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Based on results given in the recent book by Meinhardt (2013c), which presents a dual characterization of the pre-kernel by a finite union of solution sets of a family of quadratic and convex objective functions, we could derive some results related to the uniqueness of the pre-kernel. Rather than extending the knowledge of game classes for which the pre-kernel consists of a single point, we apply a different approach. We select a game from an arbitrary game class with an unique pre-kernel satisfying the non-empty interior condition of a payoff equivalence class, and then establish that the set of related and linear independent games which are derived from this pre-kernel of the default game replicate this point also as its sole pre-kernel element. In the proof we apply results and techniques employed in the above work. Namely, we prove in a first step that the linear mapping of a pre-kernel element into a specific vector subspace of balanced excesses is unique. Secondly, that there cannot exist a different and non-transversal vector subspace of balanced excesses in which a linear transformation of a pre-kernel element can be mapped. Furthermore, we establish that on the restricted subset on the game space that is constituted by the convex hull of the default and the set of related games, the pre-kernel correspondence is single-valued, and therefore continuous. Finally, we provide sufficient conditions that preserves the pre-nucleolus property for related games even when the default game has not an unique pre-kernel.

Keywords: Transferable Utility Game, Pre-Kernel, Uniqueness, Convex Analysis, Fenchel-Moreau Conjugation, Indirect Function

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1 INTRODUCTION

The coincidence of the kernel and nucleolus – that is, the kernel consists of a single point – is only known for some classes of transferable utility games. In particular, it was established by Maschler et al. (1972) that for the class of convex games – introduced by Shapley (1971) – the kernel and the nucleolus coincide. Recently, Getán et al. (2012) were able to extend this result to the class of zero-monotonic almost-convex games. However, for the class of average-convex games, there is only some evidence that both solution concepts coalesce.

In order to advance our understanding about TU games and game classes which possess an unique pre-kernel element, we propose an alternative approach to investigate this issue while applying results and techniques recently provided in the book by Meinhardt (2013c). There, it was shown that the pre-kernel of the grand coalition can be characterized by a finite union of solution sets of a family of quadratic and convex functions (Theorem 7.3.1). This dual representation of the pre-kernel is based on a Fenchel-Moreau generalized conjugation of the characteristic function. This generalized conjugation was introduced by Martinez-Legaz (1996), which he called the indirect function. Immediately thereafter, it was Meseguer-Artola (1997) who proved that the pre-kernel can be derived from an over-determined system of non-linear equations. This over-determined system of non-linear equations is equivalent to a minimization problem, whose set of global minima is equal to the pre-kernel set. However, an explicit structural form of the objective function that would allow a better and more comprehensive understanding of the pre-kernel set could not be performed.

The characterization of the pre-kernel set by a finite union of solution sets was possible due to a partition of the domain of the objective function into a finite number of payoff sets. From each payoff vector contained into a particular payoff set the same quadratic and convex function is induced. The collection of all these functions on the domain composes the objective function from which a pre-kernel element can be single out. Moreover, each payoff set creates a linear mapping that maps payoff vectors into a vector subspace of balanced excesses. Equivalent payoff sets which reflects the same underlying bargaining situation produce the same vector subspace. The vector of balanced excesses generated by a pre-kernel point is contained into the vector subspace spanned by the basis vectors derived from the payoff set that contains this pre-kernel element. In contrast, the vectors of unbalanced excesses induced from the minima of a quadratic function does not belong to its proper vector subspace. An orthogonal projection maps these vectors on this vector subspace of the space of unbalanced excesses (c.f. Meinhardt (2013c, Chap. 5-7)).

From this structure a replication result of a pre-kernel point can be attained. This is due that from the payoff set that contains the selected pre-kernel element, and which satisfies in addition the non-empty interior condition, a null space in the game space can be identified that allows a variation within the game parameter without affecting the pre-kernel properties of this payoff vector. Even though the values of the maximum surpluses have been varied, the set of most effective coalitions remains unaltered by the parameter change. Hence, a set of related games can be determined, which are linear independent, and possess the selected pre-kernel element of the default game as well as a pre-kernel point (c.f. Meinhardt (2013c, Sect. 7.6)). In the sequel of this paper, we will establish that the set of related games, which are derived from a default game exhibiting a singleton pre-kernel, must also possess the same unique pre-kernel, and therefore coincides with the pre-nucleolus. Notice, that these games need not necessarily be convex, average-convex, totally balanced, or zero-monotonic. They could belong to different subclasses of games, however, they must satisfy the non-empty interior condition. Moreover, we show that the pre-kernel correspondence in the game space restricted to the convex hull that is constituted by the extreme points, which are specified by the default and related games, is single-valued, and therefore continuous.

The structure of the paper is organized as follows: In the Section 2 we introduce some basic notations and definitions to investigate the coincidence of the pre-kernel with the pre-nucleolus. Section 3 provides the concept of the indirect function and gives a dual pre-kernel representation in terms of a solution set. In the next step, the notion of lexicographically smallest most effective coalitions is introduced in order to identify payoff equivalence classes on the domain of the objective function from which a pre-kernel element can be determined. Moreover, relevant concepts from Meinhardt (2013c) are reconsidered. Section 4 studies the uniqueness of the pre-kernel for related games. However, Section 5 investigates the continuity of the pre-kernel correspondence. In Section 6 some sufficient conditions are worked out under which the pre-nucleolus of a default game can preserve the pre-nucleolus property for related games. A few final remarks close the paper.

2 Some Preliminaries

A n-person cooperative game with side-payments is defined by an ordered pair $\langle N, v\rangle$. The set $N := \{1, 2, \ldots, n\}$ represents the player set and v is the characteristic function with $v : 2^N \to \mathbb{R}$, and the convention that $v(\emptyset) := 0$. Elements of N are denoted as players. A subset S of the player set N is called a coalition. The real number $v(S) \in \mathbb{R}$ is called the value or worth of a coalition $S \in 2^N$. However, the cardinality of the player set N is given by $n := |N|$, and that for a coalition S by $s := |S|$. We assume throughout that $v(N) > 0$ and $n \geq 2$ is valid. Formally, we identify a cooperative game by the vector $v := (v(S))_{S \subseteq N} \in \mathcal{G}^n = \mathbb{R}^{2^{|N|}}$, if no confusion can arise, whereas in case of ambiguity, we identify a game by $\langle N, v\rangle$.

A possible payoff allocation of the value $v(S)$ for all $S \subseteq N$ is described by the projection of a vector $\mathbf{x} \in \mathbb{R}^n$ on its $|S|$-coordinates such that $x(S) \leq v(S)$ for all $S \subseteq N$, where we identify the $|S|$-coordinates of the vector $\mathbf{x}$ with the corresponding measure on S, such that $x(S) := \sum_{k \in S} x_k$. The set of vectors $\mathbf{x} \in \mathbb{R}^n$ which satisfies the efficiency principle $v(N) = x(N)$ is called the **pre-imputation set** and it is defined by

$$\mathcal{I}^0(v) := \{\mathbf{x} \in \mathbb{R}^n \mid x(N) = v(N)\}, \tag{2.1}$$

where an element $\mathbf{x} \in \mathcal{I}^0(v)$ is called an pre-imputation.

Given a vector $\mathbf{x} \in \mathcal{I}^0(v)$, we define the **excess** of coalition S with respect to the pre-imputation $\mathbf{x}$ in the game $\langle N, v\rangle$ by

$$e^v(S, \mathbf{x}) := v(S) - x(S). \tag{2.2}$$

A non-negative (non-positive) excess of S at $\mathbf{x}$ in the game $\langle N, v\rangle$ represents a gain (loss) to the members of the coalition S unless the members of S do not accept the payoff distribution $\mathbf{x}$ by forming their own coalition which guarantees $v(S)$ instead of $x(S)$.

Take a game $v \in \mathcal{G}^n$. For any pair of players $i, j \in N, i \neq j$, the **maximum surplus** of player i over player j with respect to any pre-imputation $\mathbf{x} \in \mathcal{I}^0(v)$ is given by the maximum excess at $\mathbf{x}$ over the set of coalitions containing player i but not player j, thus

$$s_{ij}(\mathbf{x}, v) := \max_{S \in \mathcal{G}_{ij}} e^v(S, \mathbf{x}) \qquad \text{where } \mathcal{G}_{ij} := \{S \mid i \in S \text{ and } j \notin S\}. \tag{2.3}$$

The expression $s_{ij}(\mathbf{x}, v)$ describes the maximum amount at the pre-imputation $\mathbf{x}$ that player i can gain without the cooperation of player j. The set of all pre-imputations $\mathbf{x} \in \mathcal{I}^0(v)$ that balances the maximum surpluses for each distinct pair of players $i, j \in N, i \neq j$ is called the **pre-kernel** of the game v, and is defined by

$$\mathcal{P}r\mathcal{K}(v) := \left\{\mathbf{x} \in \mathcal{I}^0(v) \mid s_{ij}(\mathbf{x}, v) = s_{ji}(\mathbf{x}, v) \quad \text{for all } i, j \in N, i \neq j\right\}. \tag{2.4}$$

In order to define the pre-nucleolus $\nu(v)$ of a game $v \in \mathcal{G}^n$, take any $\mathbf{x} \in \mathbb{R}^n$ to define a 2^n-tuple vector $\theta(\mathbf{x})$ whose components are the excesses $e^v(S, \mathbf{x})$ of the 2^n coalitions $S \subseteq N$, arranged in decreasing order, that is,

$$\theta_i(\mathbf{x}) := e^v(S_i, \mathbf{x}) \geq e^v(S_j, \mathbf{x}) =: \theta_j(\mathbf{x}) \qquad \text{if} \qquad 1 \leq i \leq j \leq 2^n. \tag{2.5}$$

Ordering the so-called complaint or dissatisfaction vectors $\theta(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$ by the lexicographic order $\leq_L$ on $\mathbb{R}^n$, we shall write

$$\theta(\mathbf{x}) <_L \theta(\mathbf{y}) \qquad \text{if } \exists \text{ an integer } 1 \leq k \leq 2^n, \tag{2.6}$$

such that $\theta_i(\mathbf{x}) = \theta_i(\mathbf{y})$ for $1 \leq i < k$ and $\theta_k(\mathbf{x}) < \theta_k(\mathbf{y})$. Furthermore, we write $\theta(\mathbf{x}) \leq_L \theta(\mathbf{y})$ if either $\theta(\mathbf{x}) <_L \theta(\mathbf{y})$ or $\theta(\mathbf{x}) = \theta(\mathbf{y})$. Now the pre-nucleolus $\mathcal{P}r\mathcal{N}(v)$ over the pre-imputations set $\mathcal{I}^0(v)$ is defined by

$$\mathcal{P}r\mathcal{N}(v) = \left\{ \mathbf{x} \in \mathcal{I}^0(v) \mid \theta(\mathbf{x}) \leq_L \theta(\mathbf{y}) \; \forall \; \mathbf{y} \in \mathcal{I}^0(v) \right\}. \tag{2.7}$$

The **pre-nucleolus** of any game $v \in \mathcal{G}^n$ is non-empty as well as unique, and it is referred to as $\nu(v)$ if the game context is clear from the contents or $\nu(N, v)$ otherwise.

3 A Dual Pre-Kernel Representation

The concept of a Fenchel-Moreau generalized conjugation – also known as the indirect function of a characteristic function game – was introduced by Martinez-Legaz (1996), and provides the same information as the n-person cooperative game with transferable utility under consideration. This approach was successfully applied in Meinhardt (2013c) to give a dual representation of the pre-kernel solution of TU games by means of solution sets of a family of quadratic objective functions. In this section, we review some crucial results extensively studied in Meinhardt (2013c, Chap. 5 & 6) as the building blocks to investigate the single-valuedness of the pre-kernel correspondence.

Theorem 3.1 (Martinez-Legaz (1996))**.** *The indirect function $\pi : \mathbb{R}^n \to \mathbb{R}$ of any n-person TU game is a non-increasing polyhedral convex function such that*

(i) $\partial\pi(\mathbf{x}) \cap \{-1, 0\}^n \neq \emptyset \qquad \forall \mathbf{x} \in \mathbb{R}^n$,

(ii) $\{-1, 0\}^n \subset \bigcup_{\mathbf{x} \in \mathbb{R}^n} \partial\pi(\mathbf{x})$, *and*

(iii) $\min_{\mathbf{x} \in \mathbb{R}^n} \; \pi(\mathbf{x}) = 0$.

Conversely, if $\pi : \mathbb{R}^n \to \mathbb{R}$ satisfies (i)-(iii) then there exists an unique n-person TU game $\langle N, v \rangle$ having π as its indirect function, its characteristic function is given by

$$v(S) = \min_{\mathbf{x} \in \mathbb{R}^n} \left\{ \pi(\mathbf{x}) + \sum_{k \in S} x_k \right\} \qquad \forall \; S \subset N. \tag{3.1}$$

According to the above result, the associated **indirect function** $\pi : \mathbb{R}^n \to \mathbb{R}_+$ is given by:

$$\pi(\mathbf{x}) = \max_{S \subseteq N} \left\{ v(S) - \sum_{k \in S} x_k \right\} \qquad \forall \mathbf{x} \in \mathbb{R}^n, \tag{3.2}$$

whereas $\partial\pi$ is the subdifferential of the function π. Hence, $\partial\pi(\mathbf{x})$ is the set of all subgradients of π at $\mathbf{x}$, which is a closed convex set. A characterization of the pre-kernel in terms of the indirect function is due to Meseguer-Artola (1997). Here, we present this representation in its most general form, although we restrict ourselves to the the trivial coalition structure $\mathcal{B} = \{N\}$.

Proposition 3.1 (Meseguer-Artola (1997)). *For a TU game with indirect function π, a pre-imputation $\mathbf{x} \in \mathcal{I}^0(v)$ is in the pre-kernel of $\langle N, v \rangle$ for the coalition structure $\mathcal{B} = \{B_1, \ldots, B_l\}$, $\mathbf{x} \in \mathcal{P}r\mathcal{K}(v, \mathcal{B})$, if, and only if, for every $k \in \{1, 2, \ldots, l\}$, every $i, j \in B_k,\ i < j$, and some $\delta \geq \delta_1(v, \mathbf{x})$, one receives*

$$\pi(\mathbf{x}^{\,i,j,\delta}) = \pi(\mathbf{x}^{\,j,i,\delta}).$$

Meseguer-Artola (1997) was the first who recognized that based on the result of Proposition 3.1 a pre-kernel element can be derived as a solution of an over-determined system of non-linear equations. Every over-determined system can be equivalently expressed as a minimization problem. The set of global minima coalesces with the pre-kernel set. For the trivial coalition structure $\mathcal{B} = \{N\}$ the over-determined system of non-linear equations is given by

$$\begin{cases} f_{ij}(\mathbf{x}) = 0 & \forall i, j \in N, i < j \\ f_0(\mathbf{x}) = 0 \end{cases} \tag{3.3}$$

where, for some $\delta \geq \delta_1(\mathbf{x}, v)$,

$$f_{ij}(\mathbf{x}) := \pi(\mathbf{x}^{\,i,j,\delta}) - \pi(\mathbf{x}^{\,j,i,\delta}) \qquad \forall i, j \in N, i < j, \tag{3.3-a}$$

and

$$f_0(\mathbf{x}) := \sum_{k \in N} x_k - v(N). \tag{3.3-b}$$

To observe that the system above is over-determined one has to take into account that the differences f_{ij} in the maximum surpluses are residuals which define the corresponding vector functions of the system of non-linear equations. For the coalition structure $\mathcal{B} = \{N\}$, we count in total $n \cdot (n-1)/2$ residuals. This implies that the system must be over-determined, since we have $(n \cdot (n-1)/2 + 1)$ non-linear vector functions and only n unknown variables. Finally, notice that to any over-determined system an equivalent minimization problem is associated such that the set of global minima coincides with the solution set of the system. The solution set of such a minimization problem is the set of values for $\mathbf{x}$ which minimizes the following function

$$h(\mathbf{x}) := \sum_{\substack{i,j \in N \\ i<j}} (f_{ij}(\mathbf{x}))^2 + (f_0(\mathbf{x}))^2 \geq 0 \qquad \forall \mathbf{x} \in \mathbb{R}^n. \tag{3.4}$$

As we will notice in the sequel, this optimization problem is equivalent to a least squares adjustment. For further details see Meinhardt (2013c, Chap. 6).

