on bonds that mature on a future date T and also on short term bonds that mature on $t + dt$. Had we assumed an n factor model for the dynamic of the hedger's credit curve, the hedger would have to trade on $n + 1$ bonds issued by himself. Ω_t and ω_t will represent the amounts to buy back (or issue if negative) in $B(t, t + dt)$ and $B(t, T)$ respectively.

In order to ensure that the self financing condition holds:

$$V_t = \Omega_t B(t, t + dt) + \omega_t B(t, T)$$

Which implies

$$\Omega_t = \frac{V_t - \omega_t B(t, T)}{B(t, t + dt)}$$

ω_t will be determined in order to match the spread duration of the derivative and that of the debt issuance/buyback.

The hedging equation will be

$$V_t = \alpha_t H_t + \beta_t + \gamma_t CDS(t,T) + \epsilon_t \underbrace{CDS(t,t+dt)}_{=0} + \frac{V_t}{B(t,t+dt)} B(t,t+dt) + \omega_t \underbrace{\left(B(t,T) - \frac{B(t,T)}{B(t,t+dt)} B(t,t+dt)\right)}_{=0}$$

Where V_t represents the NPV from the investor's perspective, H_t represents the NPV (from the hedger's perspective) of a fully collateralized derivative (collateralized in cash) written on S_t, α_t the number of contracts to trade in H_t, β_t represents cash in collateral accounts, $CDS(t, t + dt)$ and $CDS(t, T)$ the NPVs as seen from the hedger of short term and long term credit default swaps written on the investor, γ_t and ϵ_t represent the notional to trade on each CDS.

β_t will be comprised of $-\alpha_t H_t$ and $-\gamma_t CDS(t, T)$ that has been posted to the hedger (or the opposite if either α_t or γ_t are negative). Therefore, the change in β_t will be given by:

$$d\beta_t = -c_t \alpha_t H_t dt - c_t \gamma_t CDS(t,T)dt$$

Where c_t represents the accrual rate of collateral accounts (which we assume to be the OIS rate is the deal's currency). Notice that there is no contribution of the short term CDS written on the investor since we have assumed its NPV to be zero.

Conditional on both the investor and the hedger being alive at time t, the change in V_t under every path in which the hedger remains not defaulted until $t + dt$ will be given by (applying Itô's Lemma for jump diffusion processes)

$$dV_t = \mathcal{L}_{SIH} V_t dt + \frac{\partial V_t}{\partial S_t} S_t \sigma_t^S dW_t^{S,\mathbb{P}} + \frac{\partial V_t}{\partial h_t^I} \sigma_t^I dW_t^{I,\mathbb{P}} + \frac{\partial V_t}{\partial h_t^H} \sigma_t^H dW_t^{H,\mathbb{P}} + \Delta V_t^I dN_t^{I,\mathbb{P}}$$

Where

$$\begin{aligned}\mathcal{L}_{SIH} V_t \;=\; & \tfrac{\partial V_t}{\partial t} + \mu_t^S S_t \tfrac{\partial V_t}{\partial S_t} + \mu_t^H \tfrac{\partial V_t}{\partial h_t^H} + \mu_t^I \tfrac{\partial V_t}{\partial h_t^I} + \tfrac{1}{2} \tfrac{\partial^2 V_t}{\partial S_t^2} S_t^2 (\sigma_t^S)^2 + \tfrac{1}{2} \tfrac{\partial^2 V_t}{\partial {h_t^H}^2} (\sigma_t^H)^2 + \tfrac{1}{2} \tfrac{\partial^2 V_t}{\partial {h_t^I}^2} (\sigma_t^H)^2 \\ & + \tfrac{\partial^2 V_t}{\partial S_t h_t^H} S_t \sigma_t^S \sigma_t^H \rho_t^{S,H} + \tfrac{\partial^2 V_t}{\partial S_t h_t^I} S_t \sigma_t^S \sigma_t^I \rho_t^{S,I} + \tfrac{\partial^2 V_t}{\partial h_t^I h_t^H} \sigma_t^I \sigma_t^H \rho_t^{I,H}\end{aligned}$$

The differential change in H_t (applying Itô's Lemma)

$$dH_t = \mathcal{L}_S H_t dt + \frac{\partial H_t}{\partial S_t} S_t \sigma_t^S dW_t^{S,\mathbb{P}}$$

where

$$\mathcal{L}_S H_t = \frac{\partial H_t}{\partial t} + \mu_t^S S_t \frac{\partial H_t}{\partial S_t} + \frac{1}{2} S_t^2 \left(\sigma_t^S\right)^2 \frac{\partial^2 H_t}{\partial S_t^2}$$

The differential change in $CDS(t, t+dt)$ and in $B(t, t+dt)$

$$dCDS(t, t+dt) = h_t^I dt - (1 - R_I) dN_t^{I,\mathbb{P}}$$
$$dB(t, t+dt) = f_t^H B(t, t+dt) dt$$

f_t^H represents the hedger's short term funding rate. Notice that the jump to default component of $B(t, t+dt)$ has been omitted since it will not be experienced by the hedger. In appendix A we see that f_t^H is also equal to $r_t^H + h_t^H$, where r_t^H represents the short term REPO rate written on a short term bond issued by the hedger and that matures on $t + dt$ [4]. Same as we have assumed interest rates to be non stochastic (c_t will be a deterministic function of time) we will also assume repo rates to be non stochastic, so that the dynamics of bonds issued by the hedger will be determined just by the dynamics of the short term CDS spread h_t^H.

The differential change of CDS(t,T)

$$dCDS(t,T) = \mathcal{L}_I CDS(t,T) dt + \frac{\partial CDS(t,T)}{\partial h_t^I} \sigma_t^I dW_t^{I,\mathbb{P}} + \Delta CDS(t,T) dN_t^{I,\mathbb{P}}$$

with

$$\mathcal{L}_I CDS(t,T) = \frac{\partial CDS(t,T)}{\partial t} + \mu_t^I \frac{\partial CDS(t,T)}{\partial h_t^I} + \frac{1}{2} \left(\sigma_t^I\right)^2 \frac{\partial^2 CDS(t,T)}{\partial {h_t^I}^2}$$

And finally

$$dB(t,T) = \mathcal{L}_H B(t,T) dt + \frac{\partial B(t,T)}{\partial h_t^H} \sigma_t^H dW_t^{H,\mathbb{P}}$$

[4] r_t^H is the REPO rate of a REPO that matures on $t + dt$ written on a bond that also matures on $t + dt$.

where

$$\mathcal{L}_H B(t,T) = \frac{\partial B(t,T)}{\partial t} + \mu_t^H \frac{\partial B(t,T)}{\partial h_t^H} + \frac{1}{2}\left(\sigma_t^H\right)^2 \frac{\partial^2 B(t,T)}{\partial {h_t^H}^2}$$

