On the Estimation of Supply and Demand Elasticities of Agricultural Commodites

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1. May 2014

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Abstract [1]

The report provides a literature review on the topic of estimation of demand and supply elasticities. To this end, it starts the discussion by summarizing the main facets of production theory and consumer theory to introduce the concept of elasticities, with examples of different types of elasticities most utilized in the literature. Next, it discusses the identification problem in estimating elasticities, i.e. the issue of having to solve for unique values of the parameters of the structural model from the values of the parameters of the reduced form of the model. It summarizes various methodologies employed in the literature to solve this problem and gives practical examples. These solutions include, but are not limited to, using instrumental variables, adopting a recursive structure, holding demand constant, and imposing inequality constraints in order to restrict the domain of estimates.

[1] The research was supported by African Growth and Development Policy Modeling Consortium facilitated by IFPRI. The report is also available at: http://www.agrodep.org/resource/no-10-estimation-supply-and-demand-elasticities-agricultural-commodites

Contents

1 Production theory

A preliminary description of theory of production is necessary to introduce the concept of elasticities.

The production function is the set of all possible production plans for the current state of knowledge. Factors of production can be grouped in fixed factors, which cannot be altered during a particular production period, and variable factors, those factors of production which can be altered. In the *long-run* all factors of production are variable[2].

It is common to denote inputs by x_i and outputs by y_i. More precisely, the set of inputs and outputs is defined as follows:
$\{x_1, x_2, \ldots, x_{k-1}, x_k, y_1, y_2, \ldots, y_{n-1}, y_n\}$.

Numerous assumptions are made in production theory in order to ensure a well defined problem or to make the resolution of producers' problem easier. They are as follows: monotonicity (weak or strong), concavity (or quasiconcavity), essentiality (weak or strong). Moreover, we assume the production function $f(x)$ to be finite, non-negative, single valued and real valued for $x \in R^+$. Moreover $f(x)$ is everywhere continuous and twice differentiable, homogeneous of degree θ in x. Few implications of the above mentioned assumptions are worth noting: the input set and the isoquant are convex sets; the marginal rate of technical substitution is diminishing; the Hessian matrix of second order partial derivatives is negative semi-definite[3]; Young's theo-

[2]Unlike consumer theory, estimating production functions is significantly easier since production is cardinal, unlike utility which is ordinal.

[3]From an empirical perspective, a simple rule to assess the negative semi-definiteness of the Hessian matrix consists in checking if the largest eigen value of the Hessian matrix is zero or negative.

rem implies that the Hessian matrix is symmetric. The producer problem can be stated as follows:

$$(1)\ Max_x\ \mathbf{p} * f(\mathbf{x}) - \mathbf{w} * \mathbf{x}$$

where **p** is the output price vector, and **w** the vector of input prices.

Commonly adopted production functions are the well-known Cobb-Douglas (CD) [4], the Leontief[5] and the Constant Elasticity of Substitution (CES)[6].

1.1 *On supply elasticities: theory and empirics*

In production theory, we are interested in analyzing changes in production function due to changes in input use. Those changes can be computed in discrete or continuous time. The latter is the most commonly adopted approach and relies on the use of derivatives ($\partial y/\partial x$), that is the infinitesimal changes in the objective function due to the infinitesimal change in the variable[7]. The elasticities of interest are the elasticity of output, and the elasticity of scale. The substitutability among inputs is a further measure to be considered.

In analytical terms, the elasticity of output is defined as the (infinitesimal) change in output due to the (infinitesimal) change in input use, evaluated at

[4]This is one of the most famous production functions, introduced by Charles W. Cobb and Paul H. Douglas (1928), although anticipated by Knut Wicksell (1901: p.128, 1923) and, some have argued, by J.H. von Thunen (1863). For a review of theoretical and empirical literature on the Cobb-Douglas production function, see Douglas (1934, 1967), Nerlove (1965) and Samuelson (1979). The analytical formulation is as follows: $y = A\prod_i x_i^{\alpha_i}, \alpha_i > 0$.

[5]This is also known as *perfect complements* production function in that inputs' substitutability is zero. The formulation is as follows: y=$min\{\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_n x_n\}$.

[6]It has been introduced by Arrow, Chenery, Minhas and Solow (1961), and generalized to the multiple factors case by Hirofumi Uzawa (1963) and Daniel McFadden (1963). The CES function, as its name reveals, shows a constant elasticity of inputs substitution throughout. Its formulation is as follows: $y = \beta[\alpha_1 x_1^{\frac{\sigma-1}{\sigma}} + \alpha_2 x_2^{\frac{\sigma-1}{\sigma}} + \cdots + \alpha_n x_n^{\frac{\sigma-1}{\sigma}}]^{\frac{\sigma}{\sigma-1}}$.

[7]Elasticities are a *local* concept in that they are measured at a specific point of interest: if not specified, elasticities are intended to be measured at mean values.

a specific point. The measure is analytically obtained by the first derivative of $f(x)$ with respect to x, times the input over output ratio[8]. The formula to compute the elasticity of output is as follows:

$$(2)\ \varepsilon_{i,y} = \frac{\partial f}{\partial x_i}\frac{x_i}{y}$$

which can be rewritten as $\varepsilon_{i,y} = \frac{MP_i}{AP_i}$ where MP_i and AP_i are, respectively, the marginal product and the average product.