Corollary 3.1 (Meinhardt (2013c)). *For a TU game $\langle N, v \rangle$ with indirect function π, it holds that*

$$h(\mathbf{x}) = \sum_{\substack{i,j \in N \\ i<j}} (f_{ij}(\mathbf{x}))^2 + (f_0(\mathbf{x}))^2 = \min_{\mathbf{y} \in \mathcal{I}^0(v)} h(\mathbf{y}) = 0, \tag{3.5}$$

if, and only if, $\mathbf{x} \in \mathcal{P}r\mathcal{K}(v)$.

Proof. To establish the equivalence between the pre-kernel set and the set of global minima, we have to notice that in view of Theorem 3.1 $0 = \min_{\mathbf{y}} h$ is in force. Now, we prove necessity while taking a pre-kernel element, i.e. $\mathbf{x} \in \mathcal{P}r\mathcal{K}(v)$, then the efficiency property is satisfied with $f_0(\mathbf{x}) = 0$ and the maximum surpluses $s_{ij}(\mathbf{x}, v)$ must be balanced for each distinct pair of players i, j, implying that $f_{ij}(\mathbf{x}) = 0$ for all $i, j \in N, i < j$ and therefore $h(\mathbf{x}) = 0$. Thus, we are getting $\mathbf{x} \in M(h)$. To prove sufficiency, assume that $\mathbf{x} \in M(h)$, then $h(\mathbf{x}) = 0$ with the implication that the efficiency property $f_0(\mathbf{x}) = 0$ and $f_{ij}(\mathbf{x}) = 0$ must be valid for all $i, j \in N, i < j$. This means that the difference $f_{ij}(\mathbf{x}) = (\pi(\mathbf{x}^{i,j,\delta}) - \pi(\mathbf{x}^{j,i,\delta}))$ is equalized for each distinct pair of indices $i, j \in N, i < j$. Thus, $\mathbf{x} \in \mathcal{P}r\mathcal{K}(v)$. It turns out that the minimum set coincides with the pre-kernel, i.e., we have:

$$M(h) = \{\mathbf{x} \in \mathcal{I}^0(v) \mid h(\mathbf{x}) = 0\} = \mathcal{P}r\mathcal{K}(v), \tag{3.6}$$

with this argument we are done. □

Corollary 3.1 gives an alternative characterization of the pre-kernel set in terms of a solution set. Singling out a pre-kernel element by solving the above minimization problem is, for instance, possible by a modified *Steepest Descent Method*. However, a direct method is not applicable. This is due to fact that the objective function h is the difference of two convex functions and that due to Theorem 3.1 the indirect function π is a non-increasing polyhedral convex function. This implies that function h is not continuous differentiable everywhere and that its structural form is ambiguous. Nevertheless, Proposition 6.2.2 (c.f. Meinhardt (2013c)) characterizes the objective function h as the composite of a finite family of quadratic functions. In the sequel, we do not discuss the whole details which would go beyond the scope of the paper, here we focus only on the aspect that the domain of function h can be partitioned into payoff equivalence classes. On each payoff equivalence class a quadratic and convex function can be identified. Pasting the finite collection of quadratic and convex functions together reproduces function h. For a thorough and more detailed discussion of this topic, we refer the reader to Section 5.4 and 6.2 in Meinhardt (2013c).

To understand the structural form of the objective function h, we will first identify equivalence relations on its domain. To start with, we define the set of **most effective** or **significant coalitions** for each pair of players $i, j \in N, i \neq j$ at the payoff vector $\mathbf{x}$ by

$$\mathcal{C}_{ij}(\mathbf{x}) := \left\{ S \in \mathcal{G}_{ij} \;\middle|\; s_{ij}(\mathbf{x}, v) = e^v(S, \mathbf{x}) \right\}. \tag{3.7}$$

This set determines all those coalitions of player i excluding the opponent j on which player i can rely upon to ensure his claim in a bilateral bargaining situation in order to split the gains through mutual cooperation. Gathering for all pair of player $i, j \in N, i \neq j$ all these coalitions that support the claim of a specific player over some other players, we have to consider the concept of the collection of most effective or significant coalitions w.r.t. $\mathbf{x}$, which we define as in Maschler et al. (1979, p. 315) by

$$\mathcal{C}(\mathbf{x}) := \bigcup_{\substack{i,j \in N \\ i \neq j}} \mathcal{C}_{ij}(\mathbf{x}). \tag{3.8}$$

Notice that this set generically has not cardinality one, and it might be too large to be suitable to identify an equivalence relations on the domain of function h. In order to derive an equivalence relation on the domain, we need to diminish this set while removing any form of ambiguity. By doing so, we rely on the idea that a player who has the opportunity to rely on two allies of equal strength but of different sizes for supporting his claim, has strong preference to the coalition with the smallest number of members, i.e. for those where he has to convince the fewest to support his demand.

From the set of most effective or significant coalitions of a pair of players $i, j \in N, i \neq j$ at the payoff vector $\mathbf{x}$ the smallest cardinality over the set of most effective coalitions is defined as

$$\Phi_{ij}(\mathbf{x}) := \min \left\{ |S| \;\middle|\; S \in \mathcal{C}_{ij}(\mathbf{x}) \right\}. \tag{3.9}$$

Gathering all these sets having smallest cardinality for all pairs of players $i, j \in N, i \neq j$, we end up with

$$\Psi_{ij}(\mathbf{x}) := \left\{ S \in \mathcal{C}_{ij}(\mathbf{x}) \;\middle|\; \Phi_{ij}(\mathbf{x}) = |S| \right\}. \tag{3.10}$$

For selecting a set from the collection of coalitions of equal size, we refer to the concept of a lexicographical order. Now, examine two coalitions having the same cardinality, namely coalition $S := \{i_1, \ldots, i_q\}$ and $T := \{j_1, \ldots, j_q\}$ with $2 \leq q \leq n-1$, coalition S is lexicographically smaller than coalition T if there is some integer k with $1 \leq k \leq q$ such that

$$i_l = j_l \qquad \text{for } 1 \leq l < k, \quad \text{and} \quad i_k < j_k.$$

This relation will be denoted by $S <_L T$.

With respect to an arbitrary payoff vector $\mathbf{x}$, the set of coalitions of smallest cardinality $\Psi_{ij}(\mathbf{x})$ which is minimized w.r.t. the lexicographically order $<_L$ is determined by

$$\mathcal{S}_{ij}(\mathbf{x}) := \left\{ S \in \Psi_{ij}(\mathbf{x}) \;\middle|\; S <_L T \text{ for all } S \neq T \in \Psi_{ij}(\mathbf{x}) \right\} \quad \forall i, j \in N, i \neq j. \tag{3.11}$$

We call this set, the **lexicographically smallest most effective coalitions** w.r.t. $\mathbf{x}$ of pair $i, j \in N, i \neq j$. This set is well defined and possesses cardinality one, i.e. $|\mathcal{S}_{ij}(\mathbf{x})| = 1$, which allows us to single out an unique coalition for this specific pair of players at $\mathbf{x}$. Then we are able to specify the set of lexicographically smallest most effective coalitions w.r.t. $\mathbf{x}$ through

$$\mathcal{S}(\mathbf{x}) := \left\{ \mathcal{S}_{ij}(\mathbf{x}) \;\middle|\; i, j \in N, i \neq j \right\}. \tag{3.12}$$

This set will be indicated in short as the set of **lexicographically smallest coalitions** or just more succinctly **most effective coalitions** whenever no confusion can arise. Notice that this set is never empty and can uniquely be identified. This implies that the cardinality of this set is equal to $n \cdot (n-1)$. In the following we will observe that from these type of sets equivalence relations on the domain $dom\, h$ can be identified.

To see this, consider the correspondence $\mathcal{S}$ on $dom\, h$ and two different vectors, say $\mathbf{x}$ and $\vec{\gamma}$, then both vectors are said to be equivalent w.r.t. the binary relation $\sim$ if, and only if, they induce the same set of lexicographically smallest coalitions, that is, $\mathbf{x} \sim \vec{\gamma}$ if, and only if, $\mathcal{S}(\mathbf{x}) = \mathcal{S}(\vec{\gamma})$. In case that the binary relation $\sim$ is reflexive, symmetric and transitive, then it is an **equivalence relation** and it induces **equivalence classes** $[\vec{\gamma}]$ on $dom\, h$ which we define through

$$[\vec{\gamma}] := \left\{ \mathbf{x} \in dom\ h \;\middle|\; \mathbf{x} \sim \vec{\gamma} \right\}. \tag{3.13}$$

Thus, if $\mathbf{x} \sim \vec{\gamma}$, then $[\mathbf{x}] = [\vec{\gamma}]$, and if $\mathbf{x} \nsim \vec{\gamma}$, then $[\mathbf{x}] \cap [\vec{\gamma}] = \emptyset$. This implies that whenever the binary relation $\sim$ induces equivalence classes $[\vec{\gamma}]$ on $dom\, h$, then it partitions the domain $dom\, h$ of the function h. The resulting collection of equivalence classes $[\vec{\gamma}]$ on $dom\, h$ is called the quotient of $dom\, h$ modulo $\sim$, and we denote this collection by $dom\, h/\sim$. We indicate this set as an equivalence class whenever the context is clear, otherwise we apply the term payoff set or payoff equivalence class.

Proposition 3.2. *The binary relation $\sim$ on the set $dom\, h$ defined by $\mathbf{x} \sim \vec{\gamma} \iff \mathcal{S}(\mathbf{x}) = \mathcal{S}(\vec{\gamma})$ is an equivalence relation, which forms a partition of the set $dom\, h$ by the collection of equivalence classes $\{[\vec{\gamma}_k]\}_{k \in J}$, where J is an arbitrary index set. Furthermore, for all $k \in J$, the induced equivalence class $[\vec{\gamma}_k]$ is a convex set .*

Proof. For a proof see Meinhardt (2013c, p. 59). □

The cardinality of the collection of the payoff equivalence classes induced by a TU game is finite (c.f. Meinhardt (2013c, Proposition 5.4.2.)). Furthermore, on each payoff equivalence class $[\vec{\gamma}]$ from the $dom\, h$ an unique quadratic and convex function can be identified. Therefore, there must be a finite composite of these functions that constitutes the objective function h. In order to construct such a quadratic and convex function suppose that $\vec{\gamma} \in [\vec{\gamma}]$. From this vector we attain the collection of most effective coalitions $\mathcal{S}(\vec{\gamma})$ in accordance with Proposition 3.2. Then observe that the differences in the values between a pair $\{i, j\}$ of players are defined by $\alpha_{ij} := (v(S_{ij}) - v(S_{ji})) \in \mathbb{R}$ for all $i, j \in N$, $i < j$, and $\alpha_0 := v(N) > 0$ w.r.t. $\mathcal{S}(\vec{\gamma})$. All of these q-components compose the q-coordinates of a payoff independent vector $\vec{\alpha}$, with $q = \binom{n}{2} + 1$. A vector that reflects the degree of unbalancedness of excesses for all pair of players, is denoted by $\vec{\xi} \in \mathbb{R}^q$, that is a q-column vector, which is given by

$$\begin{aligned} \xi_{ij} &:= e^v(S_{ij}, \vec{\gamma}) - e^v(S_{ji}, \vec{\gamma}) = v(S_{ij}) - \gamma(S_{ij}) - v(S_{ji}) + \gamma(S_{ji}) \quad \forall\, i, j \in N,\ i < j, \\ &= v(S_{ij}) - v(S_{ji}) + \gamma(S_{ji}) - \gamma(S_{ij}) = \alpha_{ij} + \gamma(S_{ji}) - \gamma(S_{ij}) \quad \forall\, i, j \in N,\ i < j, \\ \xi_0 &:= v(N) - \gamma(N) = \alpha_0 - \gamma(N). \end{aligned} \tag{3.14}$$

In view of Proposition 3.2, all vectors contained in the equivalence class $[\vec{\gamma}]$ induce the same set $\mathcal{S}(\vec{\gamma})$, and it holds

$$\xi_{ij} := e^v(S_{ij}, \vec{\gamma}) - e^v(S_{ji}, \vec{\gamma}) = s_{ij}(\vec{\gamma}, v) - s_{ji}(\vec{\gamma}, v) =: \zeta_{ij} \quad \forall\, i, j \in N,\ i < j. \tag{3.15}$$

The payoff dependent configurations $\vec{\xi}$ and $\vec{\zeta}$ having the following interrelationship outside its equivalence class: $\vec{\xi} \neq \vec{\zeta}$ for all $\mathbf{y} \in [\vec{\gamma}]^c$. Moreover, equation (3.15) does not necessarily mean that for $\vec{\gamma}', \vec{\gamma}^* \in [\vec{\gamma}]$, $\vec{\gamma}' \neq \vec{\gamma}^*$, it holds $\vec{\xi}' = \vec{\xi}^*$. Hence, the vector of (un)balanced excesses $\vec{\xi}$ is only equal with the vector of (un)balanced maximum surpluses $\vec{\zeta}$ if the corresponding pre-imputation $\vec{\gamma}$ is drawn from its proper equivalence class $[\vec{\gamma}]$.

In addition, we write for sake of simplicity that $\mathbf{E}_{ij} := (\mathbf{1}_{S_{ji}} - \mathbf{1}_{S_{ij}}) \in \mathbb{R}^n$, $\forall i, j \in N, i < j$, and $\mathbf{E}_0 := -\mathbf{1}_N \in \mathbb{R}^n$. Combining these q-column vectors, we can construct a $(n \times q)$-matrix in $\mathbb{R}^{n \times q}$ referred to as $\mathbf{E}$, and which is given by

$$\mathbf{E} := [\mathbf{E}_{1,2}, \ldots, \mathbf{E}_{n-1,n}, \mathbf{E}_0] \in \mathbb{R}^{n \times q}. \tag{3.16}$$

Proposition 3.3 (Quadratic Function)**.** *Let be $\langle N, v \rangle$ a TU game with indirect function π, then an arbitrary vector $\vec{\gamma}$ in the domain of h, i.e. $\vec{\gamma} \in dom\, h$, induces a quadratic function:*

$$h_\gamma(\mathbf{x}) = (1/2) \cdot \langle\, \mathbf{x}, \mathbf{Q}\, \mathbf{x} \,\rangle + \langle\, \mathbf{x}, \mathbf{a} \,\rangle + \alpha \qquad \mathbf{x} \in dom\, h, \tag{3.17}$$

where $\mathbf{a}$ is a column vector of coefficients, α is a scalar and $\mathbf{Q}$ is a symmetric $(n \times n)$-matrix with integer coefficients taken from the interval $[-n \cdot (n-1), n \cdot (n-1)]$.