Again, we have omitted the jump component in $B(t,T)$ since it will not be experienced by the hedger

So that the hedging equation in differential form will be given by

$$\begin{aligned}
&\mathcal{L}_{SIH} V_t dt + \tfrac{\partial V_t}{\partial S_t} S_t \sigma_t^S dW_t^{S,\mathbb{P}} + \tfrac{\partial V_t}{\partial h_t^I} \sigma_t^I dW_t^{I,\mathbb{P}} + \tfrac{\partial V_t}{\partial h_t^H} \sigma_t^H dW_t^{H,\mathbb{P}} + \Delta V_t^I dN_t^{I,\mathbb{P}} = \\
&= V_t f_t^H dt - c_t \alpha_t H_t dt - c_t \gamma_t CDS(t,T) dt \\
&+\alpha_t \left(\mathcal{L}_S H_t dt + \tfrac{\partial H_t}{\partial S_t} S_t \sigma_t^S dW_t^{S,\mathbb{P}}\right) \\
&+\gamma_t \left(\mathcal{L}_I CDS(t,T) dt + \tfrac{\partial CDS(t,T)}{\partial h_t^I} \sigma_t^I dW_t^{I,\mathbb{P}} + \Delta CDS(t,T) dN_t^{I,\mathbb{P}}\right) \\
&+\epsilon_t \left(h_t^I dt - (1-R_I) dN_t^{I,\mathbb{P}}\right) \\
&+\omega_t \left(\mathcal{L}_H B(t,T) dt + \tfrac{\partial B(t,T)}{\partial h_t^H} \sigma_t^H dW_t^{H,\mathbb{P}} - f_t^H B(t,T) dt\right)
\end{aligned} \tag{3}$$

In order to be hedged, the terms in $dW_t^{S,\mathbb{P}},\ dW_t^{I,\mathbb{P}},\ dW_t^{H,\mathbb{P}},\ ,dN_t^{I,\mathbb{P}}$ have to be canceled from the last equation, so that

$$\begin{aligned}
\alpha_t &= \frac{\frac{\partial V_t}{\partial S_t}}{\frac{\partial H_t}{\partial S_t}} \\
\gamma_t &= \frac{\frac{\partial V_t}{\partial h_t^I}}{\frac{\partial CDS(t,T)}{\partial h_t^I}} \\
\epsilon_t &= \gamma_t \tfrac{\Delta CDS(t,T)}{1-R_T} - \tfrac{\Delta V_t^I}{1-R_I} \\
\omega_t &= \frac{\frac{\partial V_t}{\partial h_t^H}}{\frac{\partial B(t,T)}{\partial h_t^H}}
\end{aligned} \tag{4}$$

So that every risk factor disappears from the hedging equation

$$
\begin{aligned}
&\widetilde{\mathcal{L}}_{SIH} V_t = V_t f_t^H \\
&+\alpha_t \left(\widetilde{\mathcal{L}}_S H_t - c_t H_t\right) \\
&+\gamma_t \left(\widetilde{\mathcal{L}}_I CDS(t,T) - c_t CDS(t,T)\right) \\
&+\epsilon_t h_t^I \\
&+\omega_t \left(\widetilde{\mathcal{L}}_H B(t,T) - \left(c_t + \bar{h}_t^H\right) B(t,T)\right)
\end{aligned}
$$

Where

$$
\begin{aligned}
\widetilde{\mathcal{L}}_{SIH} V_t \quad = \quad & \tfrac{\partial V_t}{\partial t} + \tfrac{1}{2}\tfrac{\partial^2 V_t}{\partial S_t^2} S_t^2 (\sigma_t^S)^2 + \tfrac{1}{2}\tfrac{\partial^2 V_t}{\partial {h_t^H}^2} (\sigma_t^H)^2 + \tfrac{1}{2}\tfrac{\partial^2 V_t}{\partial {h_t^I}^2} (\sigma_t^H)^2 \\
& + \tfrac{\partial^2 V_t}{\partial S_t h_t^H} S_t \sigma_t^S \sigma_t^H \rho_t^{S,H} + \tfrac{\partial^2 V_t}{\partial S_t h_t^I} S_t \sigma_t^S \sigma_t^I \rho_t^{S,I} + \tfrac{\partial^2 V_t}{\partial h_t^I h_t^H} \sigma_t^I \sigma_t^H \rho_t^{I,H}
\end{aligned}
$$

$$
\widetilde{\mathcal{L}}_S H_t = \frac{\partial H_t}{\partial t} + \frac{1}{2} S_t^2 \left(\sigma_t^S\right)^2 \frac{\partial^2 H_t}{\partial S_t^2}
$$

$$
\widetilde{\mathcal{L}}_I CDS(t,T) = \frac{\partial CDS(t,T)}{\partial t} + \frac{1}{2} \left(\sigma_t^I\right)^2 \frac{\partial^2 CDS(t,T)}{\partial {h_t^I}^2}
$$

$$
\widetilde{\mathcal{L}}_H B(t,T) = \frac{\partial B(t,T)}{\partial t} + \frac{1}{2} \left(\sigma_t^H\right)^2 \frac{\partial^2 B(t,T)}{\partial {h_t^H}^2}
$$

Substituting ϵ_t by its value and grouping terms

$$
\begin{aligned}
&\widetilde{\mathcal{L}}_{SIH} V_t + \tfrac{h_t^I}{1-R_I} \Delta V_t^I = V_t f_t^H \\
&+\alpha_t \left(\widetilde{\mathcal{L}}_S H_t - c_t H_t\right) \\
&+\gamma_t \left(\widetilde{\mathcal{L}}_I CDS(t,T) + \tfrac{h_t^I}{1-R_I} \Delta CDS(t,T) - c_t CDS(t,T)\right) \\
&+\omega_t \left(\widetilde{\mathcal{L}}_H B(t,T) - \left(c_t + \bar{h}_t^H\right) B(t,t+dt)\right)
\end{aligned}
$$

H_t is a cash collateralized derivative written on S_t, therefore it must meet the following PDE as seen in [14]

$$\widetilde{\mathcal{L}}_S H_t + (r_t - q_t)\, S_t \frac{\partial H_t}{\partial S_t} - c_t H_t = 0$$

$CDS(t,T)$ is a collateralized credit derivative written on I, therefore, as seen in appendix A, it must follow

$$\widetilde{\mathcal{L}}_I CDS(t,T) + \left(\mu_t^I - M_t^I \sigma_t^I\right) \frac{\partial CDS(t,T)}{\partial h_t^I} + \frac{h_t^I}{1-R_I} \Delta CDS(t,T) - c_t CDS(t,T) = 0$$

Where M_t^I is the investor's market price of credit risk and $\Delta CDS(t,T)$ is the jump in value of $CDS(t,T)$ upon default of the issuer.