The elasticity of scale provides a measure of the impacts of changes in inputs use on the scale of production. The elasticity of scale is computed as follows:

$$(3)\ \epsilon = \frac{\partial \ln(f(\lambda x))}{\partial \ln(\lambda)}|_{\lambda=1}.$$

The value of ϵ characterize the return to scale (constant, increasing, or decreasing), in that $\epsilon = 1$ implies constant returns to scale, $\epsilon < 1$ implies diminishing returns to scale, and $\epsilon > 1$ implies increasing returns to scale.

As mentioned above, it is interesting to measure the substitutability of inputs in production function.

Several elasticities can be computed. A standard measure of input substitutability is the Hicks elasticity of substitution ($\sigma_{i,j}$). Let $f_i = \frac{\partial f}{\partial x_i}$ be the change in output due to a change in input x_i, and let $i, j = \{1, 2\}, \{2, 1\}$. The Hicks elasticity of substitution is computed as follows (for the two inputs case):

$$(4)\ \sigma_{i,j} = \frac{\partial ln(x_i/x_j)}{\partial lnMRTS_{ij}} = \frac{\partial \ln(x_i/x_j)}{\partial \ln(f_i/f_j)}$$

[8]It is a common procedure to evaluate elasticities at the mean: in this specific case we compute the *elasticity at mid point*.

where $MRTS_{ij}$ (marginal rate of technical substitution for goods i and j) is the amount by which the quantity of i has to be reduced when one extra unit of the input j is used, such that the output remains constant[9]. In other terms, the Hicks elasticity of substitution measures the relative changes in output induced by (infinitesimal) changes in inputs i and j [10].

The direct elasticity of substitution ($\sigma^D_{i,j}$), or *short run* elasticity of substitution, is computed as follows:

$$(5)\ \sigma^D_{i,j} = \frac{\partial \ln(x_i/x_j)}{\partial \ln(f_i/f_j)}|_{x_k} = \bar{x}_k$$

For the two inputs case, the direct elasticity is identical to the Hicks elasticity in that it is assuming that the other factors in the production function are fixed, thus can be ignored. Roy G.D. Allen (1938) proposed a different measure. The Allen elasticity of substitution ($\sigma^A_{i,j}$), also known as the *partial* elasticity of substitution, is defined as follows:

$$(6)\ \sigma^A_{i,j} = \sum_i \frac{x_i}{x_j}\frac{f_i}{f_j}\left(\frac{F_{j,i}}{F}\right)$$

where $F_{i,j}$ represents the cofactor of H with respect to $f_{i,j}$, F is the determinant of the bordered Hessian (H) of the production function.

It is easy to recognize that if the total number of factors is two, the Allen elasticity of substitution[11] reduces to the direct elasticity of substitution ($\sigma^D_{i,j} = \sigma^A_{i,j}$.). A relevant property of the Allen elasticity of substitution is that it is symmetric (*i.e.* $\sigma^A_{i,j} = \sigma^A_{j,i}$.). In the case of two inputs, the Allen elasticity of substitution allows to distinguish among complement and substitute inputs

[9]The concept is analogous to the marginal rate of substitution in consumer theory.

[10]The elasticity can be computed for more than two inputs, holding all but two inputs constant.

[11]This is perhaps the most popular measure of the elasticity of substitution.

in that $\sigma^A_{i,j} < 0$ implies x_i and x_j are complements, while $\sigma^A_{i,j} > 0$ implies x_i and x_j are substitutes.

The Morishima elasticity of substitution (Morishima, 1967) has the seemingly unusual property of being asymmetric[12] ($\sigma^M_{i,j} \neq \sigma^M_{j,i}$.). The elasticity is computed as follows:

$$(7)\ \sigma^M_{i,j} = \frac{\partial f_i}{\partial x_i}\frac{F_{i,j}}{F} - \frac{\partial f_i}{\partial x_j}\frac{F_{i,j}}{F}.$$

As pointed by Blackorby and Russell (1981, 1989), the Morishima elasticity of substitution should be natural for a multi-factor case. The Morishima measure can be re-written in terms of the Allen measure as follows:

$$(8)\ \sigma^M_{i,j} = \frac{f_j}{f_i}\frac{x_j}{x_i}(\sigma^A_{i,j} - \sigma^A_{j,j})$$

In general, factors that are substitutes by the Allen measure, will be substitutes by the Morishima measure; but factors that are complements by the Allen measure may still be substitutes by the Morishima measure. In other terms, the Morishima measure has a bias towards treating inputs as substitutes (or, alternatively, the Allen measure has a bias towards treating them as complements). This seemingly paradoxical result reflects the fluidity of the concept of elasticity of substitution in a multiple factor world (Blackorby and Russell, 1981, 1989).

Nowadays a widely adopted functional forms to estimate supply elasticities is the the translog function (also named transcendental logarithmic function, Christiansen et al., 1971, 1973). It has been introduced by Kmenta (1967), in order to approximate the CES function with a second order Taylor

[12]It is possible for two inputs to be Hicks/Allen substitutes while Morishima complements.

expansion. Moreover, it nests the CD function for elasticity of substitution close to the unitary value.

An example of a generalized translog cost function, for the one-output and multi-input firm, is as follows:

$$(9)\ log(c) = \alpha_0 + \sum_i \alpha_i log(w_i) + \alpha_y log(y) + 0.5 \sum_i \sum_j \alpha_{ij} log(w_i) log(w_j) + \sum_i \alpha_{iy} log(w_i) log(y) + 0.5\alpha_{yy}[log(y)]^2$$

where c is the cost function, y is the output level, w_i is the input price for good i. An advantage of the translog functions is their flexibility. Using output(s) as dependent variables and input(s) as regressors, we have the translog production function. It consents the estimation of the Allen elasticities of substitution, the estimation of the production frontier or the measurement of the total factor productivity.