Proof. The proof is given in Meinhardt (2013c, pp. 66-68). □

By the above discussion, the objective function h and the quadratic as well as convex function h_γ of type (3.17) coincide on the payoff set $[\vec{\gamma}]$ (c.f. Meinhardt (2013c, Lemma 6.2.2)). However, on the complement $[\vec{\gamma}]^c$ it holds $h \neq h_\gamma$.

Proposition 3.4 (Least Squares). *A quadratic function h_γ given by equation (3.17) is equivalent to*

$$\langle\, \vec{\alpha} + \mathbf{E}^\top\,\mathbf{x}, \vec{\alpha} + \mathbf{E}^\top\,\mathbf{x}\,\rangle = \|\,\vec{\alpha} + \mathbf{E}^\top\,\mathbf{x}\,\|^2. \tag{3.18}$$

Therefore, the matrix $\mathbf{Q} \in \mathbb{R}^{n^2}$ can also be expressed as $\mathbf{Q} = 2 \cdot \mathbf{E}\;\mathbf{E}^\top$, and the column vector $\mathbf{a}$ as $2 \cdot \mathbf{E}\;\vec{\alpha} \in \mathbb{R}^n$. Finally, the scalar α is given by $\|\vec{\alpha}\|^2$, where $\mathbf{E} \in \mathbb{R}^{n\times q}, \mathbf{E}^\top \in \mathbb{R}^{q\times n}$ and $\vec{\alpha} \in \mathbb{R}^q$.

Proof. The proof can be found in Meinhardt (2013c, pp. 70-71). □

Realize that the transpose of a vector or a matrix is denoted by the symbols $\mathbf{x}^\top$, and $\mathbf{Q}^\top$ respectively.

Lemma 3.1. *Let $\mathbf{x}, \vec{\gamma} \in dom\, h, \mathbf{x} = \vec{\gamma} + \mathbf{z}$ and let $\vec{\gamma}$ induces the matrices $\mathbf{E} \in \mathbb{R}^{n\times q}, \mathbf{E}^\top \in \mathbb{R}^{q\times n}$ determined by formula (3.16), and $\vec{\alpha}, \vec{\xi} \in \mathbb{R}^q$ as in equation (3.14). If $\mathbf{x} \in M(h_\gamma)$, then*

1. $-\mathbf{E}^\top\,\mathbf{x} = \mathbf{P}\,\vec{\alpha}$.
2. $\mathbf{E}^\top\,\vec{\gamma} = \mathbf{P}\,(\vec{\xi} - \vec{\alpha}) = (\vec{\xi} - \vec{\alpha})$.
3. $-\mathbf{E}^\top\,\mathbf{z} = \mathbf{P}\,\vec{\xi}$.

In addition, let $q := \binom{n}{2} + 1$. The matrix $\mathbf{P} \in \mathbb{R}^{q^2}$ is either equal to $2 \cdot \mathbf{E}^\top\,\mathbf{Q}^{-1}\mathbf{E}$, if the matrix $\mathbf{Q} \in \mathbb{R}^{n^2}$ is non-singular, or it is equal to $2 \cdot \mathbf{E}^\top\,\mathbf{Q}^\dagger\mathbf{E}$, if the matrix $\mathbf{Q}$ is singular. Furthermore, it holds for the matrix $\mathbf{P}$ that $\mathbf{P} \neq \mathbf{I}_q$ and rank $\mathbf{P} \leq n$.

Proof. The proof is given in Meinhardt (2013c, pp. 80-81). □

Notice that $\mathbf{Q}^\dagger$ is the **Moore-Penrose** or **pseudo-inverse** matrix of matrix $\mathbf{Q}$, if matrix $\mathbf{Q}$ is singular. This matrix is unique according to the following properties:

- $\mathbf{Q}\,\mathbf{Q}^\dagger\,\mathbf{Q} = \mathbf{Q}$ (general condition),
- $\mathbf{Q}^\dagger\,\mathbf{Q}\,\mathbf{Q}^\dagger = \mathbf{Q}^\dagger$ (reflexive condition),
- $(\mathbf{Q}\,\mathbf{Q}^\dagger)^\top = \mathbf{Q}^\dagger\,\mathbf{Q}$ (normalized condition),
- $(\mathbf{Q}^\dagger\,\mathbf{Q})^\top = \mathbf{Q}\,\mathbf{Q}^\dagger$ (reversed normalized condition).

Proposition 3.5 (Orthogonal Projection Operator). *Matrix $\mathbf{P}$ is idempotent and self-adjoint, i.e. $\mathbf{P}$ is an orthogonal projection operator.*

Proof. The proof can be found in Meinhardt (2013c, p. 86). □

Lemma 3.2. *Let $\mathcal{E}$ be a subspace of $\mathbb{R}^q$ with basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ derived from the linear independent vectors of matrix $\mathbf{E}^\top$ having rank m, with $m \leq n$, and let $\{\mathbf{w}_1, \ldots, \mathbf{w}_{q-m}\}$ be a basis of $\mathcal{W} := \mathcal{E}^\perp$. In addition, define matrix $E^\top := [\mathbf{e}_1, \ldots, \mathbf{e}_m] \in \mathbb{R}^{q\times m}$, and matrix $W^\top := [\mathbf{w}_1, \ldots, \mathbf{w}_{q-m}] \in \mathbb{R}^{q\times(q-m)}$, then for any $\vec{\beta} \in \mathbb{R}^q$ it holds*

1. $\vec{\beta} = [E^\top\; W^\top] \cdot \mathbf{c}$ *where $\mathbf{c} \in \mathbb{R}^q$ is a coefficient vector, and*
2. *the matrix $[E^\top\; W^\top] \in \mathbb{R}^{q\times q}$ is invertible.*

Proof. For a proof see Meinhardt (2013c, pp. 90-91). □

Notice that $\mathcal{E}$ can be interpreted as indicating a vector subspace of balanced excesses. A pre-imputation will be mapped into its proper vector subspace of balanced excesses $\mathcal{E}$, i.e. the vector subspace induced by the pre-imputation. However, the corresponding vector of (un)balanced excesses generated by this pre-imputation is an element of this vector subspace of balanced excesses, if the pre-imputation is also a pre-kernel point. Hence, the vector of balanced excesses coincides with the vector of balanced maximum surpluses. This is a consequence of Lemma 3.1 or see Proposition 8.4.1 in Meinhardt (2013c). Otherwise, this vector of unbalanced excesses will be mapped by the orthogonal projection $\mathbf{P}$ on $\mathcal{E}$. More information about the properties of this kind of vector subspace can be found in Meinhardt (2013c, pp. 87-113 and 138-168).

Proposition 3.6 (Positive General Linear Group)**.** *Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ as well as $\{\mathbf{e}^1_1, \ldots, \mathbf{e}^1_m\}$ be two ordered bases of the subspace $\mathcal{E}$ derived from the payoff sets $[\vec{\gamma}]$ and $[\vec{\gamma}_1]$, respectively. In addition, define the associated basis matrices $E^\top, E_1^\top \in \mathbb{R}^{q \times m}$ as in Lemma 3.2, then the unique transition matrix $X \in \mathbb{R}^{m^2}$ such that $E_1^\top = E^\top X$ is given, is an element of the positive general linear group, that is $X \in GL^+(m)$.*

Proof. The proof can be found in Meinhardt (2013c, p. 101). □

Proposition 3.6 denotes two payoff sets $[\vec{\gamma}]$ and $[\vec{\gamma}_1]$ as equivalent, if there exists a transition matrix X from the positive general linear group, that is $X \in \mathrm{GL}^+(m)$, such that $E_1^\top = E^\top X$ is in force. Notice that the transition matrix X must be unique (c.f. Meinhardt (2013c, p. 102)). The underlying group action (c.f. Meinhardt (2013c, Corollary 6.6.1)) can be interpreted that a bargaining situation is transformed into an equivalent bargaining situation. For a thorough discussion of a group action onto the set of all ordered bases, the interested reader should consult Meinhardt (2013c, Sect. 6.6).

The vector space $\mathbb{R}^q$ is an orthogonal decomposition by the subspaces $\mathcal{E}$ and $\mathcal{N}_{\mathbf{E}}$. We denote in the sequel a basis of the orthogonal complement of space $\mathcal{E}$ by $\{\mathbf{w}_1, \ldots, \mathbf{w}_{q-m}\}$. This subspace of $\mathbb{R}^q$ is identified by $\mathcal{W} := \mathcal{N}_{\mathbf{E}} = \mathcal{E}^\perp$. In addition, we have $\mathbf{P}\,\mathbf{w}_k = \mathbf{0}$ for all $k \in \{1, \ldots, q-m\}$. Thus, we can obtain the following corollary

Corollary 3.2 (Meinhardt (2013c))**.** *If $\vec{\gamma}$ induces the matrices $\mathbf{E} \in \mathbb{R}^{n \times q}, \mathbf{E}^\top \in \mathbb{R}^{q \times n}$ determined by formula (3.16), then with respect to the Euclidean inner product, getting*

1. $\mathbb{R}^q = \mathcal{E} \oplus \mathcal{W} = \mathcal{E} \oplus \mathcal{E}^\perp$.

A consequence of the orthogonal projection method presented by the next theorem and corollary is that every payoff vector belonging to the intersection of the minimum set of function h_γ and its payoff equivalence class $[\vec{\gamma}]$ is a pre-kernel element. This due to $h_\gamma = h$ on $[\vec{\gamma}]$.

Theorem 3.2 (Orthogonal Projection Method)**.** *Let $\vec{\gamma}_k \in [\vec{\gamma}]$ for $k = 1,2,3$. If $\vec{\gamma}_2 \in M(h_\gamma)$ and $\vec{\gamma}_k \notin M(h_\gamma)$ for $k = 1,3$, then $\vec{\zeta}_2 = \vec{\xi}_2 = \mathbf{0}$, and consequently $\vec{\gamma}_2 \in \mathcal{P}r\mathcal{K}(v)$.*

Proof. For a proof see Meinhardt (2013c, pp. 109-111). □

Corollary 3.3 (Meinhardt (2013c))**.** *Let be $[\vec{\gamma}]$ an equivalence class of dimension $3 \leq m \leq n$, and $\mathbf{x} \in M(h_\gamma) \cap [\vec{\gamma}]$, then $\vec{\alpha} = \mathbf{P}\,\vec{\alpha}$, and consequently $\mathbf{x} \in \mathcal{P}r\mathcal{K}(v)$.*

4 The Uniqueness of the Pre-Kernel

To study the uniqueness of the pre-kernel solution of a related TU game derived from a pre-kernel element of a default game, we need to know: (1) if the linear mapping of a pre-kernel element into a specific vector subspace of balanced excesses $\mathcal{E}$ consists of a single point, and (2) that there cannot exist any other non-transversal vector subspace of balanced excesses $\mathcal{E}_1$ in which a linear transformation of pre-kernel element can be mapped. (3) It must be shown that the pre-kernel coincides with the pre-nucleolus of the set of related games, otherwise, it is obvious that there must exist at least a second pre-kernel point, namely the pre-nucleolus.

For conducting this line of investigation some additional concepts are needed. In a first step we introduce the definition of a **unanimity game**, which is indicated by

$$\mathbf{u}_T(S) := \begin{cases} 1 & T \subseteq S \\ 0 & \text{otherwise,} \end{cases}$$

whereas $T \subseteq N, T \neq \emptyset$, which forms a **unanimity/game basis**. A formula to express the coordinates of this basis is given by

$$v = \sum_{\substack{T \subset N, \\ T \neq \emptyset}} \lambda_T^v \, \mathbf{u}_T \iff \lambda_T^v = \sum_{\substack{S \subset T, \\ S \neq \emptyset}} (-1)^{t-s} \cdot v(S),$$

if $\langle N, v\rangle$, where $|S| = s$, and $|T| = t$. A coordinate λ_T^v is said to be an unanimity coordinate of game $\langle N, v\rangle$, and vector λ^v is called the unanimity coordinates of game $\langle N, v\rangle$. Notice that we assume here that the game is defined in $\mathbb{R}^{2^n-1}$ rather than $\mathbb{R}^{2^n}$, since we want to write for sake of convenience the **game basis** in matrix form without a column and row of zeros. Thus we write

$$v = \sum_{\substack{T \subset N, \\ T \neq \emptyset}} \lambda_T^v \, \mathbf{u}_T = [\mathbf{u}_{\{1\}}, \ldots, \mathbf{u}_{\{N\}}] \, \lambda^v = \boldsymbol{\mathcal{U}} \, \lambda^v$$

where the unanimity basis $\boldsymbol{\mathcal{U}}$ is in $\mathbb{R}^{p' \times p'}$ with $p' = 2^n - 1$. In addition, define the **unity games (Dirac games)** $\mathbf{1}^T$ for all $T \subseteq N$ by

$$\mathbf{1}^T(S) := \begin{cases} 1 & \text{if } T = S, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

In the next step, we select a payoff vector $\vec{\gamma}$, which also determines its payoff set $[\vec{\gamma}]$. With regard to Proposition 3.2, this vector induces in addition a set of lexicographically smallest most effective coalitions indicated by $\mathcal{S}(\vec{\gamma})$. Implying that we get the configuration $\vec{\alpha}$ by the q-coordinates $\alpha_{ij} := (v(S_{ij}) - v(S_{ji})) \in \mathbb{R}$ for all $i, j \in N, i < j$, and $\alpha_0 := v(N)$. Furthermore, we can also define a set of vectors as the differences of unity games (4.1) w.r.t. the set of lexicographically smallest most effective coalitions, which is given by

$$\mathbf{v}_{ij} := \mathbf{1}^{S_{ij}} - \mathbf{1}^{S_{ji}} \quad \text{for } S_{ij}, S_{ji} \in \mathcal{S}(\vec{\gamma}) \quad \text{and} \quad \mathbf{v}_0 := \mathbf{1}^N, \tag{4.2}$$

whereas $\mathbf{v}_{ij}, \mathbf{v}_0 \in \mathbb{R}^{p'}$ for all $i, j \in N, i < j$. With these column vectors, we can identify matrix $\boldsymbol{\mathcal{V}} := [\mathbf{v}_{1,2}, \ldots, \mathbf{v}_{n-1,n}, \mathbf{v}_0] \in \mathbb{R}^{p' \times q}$. Then we obtain $\vec{\alpha} = \boldsymbol{\mathcal{V}}^\top v$ with $v \in \mathbb{R}^{p'}$ due to the removed empty set. Moreover, by the measure $y(S) := \sum_{k \in S} y_k$ for all $\emptyset \neq S \subseteq N$, we extend every payoff vector $\mathbf{y}$ to a vector $\overline{\mathbf{y}} \in \mathbb{R}^{p'}$, and define the excess vector at $\mathbf{y}$ by $\overline{e}_{\mathbf{y}} := v - \overline{\mathbf{y}} \in \mathbb{R}^{p'}$, then we get $\vec{\xi}_{\mathbf{y}} = \boldsymbol{\mathcal{V}}^\top \overline{e}_{\mathbf{y}}$. From matrix $\boldsymbol{\mathcal{V}}^\top$, we can also derive an orthogonal projection $\mathbf{P}_{\mathcal{V}}$ specified by $\boldsymbol{\mathcal{V}}^\top (\boldsymbol{\mathcal{V}}^\top)^\dagger \in \mathbb{R}^{q \times q}$ such that $\mathbb{R}^q = \mathcal{V} \oplus \mathcal{V}^\perp$ is valid, i.e. the rows of matrix $\boldsymbol{\mathcal{V}}^\top$ are a spanning system of the vector subspace $\mathcal{V} \subseteq \mathbb{R}^{q \times q}$, thus $\mathcal{V} := span\{\mathbf{v}_{1,2}^\top, \ldots, \mathbf{v}_{n-1,n}^\top, \mathbf{v}_0^\top\}$. Vector subspace $\mathcal{V}$ reflects the power of the set of lexicographically smallest most effective coalitions. In contrast, vector subspace $\mathcal{E}$ reflects the ascribed unbalancedness in the coalition power w.r.t. the bilateral bargaining situation attained at $\vec{\gamma}$ through $\mathcal{S}(\vec{\gamma})$. The next results show how these vector subspaces are intertwined.