And $B(t,T)$, as also seen in appendix A for the hedger's own debt, must follow

$$\widetilde{\mathcal{L}}_H B(t,T) + \left(\mu_t^H - M_t^H \sigma_t^H\right) \frac{\partial B(t,T)}{\partial h_t^H} - f_t^H B(t,T) = 0$$

Where M_t^H is the hedger's market price of credit risk.

So that the hedging equation is given by

$$\begin{aligned}
&\widetilde{\mathcal{L}}_{SIH} V_t + \tfrac{h_t^I}{1-R_I} \Delta V_t^I = V_t f_t^H \\
&+ \tfrac{\frac{\partial V_t}{\partial S_t}}{\frac{\partial H_t}{\partial S_t}} \left(- (r_t - q_t)\, S_t \tfrac{\partial H_t}{\partial S_t} \right) \\
&+ \tfrac{\frac{\partial V_t}{\partial h_t^I}}{\frac{\partial CDS(t,T)}{\partial h_t^I}} \left(- \left(\mu_t^I - M_t^I \sigma_t^I\right) \tfrac{\partial CDS(t,T)}{\partial h_t^I} \right) \\
&+ \tfrac{\frac{\partial V_t}{\partial h_t^H}}{\frac{\partial B(t,T)}{\partial h_t^H}} \left(- \left(\mu_t^H - M_t^H \sigma_t^H\right) \tfrac{\partial B(t,T)}{\partial h_t^H} \right)
\end{aligned}$$

Which implies

$$\widehat{\mathcal{L}}_{SIH} V_t + \tfrac{h_t^I}{1-R_I} \Delta V_t^I = V_t f_t^H$$

Where

$$
\begin{aligned}
\widehat{\mathcal{L}}_{SIH} V_t &= \tfrac{\partial V_t}{\partial t} + (r_t - q_t) S_t \tfrac{\partial V_t}{\partial S_t} + (\mu_t^H - M_t^H \sigma_t^H) \tfrac{\partial V_t}{\partial h_t^H} + (\mu_t^I - M_t^I \sigma_t^I) \tfrac{\partial V_t}{\partial h_t^I} \\
&\quad + \tfrac{1}{2} \tfrac{\partial^2 V_t}{\partial S_t^2} S_t^2 (\sigma_t^S)^2 + \tfrac{1}{2} \tfrac{\partial^2 V_t}{\partial {h_t^H}^2} (\sigma_t^H)^2 + \tfrac{1}{2} \tfrac{\partial^2 V_t}{\partial {h_t^I}^2} (\sigma_t^H)^2 \qquad (5) \\
&\quad + \tfrac{\partial^2 V_t}{\partial S_t h_t^H} S_t \sigma_t^S \sigma_t^H \rho_t^{S,H} + \tfrac{\partial^2 V_t}{\partial S_t h_t^I} S_t \sigma_t^S \sigma_t^I \rho_t^{S,I} + \tfrac{\partial^2 V_t}{\partial h_t^I h_t^H} \sigma_t^I \sigma_t^H \rho_t^{I,H}
\end{aligned}
$$

The solution to (5) with terminal condition given by $V_T = g(S_T)$ is equal to calculating the following expected value

$$
\begin{aligned}
V_t &= \underbrace{E_{\mathbb{Q}}\left[V_T \exp\left(-\int_{s=t}^{T} c_s ds \right) \Big| \mathcal{F}_t \right]}_{\text{Fully collateralized price}} \\
&\quad \underbrace{- E_{\mathbb{Q}}\left[\int_{s=t}^{T} 1_{\{\tau^I > s\}} \exp\left(-\int_{h=t}^{s} c_h dh \right) \bar{h}_s^H V_s ds \Big| \mathcal{F}_t \right]}_{\text{Funding value adjustment}} \qquad (6) \\
&\quad + \underbrace{E_{\mathbb{Q}}\left[\int_{s=t}^{T} 1_{\{\tau^I > s\}} \exp\left(-\int_{h=t}^{s} c_h dh \right) (R_I - 1) \left(V_s^C \right)^- dN_s^{I,\mathbb{Q}} \Big| \mathcal{F}_t \right]}_{\text{CVA}}
\end{aligned}
$$

Where $\bar{h}_t^H$ is the hedger's funding spread over the collateral rate c_t.

In a measure $\mathbb{Q}$ in which the drifts of S_t, h_t^H and h_t^I are given by $(r_t - q_t)S_t$, $\mu_t^H - M_t^H \sigma_t^H$ and $\mu_t^I - M_t^I \sigma_t^I$ respectively. Under this measure, the default intensity of the default event of the investor is $\frac{h_t^I}{1-R_I}$. V_t^C is the value of the cash collateralized transaction (from the investor's perspective). We have assumed that upon default of the investor V_t jumps to $R_I V_t^C$ if $V_t^C < 0$ and to V_t^C if $V_t^C \geq 0$.

The solution to (5) can also be written as

$$
\begin{aligned}
V_t &= \underbrace{E_{\mathbb{Q}}\left[V_T \exp\left(-\int_{s=t}^{T} f_s^H ds\right)\Big|\mathcal{F}_t\right]}_{\text{Price with funding adjustment and no counterparty credit risk}} \\
&+ \underbrace{E_{\mathbb{Q}}\left[\int_{s=t}^{T} 1_{\{\tau^I > s\}} \exp\left(-\int_{h=t}^{s} f_h^H dh\right)\left((V_s^{rf})^- R_I + (V_s^{rf})^+ - V_s^f\right) dN_s^{I,\mathbb{Q}}\Big|\mathcal{F}_t\right]}_{\text{CVA over price with funding}}
\end{aligned} \tag{7}
$$

Where

$$
V_s^f := E_{\mathbb{Q}}\left[V_T \exp\left(-\int_{s=t}^{T} f_s^H ds\right)\Big|\mathcal{F}_s\right]
$$

So that full replication implies that CVA (a unilateral version of it that does not depend on the hedger's funding curve) and FVA (a funding adjustment that does only depend on the investor's default indicator and not on the hedger's) are the only components to be incorporated in the price of financial derivatives.

4 DVA hedging vs FVA hedging: a simplified example

In this section we explore DVA vs FVA hedging in a simplified framework. We will assume:

- We want to replicate a forward on a particular underlying asset (S_t) such that at maturity (5 years) the investor (risk taker) receives $S_T - K$.
- The underlying asset pays no dividends.
- Interest rates are assumed to be zero (OIS and REPO rates).
- The investor is default free.
- The hedger is defaultable with a short term funding spread z_t.
- The recovery rate for the hedger is 0.
- The underlying asset follows Black-Scholes.
- z_t^H follows an Ornstein-Uhlenbeck process.
- We assume no correlation between S_t and z_t.