Acreage response to prices is a further information of great interest. A simplified model à la Nerlove may clarify how to proceed:

$$(10)\ A^* = \alpha + \beta P_t$$

$$(11)\ A_t - A_{t-1} = (1-\gamma)(A_t - A^*_{t-1})$$

where equations (10) refers to the desired acreage and equation (11) models the actual acreage,respectively. Through substitution and simplifications we can derive the following specification:

$$(12)\ A_t = (1-\gamma)\alpha + (1-\gamma)\beta P_t + \gamma A_{t-1}$$

where A_t is the acreage (or land size), P_t is the grower price per pound. The specification (12) incorporates the behavior of producers that adjust their acreage when they realize that the desired acreage differs from the acreage realized in the previous year. The adjustment coefficient $(1 - \gamma)$ indicates the rate of adjustment of actual acreage to desired acreage (Kmenta, 1986).

Finally, we may be interested in estimating the supply elasticity to prices. A simple specification is to regress prices on output. Nerlove (1956) suggested to regress logarithms of quantities on current and (or) lagged output and input prices, also in logarithms[13]. A simple specification may be adopted:

$$(13)\ ln(Q_t) = \alpha + \beta ln(P_{t-1}) + \gamma ln(Z_t) + \delta ln(Q_{t-1})$$

where β represents the price elasticity, Z_t collects the control variables, and Q_{t-1} allows to control for possible temporal correlation of production [14].

[13]Askari and Cummings (1977) provide an excellent survey.

[14]The model should provide an estimate for δ less than $|1|$. If $\delta = |1|$ we should use a model for non-stationary series (e.g. use a first-order difference model)

2 Consumer theory

In analogy with the previous section, we will first review the theory and afterward we will describe the empirics of demand estimation.

The problem faced by the consumer is to maximize its utility function $U(\cdot)$ subject to a budget constraint. The utility derives from consumption of the goods that are included in the economy. It is assumed to be a strictly quasi-concave, monotonic increasing function. The utility function is an ordinal indicator of preferences over a given set of n commodities. It is derived from the following set of axioms: 1) reflexivity; 2) completeness; 3) transitivity; 4) continuity; 5) non-satiation; 6) convexity [15].

The (primal) problem of utility maximization is postulated as follows:

$$(14)\ Max_q\ U(q)$$

$$s.t.\ \sum_k p_k q_k \leq x$$

$$q_k \geq 0, \forall k$$

The solution (q_i) to the maximization problem provide the so called Marshallian (uncompensated) demand functions (g_i), $x(\mathbf{P}, M)$, where $\mathbf{P}$ and M are respectively the vector of prices and the consumer's income. Marshallian demand functions satisfy the following properties: 1) adding-up; 2) homogeneity. Those properties are better known if expressed in elasticity form: the Engel aggregation states that the uncompensated elasticities, weighted by expenditure shares, sum to unity; the Euler aggregation states that the

[15] A detailed description of the axioms and their implications is beyond the scope of this report. An excellent review is provided in Mas Colell et al. (1995)

uncompensated cross-elasticities plus own-elasticity sum to zero; the Cournot aggregation states that expenditure shares times the uncompensated cross-elasticities plus the own expenditure share sum to zero. Analytically, the properties are expressed as follows[16]:

i) $\sum_k w_k e_k = 1$;

ii) $\sum_k e_{ik} + e_i = 0$;

iii) $\sum_k w_k e_{ki} + w_i = 0$

where w_k represent the budget shares, e_k stand for the income (or expenditure) elasticities for good k, and e_{ik} are the uncompensated cross-elasticities.

Those properties can be derived through direct differentiation of the first order conditions (FOC) or through the use of duality theory.

The dual problem is postulated as follows:

$$(15)\ Min_q\ p\ q$$

$$s.t.\ u = V(q)$$

where $V(q)$ represents the indirect utility function. The solution to the dual problem are the so called Hicksian (compensated) demand functions (h_i), which are functions of prices and utility. The Marshallian and Hicksian demand functions are tightly linked in that the following identity holds[17]:

$$(16)\ q_i = h_i[U, p] = h_i[\psi(M, p), p] = g_i(M, p)$$

More intuitively, the fundamental difference between the Marshallian and the Hicksian demand functions is that if we consider a change in the Hicksian demand at a price increase, the consumer will preserve the same utility

[16]The reader may refer to Deaton and Muellbauer (1980) for further details.

[17]Moreover, the Marshallian and Hicksian demand functions are linked through the *Roy's identity*: $q_i = g_i(M, p) = \frac{-\partial\psi(M,p)/\partial p_i}{\partial\psi(M,p)/\partial x}$.

level before and after the price increase. In other terms we assume that the consumer gets compensated for the price increase through a rise of income: the income effect is disregarded so that only the substitution effect is left. The opposite applies to the Marshallian demand, that is the income is held constant while the utility level might change (Varian, 1992).

2.1 *On demand elasticities: theory and empirics*

In consumer theory we are interested in several elasticities, namely the own-price elasticity, the income elasticity, the cross-price elasticity, the elasticity of substitution, and the compensated and uncompensated elasticities. We will review them in more detail.

The own-price elasticity (also called price elasticity) measures the percentage change in the demand at a percentage rise in the price of the good itself. As the demand curves is downward sloping the own-price elasticity is negative too: a price increase causes a decline in the quantity demanded (the well known Law of Demand) (Marshall, 1890)[18].