Lemma 4.1. *Let $\mathbf{E}^{\top} \in \mathbb{R}^{q \times n}$ be defined as in Equation (3.16), $\mathcal{V}^{\top} \in \mathbb{R}^{q \times p'}$ as by Equation (4.2), then there exists a matrix $\mathbf{Z}^{\top} \in \mathbb{R}^{p' \times n}$ such that $\mathbf{E}^{\top} = \mathcal{V}^{\top}\,\mathbf{Z}^{\top}$ if, and only if, $\mathcal{R}_{\mathbf{E}^{\top}} \subseteq \mathcal{R}_{\mathcal{V}^{\top}}$, that is, $\mathcal{E} \subseteq \mathcal{V}$.*

Proof. The proof is given in Meinhardt (2013c, p. 141). □

Notice that the minimal rank of matrix $\mathcal{V}^{\top}$ is bounded by $\mathbf{E}^{\top}$ which is equal to $m < n$ with the consequence that we get in this case $\mathcal{V} = \mathcal{E}$. However, the maximal rank is equal to q, and then $\mathcal{V} = \mathbb{R}^{q}$ (c.f. Meinhardt (2013c, Corollary 7.4.1)).

Lemma 4.2. *Let $\vec{\alpha}, \vec{\xi} \in \mathbb{R}^{q}$ as in Equation (3.14), then the following relations are satisfied on the vector space $\mathcal{V}$:*

1. $\mathbf{P}_{\mathcal{V}}\,\vec{\alpha} = \vec{\alpha} \in \mathcal{V}$
2. $\mathbf{P}_{\mathcal{V}}\,\vec{\xi} = \vec{\xi} \in \mathcal{V}$
3. $\mathbf{P}_{\mathcal{V}}\,(\vec{\xi} - \vec{\alpha}) = (\vec{\xi} - \vec{\alpha}) \in \mathcal{V}$
4. $\mathbf{P}_{\mathcal{V}}\,\mathbf{E}^{\top} = \mathbf{P}\,\mathbf{E}^{\top} = \mathbf{E}^{\top}$, *hence* $\mathcal{E} \subseteq \mathcal{V}$
5. $\mathbf{P}_{\mathcal{V}}\,\mathbf{P} = \mathbf{P}$, *hence* $\mathcal{E} \subseteq \mathcal{V}$
6. $\mathbf{E}\,\mathbf{P}_{\mathcal{V}} = \mathbf{E}\,\mathbf{P} = \mathbf{E}$, *hence* $\mathcal{R}_{\mathbf{E}} \subseteq \mathcal{V}$
7. $\mathbf{P}\,\mathbf{P}_{\mathcal{V}} = \mathbf{P}$, *hence* $\mathcal{E} \subseteq \mathcal{V}$.

Proof. For a proof see Meinhardt (2013c, p. 142). □

It was worked out by Meinhardt (2013c, Sect. 7.6) that a pre-kernel element of a specific game can be replicated as a pre-kernel element of a related game whenever the non-empty interior property of the payoff set, in which the pre-kernel element of default game is located, is satisfied. In this case, a full dimensional ellipsoid can be inscribed from which some bounds can be specified within the game parameter can be varied without destroying the pre-kernel properties of the payoff vector of the default game. These bounds specify a redistribution of the bargaining power among coalitions while supporting the selected pre-imputation still as a pre-kernel point. Although the values of the maximum excesses have been changed by the parameter variation, the set of lexicographically smallest most significant coalitions remains unaffected.

Theorem 4.1 (Replication). *If $[\vec{\gamma}]$ has non-empty interior and $\mathbf{x} \in \mathcal{P}r\mathcal{K}(v) \subset [\vec{\gamma}]$, then $\mathbf{x} \in \mathcal{P}r\mathcal{K}(v^{\mu})$ for all $\mu \cdot v^{\Delta} \in [-\mathsf{C}, \mathsf{C}]^{p'}$, where $v^{\mu} = v + \mu \cdot v^{\Delta} \in \mathbb{R}^{p'}$, $\mu \in \mathbb{R}$*

$$\mathsf{C} := \min_{i,j \in N, i \neq j} \left\{ \left| \frac{\pm\sqrt{\bar{c}}}{\|\mathbf{E}^{\top}(\mathbf{1}_j - \mathbf{1}_i)\|} \right| \right\}, \tag{4.3}$$

and $\mathbf{0} \neq \Delta \in \mathcal{N}_{\mathcal{W}} = \{\Delta \in \mathbb{R}^{p'} \mid \mathcal{W}\Delta = \mathbf{0}\}$ with matrix $\mathcal{W} := \mathcal{V}^{\top}\,\mathcal{U}$.

Proof. The proof is given in Meinhardt (2013c, p. 156). □

It was also shown there by some examples that the specified bounds by Theorem 4.1 are not tight, in the sense that pre-kernel points belonging to the relative interior of a payoff set can also be the object of a replication. However, pre-kernel elements which are located on the relative boundary of a payoff set are probably not replicable. Therefore, there must exist a more general rule to reproduce a pre-kernel element for a related game v^{μ}.

In the course of our discussion, we establish that the single pre-kernel element of a default game which is an interior point of a payoff set is also the singleton pre-kernel of the derived related games. In a first step, we show that there exists an unique linear transformation of the pre-kernel point of a related game into the vector subspace of balanced excesses $\mathcal{E}$. This means, there is no other pre-kernel element in a payoff equivalence class that belongs to the same set of ordered bases, i.e. reflecting an equivalent bargaining situation with a division of the proceeds of mutual cooperation in accordance with the pre-kernel solution. Secondly, we prove that there cannot exist any other vector subspace of balanced excesses $\mathcal{E}_1$ non-transversal to $\mathcal{E}$ in which a pre-kernel vector can be mapped by a linear transformation. That is, there exists no other non-equivalent payoff set in which an other pre-kernel point can be located.

Lemma 4.3 (Meinhardt (2013c)). *Let $\vec{\gamma}$ induces matrix $\mathbf{E}$, then*

$$(\mathbf{E}^\top)^\dagger = 2 \cdot \mathbf{Q}^\dagger \mathbf{E} \in \mathbb{R}^{n \times q}.$$

Proof. Remind from Lemma 3.1 that $\mathbf{P} = 2 \cdot \mathbf{E}^\top \, \mathbf{Q}^\dagger \mathbf{E}$ holds. In addition, note that we have the following relation $\mathbf{Q}^\dagger \mathbf{Q} = (\mathbf{E}^\top)^\dagger \, \mathbf{E}^\top$ which is an orthogonal projection onto $\mathcal{R}_{\mathbf{E}}$. Then attaining

$$\begin{aligned} 2 \cdot \mathbf{Q}^\dagger \mathbf{E} &= 2 \cdot \mathbf{Q}^\dagger \mathbf{Q} \mathbf{Q}^\dagger \mathbf{E} = 2 \cdot (\mathbf{E}^\top)^\dagger \mathbf{E}^\top \mathbf{Q}^\dagger \mathbf{E} \\ &= (\mathbf{E}^\top)^\dagger (2 \cdot \mathbf{E}^\top \mathbf{Q}^\dagger \mathbf{E}) = (\mathbf{E}^\top)^\dagger \mathbf{P} = (\mathbf{E}^\top)^\dagger. \end{aligned}$$

The last equality follows from Lemma 4.2. This argument terminates the proof. □

Proposition 4.1 (Meinhardt (2013c)). *Let $E_1^\top = E^\top \, X$ with $X \in SO(n)$, that is $[\vec{\gamma}] \sim [\vec{\gamma}_1]$, and suppose $\vec{\alpha}_1 = \mathcal{V}^\top \, v^\mu$. In addition, assume that the payoff equivalence class $[\vec{\gamma}]$ induced from TU game $\langle\, N, v \,\rangle$ has non-empty interior such that $\{\mathbf{x}\} = \mathcal{P}r\mathcal{K}(v) \subset [\vec{\gamma}]$ is satisfied, then there exists no other pre-kernel element in payoff equivalence class $[\vec{\gamma}_1]$ for a related TU game $\langle\, N, v^\mu \,\rangle$, where $v^\mu = v + \mu \cdot v^\Delta \in \mathbb{R}^{p'}$, as defined by Theorem 4.1.*

Proof. By the way of contradiction suppose that $\mathbf{x}, \mathbf{y} \in \mathcal{P}r\mathcal{K}(v^\mu)$ with $\mathbf{y} \in [\vec{\gamma}_1]$ is valid. Then we get

$$h^{v^\mu}(\mathbf{x}) = h^{v^\mu}_{\gamma}(\mathbf{x}) = \|\mathbf{E}^\top \, \mathbf{x} + \vec{\alpha}\|^2 = 0 \quad \text{and} \quad h^{v^\mu}(\mathbf{y}) = h^{v^\mu}_{\gamma_1}(\mathbf{y}) = \|\mathbf{E}_1^\top \, \mathbf{y} + \vec{\alpha}_1\|^2 = 0,$$

implying that

$$\mathbf{P}\,\vec{\alpha} = \vec{\alpha} \in \mathcal{E} \qquad \text{and} \qquad \mathbf{P}\,\vec{\alpha}_1 = \vec{\alpha}_1 \in \mathcal{E}.$$

Hence, we have

$$\mathbf{P}\,\vec{\alpha} - \vec{\alpha} = \mathbf{P}\,\vec{\alpha}_1 - \vec{\alpha}_1 = \mathbf{0} \in \mathcal{E} \iff \mathbf{P}\,(\vec{\alpha} - \vec{\alpha}_1) = (\vec{\alpha} - \vec{\alpha}_1) \in \mathcal{E}.$$

Therefore, obtaining the equivalent expression

$$\mathbf{E}^\top \, (X\,\mathbf{y} - \mathbf{x}) = (\vec{\alpha} - \vec{\alpha}_1) = \mathcal{V}^\top \, v - \mathcal{V}^\top \, (v + \mu \cdot v^\Delta) = \mathbf{0},$$

then $\mathbf{x} = X\,\mathbf{y}$, since matrix $\mathbf{E}^\top$ has full rank due to $\{\mathbf{x}\} = \mathcal{P}r\mathcal{K}(v)$. Furthermore, notice that

$$\langle\, \mathbf{x}, \mathbf{y} \,\rangle = \langle\, (\mathbf{E}^\top)^\dagger \, \vec{\alpha}, (\mathbf{E}_1^\top)^\dagger \, \vec{\alpha}_1 \,\rangle = \langle\, (\mathbf{E}^\top)^\dagger \, \vec{\alpha}, X^{-1} \, (\mathbf{E}^\top)^\dagger \, \vec{\alpha} \,\rangle = \langle\, 2\, \mathbf{Q}^\dagger \, \mathbf{E}\,\vec{\alpha}, 2\, X^{-1} \, \mathbf{Q}^\dagger \, \mathbf{E}\,\vec{\alpha} \,\rangle \neq \mathbf{0}$$

Matrix $\mathbf{E}^\top$ has full rank, and $\mathbf{Q}$ is symmetric and positive definite, hence $\mathbf{Q}^\dagger = \mathbf{Q}^{-1}$, and the above expression can equivalently be written as

$$\begin{aligned} \langle\, \mathbf{Q}^\dagger \, \mathbf{a}, X^{-1} \, \mathbf{Q}^\dagger \, \mathbf{a} \,\rangle &= \langle\, \mathbf{Q}^{-1} \, \mathbf{a}, X^{-1} \, \mathbf{Q}^{-1} \, \mathbf{a} \,\rangle = \langle\, \mathbf{a}, \mathbf{Q}\, X^{-1} \mathbf{Q}^{-1} \, \mathbf{a} \,\rangle \\ &= \langle\, \mathbf{a}, X_1 \mathbf{a} \,\rangle = \langle\, \mathbf{a}, \mathbf{a}_1 \,\rangle \neq \mathbf{0}, \end{aligned} \tag{4.4}$$

while using $\mathbf{a} = 2\,\mathbf{E}\,\vec{\alpha}$ from Proposition 3.4, and with similar matrix $X_1 = \mathbf{Q}\,X^{-1}\mathbf{Q}^{-1}$ as well as $\mathbf{a}_1 = X_1\,\mathbf{a}$. According to $\mathbf{E}_1^\top = \mathbf{E}^\top\,X$ with $X \in \text{SO}(n)$, we can write $X = \mathbf{Q}^{-1}(2\,\mathbf{E}\,\mathbf{E}_1^\top)$. But then

$$X_1 = \mathbf{Q}\,X^{-1}\mathbf{Q}^{-1} = \mathbf{Q}\,(2\,\mathbf{E}\,\mathbf{E}_1^\top)^{-1}.$$

Since we have $X \in \text{SO}(n)$, it holds $X^{-1} = X^\top$ implying that

$$X_1^\top = X^{-1} = (2\,\mathbf{E}\,\mathbf{E}_1^\top)^{-1}\,\mathbf{Q} = (2\,\mathbf{E}\,\mathbf{E}_1^\top)\,\mathbf{Q}^{-1} = X^\top = X_1^{-1},$$

which induces $X = \mathbf{Q}^{-1}\,(2\,\mathbf{E}\,\mathbf{E}_1^\top) = \mathbf{Q}\,(2\,\mathbf{E}\,\mathbf{E}_1^\top)^{-1} = X_1$. Now, observe

$$\begin{aligned} X_1 &= \mathbf{Q}\,X^{-1}\mathbf{Q}^{-1} = \mathbf{Q}\,X^\top\mathbf{Q}^{-1} = \mathbf{Q}\,(2\,\mathbf{E}\,\mathbf{E}_1^\top)\,\mathbf{Q}^{-1}\,\mathbf{Q}^{-1} \\ &= \mathbf{Q}\,(2\,\mathbf{E}\,\mathbf{E}^\top\,X)\,\mathbf{Q}^{-2} = \mathbf{Q}^2\,X\,\mathbf{Q}^{-2}, \end{aligned}$$

hence, we can conclude that $X = \mathbf{I}$ implying $X_1 = \mathbf{I}$ as well. We infer that $\mathbf{x} = \mathbf{y}$ contradicting the assumption $\mathbf{x} \neq \mathbf{y}$ due to $\mathbf{x} \in [\vec{\gamma}]$, and $\mathbf{y} \in [\vec{\gamma}_1]$. With this argument we are done. $\square$