So that the SDEs of the two processes under the real world measure are:

$$dS_t = \mu_t^{\mathbb{P}} S_t dt + \sigma_t^S dW_t^{S,\mathbb{P}}$$
$$dz_t = \kappa \left(\theta_t^{\mathbb{P}} - z_t\right) dt + \sigma_t^z dW_t^{z,\mathbb{P}}$$
$$dW_t^{S,\mathbb{P}} dW_t^{z,\mathbb{P}} = 0$$

We have chosen the following set of parameters:

$\mu_t^{\mathbb{P}}$	10%
σ_t^S	20%
$\theta_t^{\mathbb{P}}$	4%
κ	0.5
σ_t^z	1%
S_0	1
z_0	3%
K	1

We assume that at $t = 0$, the funding curve is flat at a level of 3%.

Before exploring the effects of hedging, we will analyze the sensitivities with respect to spread changes of both approaches (Risk free price + FVA vs Risk free price + DVA).

Notice that the sensitivity of the DVA adjusted price is always positive (the well known effect of DVA), whereas for the case of the FVA adjusted price, the sensitivity is positive when $K < S_0$ and negative when $K > S_0$.

- Under a DVA approach, the hedger always benefits from an increase in his funding spread.
- Under a FVA approach, the hedger benefits from an increase in the spread when the NPV is positive for the risk taker, that is, the hedger borrows funds from the client.
- Under a FVA approach, the hedger experiences a loss from an increase in the spread when the NPV is negative for the risk taker, that is, the hedger lends funds to the client.

FVA Hedging

In figure 2 we can observe the evolution of both the FVA adjusted price and the hedging portfolio. We can see that the P&L is negligible and deviations from

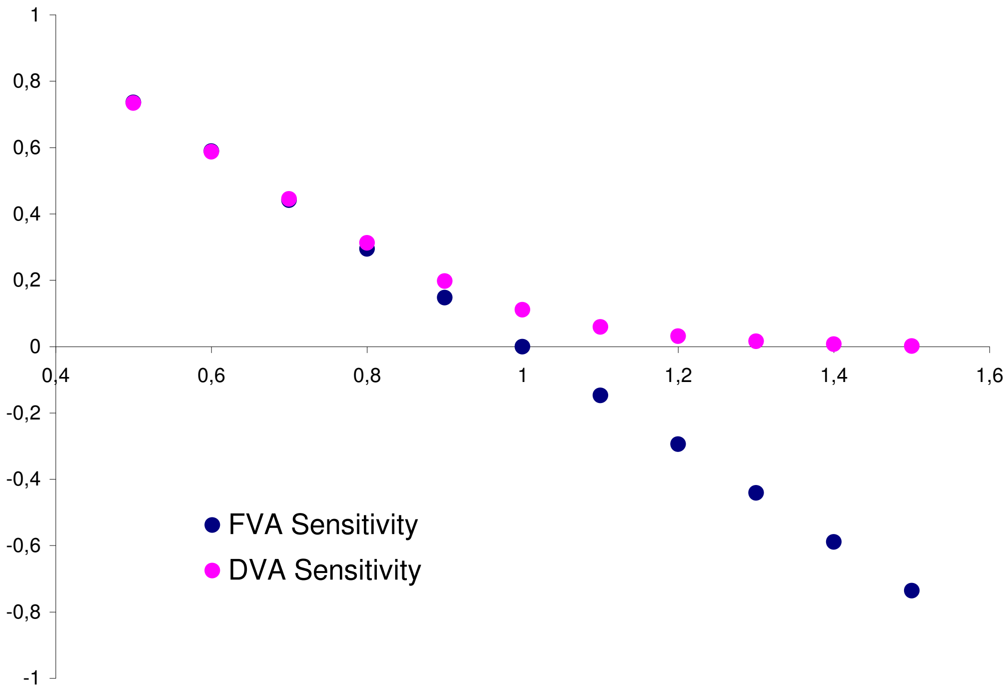

Figure 1: Sensitivities to spread changes of both FVA and DVA adjusted price. x-axis represents K (fix payment to be paid by the risk taker in the forward contract.)

zero seem noisy and due to the discrete rebalancing frequency of the hedge. This confirms the theoretical results that we have seen in the previous sections.

DVA Hedging

Now we explore DVA hedging.

As a hedging strategy we use the same one used to try to hedge FVA. That is, if the price for the risk taker is positive, we receive funds from the risk taker

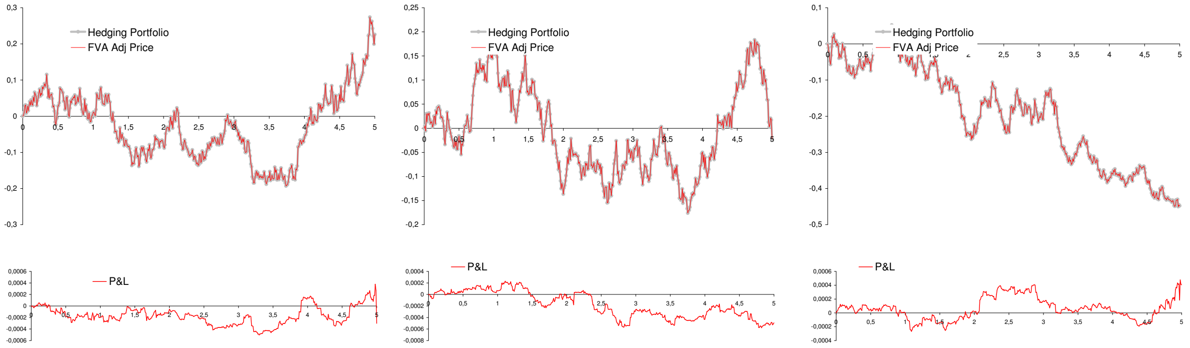

Figure 2: FVA adjusted price vs hedging portfolio (graphs above) and P&L (graphs below)

with which we buy back our own debt.

If the price is negative, we need to issue new debt.

In either case, we impose that the sensitivity to spread changes of the debt issued (or bought back) matches that of the incoming derivative.