The own-price elasticity is given by the following formula:

$$(17)\ \epsilon_p = \frac{\partial q_i p_i}{\partial p_i q_i}.$$

If $|\epsilon_p| = 1$, the demand is defined as being unit elastic, while the demand is defined as being elastic if $|\epsilon_p| > 1$ and inelastic if $|\epsilon_p| < 1$. If the demand is inelastic, a price increase means that the decrease in the purchased quantity will be relatively smaller than the increase in price in % terms. So the

[18]Probably worthy to recall that the negative slope of the demand curve is due to the fact that the demand is function of price and the curve is depicted in a coordinate system with the price on the y-axis and the quantity on the x-axis.

consumer's total expense for the good in question increases. The opposite is the case at a price increase of a good where the demand is elastic.

The income elasticity measures the percentage change in the demand for a given good as a result of a percentage change in income:

$$(18)\ \epsilon_M = \frac{\partial q_i M}{\partial M q_i}$$

where M is the income. Generally, the income elasticity for necessaries is smaller than for luxury goods, that is a reduction in income will not reduce the consumption of bread just as much as the consumption of exotic fruits. Moreover, the so called *Engel's law* states that consumers increase their expenditures for food products (in % terms) less than their increases in income (Timmer et al., 1983).

The cross-price elasticity shows the percentage change in demand for good i as a result of a percentage change in the price of good j. The analytical formulation is as follows:

$$(19)\ \epsilon_{ik} = \frac{\partial q_i p_k}{\partial p_k q_i}$$

Cross-price elasticity for a good having a close substitution or complementarity would numerically be relatively big. If there is a close substitution the cross-price elasticities will be positive as a price increase of good i (e.g. wine) will make the consumers substitute towards demanding good j (e.g. beer). If i and j are complementary goods (e.g. beer and pizza) the cross-price elasticity will be negative. For goods that are neither close substitutes nor complementary goods, the cross-price elasticity will be insignificant (e.g. pizza and exotic fruits).

The elasticity of substitution measures the percentage change in the relative consumption of two goods as a consequence of a change in the relative prices of the goods. Thus, an increase of 1% in the relation between the two commodity prices will push the relation between the commodities ϕ % in the direction of the commodity which has become relatively cheaper. The elasticity of substitution is computed as follows:

$$(20)\ \sigma_{ij} = -\ \frac{(\partial q_i/q_j)MRS_{ij}}{\partial MRS_{ij}(q_i/q_j)}$$

MRS indicates the marginal substitution relationship between good i and good j which corresponds to the slope of the indifference curve. At the optimum bundle, the slope of the indifference curve is equal to the price relation among the goods. The bigger the elasticity of substitution between good i and good j the more substitutional the goods (e.g. wine and beer) are. Opposite, if the elasticity of substitution approaches zero, the good i and j will be complementary goods (e.g. beer and pizza). The size of the elasticity of substitution determines the slope of the indifference curve: they converge towards a straight line when the elasticity of substitution approaches infinity (and the goods are sad to be *perfect substitutes*), while the curve converges towards making a 90 degree kink when the elasticity of substitution approaches zero (and the goods are said *perfect complements* (Figure 1), while the IC are said *Leontief*).

The elasticity of substitution and the cross-price elasticity are closely related in that they allow to characterize the change in the demand function of good i due to the price change of good j. However, the cross-price elasticity does not take into account the price sensitivity of the good whose price has

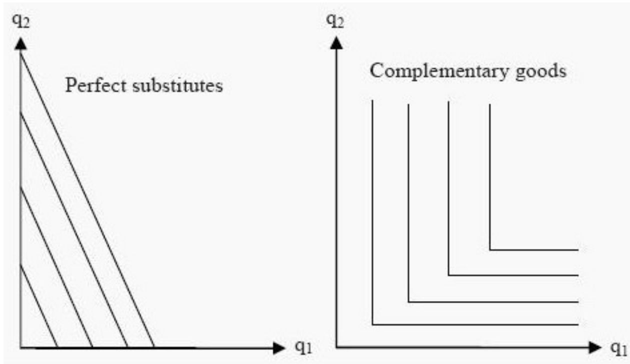

Figure 1: Perfect substitutes and perfect complements

been changed (say good j), while the elasticity of substitution takes this into account. Therefore, equal cross-price elasticities does not imply equal elasticities of substitution [19]. From an empirical point of view, a functional form which implies a constant elasticity of substitution will have the advantage that the percentage change in the relative consumption of the two goods is independent of the level of prices and consumption. Furthermore, the elasticity of substitution will be independent of the price development of other goods. On the contrary, if the function does not have a constant elasticity of substitution the elasticity has to be measured at a specific point on the demand curve (usually it is computed at the mean or median value) [20].

We turn now to the empirical estimation of demand functions. The sim-

[19] A clarificatory example. Let the cross-price elasticity between good x and good z be the same as between good y and good z, that is a price increase of good x, or good y, will have the same influence on the consumption of good z. If the price sensitivity for good x is larger than for good y, the consumption of good x will decrease more than the consumption of good y when the price of the two goods increases. Therefore, the elasticity of substitution becomes bigger between good x and good z than between good y and good z.

[20] The great advantage of the family of CES functions is that it is easy to switch from the direct utility function to the indirect utility function, and vice-versa. When we use the Hicksian demand functions, we consider the compensated elasticities which are the Hicksian-equivalent uncompensated (or general) elasticities. The latter are computed from the Marshallian demand function.

plest model to estimate demand elasticities is the log-log single equation model:

$$(21)\ log(q_i) = \alpha_i + e_i log(M) + \sum_{k=1}^{n} e_{ik} log(p_k)$$

where q_i represent per capita consumption, M is the per capita total expenditure, and p stand for prices [21].