Proposition 4.2. *Impose the same conditions as under Proposition 4.1 with the exception that $X \in GL^+(n)$, then there exists no other pre-kernel element in payoff equivalence class $[\vec{\gamma}_1]$ for a related TU game $\langle\, N, v^\mu \,\rangle$.*

Proof. By the proof of Proposition 4.1 the system of linear equations $\mathbf{E}^\top\,(X\,\mathbf{y} - \mathbf{x}) = \mathbf{0}$ is consistent, then we get $\mathbf{x} = X\,\mathbf{y}$ by the full rank of matrix $\mathbf{E}^\top$. By Equation 4.4 we obtain similar matrix $X_1 = \mathbf{Q}\,X^{-1}\mathbf{Q}^{-1}$, hence the matrix X_1 is in the same orbit (conjugacy class) as matrix X^{-1}, this implies that $E^\top = E_1^\top\,X^{-1} = E_1^\top\,X_1$ must be in force. But then $E^\top = E^\top\,X\,X_1$, which requires that $X\,X_1 = \mathbf{I}$ must be satisfied in accordance with the uniqueness of the transition matrix $X \in \text{GL}^+(m)$ (c.f. Meinhardt (2013c, p. 102)). In addition, we have $\mathbf{a}_1 = X_1\,\mathbf{a}$ as well as $\mathbf{a}_1 = 2\,\mathbf{E}_1\,\vec{\alpha} = X\,\mathbf{a}$. Therefore, we obtain $X\,\mathbf{a}_1 = \mathbf{a} = X^2\,\mathbf{a}$. From this we draw the conclusion in connection with the uniqueness of the transition matrix X that $X = \mathbf{I}$ is valid. Hence, $\mathbf{x} = \mathbf{y}$ as required. $\square$

Proposition 4.3. *Assume $[\vec{\gamma}] \sim [\vec{\gamma}_1]$, and that the payoff equivalence class $[\vec{\gamma}]$ induced from TU game $\langle\, N, v \,\rangle$ has non-empty interior such that $\{\mathbf{x}\} = \mathcal{P}r\mathcal{K}(v) \subset [\vec{\gamma}]$ is satisfied, then there exists no other pre-kernel element in payoff equivalence class $[\vec{\gamma}_1]$ for a related TU game $\langle\, N, v^\mu \,\rangle$, where $v^\mu = v + \mu \cdot v^\Delta \in \mathbb{R}^{p'}$, as defined by Theorem 4.1.*

Proof. We have to establish that there is no other element $\mathbf{y} \in \mathcal{P}r\mathcal{K}(v^\mu)$ such that $\mathbf{y} \in [\vec{\gamma}_1]$ is valid, whereas $\mathbf{y} \notin \mathcal{P}r\mathcal{K}(v)$ in accordance with the uniqueness of the pre-kernel for game v. In view of Theorem 4.1 the pre-kernel $\{\mathbf{x}\} = \mathcal{P}r\mathcal{K}(v)$ of game $\langle\, N, v \,\rangle$ is also a pre-kernel element of the related game $\langle\, N, v^\mu \,\rangle$, i.e. $\mathbf{x} \in \mathcal{P}r\mathcal{K}(v^\mu)$ with $\mathbf{x} \in [\vec{\gamma}]$ due to Corollary 3.2.

Extend the payoff element $\mathbf{y}$ to a vector $\overline{\mathbf{y}}$ by the measure $y(S) := \sum_{k \in S} y_k$ for all $S \in 2^n \backslash \{\emptyset\}$, then define the excess vector by $\overline{e}^\mu := v^\mu - \overline{\mathbf{y}}$. Moreover, compute the vector of (un)balanced excesses $\vec{\xi}^{v^\mu}$ at $\mathbf{y}$ for game v^μ by $\mathcal{V}_1^\top\,\overline{e}^\mu$. This vector is also the vector of (un)balanced maximum surpluses, since $\mathbf{y} \in [\vec{\gamma}_1]$, and therefore $h^{v^\mu} = h^{v^\mu}_{\gamma_1}$ on $[\vec{\gamma}_1]$ in view of Lemma 6.2.2 by Meinhardt (2013c). Notice that in order to have a pre-kernel element at $\mathbf{y}$ for the related game v^μ it must hold $\vec{\xi}^{v^\mu} = \mathbf{0}$. In addition, by hypothesis $[\vec{\gamma}] \sim [\vec{\gamma}_1]$, it must hold $\mathbf{E}^\top = \mathcal{V}^\top\,\mathbf{Z}^\top$ and $\mathbf{E}_1^\top = \mathcal{V}_1^\top\,\mathbf{Z}^\top$ in view of Lemma 4.1, thus $E_1^\top \neq E^\top\,X$ for all $X \in \text{GL}^+(n)$. This implies that we derive the corresponding matrices $\mathcal{W} := \mathcal{V}^\top\,\mathcal{U}$ and $\mathcal{W}_1 := \mathcal{V}_1^\top\,\mathcal{U}$, respectively.

We have to consider two cases, namely $\Delta \in \mathcal{N}_{\mathcal{W}} \cap \mathcal{N}_{\mathcal{W}_1}$ and $\Delta \in \mathcal{N}_{\mathcal{W}} \backslash \mathcal{N}_{\mathcal{W}_1}$.

1. Suppose $\Delta \in \mathcal{N}_{\mathcal{W}} \cap \mathcal{N}_{\mathcal{W}_1}$, then we get

$$\vec{\xi}^{v^\mu} = \mathcal{V}_1^\top \, \overline{e}^\mu = \mathcal{V}_1^\top \, (v^\mu - \overline{\mathbf{y}}) = \mathcal{V}_1^\top \, (v - \overline{\mathbf{y}} + \mu \cdot v^\Delta) = \mathcal{V}_1^\top \, (v - \overline{\mathbf{y}}) = \mathcal{V}_1^\top \, \overline{e} = \vec{\xi}^v \neq \mathbf{0}.$$

Observe that $\vec{\xi}^v = \mathcal{V}_1^\top \, (v - \overline{\mathbf{y}}) \neq \mathbf{0}$, since vector $\mathbf{y} \in [\vec{\gamma}_1]$ is not a pre-kernel element of game v.

2. Now suppose $\Delta \in \mathcal{N}_{\mathcal{W}} \backslash \mathcal{N}_{\mathcal{W}_1}$, then

$$\vec{\xi}^{v^\mu} = \mathcal{V}_1^\top \, \overline{e}^\mu = \mathcal{V}_1^\top \, (v^\mu - \overline{\mathbf{y}}) = \mathcal{V}_1^\top \, (v - \overline{\mathbf{y}} + \mu \cdot v^\Delta) = \mathcal{V}_1^\top \, \overline{e} + \mu \cdot \mathcal{V}_1^\top \, v^\Delta = \vec{\xi}^v + \mu \cdot \mathcal{V}_1^\top \, v^\Delta \neq \mathbf{0}.$$

Since, we have $\mathcal{V}_1^\top \, (v - \overline{\mathbf{y}}) \neq \mathbf{0}$ as well as $\mathcal{V}_1^\top \, v^\Delta \neq \mathbf{0}$, and $\mathcal{V}_1^\top \, v^\Delta$ cannot be expressed by $-\mathcal{V}_1^\top \, (v - \overline{\mathbf{y}})$ in accordance with our hypothesis. To see this, suppose that the vector Δ is expressible in this way, then it must hold

$$\Delta = -\frac{1}{\mu} \, (\mathcal{W}_1)^\dagger \, \vec{\xi}^v.$$

However, this implies

$$\mathcal{W} \, \Delta = -\frac{1}{\mu} \, \mathcal{W} \, (\mathcal{W}_1)^\dagger \, \vec{\xi}^v = -\frac{1}{\mu} \, (\mathcal{V}^\top \, \mathcal{U}) \, (\mathcal{V}_1^\top \, \mathcal{U})^\dagger \, \vec{\xi}^v = -\frac{1}{\mu} \, \mathcal{V}^\top \, (\mathcal{V}_1^\top)^\dagger \, \vec{\xi}^v \neq \mathbf{0}.$$

This argument terminates the proof. □

To complete our uniqueness investigation, we need to establish that the single pre-kernel element of the default game preserves also the pre-nucleolus property for the related games, otherwise we can be sure that there must exist at least a second pre-kernel point for the related game different form the first one. For doing so, we introduce the following set:

Definition 4.1. *For every* $\mathbf{x} \in \mathbb{R}^n$*, and* $\psi \in \mathbb{R}$ *define the set*

$$\mathcal{D}^v(\psi, \mathbf{x}) := \{S \subseteq N \,|\, e^v(S, \mathbf{x}) \geq \psi\}, \tag{4.5}$$

and let $\mathcal{B} = \{S_1, \ldots, S_m\}$ be a collection of non-empty sets of N. We denote the collection $\mathcal{B}$ as balanced whenever there exist positive numbers w_S for all $S \in \mathcal{B}$ such that we have $\sum_{S \in \mathcal{B}} w_S \mathbf{1}_S = 1_N$. The numbers w_S are called weights for the balanced collection $\mathcal{B}$ and $\mathbf{1}_S$ is the **indicator function** or **characteristic vector** $\mathbf{1}_S : N \mapsto \{0, 1\}$ given by $\mathbf{1}_S(k) := 1$ if $k \in S$, otherwise $\mathbf{1}_S(k) := 0$.

A characterization of the pre-nucleolus in terms of balanced collections is due to Kohlberg (1971).

Theorem 4.2. *Let* $\langle\, N, v \,\rangle$ *be a TU game and let be* $\mathbf{x} \in \mathcal{I}^0(v)$*. Then* $\mathbf{x} = \nu(N, v)$ *if, and only if, for every* $\psi \in \mathbb{R}$*,* $\mathcal{D}^v(\psi, \mathbf{x}) \neq \emptyset$ *implies that* $\mathcal{D}^v(\psi, \mathbf{x})$ *is a balanced collection over N.*

Proof. For a proof see Peleg and Sudhölter (2007, pp. 108-109). □

Theorem 4.3. *Let* $\langle\, N, v \,\rangle$ *be a TU game that has a singleton pre-kernel such that* $\{\mathbf{x}\} = \mathcal{P}r\mathcal{K}(v) \subset [\vec{\gamma}]$*, and let* $\langle\, N, v^\mu \,\rangle$ *be a related game of* v *derived from* $\mathbf{x}$*, then* $\mathbf{x} = \nu(N, v^\mu)$*, whereas the payoff equivalence class* $[\vec{\gamma}]$ *has non-empty interior.*

Proof. By our hypothesis, $\mathbf{x}$ is an interior point of an inscribed ellipsoid with maximum volume $\varepsilon := \{\mathbf{y}' \,|\, h^v_\gamma(\mathbf{y}') \leq \bar{c}\} \subset [\vec{\gamma}]$, whereas h^v_γ is of type (3.17) and $\bar{c} > 0$ (cf. Lemma 7.6.2 by Meinhardt (2013c)). This implies by Theorem 4.1 that this point is also a pre-kernel point of game v^μ, there is no change in set of lexicographically smallest most effective coalitions $\mathcal{S}(\mathbf{x})$ under v^μ. Moreover, matrix $\mathbf{E}^\top$ induced from

$\mathcal{S}(\mathbf{x})$ has full rank, therefore, the column vectors of matrix $\mathbf{E}^\top$ are a spanning system of $\mathbb{R}^n$. Hence, we get *span* $\{\mathbf{1}_S \mid S \in \mathcal{S}(\mathbf{x})\} = \mathbb{R}^n$, which implies that the corresponding matrix $[\mathbf{1}_S]_{S\in\mathcal{S}(\mathbf{x})}$ must have rank n, therefore collection $\mathcal{S}(\mathbf{x})$ is balanced (see Lemma 6.1.2 Peleg and Sudhölter (2007)). The vector $\mathbf{x}$ is also the pre-nucleolus of the game v, therefore we can choose the largest $\psi \in \mathbb{R}$ s.t. $\emptyset \neq \mathcal{D}^v(\psi, \mathbf{x}) \subseteq \mathcal{S}(\mathbf{x})$ is valid, which is a balanced set. Moreover, we have $\mu \cdot v^\Delta \in [-\mathsf{C}, \mathsf{C}]^{p'}$. Since $\mathsf{C} > 0$, the set $\mathcal{D}^v(\psi - 2\,\mathsf{C}, \mathbf{x}) \neq \emptyset$ is balanced as well. Now observe that $e^v(S,\mathbf{x}) - \mathsf{C} \leq e^v(S,\mathbf{x}) + \mu \cdot v^\Delta(S) \leq e^v(S,\mathbf{x}) + \mathsf{C}$ for all $S \subseteq N$. This implies $\mathcal{D}^v(\psi,\mathbf{x}) \subseteq \mathcal{S}(\mathbf{x}) \subseteq \mathcal{D}^{v^\mu}(\psi - \mathsf{C}, \mathbf{x}) \subseteq \mathcal{D}^v(\psi - 2\,\mathsf{C}, \mathbf{x})$, hence, $\mathcal{D}^{v^\mu}(\psi - \mathsf{C}, \mathbf{x})$ is balanced. Let $c \in [-\mathsf{C}, \mathsf{C}]$, and from the observation $\lim_{c\uparrow 0} \mathcal{D}^{v^\mu}(\psi + c, \mathbf{x}) = \mathcal{D}^{v^\mu}(\psi, \mathbf{x}) \supseteq \mathcal{D}^v(\psi, \mathbf{x})$, we draw the conclusion $\mathbf{x} = \nu(N, v^\mu)$. □

Theorem 4.4. *Assume that the payoff equivalence class $[\vec{\gamma}]$ induced from TU game $\langle N, v \rangle$ has non-empty interior. In addition, assume that game $\langle N, v \rangle$ has a singleton pre-kernel such that $\{\mathbf{x}\} = \mathcal{P}r\mathcal{K}(v) \subset [\vec{\gamma}]$ is satisfied, then the pre-kernel $\mathcal{P}r\mathcal{K}(v^\mu)$ of a related TU game $\langle N, v^\mu \rangle$, as defined by Theorem 4.1, consists of a single point, which is given by $\{\mathbf{x}\} = \mathcal{P}r\mathcal{K}(v^\mu)$.*

Proof. This result follows from Theorems 4.1, 4.3, and Propositions 4.2, 4.3. □

Example 4.1. In order to illuminate the foregoing discussion of replicating a pre-kernel element consider a four person average-convex but non-convex game that is specified by

$$\begin{aligned} v(N) &= 16, v(\{1,2,3\}) = v(\{1,2,4\}) = v(\{1,3,4\}) = 8, \\ v(\{1,3\}) &= 4, v(\{1,4\}) = 1, v(\{1,2\}) = 16/3, \\ v(S) &= 0 \quad \text{otherwise}, \end{aligned}$$

with $N = \{1,2,3,4\}$. For this game the pre-kernel coalesces with the pre-nucleolus, which is given by the point: $\nu(v) = \mathcal{P}r\mathcal{K}(v) = \{44/9, 4, 32/9, 32/9\}$. Moreover, this imputation is even an interior point, thus the non-empty interior condition is valid, hence by Theorem 4.1 a redistribution of the bargaining power among coalitions can be attained while supporting the imputation $\{44/9, 4, 32/9, 32/9\}$ still as a pre-kernel element for a set of related games. In order to get a null space $\mathcal{N}_{\mathcal{W}}$ with maximum dimension we set the parameter μ to 0.9. In this case, the rank of matrix $\mathcal{W}$ must be equal to 4, and we could derive at most 11-linear independent games which replicate the element $\{44/9, 4, 32/9, 32/9\}$ as a pre-kernel element. Theorem 4.4 even states that this point is also the sole pre-kernel point, hence the pre-kernel coincide with the pre-nucleolus for these games (see Table 4.1).