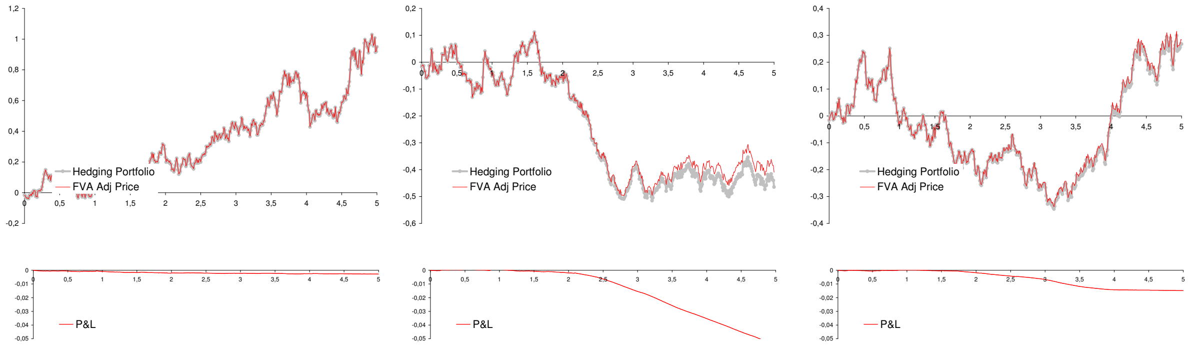

Figure 3: DVA adjusted price vs hedging portfolio (graphs above) and P&L (graphs below)

In figure 3 we observe that the P&L becomes negative (although the evolution is smooth) and that it depends on the path. It seems that there is a theta mismatch between the DVA adjusted price and the hedging portfolio.

In figure 4 we plot many different scenarios and see that the P&L seems to be always negative, and that it is path dependent.

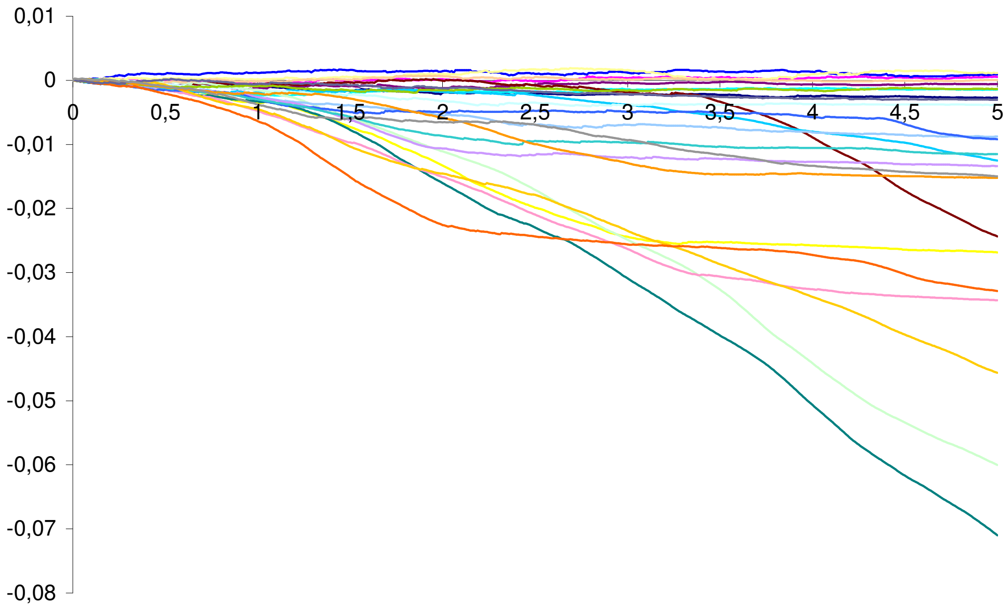

Figure 4: P&L for DVA adjusted price hedging (various paths.)

What is the main driver of the path dependent P&L?

The hedging portfolio minus the DVA adjusted price is given by (we assume that we are initially hedged)

$$\alpha_t H_t + \beta_t + \Omega_t B(t, t+dt) + \omega_t B(t,T) - V_t$$

The differential change is given by (assuming the hedger does not default):

$$\begin{aligned} d\Pi_t &= \alpha_t \left(\mathcal{L}_s H_t dt + \tfrac{\partial H_t}{\partial S_t} dS_t - c_t H_t dt\right) \\ &\quad +\omega_t \left(\mathcal{L}_{sh} B(t,T) dt + \tfrac{\partial B(t,T)}{\partial S_t} dS_t + \tfrac{\partial B(t,T)}{\partial h_t} dh_t - f_t B(t,T) dt\right) \\ &\quad - \left(\mathcal{L}_{sh} V_t dt + \tfrac{\partial V_t}{\partial S_t} dS_t + \tfrac{\partial V_t}{\partial h_t} dh_t - f_t V_t dt\right) \end{aligned} \tag{8}$$

Where

$$\mathcal{L}_S = \tfrac{\partial}{\partial t} + \tfrac{1}{2} S_t^2 \left(\sigma_t^S\right)^2 \tfrac{\partial^2}{\partial S_t^2}$$

$$\mathcal{L}_h = \tfrac{\partial}{\partial t} + \tfrac{1}{2} \left(\sigma_t^h\right)^2 \tfrac{\partial^2}{\partial {h_t}^2} \tag{9}$$

$$\mathcal{L}_{Sh} = \tfrac{\partial}{\partial t} + \tfrac{1}{2} \tfrac{\partial^2}{\partial S_t^2} S_t^2 (\sigma_t^S)^2 + \tfrac{1}{2} \tfrac{\partial^2}{\partial {h_t}^2} (\sigma_t^h)^2 + \tfrac{\partial^2}{\partial S_t \partial h_t} S_t \sigma_t^S \sigma_t^h \rho_t$$

Where we have taken into account that $V_t = \Omega_t B(t, t+dt) + \omega_t B(t,T)$. That is, funds exchanged with the investor are matched with the issuance or buy back of debt.

In order to be hedged to the two risk factors on every scenario under which the hedger has not defaulted:

$$\begin{aligned} d\Pi_t &= \tfrac{\frac{\partial V_t}{\partial S_t}}{\frac{\partial H_t}{\partial S_t}} \left(\mathcal{L}_s H_t dt - c_t H_t dt\right) \\ &\quad + \tfrac{\frac{\partial V_t}{\partial h_t}}{\frac{\partial B(t,T)}{\partial h_t}} \left(\mathcal{L}_h B(t,T) dt - f_t B(t,T) dt\right) \\ &\quad - \left(\mathcal{L}_{sh} V_t dt - f_t V_t dt\right) \end{aligned} \tag{10}$$

We assume that DVA is discounted at the Eonia rate, therefore its PDE would be:

$$\mathcal{L}_{Sh} V_t + (r_t - q_t) S_t \frac{\partial V_t}{\partial S_t} + \left(\mu_t^h - M_t^h \sigma_t^h\right) \frac{\partial V_t}{\partial h_t} + \Delta V_t \frac{h_t}{1-R} - c_t V_t = 0 \tag{11}$$