The model has been extensively used to estimate own-price and income elasticities for individual when the systems of demand functions were not yet developed. Nowadays the model is still applied for empirical analysis (Russo et al.,2008) due to its advantages: the model is linear, assume a "nicely shaped" demand function, and it is simple to be estimated. Furthermore, the model interpretation is straightforward in that the coefficients for income and prices are directly interpretable as elasticities.

A more structured model, thus more consistent with the theory, is the Stone model (Stone, 1954). Its functional form is as follows:

$$(22)\ log(q_i) = \alpha_i + e_i log(M/P) + \sum_{k=1}^{n} e^*_{ik} log(p_k/P)$$

where P is the so called Stone index, a geometric price index[22], and e_i is the income elasticity, and e^*_{ik} represent the compensated elasticities.

A drawback of the Stone model is that there is no guarantee symmetry will be satisfied. The Linear Expenditure System (LES) is able to overcome these problems (Stone, 1954). The LES is derived from a linear demand functional form ($p_i q_i = \beta_i M + \sum_i \beta_{ik} p_k$) by imposing the general restrictions of consumer

[21] The model is expressed in per capita terms in order to neutralize the effects of changes in population. Therefore, the elasticities refer to the "representative" consumer.

[22] The Stone price index is defined as follows: $log(P) = \sum_{k=1}^{n} w_k log(p_k)$, where w_k is the budget share of good k.

behavior, namely homogeneity, adding-up property and symmetry[23]:

$$(23)\ p_i q_i = p_i \gamma_i + \beta_i (M - \textstyle\sum_i p_k \gamma_k)$$

where γ_i is interpreted to be the minimum quantity demanded of commodity i, the first term on the RHS ($p_i \gamma_i$) is referred to as "committed expenditure" in that it represents expenditures on a fixed bundle of consumption. The second term on the right-hand side ($M - \sum_i p_k \gamma_k$) of the equation is referred to as "supernumerary income": it represents expenditures above the committed expenditures.

The number of parameters to estimate for the LES is rather limited. This desirable characteristic is due to the restrictiveness of the linearity. However, there is no free lunch. The LES suffers of several limitations: it does not admit inferior goods; it implies that all goods must be net substitutes; it imposes additivity restriction on the direct utility function; absolute values of all uncompensated own-price elasticities are constrained to be less than their income elasticities. On the contrary, a nice feature of the LES is that both the Cobb-Douglas and Constant Elasticity of Demand functional forms are special cases of the LES model.

The Rotterdam model (Theil, 1965; Barten, 1969) also deserves particular attention. The RM is the differential form of Stone's model. However, expressed in differential form the model is more general, in that restrictions have to hold on deviations, not in level. The empirical formulation of the Rotterdam model is as follows:

$$(24)\ \overline{w}_{it} \Delta log q_{it} = b_i \Delta log \overline{x}_t + \textstyle\sum_k c_{ik} \Delta log p_{kt} + \epsilon_{it}$$

[23] For further details the reader may refer to Deaton (1986).

where Δ is the first-order differential operator (i.e. $\Delta logq_{it} = logq_{it} - logq_{it-1}$), $\overline{w}_{it} = 0.5(w_t - w_{t-1})$, and $\Delta log\overline{x}_t = \Delta logx_t - \sum_k \overline{w}_{kt}\Delta logp_{kt}$. The income and compensated price elasticities of the RM model can be obtained by dividing estimated coefficients by the budget share of the good in question: $e^*_{ik} = \frac{c_{ik}}{w_i}$. In turn, the Slutsky equation ($e^*_{ik} = e_{ik} + w_k e_i$) can be used to estimate uncompensated price elasticities.

Nowadays the most commonly used form in demand analysis is the Almost Ideal Demand System (AIDS) proposed by Deaton and Muellbauer (1980). With more than three thousands citations, their paper has become a *must read* reference in applied economics. The empirical formulation is as follows:

$$(25)\ w_i = \alpha_i + \sum_k \gamma_{ki} log(p_k) + \beta_i log(M/P)$$

where $logP = \alpha_0 + \sum_k \alpha_k log(p_k) + 0.5 \sum_k \sum_l \gamma_{ki} log(p_k) log(p_l)$. The AIDS model has some nice properties: satisfies axioms of consumer choice exactly; aggregates perfectly from a representative consumer; its estimate implies a Engel curves consistent with household data; can be viewed as a first-order approximation to arbitrary demand system; allow to test for homogeneity and symmetry. In the AIDS model, the elasticities are obtained through the following formulas: $e_{ik} = \frac{\gamma_{ik} - \beta_i[w_k - \beta_k log(M/P)]}{w_i} - \delta_{ik}$; $e_i = \frac{\beta_i}{w_i} + 1$.

3 Identification

Identification is a main issue in econometrics, the branch of economics that aims to answer to empirical questions based on economic models. Econometrics models are always based on some assumptions, sometimes testable, sometimes not. In this framework, identification deals with the relationship between the assumptions of an econometric model and the possibility of answering or not, an empirical question using that model. More specifically, the area of identification studies the necessary and sufficient conditions to estimate (consistently) parameters of interest[24].

From a different perspective, the identification problem in econometrics is the issue of having to solve for unique values of the parameters of the structural model from the values of the parameters of the reduced form of the model (i.e. a single estimate of the structural parameters from the reduced form parameters for each structural equation, *cfr.* Maddala, 1992) [25]. Therefore, if there are multiple solutions which make the reduced form coefficients compatible with the structural coefficients, the model is underidentified. Instead if there are no compatible solutions, the model is said to be overidentified. Finally, if a solution exists and is unique, the model is said to be just identified or exactly identified[26].