Table 4.1: List of Games[d] which possess the same unique Pre-Kernel[a] as v

			$\mu = 0.9$					
Game	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{3\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$	$\{4\}$
v	0	0	16/3	0	4	0	8	0
v_1	18/49	32/95	127/24	-1/24	256/59	4/13	175/22	-1/24
v_2	-9/25	21/38	89/16	11/48	231/58	42/71	385/47	11/48
v_3	-14/45	-1/40	201/41	-28/65	39/11	-19/44	142/19	-28/65
v_4	0	0	16/3	0	159/47	16/33	107/14	0
v_5	0	0	16/3	0	149/40	-37/102	497/66	0
v_6	0	0	16/3	0	4	-5/47	143/19	0
v_7	0	0	16/3	0	4	-5/47	143/19	0
v_8	0	0	16/3	0	149/40	-37/102	497/66	0
v_9	0	0	16/3	0	149/40	-37/102	497/66	0
v_{10}	0	0	16/3	0	4	-5/47	143/19	0
v_{11}	0	0	16/3	0	4	-5/47	143/19	0

Continued on next page

Table 4.1 – continued from previous page

				$\mu = 0.9$					
Game	$\{1,4\}$	$\{2,4\}$	$\{1,2,4\}$	$\{3,4\}$	$\{1,3,4\}$	$\{2,3,4\}$	N	ACV [b]	ZM [c]
v	1	0	8	0	8	0	16	Y	Y
v_1	79/59	4/13	175/22	-4/57	792/95	10/33	16	N	Y
v_2	57/58	42/71	385/47	4/7	325/38	31/56	16	N	N
v_3	6/11	-19/44	142/19	-27/47	319/40	-29/55	16	N	Y
v_4	41/34	-3/46	428/53	7/34	8	14/25	16	N	N
v_5	203/120	2/41	167/19	-5/24	8	-9/19	16	N	N
v_6	1	23/29	139/16	0	8	18/31	16	N	N
v_7	1	-5/47	139/16	0	8	-8/25	16	N	N
v_8	19/24	2/41	71/9	83/120	8	26/61	16	N	N
v_9	19/24	2/41	71/9	-5/24	8	-9/19	16	N	N
v_{10}	1	-5/47	475/61	0	8	18/31	16	N	N
v_{11}	1	-5/47	475/61	0	8	-8/25	16	N	N

[a] Pre-Kernel and Pre-Nucleolus: $\{44/9, 4, 32/9, 32/9\}$
[b] ACV: Average-Convex Game
[c] ZM: Zero-Monotonic Game
[d] Note: Computation performed with MatTuGames.

Notice that non of these 11-linear independent related games is average-convex. Only two games, namely v_1 and v_3 are zero-monotonic and super-additive. Nevertheless, all games have a non-empty core and are semi-convex. The cores of the games have between 16 and 24-vertices, and have volumes that range from approximately 80 to 127 percent of the default core. TU game v_2 has the smallest and v_3 the largest core.[1] #

5 On the Continuity of the Pre-Kernel

In the previous section, we have established uniqueness on the set of related games. Here, we generalize these results while showing that even on the convex hull comprising the default and related games in the game space, the pre-kernel must be unique and is identical with the point specified by the default game. Furthermore, the pre-kernel correspondence restricted on this convex subset in the game space must be single-valued, and therefore continuous.

Define $\mathcal{G}(N) := \{v \in \mathcal{G}^n \,|\, v(\emptyset) = 0\}$ and

$$\mathcal{G}^n_{\mu,v} := \left\{ v^{\mu} \in \mathcal{G}(N) \,|\, \mu \cdot v^{\Delta} \in [-\mathsf{C}, \mathsf{C}]^{p'} \right\},$$

this set is the translate of a convex set by v, which is also convex and non-empty with dimension $p' - m'$, if matrix $\mathcal{W}$ has rank $m' \leq q < p'$. Then we can construct a convex set in the game space $\mathcal{G}(N)$ by taking the convex hull of game v and the convex set $\mathcal{G}^n_{\mu,v}$, thus

$$\mathcal{G}^n_c := conv\ \{v, \mathcal{G}^n_{\mu,v}\}.$$

Theorem 5.1. *The pre-kernel $\mathcal{P}r\mathcal{K}(v^{\mu^*})$ of game v^{μ^*} belonging to $\mathcal{G}^n_c$ is unique, and is equal to $\{\mathbf{x}\}$.*

Proof. Let be $\{\mathbf{x}\} = \mathcal{P}r\mathcal{K}(v)$ for game v. Take a convex combination of games in $\mathcal{G}^n_c$, hence

$$v^{\mu^*} = \sum_{k=1}^{m} t_k \cdot v^{\mu}_k + t_{m+1} \cdot v = \sum_{k=1}^{m} t_k \cdot (v + \mu \cdot v^{\Delta}_k) + t_{m+1} \cdot v = v + \mu \sum_{k=1}^{m} t_k \cdot v^{\Delta}_k + \mu\, t_{m+1} \cdot \mathbf{0} = v + \mu \cdot v^{\Delta^*},$$

[1] The example can be reproduced while using our MATLAB toolbox *MatTuGames* 2013b. The results can also be verified with our Mathematica package *TuGames* 2013a.

with $v^{\Delta^{*}} := \sum_{k=1}^{m} t_k \cdot v_k^{\Delta} + t_{m+1} \cdot \mathbf{0}$, where $0 \leq t_k \leq 1, \forall k \in \{1,2,\ldots,m+1\}$, and $\sum_{k=1}^{m+1} t_k = 1$. Then $\mu\, v^{\Delta^{*}} \in [-\mathsf{C}, \mathsf{C}]^{p'}$, thus the set of lexicographically smallest coalitions $\mathcal{S}(\mathbf{x})$ does not change. By Theorem 4.1 the vector $\{\mathbf{x}\} = \mathcal{P}r\mathcal{K}(v)$ is also a pre-kernel element of game $v^{\mu^{*}}$. But then by Theorem 4.4 the pre-kernel of game $v^{\mu^{*}}$ consists of a single point, therefore $\{\mathbf{x}\} = \mathcal{P}r\mathcal{K}(v^{\mu^{*}})$. $\square$

Example 5.1. To see that even on the convex hull $\mathcal{G}_c^4$, which is constituted by the default and related games of Table 4.1, a particular TU game has the same singleton pre-kernel, we choose the following vector of scalars $\vec{t} = \{1,3,8,1,2,4,3,5,7,9,2,3\}/48$ such that $\sum_{k=1}^{12} t_k = 1$ is given to construct by the convex combination of games presented by Table 4.1 a TU game $v^{\mu^{*}}$ that reproduces the imputation $\{44/9, 4, 32/9, 32/9\}$ as its unique pre-kernel. The TU game $v^{\mu^{*}}$ on this convex hull in the game space that replicates this pre-kernel is listed through Table 5.1:

Table 5.1: A TU Game $v^{\mu^{*}}$ on the Convex Hull $\mathcal{G}_c^4$ with the same singleton Pre-Kernel as v [a,b]

Game	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{3\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$	$\{4\}$
$v^{\mu^{*}}$	-1/23	8/71	134/25	2/75	530/137	-8/157	1436/187	2/75
Game	$\{1,4\}$	$\{2,4\}$	$\{1,2,4\}$	$\{3,4\}$	$\{1,3,4\}$	$\{2,3,4\}$	N	
$v^{\mu^{*}}$	179/178	173/1125	1946/239	19/144	576/71	15/232	16	

[a] Pre-Kernel and Pre-Nucleolus: $\{44/9, 4, 32/9, 32/9\}$
[b] Note: Computation performed with MatTuGames.

This game is neither average-convex nor zero-monotonic, however, it is again semi-convex and has a rather large core with a core volume of 97 percent w.r.t. the core of the average-convex game, and 20 vertices in contrast to 16 vertices respectively. #

Let $\mathcal{X}$ and $\mathcal{Y}$ be two metric spaces. A set-valued function or correspondence σ of $\mathcal{X}$ into $\mathcal{Y}$ is a rule that assigns to every element $x \in \mathcal{X}$ a non-empty subset $\sigma(x) \subset \mathcal{Y}$. Given a correspondence $\sigma : \mathcal{X} \twoheadrightarrow \mathcal{Y}$, the corresponding graph of σ is defined by

$$Gr(\sigma) := \{(x,y) \in \mathcal{X} \times \mathcal{Y} \,|\, y \in \sigma(x)\}\,. \tag{5.1}$$

Definition 5.1. *A set-valued function $\sigma : \mathcal{X} \twoheadrightarrow \mathcal{Y}$ is closed, if $Gr(\sigma)$ is a closed subset of $\mathcal{X} \times \mathcal{Y}$*

The graph of the pre-kernel $\mathcal{P}r\mathcal{K}(v)$ is given by

$$Gr(\mathcal{P}r\mathcal{K}) := \left\{(v,\mathbf{x}) \,|\, v \in \mathcal{G}_c^n, \mathbf{x} \in \mathcal{I}^0(v), \;\; s_{ij}(\mathbf{x},v) = s_{ji}(\mathbf{x},v) \quad \text{for all } i,j \in N, i \neq j\right\}.$$

Similar, the graph of the solution set of function h of type (3.4) is specified by

$$\begin{aligned} Gr(M(h)) &:= \left\{(v,\mathbf{x}) \,|\, v \in \mathcal{G}_c^n, \mathbf{x} \in \mathcal{I}^0(v), \;\; h^v(\mathbf{x}) = 0\right\} \\ &= \bigcup_{k \in \mathcal{J}'} \left\{(v,\mathbf{x}) \,|\, v \in \mathcal{G}_c^n, \mathbf{x} \in \overline{[\vec{\gamma}_k]}, \;\; h^v_{\gamma_k}(\mathbf{x}) = 0\right\} = \bigcup_{k \in \mathcal{J}'} Gr(M(h_{\gamma_k}, \overline{[\vec{\gamma}_k]})), \end{aligned}$$

with $\mathcal{J}' := \{k \in \mathcal{J} \,|\, g(\vec{\gamma}_k) = 0\}$. This graph is equal to the finite union of graphs of the restricted solution sets of quadratic and convex functions h_{γ_k} of type (3.17). The restriction of each solution set of function

h_{γ_k} to $\overline{[\vec{\gamma}_k]}$ is bounded, closed, and convex (cf. Meinhardt (2013c, Lemmata 7.1.3, 7.3.1)), hence each graph $Gr(M(h_{\gamma_k}, \overline{[\vec{\gamma}_k]}))$ from the finite index set $\mathcal{J}'$ is bounded, closed and convex.

Proposition 5.1. *The following relations are satisfied between the above graphs:*

$$Gr(\mathcal{P}r\mathcal{K}) = Gr(M(h^v)) = \bigcup_{k \in \mathcal{J}'} Gr(M(h_{\gamma_k}, \overline{[\vec{\gamma}_k]})). \tag{5.2}$$

Hence, the pre-kernel correspondence $\mathcal{P}r\mathcal{K} : \mathcal{G}(N) \twoheadrightarrow \mathbb{R}^N$ is closed and bounded.

Proof. The equality of the graph of the pre-kernel and the solution set of function h follows in view of Corollary 3.1. Finally, the last equality is a consequence of Theorem 7.3.1 by Meinhardt (2013c). From this argument boundedness and closedness follows. □

Definition 5.2. *The correspondence $\sigma : \mathcal{X} \twoheadrightarrow \mathcal{Y}$ is said to be upper hemi-continuous (**uhc**) at x if for every open set $\mathcal{O}$ containing $\sigma(x) \subseteq \mathcal{O}$ it exists an open set $\mathcal{Q} \subseteq \mathcal{Y}$ of x such that $\sigma(x') \subseteq \mathcal{O}$ for every $x' \in \mathcal{Q}$. The correspondence σ is **uhc**, if it is **uhc** for each $x \in \mathcal{X}$.*

Definition 5.3. *The correspondence $\sigma : \mathcal{X} \twoheadrightarrow \mathcal{Y}$ is said to be lower hemi-continuous (**lhc**) at x if for every open set $\mathcal{O}$ in $\mathcal{Y}$ with $\sigma(x) \cap \mathcal{O} \neq \emptyset$ it exists an open set $\mathcal{Q} \subseteq \mathcal{Y}$ of x such that $\sigma(x') \cap \mathcal{O} \neq \emptyset$ for every $x' \in \mathcal{Q}$. The correspondence σ is **lhc**, if it is **lhc** for each $x \in \mathcal{X}$.*

Lemma 5.1 (Peleg and Sudhölter (2007)). *Let $\mathcal{X}$ be a non-empty and convex polyhedral subset of $\mathbb{R}^{\tilde{p}}$, and $\mathcal{Y} \subseteq \mathbb{R}^{\tilde{n}}$. If $\sigma : \mathcal{X} \twoheadrightarrow \mathcal{Y}$ is a bounded correspondence with a convex graph, then σ is lower hemi-continuous.*

Proof. For a proof see Peleg and Sudhölter (2007, pp. 185-186). □

Theorem 5.2. *The pre-kernel correspondence $\mathcal{P}r\mathcal{K} : \mathcal{G}(N) \twoheadrightarrow \mathbb{R}^N$ is on $\mathcal{G}_c^n$ upper hemi-continuous as well as lower hemi-continuous, that is, continuous.*

Proof. The non-empty set $\mathcal{G}_c^n$ is a bounded polyhedral set, which is convex by construction. We draw from Proposition 5.1 the conclusion that the graph of the pre-kernel correspondence is bounded and closed. Form Theorem 5.1 it follows $|\,\mathcal{J}'\,| = 1$ on $\mathcal{G}_c^n$, this implies that the graph of the pre-kernel correspondence is also convex on $\mathcal{G}_c^n$. The sufficient conditions of Lemma 5.1 are satisfied, hence $\mathcal{P}r\mathcal{K}$ is lower hemi-continuous on $\mathcal{G}_c^n$. It is known from Theorem 9.1.7. by Peleg and Sudhölter (2007) that $\mathcal{P}r\mathcal{K}$ is upper hemi-continuous on $\mathcal{G}(N)$. Hence, on the restricted set $\mathcal{G}_c^n$, the set-valued function $\mathcal{P}r\mathcal{K}$ is upper and lower hemi-continuous, and therefore continuous. Actual, it is a continuous function on $\mathcal{G}_c^n$ in accordance with $|\,\mathcal{J}'\,| = 1$. □