The PDEs followed by H_t and $B(t,T)$

$$\mathcal{L}_S H_t + (r_t - q_t) S_t \frac{\partial H_t}{\partial S_t} - c_t H_t = 0 \tag{12}$$

$$\mathcal{L}_h B(t,T) + \left(\mu_t^h - \sigma_t^h M_t^h\right) \frac{\partial B(t,T)}{\partial h_t} + \frac{h_t}{1-R}\Delta B(t,T) - r_t^T B(t,T) = 0 \tag{13}$$

So that

$$\begin{aligned} d\Pi_t &= -\frac{\frac{\partial V_t}{\partial S_t}}{\frac{\partial H_t}{\partial S_t}} (r_t - q_t) S_t \frac{\partial H_t}{\partial S_t} dt \\ &+ \frac{\frac{\partial V_t}{\partial h_t}}{\frac{\partial B(t,T)}{\partial h_t}} \left(-\left(\mu_t^h - \sigma_t M_t\right) \frac{\partial B(t,T)}{\partial h_t} - \frac{h_t}{1-R}\Delta B(t,T) + r_t^T B(t,T) - f_t B(t,T)\right) dt \\ &- \left(-(r_t - q_t) S_t \frac{\partial V_t}{\partial S_t} - \left(\mu_t^h - M_t^h \sigma_t^h\right) \frac{\partial V_t}{\partial h_t} - \Delta V_t \frac{h_t}{1-R} + c_t V_t - f_t V_t \right) dt \end{aligned} \tag{14}$$

Canceling terms:

$$\begin{aligned} d\Pi_t &= \frac{\frac{\partial V_t}{\partial h_t}}{\frac{\partial B(t,T)}{\partial h_t}} \left(-\frac{h_t}{1-R}\Delta B(t,T) + r_t^T B(t,T) - f_t B(t,T)\right) dt \\ &+ \left(\Delta V_t \frac{h_t}{1-R} + z_t V_t\right) dt \end{aligned} \tag{15}$$

Reordering terms

$$d\Pi_t = \underbrace{\frac{h_t}{1-R}\left(\Delta V_t - \frac{\frac{\partial V_t}{\partial h_t}}{\frac{\partial B(t,T)}{\partial h_t}}\Delta B(t,T)\right)}_{\text{Jump to default mismatch}} + \underbrace{\left(z_t V_t - (f_t - r_t^T) B(t,T) \frac{\frac{\partial V_t}{\partial h_t}}{\frac{\partial B(t,T)}{\partial h_t}}\right)}_{\text{Funding mismatch}} \tag{16}$$

If $\Delta B(t,T) = (1-R)B(t,T),\ r_t^T = r_t$

$$d\Pi_t = +\left(\Delta V_t \frac{h_t}{1-R} + z_t V_t\right) dt$$

$$\begin{aligned} &\Delta V_t = R(V_t^{rf})^+ + (V_t^{rf})^- - V_t = V_t^{rf} - (1-R)V_t^{rf} - V_t \\ &= DVA_t - (1-R)V_t rf = -JTD_{DVA} \end{aligned} \tag{17}$$

$$\Downarrow$$

$$d\Pi_t = \left(\Delta V_t \frac{h_t}{1-R} + z_t V_t\right) dt = -\left(\frac{JTD_{DVA}}{1-R} + z_t V_t\right) dt$$

The first term is due to the jump to default component of DVA that cannot be hedged. The second term is due to the funding adjustment not made in the pricing.

Which is generally negative .

In our case:

$R = 0$; $r_t^T = 0$; $f_t = z_t = h_t$. Therefore

$$d\Pi_t = \min(V_t^{rf}, 0) z_t dt$$

In figure 5 we compare the evolution of $\int_{s=0}^{t} \exp\left(\int_{u=0}^{s} z_s ds\right) \min(V_s^{rf}, 0) z_s ds$

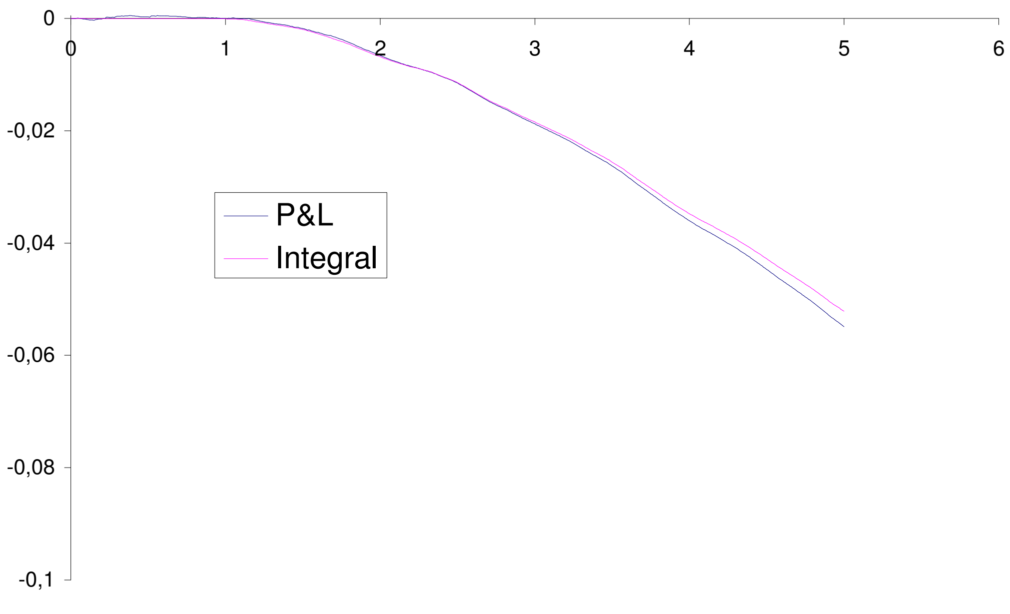

Figure 5: P&L vs integral

5 Conclusions

We have seen that assuming that the derivative's price incorporates the hedging costs borne by the hedger (and not those of the investor if he was to hedge the derivative) and that the hedger has only the incentive to hedge the risks that he will be exposed to while he remains not defaulted, the only adjustments to be made to the risk free price (that is, the price of a fully colateralized transaction) are an unilateral CVA that does not depend on the hedger's default indicator and a funding adjustment (FVA) where just the investor default indicator is present.

We have also seen that both components can be hedged under reasonable assumptions and that the hedging of those components leaves the sensitivity of the hedger's debt with respect changes in his credit spread unchanged after a new uncollateralized transaction is traded and during its replication. Regarding the hedging to the day to day changes of the hedger's credit curve, it is done by imposing that the debt issued or bought back matches the same spread duration of the uncollateralized derivative. We have carried the analysis under the realistic assumption of stochastic spreads.

Notice that due to the fact that the hedger does not try to hedge the jump in the derivative's value upon his own default (first because it can not be hedged in general and second because there is no incentive to hedge it), default indicators of the hedger are not present in the valuation of uncollateralized derivatives and neither is the jump in the derivative's value upon default of the hedger.