[24] A formal, and simple, definition is provided by A. M. Shaikh. See http://home.uchicago.edu/ amshaikh/webfiles/ident.pdf. We provide a shorter version. Let P denote the true distribution of the observed data X. Denote by $\mathbf{P} = \{P_\theta : \theta \in \Theta\}$ a model for the distribution of the observed data, assuming correctly specification, that is $P \in \mathbf{P}$. We know that $\theta \in \Theta_0(P)$, where $\Theta_0(P) = \{\theta \in \Theta : P_\theta = P\}$ where $\Theta_0(P)$ is the identified set and θ is identified if the identified set is a singleton for all $P \in \mathbf{P}$

[25] The reduced form of a model is the one in which the endogenous variables are expressed as functions of the exogenous variables.

[26] In general, a linear system of M equations, with $M > 1$, cannot be identified from the data if less than $M-1$ variables are excluded from that equation. This is a particular form

All in all, the identification problem can be viewed as the (unresolved) dilemma of economists to make (correct) inference by reducing at most the number and strength of (necessary) assumptions. A major criticism related to this puzzle is the well known *Law of Decreasing Credibility* (Manski, 2003) which states that "the credibility of inference decreases with the strength of the assumptions maintained". Let us provide a practical example of the identification problem: the estimation of a system of demand and supply equations (Koopmans, 1949).

Consider a linear model for the supply and demand: the former will be upward sloping with respect to price and the latter is expected to be downward sloping. We observe data on both the price (P) and the traded quantity (Q) of this good for several years. Unfortunately this information does not suffice to identify both demand and supply by using regression analysis on observations of Q and P. In fact it is impossible to estimate a downward slope and an upward slope with one linear regression line involving only two variables. Indeed, additional variables solve this issue and help to identify the individual relations. Figure 2 depicts the identification problem in supply and demand and clarifies how the use of additional information can solve the puzzle. In particular, the graph shows that by observing shifts in the demand curve, due to an exogenous variable (Z) we are able to identify the (positive) slope of the supply equation.

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of the order condition for identification, which is necessary but not sufficient for identification. The rank condition is a necessary and sufficient condition for identification. In the case of only exclusion restrictions, it must "be possible to form at least one nonvanishing determinant of order $M-1$ from the columns of A corresponding to the variables excluded a priori from that equation", where A is the matrix of coefficients of the equations (Fisher 1966, p. 40).

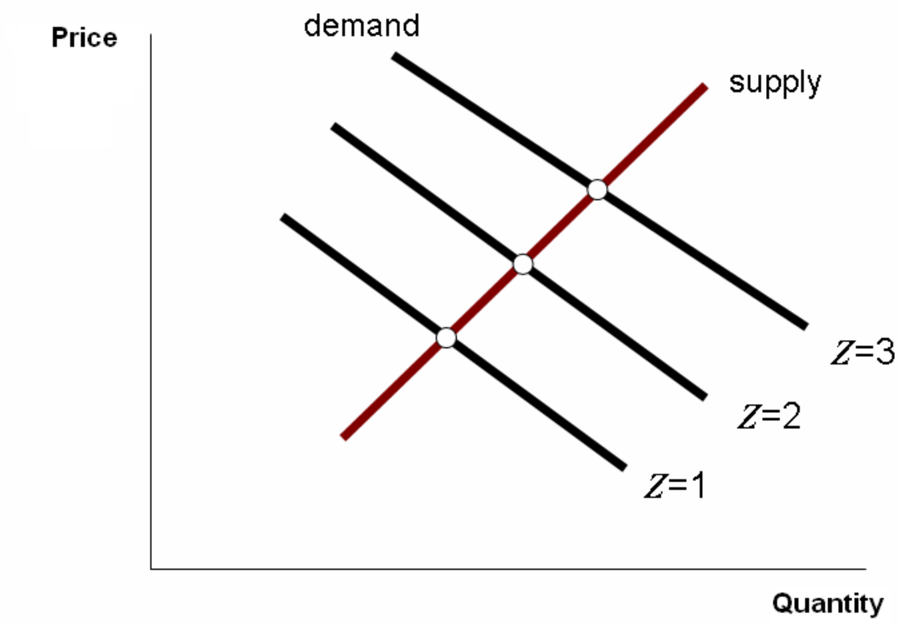

Figure 2: Identification of supply and demand

It should be also noted that the (negative) slope parameter of the demand equation cannot be identified. More generally, we are able to identify the parameters of the equation (in our case the supply) not affected by the exogenous variable (Z). In order to identify both the supply and the demand equation, we would need both a variable (or *shifter*) Z entering the demand equation but not the supply equation[27], and X entering the supply equation but not the demand equation[28]:

demand: $Q = a_1 + b_1 P + c_1 Z$; supply: $Q = a_2 + b_2 P + c_2 X$

with positive b_2 and negative b_1. Here both equations are identified if c_1 and c_2 are nonzero.

[27]In agricultural economics it is common to use weather variables. More details are provided in subsequent sections.

[28]In agricultural economics a common approach is to introduce income as demand shifter. More details are provided in subsequent sections.