Corollary 5.1. *The pre-kernel correspondence $\mathcal{P}r\mathcal{K} : \mathcal{G}(N) \twoheadrightarrow \mathbb{R}^N$ is on $\mathcal{G}_c^n$ single-valued and constant.*

Example 5.2. To observe that on the restricted set $\mathcal{G}_c^4$ the pre-kernel correspondence $\mathcal{P}r\mathcal{K} : \mathcal{G}(N) \twoheadrightarrow \mathbb{R}^N$ is single-valued and continuous, we exemplarily select a line segment in $\mathcal{G}_c^4$ to establish that all games on this segment have the same singleton pre-kernel. For this purpose, we resume Example 4.1 and 5.1. Then we choose a vector of scalars $\vec{t}^{\epsilon} := \{1, 3, 8, 1, 2, 4 + \epsilon, 3, 5, 7, 9, 2 - \epsilon, 3\}/48$ with $t_k^{\epsilon} \geq 0$ for each k such that $\sum_{k=0}^{11} t_k^{\epsilon} = 1$ and $\epsilon \in [-2, 2]$. Thus, we define the line segment in $\mathcal{G}_c^4$ through TU game v^{μ^*} from Example 5.1 by

$$\mathcal{G}_c^{4,l} := \left\{ \sum_{k=0}^{11} t_k^{\epsilon} \cdot v_k^{\mu} \;\middle|\; v_k^{\mu} \in \mathcal{G}_c^4, \epsilon \in [-2, 2] \right\}.$$

Therefore, for each game in the line segment $\mathcal{G}_c^{4,l}$, we can write

$$v^{\epsilon} := \sum_{k=1}^{11} t_k^{\epsilon} \cdot v_k^{\mu} + t_0^{\epsilon} \cdot v = \sum_{k=1}^{11} t_k \cdot v_k^{\mu} + t_0 \cdot v + \frac{\epsilon}{48}\,(v_6^{\mu} - v_{11}^{\mu}) = v^{\mu^*} + \frac{\epsilon}{48}\,(v_6^{\mu} - v_{11}^{\mu})$$
$$= v + \mu \cdot v^{\Delta^*} + \frac{\epsilon\,\mu}{48}\,(v_6^{\Delta} - v_{11}^{\Delta}).$$

We extend the pre-kernel element $\mathbf{x} = \{44/9, 4, 32/9, 32/9\}$ to a vector $\overline{\mathbf{x}}$ in order to define the excess vector under game v as $\overline{e} := v - \overline{\mathbf{x}}$, and for game v^{ϵ} as $\overline{e}^{\,v^{\epsilon}} := v^{\epsilon} - \overline{\mathbf{x}}$, respectively. According to these definitions, we get for $\vec{\zeta}^{v^{\epsilon}} = \vec{\xi}^{v^{\epsilon}}$ at $\mathbf{x}$ the following chain of equalities:

$$\vec{\xi}^{v^{\epsilon}} = \boldsymbol{\mathcal{V}}^{\top}\,\overline{e}^{\,v^{\epsilon}} = \boldsymbol{\mathcal{V}}^{\top}\left(v - \overline{\mathbf{x}} + \mu \cdot v^{\Delta^*} + \frac{\epsilon\,\mu}{48}\,(v_6^{\Delta} - v_{11}^{\Delta})\right) = \boldsymbol{\mathcal{V}}^{\top}\,(v - \overline{\mathbf{x}}) = \boldsymbol{\mathcal{V}}^{\top}\,\overline{e} = \vec{\xi} = \vec{\zeta} = \mathbf{0},$$

The last equality is satisfied, since $\mathbf{x}$ is the pre-kernel of game v. Recall that it holds $\mu\,v^{\Delta^*}, \mu\,v_6^{\Delta}, \mu\,v_{11}^{\Delta} \in [-\mathsf{C}, \mathsf{C}]^{15}$, whereas $\boldsymbol{\mathcal{V}}^{\top}\,v^{\Delta^*} = \boldsymbol{\mathcal{V}}^{\top}\,v_6^{\Delta} = \boldsymbol{\mathcal{V}}^{\top}\,v_{11}^{\Delta} = \mathbf{0}$ is in force. Therefore, for each TU game $v^{\epsilon} \in \mathcal{G}_c^{4,l}$ we attain

$$\mathcal{P}r\mathcal{K}(v^{\epsilon}) = \{44/9, 4, 32/9, 32/9\}.$$

The pre-kernel correspondence $\mathcal{P}r\mathcal{K}$ is a single-valued and constant mapping on $\mathcal{G}_c^{4,l}$. Hence its is continuous on the restriction $\mathcal{G}_c^{4,l}$, and due to Theorem 5.2 a fortiori on $\mathcal{G}_c^{4}$. #

6 Preserving the Pre-Nucleolus Property

In this section we study some conditions under which a pre-nucleolus of a default can preserve the pre-nucleolus property in order to generalize the above results in the sense to identify related games with an unique pre-kernel point even when the default game has not a single pre-kernel point. This question can only be addressed with limitation, since we are not able to make it explicit while giving only sufficient conditions under which the pre-kernel point must be at least disconnected, otherwise it must be a singleton. However, a great deal of our investigation is devoted to work out explicit conditions under which the pre-nucleolus of a default game will loose this property under a related game.

For the next result remember that a balanced collection $\mathcal{B}$ is called minimal balanced, if it does not contain a proper balanced sub-collection.

Theorem 6.1. *Let $\langle\, N, v \,\rangle$ be a TU game that has a non unique pre-kernel such that $\mathbf{x} \in \mathcal{P}r\mathcal{K}(v)$, $\mathbf{y} = \nu(v)$ with $\mathbf{x}, \mathbf{y} \in [\vec{\gamma}]_v$, and $\mathbf{x} \neq \mathbf{y}$ is satisfied. In addition, let $\langle\, N, v^{\mu} \,\rangle$ be a related game of v with $\mu \neq 0$ derived from $\mathbf{x}$ such that $\mathbf{x} \in \mathcal{P}r\mathcal{K}(v^{\mu}) \cap [\vec{\gamma}]_{v^{\mu}}$, and $\mathbf{y} \notin [\vec{\gamma}]_{v^{\mu}}$ holds. If the collection $\mathcal{S}^{v}(\mathbf{x})$ as well as its sub-collections are not balanced,*

1. *then $\mathbf{y} \notin \mathcal{P}r\mathcal{N}(v^{\mu})$.*
2. *Moreover, if in addition $\mathbf{x} = \mathbf{y} \notin [\vec{\gamma}]_{v^{\mu}}$, then $\mathbf{x} \notin \mathcal{P}r\mathcal{N}(v^{\mu})$.*

Proof. The proof starts with the first assertion.

1. By our hypothesis, $\mathbf{x}$ is a pre-kernel element of game v and a related game v^{μ} that is derived from $\mathbf{x}$. There is no change in set of lexicographically smallest most effective coalitions $\mathcal{S}^{v}(\mathbf{x})$ under v^{μ} due to $\mathbf{x} \in [\vec{\gamma}]_{v^{\mu}}$, hence $\mathcal{S}^{v}(\mathbf{x}) = \mathcal{S}^{v^{\mu}}(\mathbf{x})$. Moreover, we have $\mu \cdot v^{\Delta} \in \mathbb{R}^{p'}$. Furthermore, it holds $\mathbf{y} = \nu(v)$ by our assumption. Choose a balanced collection $\mathcal{B}$ that contains $\mathcal{S}^{v}(\mathbf{x})$ such that $\mathcal{B}$ is minimal. Then single out any $\psi \in \mathbb{R}$ such that the balanced set $\mathcal{D}^{v}(\psi, \mathbf{y})$ satisfies $\mathcal{S}^{v}(\mathbf{x}) \subseteq$

$\mathcal{B} \subseteq \mathcal{D}^{v}(\psi, \mathbf{y}) \neq \emptyset$. Now choose $\epsilon > 0$ such that $\mathcal{D}^{v}(\psi, \mathbf{y}) = \mathcal{D}^{v}(\psi - 2\,\epsilon, \mathbf{y})$ is given. The set $\mathcal{D}^{v}(\psi - 2\,\epsilon, \mathbf{y})$ is balanced as well. Observe that due to $\mathbf{x} \in [\vec{\gamma}]_{v^{\mu}}$ we get $\mu \cdot v^{\Delta}(S) \leq \epsilon$ for all $S \subset N$. However, it exists some coalitions $S \in \mathcal{S}^{v}(\mathbf{x})$ such that $e^{v}(S, \mathbf{y}) - \epsilon \nleq e^{v}(S, \mathbf{y}) + \mu \cdot v^{\Delta}(S)$ holds. Let $c \in [-\epsilon, \epsilon]$, now as $\lim_{c \uparrow 0}\, \mathcal{D}^{v^{\mu}}(\psi + c, \mathbf{y}) = \mathcal{D}^{v^{\mu}}(\psi, \mathbf{y})$ we have $\mathcal{D}^{v^{\mu}}(\psi, \mathbf{y}) \subseteq \mathcal{D}^{v}(\psi, \mathbf{y})$. Furthermore, we draw the conclusion that $\mathcal{S}^{v}(\mathbf{x}) \nsubseteq \mathcal{D}^{v^{\mu}}(\psi, \mathbf{y})$ is given due to $\mathcal{S}^{v}(\mathbf{x}) = \mathcal{S}^{v}(\mathbf{y}) \neq \mathcal{S}^{v^{\mu}}(\mathbf{y})$. Therefore, we obtain $\mathcal{D}^{v^{\mu}}(\psi, \mathbf{y}) \subset \mathcal{B} \subseteq \mathcal{D}^{v}(\psi - 2\,\epsilon, \mathbf{y})$, but then the set $\mathcal{D}^{v^{\mu}}(\psi, \mathbf{y})$ can not be balanced. Hence, $\mathbf{y} \notin \mathcal{P}r\mathcal{N}(v^{\mu})$.

2. Finally, if $\mathbf{x} = \mathbf{y}$, then $\mathbf{x}$ is the pre-nucleolus of game v, but it does not belong anymore to payoff equivalence class $[\vec{\gamma}]$ under v^{μ}, that is, $[\vec{\gamma}]$ has shrunk. Therefore, $\mathcal{S}^{v}(\mathbf{x}) \neq \mathcal{S}^{v^{\mu}}(\mathbf{x})$. Define from the set $\mathcal{S}^{v}(\mathbf{x})$ a minimal balanced collection $\mathcal{B}$ that contains $\mathcal{S}^{v}(\mathbf{x})$. In the next step, we can single out any $\psi \in \mathbb{R}$ such that the balanced set $\mathcal{D}^{v}(\psi, \mathbf{x})$ satisfies $\mathcal{S}^{v}(\mathbf{x}) \subseteq \mathcal{B} \subseteq \mathcal{D}^{v}(\psi, \mathbf{x}) \neq \emptyset$. In accordance with $\mathbf{x} \in \mathcal{P}r\mathcal{K}(v^{\mu})$, it must exist an $\epsilon > 0$ within the maximum values can be varied without effecting the pre-kernel property of $\mathbf{x}$ even when $\mathbf{x} \notin [\vec{\gamma}]_{v^{\mu}}$, thus we have $\mu \cdot v^{\Delta}(S) \leq \epsilon$ for all $S \subset N$. This implies that $\mathcal{D}^{v}(\psi, \mathbf{x}) \subseteq \mathcal{D}^{v}(\psi - 2\,\epsilon, \mathbf{x})$ is in force. The set $\mathcal{D}^{v}(\psi - 2\,\epsilon, \mathbf{x})$ is balanced as well. However, it exists some coalitions $S \in \mathcal{S}^{v}(\mathbf{x})$ such that $e^{v}(S, \mathbf{x}) - \epsilon \nleq e^{v}(S, \mathbf{x}) + \mu \cdot v^{\Delta}(S)$ is valid. Let $c \in [-\epsilon, \epsilon]$, now as $\lim_{c \uparrow 0}\, \mathcal{D}^{v^{\mu}}(\psi + c, \mathbf{x}) = \mathcal{D}^{v^{\mu}}(\psi, \mathbf{x})$ we have $\mathcal{D}^{v^{\mu}}(\psi, \mathbf{x}) \subseteq \mathcal{D}^{v}(\psi, \mathbf{x})$. Furthermore, we draw the conclusion that $\mathcal{S}^{v}(\mathbf{x}) \nsubseteq \mathcal{D}^{v^{\mu}}(\psi, \mathbf{x})$ is given due to $\mathcal{S}^{v}(\mathbf{x}) \neq \mathcal{S}^{v^{\mu}}(\mathbf{x})$. Therefore, we obtain $\mathcal{D}^{v^{\mu}}(\psi, \mathbf{x}) \subset \mathcal{B} \subseteq \mathcal{D}^{v}(\psi - 2\,\epsilon, \mathbf{x})$, but then the set $\mathcal{D}^{v^{\mu}}(\psi, \mathbf{x})$ can not be balanced. Hence, $\mathbf{x} \notin \mathcal{P}r\mathcal{N}(v^{\mu})$.