Under these assumptions the price of a derivative between two counterparties will depend on what counterparty acts as the investor (risk taker) and what as the hedger (risk taker) since the derivative's price will reflect the replication costs from the hedger's perspective, which will not be equal to the replication price from the investor's perspective.

A Modeling credit in a PDE framework

In this section our aim is to derive the PDE followed by both bonds issued by and collateralized credit derivatives written on a generic credit reference. We will assume a one factor model assumed for credit spreads and non stochastic interest rates.

Let's assume that we wanted to hedge a credit derivative written on a particular credit reference. h_t represents the credit reference short term credit default swap spread. We assume that under the real world measure $\mathbb{P}$ h_t follows

$$dh_t = \mu_t^{\mathbb{P}} dt + \sigma_t dW_t^{\mathbb{P}}$$

$\mu_t^{\mathbb{P}} t$ represents the drift and σ_t the volatility. $W_t^{\mathbb{P}}$ is a $\mathbb{P}$ brownian process.

E_t will represent the value of a credit derivative written on the credit reference from the investor perspective. It will both depend on the spread h_t and of the default indicator function $N_t^{\mathbb{P}} = 1_{\{\tau \le t\}}$, where τ is the default time of the credit reference. Therefore

$$dE_t = \frac{\partial E_t}{\partial t} dt + \frac{\partial E_t}{\partial h_t} dh_t + \frac{1}{2}\sigma_t^2 \frac{\partial^2 E_t}{\partial h_t^2} dt + \Delta E_t dN_t^{\mathbb{P}}$$

Where ΔE_t represents the change in E_t on default.

The two sources of randomness will have to be hedged with two different credit derivatives. One of them will be a short term credit default swap whose value from the protection seller will be represented by $CDS(t, t+dt)$. h_t will be such that $CDS(t, t+dt) = 0$. Its differential change will be given by:

$$dCDS(t, t+dt) = h_t dt - (1-R) dN_t^{\mathbb{P}}$$

R will represent the recovery rate.

Appart from trading on $CDS(t, t+dt)$, that will only have sensitivity to the default events, the hedger should also trade on another collateralized credit derivative H_t (NPV as seen by the hedger) such that

$$dH_t = \frac{\partial H_t}{\partial t} dt + \frac{\partial H_t}{\partial h_t} dh_t + \frac{1}{2}\sigma_t^2 \frac{\partial^2 H_t}{\partial h_t^2} dt + \Delta H_t dN_t^{\mathbb{P}}$$

Where ΔH_t represents the change in H_t on default.

The hedging equation will be

$$E_t = \alpha_t H_t + \gamma_t CDS(t, t+dt) + \beta_t$$

Where β_t represents cash held in collateral accounts. We assume both E_t and H_t to be collateralized in cash, so that:

$$d\beta_t = c_t E_t dt - c_t \alpha_t H_t dt$$

So that the hedging equation in differential form is

$$\begin{aligned}
&\tfrac{\partial E_t}{\partial t} dt + \tfrac{\partial E_t}{\partial h_t} dh_t + \tfrac{1}{2}\sigma_t^2 \tfrac{\partial^2 E_t}{\partial h_t^2} dt + \Delta E_t dN_t^{\mathbb{P}} = \\
&\alpha_t \left(\tfrac{\partial H_t}{\partial t} dt + \tfrac{\partial H_t}{\partial h_t} dh_t + \tfrac{1}{2}\sigma_t^2 \tfrac{\partial^2 H_t}{\partial h_t^2} dt + \Delta H_t dN_t^{\mathbb{P}} \right) \\
&+ \gamma_t \left(h_t dt - (1-R) dN_t^{\mathbb{P}} \right) \\
&+ c_t E_t dt - c_t \alpha_t H_t dt
\end{aligned} \tag{18}$$

In order to be hedged, the random terms dh_t and $dN_t^{\mathbb{P}}$ should be canceled. In order to do so

$$\alpha_t = \frac{\frac{\partial E_t}{\partial h_t}}{\frac{\partial H_t}{\partial h_t}} \quad \gamma_t = \alpha_t \frac{\Delta H_t}{1-R} - \frac{\Delta E_t}{1-R}$$

So that

$$\frac{\frac{\partial E_t}{\partial t} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 E_t}{\partial h_t^2} + \frac{h_t}{1-R}\Delta E_t - c_t E_t}{\frac{\partial E_t}{\partial h_t}} = \frac{\frac{\partial H_t}{\partial t} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 H_t}{\partial h_t^2} + \frac{h_t}{1-R}\Delta H_t - c_t H_t}{\frac{\partial H_t}{\partial h_t}} \tag{19}$$

Adding $\mu_t^{\mathbb{P}}$ and dividing by σ_t both sides of the last equation we obtain what could be interpreted as the expected excess return of the derivative over the collateral rate divided by the the derivatives volatility factor, therefore

$$\frac{\frac{\partial E_t}{\partial t} + \mu_t^{\mathbb{P}} \frac{\partial E_t}{\partial h_t} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 E_t}{\partial h_t^2} + \frac{h_t}{1-R}\Delta E_t - c_t E_t}{\sigma_t \frac{\partial E_t}{\partial h_t}} = \frac{\frac{\partial H_t}{\partial t} + \mu_t^{\mathbb{P}} \frac{\partial H_t}{\partial h_t} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 H_t}{\partial h_t^2} + \frac{h_t}{1-R}\Delta H_t - c_t H_t}{\sigma_t \frac{\partial H_t}{\partial h_t}} = M(t, h_t) \tag{20}$$

Since the ratio must be valid for any credit derivative (H_t and E_t are two generic payoffs), then it must be just a function of t and h_t. $M_t = M(t, h_t)$ will be called the market price of credit risk. Therefore, the PDE followed by any credit derivative must be

$$\frac{\partial E_t}{\partial t} + \left(\mu_t^{\mathbb{P}} - \sigma_t M_t\right) \frac{\partial E_t}{\partial h_t} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 E_t}{\partial h_t^2} + \frac{h_t}{1-R}\Delta E_t - c_t E_t = 0$$

When dealing with bonds, things are a little bit different. First we have to establish a relationship between the short term financing rate f_t and the short term CDS rate h_t. In order to do so, we compare two different strategies:

- Selling protection at time t with maturity $t + dt$.
- Buying a bond at t maturing at time $t + dt$ through a REPO transaction maturing also at time $t + dt$.