Solving for P and Q we obtain the reduced-form equations:

$Q = \frac{a_1 b_2 - a_2 b_1}{b_2 - b_1} + \frac{c_1 b_2}{b_2 - b_1} Z - \frac{c_2 b_1}{b_2 - b_1} X$

$Q = \frac{a_1 - a_2}{b_2 - b_1} + \frac{c_1}{b_2 - b_1} Z - \frac{c_2}{b_2 - b_1} X$

where $\pi_1 = \frac{a_1 b_2 - a_2 b_1}{b_2 - b_1}$, $\pi_2 = \frac{c_1 b_2}{b_2 - b_1}$, etc. are the reduced-form parameters. Suppose we observe Z, but not X. In this case we have two estimates for b_2, and a_2: the supply is said over-identified; the demand is under-identified. When we have unique estimates for the structural parameters, the equations are said exactly identified; multiple estimates imply over-identification; no estimates imply under-identification.

3.1 *Solutions to the identification problem*

As previously described, the identification problem arises when we try to identify parameters using a reduced form. In the example of supply and demand, we may solve the problem by using an instrumental variable. Few points need to be recalled. More precisely, an instrument will be valid if the variable is correlated with the endogenous regressor and uncorrelated with the regression error.

Maddala (1977) pointed it is very difficult to have such kind of a variable, and econometrics textbooks do not provide clear guidelines. Angrist and Krueger (2001, p. 73) argue that "good instruments often come from detailed knowledge of the economic mechanism and institutions determining the regressor of interest". For example, a valid instrument shifts only one "curve" (e.g. supply, but not demand). In agricultural markets, the instrument may be rainfall or weather shocks.

Wright (1928) has pioneered the use of instrumental variables. He estimated they supply and demand for flaxseed and used prices of substituted goods as instrumental variables for demand, and yield per acre as instruments for supply. He averaged out the estimates obtained using different instruments. Current researches have shown that a more efficient way to rely on multiple instruments is to use a two-stage least squares (2SLS) procedure. The method is described below.

First we provide a chronological review of the solutions have been proposed to solve the identification problem.

A simple, probably too naive, solution is to ignore the problem. Indeed this solution is not lacking of a theoretical justification. As Wright (1929) pointed in JASA, ignoring the issue is a valid solution if “it may be assumed that the dynamic forces will continue to operate thereafter in the same manner as they have been operating during that period”.

Another solution is to adopt a recursive structure:

$$\text{(i) } p_t = \beta_1 q_t + u^D \ ;$$
$$\text{(ii) } q_t = \beta_2 p_{t-1} + u^S.$$

In this formulation p_{t-1} is exogenous in the supply equation, u^S is uncorrelated with u^D (therefore there are no common shifters), and q_t is exogenous in the demand equation with p_t on the left hand side.

Frisch and Waugh (1933) have proposed another approach. They suggested to hold demand constant. Given that the observed quantity demanded differs from the true (or latent) demand, the approach consists of estimating the observed demand and correcting for the bias. We clarify with an example. Suppose that quantity is measured with error ϵ_t, that is:

$$\text{(true/latent demand)} \;\; q_t^* = \beta p_t + \gamma W_t$$
$$\text{(observed quantity)} \;\; q_t = q_t^* + \epsilon_t$$

where W_t represents all determinants of demand and ϵ_t is pure independent measurement error. Solving for observed demand:

$$\text{(observed demand)} \; q_t = \beta p_t + \gamma W_t + \epsilon_t$$

where $E(p_t \epsilon_t) = 0$. The approach suggested by Frisch and Waugh (1933) is to adjust for the bias, given the "known" γ and W_t. In this case, as they prove, OLS estimates are consistent.

Another approach is to use an instrumental variables (IV) regression. In the case of a single equation, the Limited Information Maximum Likelihood method (LIML) is a valid alternative. The method has been proposed by Anderson and Rubin (1949), and has been popular until the introduction of the 2SLS by Theil (1965). The LIML consists in minimizing the residual sum of squares (RSS)to select the regressors. More precisely, the LIML minimize the ratio of RSS under the restricted and unrestricted model (Maddala, 1992). The analogy with the 2SLS is very strong in that the latter minimize the difference of RSS under the restricted and unrestricted model. As a consequence, if the model is exactly identified the 2SLS and LIML provide identical. estimates. Sargan (1958) has extended the IV approach to multiple instruments through the 2SLS method.

In a nutshell, the approach is as follows. In the first stage, each explanatory variable that is an endogenous covariate in the equation of interest is regressed on all of the exogenous variables in the model (including both exogenous covariates in the equation of interest and the excluded instruments). This first stage allows us to obtain the predicted values. In the second stage, the regression of interest is estimated as usual, except that in this stage each endogenous covariate is replaced with the predicted values from the first stage (Wooldridge, 2002).

Empirically, the 2SLS is performed as follows. Let y be the dependent variable, $x_1, ..., x_{k-1}$ the explanatory variables, x_k the endogenous regressor, $z_1, ..., z_M$ the set of instruments.

(I) First stage: compute $\hat{x}_k$ regressing x_k on regressors and instruments.

$$(26)\ x_k = \alpha + \textstyle\sum_{i=1}^{k-1} \beta_i x_i + \sum_{j=1}^{N} \gamma_j z j$$

(II) Second stage: estimate the model replacing x_k with $\hat{x}_k$.

$$(27)\ y = \alpha + \textstyle\sum_{i=1}^{k-1} \beta_i x_i + \beta_k \hat{x}_k$$

From an empirical point of view, it is worth recalling the pitfalls of instrumental variables approach. The 2SLS provides consistent, but not unbiased estimates, therefore researchers that use this approach should always aspire to use large datasets. Moreover, an instrumental variable correlated with omitted variables can lead to biased estimates that is much greater than the bias in ordinary least squares estimates. However, the bias is proportional to the degree of overidentification, hence using fewer instruments would reduce the bias. Moreover, it is wise to test for the validity of instruments. Many tests have been proposed and some are implement in common packages (see

Berkowitz et al., 2012)[29].