□

Theorem 6.2. *Let $\langle\, N, v\,\rangle$ be a TU game that has a non unique pre-kernel such that $\mathbf{x} \in \mathcal{P}r\mathcal{K}(v) \cap [\vec{\gamma}]$, $\{\mathbf{y}\} = \mathcal{P}r\mathcal{N}(v) \cap [\vec{\gamma}_1]$ is satisfied, and let $\langle\, N, v^{\mu}\,\rangle$ be a related game of v with $\mu \neq 0$ derived from $\mathbf{x}$ such that $\mathbf{x} \in \mathcal{P}r\mathcal{K}(v^{\mu}) \cap [\vec{\gamma}]$ holds. If $\Delta \in \mathcal{N}_{\boldsymbol{\mathcal{W}}} \backslash \mathcal{N}_{\boldsymbol{\mathcal{W}}_1}$, then $\mathbf{y} \notin \mathcal{P}r\mathcal{K}(v^{\mu})$ and a fortiori $\mathbf{y} \notin \mathcal{P}r\mathcal{N}(v^{\mu})$.*

Proof. From the payoff equivalence classes $[\vec{\gamma}]$ and $[\vec{\gamma}_1]$ we derive the corresponding matrices $\boldsymbol{\mathcal{W}} := \boldsymbol{\mathcal{V}}^{\top}\, \boldsymbol{\mathcal{U}}$ and $\boldsymbol{\mathcal{W}}_1 := \boldsymbol{\mathcal{V}}_1^{\top}\, \boldsymbol{\mathcal{U}}$, respectively. By assumption, it is $\Delta \in \mathcal{N}_{\boldsymbol{\mathcal{W}}} \backslash \mathcal{N}_{\boldsymbol{\mathcal{W}}_1}$ satisfied. From this argument, we can express the vector of unbalanced excesses $\vec{\xi}^{\,v^{\mu}}$ at $\mathbf{y}$ by

$$\vec{\xi}^{\,v^{\mu}} = \boldsymbol{\mathcal{V}}_1^{\top}\, \overline{e}^{\mu} = \boldsymbol{\mathcal{V}}_1^{\top}\, (v^{\mu} - \overline{\mathbf{y}}) = \boldsymbol{\mathcal{V}}_1^{\top}\, (v - \overline{\mathbf{y}} + \mu \cdot v^{\Delta}) = \vec{\xi}^{\,v} + \mu \cdot \boldsymbol{\mathcal{V}}_1^{\top}\, v^{\Delta} = \mu \cdot \boldsymbol{\mathcal{V}}_1^{\top}\, v^{\Delta} \neq \mathbf{0}.$$

Observe that $\vec{\xi}^{\,v} = \boldsymbol{\mathcal{V}}_1^{\top}\, (v - \overline{\mathbf{y}}) = \mathbf{0}$, since vector $\mathbf{y} \in [\vec{\gamma}_1]$ is a pre-kernel element of game v. However, due to $\Delta \in \mathcal{N}_{\boldsymbol{\mathcal{W}}} \backslash \mathcal{N}_{\boldsymbol{\mathcal{W}}_1}$, we obtain $\boldsymbol{\mathcal{V}}_1^{\top}\, v^{\Delta} \neq \mathbf{0}$, it follows that $\mathbf{y} \notin \mathcal{P}r\mathcal{K}(v^{\mu})$. The conclusion follows that $\mathbf{y} \notin \mathcal{P}r\mathcal{N}(v^{\mu})$ must hold. □

Theorem 6.3. *Let $\langle\, N, v\,\rangle$ be a TU game that has a non unique pre-kernel such that $\mathbf{x} \in \mathcal{P}r\mathcal{K}(v) \backslash \mathcal{P}r\mathcal{N}(v)$ and $\mathbf{x} \in [\vec{\gamma}]$. If $\langle\, N, v^{\mu}\,\rangle$ is a related game of v with $\mu \neq 0$ derived from $\mathbf{x}$ such that $\mathbf{x} \in \mathcal{P}r\mathcal{K}(v^{\mu}) \cap [\vec{\gamma}]$ holds, then $\mathbf{x} \notin \mathcal{P}r\mathcal{N}(v^{\mu})$.*

Proof. According to our assumption $\mathbf{x}$ is not the pre-nucleolus of game v, this implies that there exists some $\psi \in \mathbb{R}$ such that $\mathcal{D}^{v}(\psi, \mathbf{x}) \neq \emptyset$ is not balanced. Recall that the set of lexicographically smallest most effective coalitions $\mathcal{S}^{v}(\mathbf{x})$ has not changed under v^{μ}, since $\mathbf{x}$ is a pre-kernel element of game v^{μ} which still belongs to the payoff equivalence class $[\vec{\gamma}]$. Then exists a bound $\epsilon > 0$ within the maximum surpluses can be varied without effecting the pre-kernel property of $\mathbf{x}$. Thus, we get $\mathcal{D}^{v}(\psi, \mathbf{x}) = \mathcal{D}^{v}(\psi - 2\,\epsilon, \mathbf{x}) \neq \emptyset$ is satisfied. Then $e^{v}(S, \mathbf{x}) - \epsilon \leq e^{v}(S, \mathbf{x}) + \mu \cdot v^{\Delta}(S) \leq e^{v}(S, \mathbf{x}) + \epsilon$ for all $S \subseteq N$, therefore, this implies $\mathcal{D}^{v^{\mu}}(\psi - \epsilon, \mathbf{x}) = \mathcal{D}^{v}(\psi, \mathbf{x})$. The set $\mathcal{D}^{v^{\mu}}(\psi - \epsilon, \mathbf{x})$ is not balanced, we conclude that $\mathbf{x} \notin \mathcal{P}r\mathcal{N}(v^{\mu})$. □

Theorem 6.4. *Assume that the payoff equivalence class* $[\vec{\gamma}]$ *induced from TU game* $\langle\, N, v \,\rangle$ *has non-empty interior. In addition, assume that the pre-kernel of game* $\langle\, N, v \,\rangle$ *constitutes a line segment such that* $\mathbf{x} \in \mathcal{P}r\mathcal{N}(v) \cap \partial\overline{[\vec{\gamma}]}$, $\mathcal{P}r\mathcal{K}(v) \cap \overline{[\vec{\gamma}_1]}$, *and* $\mathbf{x} \in \mathcal{P}r\mathcal{K}(v^{\mu}) \cap [\vec{\gamma}]$ *is satisfied, then the pre-kernel* $\mathcal{P}r\mathcal{K}(v^{\mu})$ *of a related TU game* $\langle\, N, v^{\mu} \,\rangle$ *with* $\mu \neq 0$ *derived from* $\mathbf{x}$ *is at least disconnected, otherwise unique.*

Proof. In the fist step, we have simply to establish that for game v^{μ} the pre-imputations lying on the part of line segment included in payoff equivalence class $[\vec{\gamma}_1]$ under game v will loose their pre-kernel properties due to the change in the game parameter. In the second step, we have to show that the pre-nucleolus $\mathbf{x}$ under game v is also the pre-nucleolus of the related game v^{μ}.

1. First notice that the payoff equivalence class $[\vec{\gamma}]$ has full dimension in accordance with its non-empty interior condition. This implies that the vector $\mathbf{x}$ must be the sole pre-kernel element in $\overline{[\vec{\gamma}]}$ (c.f. with the proof of Theorem 7.8.1 in Meinhardt (2013c)). By our hypothesis, it is even a boundary point of the payoff equivalence class under game v. Moreover, it must hold $[\vec{\gamma}] \not\sim [\vec{\gamma}_1]$, since the rank of the induced matrix $\mathbf{E}^{\top}$ is n, and that of $\mathbf{E}_1^{\top}$ is $n-1$, therefore, we have $E_1^{\top} \neq E^{\top}\, X$ for all $X \in \mathrm{GL}^{+}(n)$.

   In the next step, we select an arbitrary pre-kernel element from $\mathcal{P}r\mathcal{K}(v) \cap \overline{[\vec{\gamma}_1]}$, say $\mathbf{y}$. By hypothesis, there exists a related game v^{μ} of v such that $\mathbf{x} \in \mathcal{P}r\mathcal{K}(v^{\mu}) \cap [\vec{\gamma}]$ holds, that is, there is no change in matrix $\mathbf{E}$ and vector $\vec{\alpha}$ implying $h^{v^{\mu}}(\mathbf{x}) = h^{v^{\mu}}_{\gamma}(\mathbf{x}) = 0$. This implies that for game v^{μ} the payoff equivalence class $[\vec{\gamma}]$ has been enlarged in such a way that we can inscribe an ellipsoid with maximum volume $\varepsilon := \{\mathbf{y}' \,|h^{v^{\mu}}_{\gamma}(\mathbf{y}') \leq \bar{c}\}$, whereas $h^{v^{\mu}}_{\gamma}$ is of type (3.17) and $\bar{c} > 0$ (cf. Lemma 7.6.2 by Meinhardt (2013c)). It should be obvious that element $\mathbf{x}$ is an interior point of ε, since $\mathbf{x} = M(h^{v^{\mu}}_{\gamma}) \subset \varepsilon \subset [\vec{\gamma}]$. We single out a boundary point $\mathbf{x}'$ in $\partial\overline{[\vec{\gamma}]}$ under game v^{μ} which was a pre-kernel element under game v, and satisfying after the parameter change the following properties: $\mathbf{x}' \in \partial\overline{[\vec{\gamma}]} \cap \overline{[\vec{\gamma}_1]}$ with $\mathbf{x}' = \mathbf{x} + \mathbf{z}$, and $\mathbf{z} \neq \mathbf{0}$. This is possible due to the fact that the equivalence class $[\vec{\gamma}]$ has been enlarged at the expense of equivalence class $[\vec{\gamma}_1]$, which has shrunk or shifted by the change in the game parameter. Observe now that two cases may happen, that is, either $\mathbf{x}' \in \varepsilon$ or $\mathbf{x}' \notin \varepsilon$. In the former case, we have $h^{v^{\mu}}_{\gamma}(\mathbf{x}') = h^{v^{\mu}}(\mathbf{x}') = h^{v^{\mu}}_{\gamma_1}(\mathbf{x}') = \bar{c} > 0$, and in the latter case, we have $h^{v^{\mu}}_{\gamma}(\mathbf{x}') = h^{v^{\mu}}(\mathbf{x}') = h^{v^{\mu}}_{\gamma_1}(\mathbf{x}') > \bar{c} > 0 = h^{v}(\mathbf{x}') = h^{v}_{\gamma_1}(\mathbf{x}')$.

   From $h^{v^{\mu}}_{\gamma_1}(\mathbf{x}') > 0$ and notice that the vector of unbalanced excesses at $\mathbf{x}'$ is denoted as $\vec{\xi}^{\,v^{\mu}}$, we derive the following relationship

$$h^{v^{\mu}}_{\gamma_1}(\mathbf{x}') = \|\, \vec{\xi}^{\,v^{\mu}} \,\|^2 = \|\, \vec{\xi}^{\,v} + \mu \cdot \mathcal{V}_1^{\top}\, v^{\Delta} \,\|^2 = \|\, \mu \cdot \mathcal{V}_1^{\top}\, v^{\Delta} \,\|^2 = \mu^2 \cdot \|\, \mathcal{V}_1^{\top}\, v^{\Delta} \,\|^2 > 0,$$

   with $\mu \neq 0$. Thus, we have $\mathcal{V}_1^{\top}\, v^{\Delta} \neq \mathbf{0}$, and therefore $\Delta \in \mathcal{N}_{\mathcal{W}} \backslash \mathcal{N}_{\mathcal{W}_1}$. Observe that $\vec{\xi}^{\,v} = \mathcal{V}_1^{\top}\,(v - \overline{\mathbf{x}'}) = \mathbf{0}$, since vector $\mathbf{x}' \in \overline{[\vec{\gamma}_1]}$ is a pre-kernel element of game v. Take the vector $\mathbf{y} \in [\vec{\gamma}_1]$ from above that was on the line segment as vector $\mathbf{x}'$ under game v which constituted a part of the pre-kernel of game v, we conclude that $\mathbf{y} \notin \mathcal{P}r\mathcal{K}(v^{\mu})$ in accordance with $\mathcal{V}_1^{\top}\, v^{\Delta} \neq \mathbf{0}$.

2. By our hypothesis, $\mathbf{x}$ is the pre-nucleolus of game v, and an interior point of equivalence class $[\vec{\gamma}]$ of the related game v^{μ}. Using a similar argument as under (1) we can inscribe an ellipsoid with maximum volume ε, whereas $h^{v^{\mu}}_{\gamma}$ is of type (3.17) and $\bar{c} > 0$. In accordance with the assumption that $\mathbf{x}$ is also pre-kernel element of game v^{μ}, we can draw the conclusion that the set of lexicographically smallest most effective coalitions $\mathcal{S}(\mathbf{x})$ has not changed under v^{μ}. But then, we have $\mu \cdot v^{\Delta} \in [-\mathsf{C}, \mathsf{C}]^{p'}$. Moreover, matrix $\mathbf{E}^{\top}$ induced from $\mathcal{S}(\mathbf{x})$ has full rank, therefore, the column vectors of matrix $\mathbf{E}^{\top}$ are a spanning system of $\mathbb{R}^n$. Hence, we get *span* $\{\mathbf{1}_S \,|\, S \in \mathcal{S}(\mathbf{x})\} = \mathbb{R}^n$ as well, which implies that matrix $[\mathbf{1}_S]_{S \in \mathcal{S}(\mathbf{x})}$ has rank n, the collection $\mathcal{S}(\mathbf{x})$ must be balanced (c.f. Lemma 6.1.2 Peleg and Sudhölter (2007)). In accordance with vector $\mathbf{x}$ as the pre-nucleolus of game v,

we can choose the largest $\psi \in \mathbb{R}$ s.t. $\emptyset \neq \mathcal{D}^{v}(\psi, \mathbf{x}) \subseteq \mathcal{S}(\mathbf{x})$ is valid, which is a balanced set. Since $\mathsf{C} > 0$, the set $\mathcal{D}^{v}(\psi - 2\,\mathsf{C}, \mathbf{x}) \neq \emptyset$ is balanced as well. Now observe that $e^{v}(S, \mathbf{x}) - \mathsf{C} \leq e^{v}(S, \mathbf{x}) + \mu \cdot v^{\Delta}(S) \leq e^{v}(S, \mathbf{x}) + \mathsf{C}$ for all $S \subseteq N$. This implies $\mathcal{D}^{v}(\psi, \mathbf{x}) \subseteq \mathcal{S}(\mathbf{x}) \subseteq \mathcal{D}^{v^{\mu}}(\psi - \mathsf{C}, \mathbf{x}) \subseteq \mathcal{D}^{v}(\psi - 2\,\mathsf{C}, \mathbf{x})$, hence, $\mathcal{D}^{v^{\mu}}(\psi - \mathsf{C}, \mathbf{x})$ is balanced. To conclude, let $c \in [-\mathsf{C}, \mathsf{C}]$, and from the observation $\lim_{c \uparrow 0} \mathcal{D}^{v^{\mu}}(\psi + c, \mathbf{x}) = \mathcal{D}^{v^{\mu}}(\psi, \mathbf{x}) \supseteq \mathcal{D}^{v}(\psi, \mathbf{x})$, we draw the implication $\mathbf{x} = \nu(N, v^{\mu})$.

Finally, recall that the vector $\mathbf{x}$ is also the unique minimizer of function $h_{\gamma}^{v^{\mu}}$, which is an interior point of payoff equivalence class $[\vec{\gamma}]$, therefore the pre-kernel of the related game v^{μ} cannot be connected. Otherwise the pre-kernel of the game consists of a single point. $\square$

Corollary 6.1. *Let $\langle\, N, v \,\rangle$ be a TU game that has a non single-valued pre-kernel such that $\mathbf{x} \in \mathcal{P}r\mathcal{N}(v) \cap \partial\overline{[\vec{\gamma}]}$ and let $\langle\, N, v^{\mu} \,\rangle$ be a related game of v derived from $\mathbf{x}$, whereas $\mathbf{x} \in int\, [\vec{\gamma}]_{v^{\mu}}$, then $\mathbf{x} = \nu(N, v^{\mu})$.*

7 CONCLUDING REMARKS

In this paper we have established that the set of related games derived from a default game with an unique pre-kernel must also possess this pre-kernel element as its single pre-kernel point. Moreover, we have shown that the pre-kernel correspondence in the game space restricted to the convex hull comprising the default and related games is single-valued and constant, and therefore continuous. Although, we could provide some sufficient conditions under which the pre-nucleolus of a default game – whereas the pre-kernel constitutes a line segment – induces at least a disconnected pre-kernel for the set of related games, it is, however, still an open question if it is possible to obtain from a game with a non-unique pre-kernel some related games that have an unique pre-kernel. In this respect, the knowledge of more general conditions that preserve the pre-nucleolus property is of particular interest.

Even though, we have not provided a new set of game classes with a sole pre-kernel element, we nevertheless think that the presented approach is also very useful to bring forward our knowledge about the classes of transferable utility games where the pre-kernel coalesces with the pre-nucleolus. To answer this question, one need just to select boundary points of the convex cone of the class of convex games to enlarge the convex cone within the game space to identify game classes that allow for a singleton pre-kernel.

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