Both strategies imply a net cash flow at time t equal to 0. At time $t + dt$, the net cash flows are (assuming $\tau > t$):

CDS: $h_t dt - (1-R)1_{\{\tau \leq t+dt\}}$

REPO: $(1 + f_t dt)1_{\{\tau > t+dt\}} + R1_{\{\tau \leq t+dt\}} - (1 + r_t dt) =$

$= (1 + f_t dt) - (1 + r_t dt) - (1 - R + f_t dt)1_{\{\tau \leq t+dt\}} = (f_t - r_t)dt - (1-R)1_{\{\tau \leq t+dt\}}$

Where r_t is a short term REPO rate on a short term bond maturing at time $t + dt$. Therefore:

$$h_t = f_t - r_t$$

In order to obtain the PDE followed by defaultable bonds and derivatives that are replicated with bonds we should keep in mind that collateralized credit derivatives are financed at the collateral rate used to remunerate collateral accounts in cash no matter the volatility of the underlying derivative, whereas bonds are purchased at REPO rates that might differ between different bonds. Therefore the PDE will be

$$\frac{\partial B_t}{\partial t} + \left(\mu_t^{\mathbb{P}} - \sigma_t M_t\right) \frac{\partial B_t}{\partial h_t} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 B_t}{\partial h_t^2} + \frac{h_t}{1-R}\Delta B_t - r_t^B B_t = 0 \tag{21}$$

Where r_t^B represents the short term REPO rate for bond B_t. Notice that h_t is again the short term CDS spread and not the financing spread over EONIA.

The results obtained so far are valid when we are trading on someone else's debt.

When trading on our own debt:

- We will have no access to the CDS market written on our debt (We won't be able to sell protection on ourselves).
- We will have no access to the REPO market (We won't be able to get financing leaving our own bonds as collateral.)
- We will have no access to the recovery lock market written on our debt.

Therefore the risk neutral dynamics imposed by (21), that depend on magnitudes implied by markets to which we do not have access, seem not to work when we are managing our own debt.

What do we mean by managing our own debt?

Cash flow matching of our assets and liabilities such that the bank meets its current and future cash-flow obligations and collateral needs (assets / liabilities management).

Let's assume that a bank has issued debt with both short term maturity $(B(t, t + dt))$ and long term maturity $B^C(t, T)$.

Let's assume that we needed to issue (or buy back) debt with a given coupon and maturity S with a notional N.

Can we dynamically replicate the issuance (or buy back) of a bond with maturity $S \neq T$ with a net issuance (or bay back) in $B(t, t + dt)$ and $B^C(t, T)$?

In a one factor world, yes. In a n factor world, we will have to trade on $n + 1$ issued bonds.

The hedging equation would be:

$$NB^C(t,S) = N\left(\omega_t B(t,t+dt) + \Omega_t B^C(t,T)\right) \tag{22}$$

In (22), $N > 0$ represents a buy back and $N < 0$ an issuance.

The differential change of both sides of the hedging equation under the real world measure would be given by

$$\frac{\partial B^C(t,S)}{\partial t}dt + \frac{\partial B^C(t,S)}{\partial h_t}dh_t + \frac{1}{2}\sigma_t^2\frac{\partial^2 B^C(t,S)}{\partial h_t^2}dt + \Delta B^C(t,S)dN_t^{\mathbb{P}}$$

and

$$\omega_t B(t,t+dt)\left(f_t dt - (1-R)dN_t^{\mathbb{P}}\right) + \Omega_t\left(\frac{\partial B^C(t,T)}{\partial t}dt + \frac{\partial B^C(t,T)}{\partial h_t}dh_t + \frac{1}{2}\sigma_t^2\frac{\partial^2 B^C(t,T)}{\partial h_t^2}dt + \Delta B^C(t,T)dN_t^{\mathbb{P}}\right)$$

Notice that in (22) there is only one free parameter. Therefore we won't be able to hedge both the spread and the jump to default risks simultaneously.

In addition, the jump to default risk will not be experienced by ourselves. Therefore, leaving the $dN_t^{\mathbb{P}}$ term unhedged is not a concern.

We will remain hedged on every path under which we remain not defaulted.

$$\begin{aligned}&\tfrac{\partial B^C(t,S)}{\partial t}dt + \tfrac{\partial B^C(t,S)}{\partial h_t}dh_t + \tfrac{1}{2}\sigma_t^2\tfrac{\partial^2 B^C(t,S)}{\partial h_t^2}dt \\ &= \omega_t B(t,t+dt)f_t dt + \Omega_t\left(\tfrac{\partial B^C(t,T)}{\partial t}dt + \tfrac{\partial B^C(t,T)}{\partial h_t}dh_t + \tfrac{1}{2}\sigma_t^2\tfrac{\partial^2 B^C(t,T)}{\partial h_t^2}dt\right)\end{aligned} \tag{23}$$

In order to hedge the spread risk:

$$\Omega_t = \frac{\frac{\partial B^C(t,S)}{\partial h_t}}{\frac{\partial B^C(t,T)}{\partial h_t}}$$

Which together with (22) and (23) imply

$$\frac{\frac{\partial B^C(t,T)}{\partial t}+\mu_t^{\mathbb{P}}\frac{\partial B^C(t,T)}{\partial h_t}+\frac{1}{2}\sigma_t^2\frac{\partial^2 B^C(t,T)}{\partial h_t^2}-f_tB^C(t,T)}{\sigma_t\frac{\partial B^C(t,T)}{\partial h_t}} = \frac{\frac{\partial B^C(t,S)}{\partial t}+\mu_t^{\mathbb{P}}\frac{\partial B^C(t,S)}{\partial h_t}+\frac{1}{2}\sigma_t^2\frac{\partial^2 B^C(t,S)}{\partial h_t^2}-f_tB^C(t,S)}{\sigma_t\frac{\partial B^C(t,S)}{\partial h_t}} = M^{\mathrm{OD}}(t,h_t) \tag{24}$$

Where $M^{\mathrm{OD}}(t,h_t)$ represents the market price of risk of our own debt. So that the PDE followed by our bonds is:

$$\frac{\partial B^C(t,T)}{\partial t} + \left(\mu_t^{\mathbb{P}} - \sigma_t M_t^{\mathrm{OD}}\right)\frac{\partial B^C(t,T)}{\partial h_t} + \frac{1}{2}\sigma_t^2\frac{\partial^2 B^C(t,T)}{\partial h_t^2} - f_tB^C(t,T) = 0 \tag{25}$$

So that the price of a particular bond would be given by

$$B(t,T) \quad = \quad \underbrace{E_{\mathbb{Q}}\left[C\sum_{j=1}^{n}\gamma_j \exp\left(-\int_{s=t}^{t_j} f_s ds\right) + \exp\left(-\int_{s=t}^{t_n} f_s ds\right)\Big|\mathcal{F}_t\right]}_{\text{Bond coupons \& notional}} \tag{26}$$

Notice that the risk free dynamics of our own debt reflected in (25) do not depend on REPO rates, recovery rates and has no default indicators.

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