For the above mentioned approaches we have implicitly assumed to deal only with a single equation. Special attention needs the case in which we consider a simultaneous equation model. An efficient way to estimate a full system of equations is to use Generalized Method of Moments (GMM) estimation. Unfortunately, GMM is usually unfeasible, unless the system covariance matrix (Σ) is known. Alternative approaches consist in estimating the system by using a three stage least squares (3SLS) procedure, or by adopting a full information maximum likelihood (FIML) estimator. The former consists in estimating a 2SLS (or equation-by-equation) and then using the residuals to compute Σ. Using $\widehat{\Sigma}$ the estimation of the third stage will be consistent. Alternatively a FIML estimator can be adopted. The estimator uses information about all the equations in the system, providing consistent estimates. Although asymptotically equivalent, the FIML is not equal to the continuously updated 3SLS estimator (unless the system is just-identified). Empirically, the 3SLS estimator is much easier to be computed than the FIML estimator (Davidson and MacKinnon, 2004).

Alternative approaches have been proposed. Leamer (1981) has suggested to solve the identification problem by imposing inequality constraints in order to restrict the domain of estimates. His words are self-explanatory: "when the regression of quantity on price yields a positive estimate, we may assume that this is an attenuated estimate of the supply curve and that the data contain no useful information about the demand curve [30]. If the the estimate

[29] The FAR test, recently developed, does not overreject the null hypothesis when we use half of the sample without replacement. The test is implemented in STATA.

[30] Thurman and Wohlgenant (1989) provide an empirical application of Leamer's method

is negative, th number may be treated as an attenuated estimate of the demand slope, and we may assert that the data contain no useful information about the supply curve" (Leamer, 1981, p. 321).

Rigobon (2003) exploits the intuition in Wright (1928) suggesting to restrict the parametric space using the information provided by the heteroskedasticity in the data (e.g. due to crises, policy shifts, changes in collecting the data, etc.). He provides necessary and sufficient conditions for identification of a system of simultaneous equations. In particular, Rigobon suggests to use the second moments to increase the number of relations between the parameters in the reduced and structural forms. An appealing feature of his approach is that it only requires the existence of heteroskedasticity in that the direct modeling of the source of heteroskedasticity can be ignored for the identification purpose. The approach is as follows. First, Rigobon (2003) estimated a vector autoregressive model of interest rates (prices may be used for agricultural markets); second, he defined subsamples according to different volatility; finally he computed the covariances matrices that have been used in the GMM estimation of contemporaneous shocks. Although the intuition to use the variance of the shocks to reduce the bias in OLS estimates has to be attributed to Wright (1928), Rigobon (2003) generalized the intuition and provided the conditions to identify the system [31].

More recently, Roberts and Schlenker (2013) have revisited the problem of identification of supply and demand for agricultural commodities. The authors use theory of storage to derive the following empirical model:

in agricultural markets for the estimation of demand. Renuka Kalirajan (2002) applied the method to the demand for services.

[31] See Okumura (2011), and Lütkepohl and Netšunajev (2013) for recent applications.

$$(28)\ \text{(Supply)}\ log(s_t) = \alpha_s + \beta_s log(E[p_t|\Omega_{t-1}]) + \gamma_s w_t + f(t) + u_t$$

$$(29)\ \text{(Demand)}\ log(c_t) = \alpha_d + \beta_d log(p_t) + g(t) + v_t$$

and $c_t = s_t - z_t$ (consumption, c_t, is the difference of supply, s_t, and storage, z_t), α_s and α_d are intercepts for supply and demand, the Ω is the information set, w_t stands for the random weather-induced yield shocks, $f(t)$ and $g(t)$ capture time trends in supply and demand, u_t and v_t are the error terms. The rationale for (24) and (25) is that weather-induced shocks (current and lagged) are expected to shift only the supply curve, and to leave the demand unchanged. The model is solved in two stages. The first stage consists in estimating $log(p_t)$ and $log(E[p_t|\Omega_{t-1}])$. The authors suggest to use a distributed lag model of yield shocks and a polynomial time trend. The reduced forms are as follows:

$$(30)\ log(p_t) = \pi_{d0} + \sum_{k=1}^{K-1} \mu_d w_{t-k} + f(t) + \epsilon_{dt}$$

$$(31)\ log(E[p_t|\Omega_{t-1}]) = \pi_{s0} + \sum_{k=1}^{K} \mu_s w_{t-k} + f(t) + \epsilon_{st}$$

where $f(t)$ and $g(t)$ represent the polynomial time trend functions,ϵ_{dt} and ϵ_{st} are the error terms. In the second stage the lagged yield shocks are used as instruments. In particular the supply is estimated as follows:

$$(32)\ log(s_t) = \alpha_s + \beta_s log(\widehat{E[p_t|\Omega}_{t-1}]) + \lambda_{s0}\gamma_{s0} w_t + f(t) + u_t$$

and demand is obtained as follows:

$$(33)\ \text{(Demand)}\ log(c_t) = \alpha_d + \beta_d \widehat{log(p_t)} + g(t) + v_t$$

The novelty of this approach is that Roberts and Schlenker (2013) have considered simultaneously four commodities that are substitutes in supply and demand. Using weather as natural instrument has returned to the frontier in identification of supply and demand for agricultural commodities.

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