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Abstract

In this paper we study a multi-stage elimination contest with non-sunk bids: differently from existing literature, we realize that when players are budget-constrained, they do not regard past bids as strategically irrelevant in their decision of how much to bid in following stages. This happens because they face a basic trade-off when allocating scarce resources over stages. We believe that although non-sunk bids make the analysis more complex, they allow to improve the quality of the modelization for many real scenarios, like R&D contests and sport tournaments. In our simple two-stage framework with complete information and asymmetric players, we find that: (i) there is a unique SPNE where in the first stage only the strongest player bids positive, while forcing the others to bid zero; in the second stage shortlisted bidders play mixed strategies, and the strongest player wins the game on average; (ii) relative ex-ante strengths of players are relatively more important, in determining the outcome of the game, than their relative abilities of allocating limited resources over the stages; (iii) the two-stage contest yields a lower expected revenue than the one-stage one, due to the fact that the first stage yields basically no revenue and that shortlisting to the second stage is inefficient. On the basis of these results, our elimination contest does not seem to be a very advantageous allocation mechanism for the contest sponsor.

Keywords: All-pay auctions, Elimination contests, Non-sunk bids.

JEL Classification Codes: C72, C73, D44, D72, O31

1. Introduction

Many real world competitive scenarios can be modeled as contests, i.e., as games where agents spend unrecoverable resources (typically money or effort) in order to win a prize. Immediate examples of such kind of competitions are R&D contests, political lobbying, political campaigns, job promotion tournaments, litigation, marketing by firms, sports races and wars.

Given the pervasiveness of contest-like strategic interactions in real life, a large body of economic literature has been developed about contests. Starting from the seminal work by Tullock (1980), early contributions mostly focused on single-stage contests (see Nitzan (1994) for a survey). More recently, however, it has been acknowledged that many real contests are multi-stage in nature: in many sport competitions, athletes first compete in preliminary rounds and then the best are selected to compete in subsequent rounds; in R&D contests, firms need to spend resources in each phase of the product development (e.g., solution exploration, prototyping, product testing); in political competition, candidates for the country premiership need first to expend resources to get

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their party nomination and then incur additional expenses for the electoral campaign.
Therefore, a growing body of literature has been focusing on multi-stage contests, or *elimination contests*, where in each stage a subset of contestants is shortlisted to compete in subsequent stages, while the others are eliminated, and the final winner is the contestant who proceeds successfully through all the stages (see e.g., Rosen (1986), Baye et al. (1993), Gradstein and Konrad (1999), Amegashie (1999), Fullerton and McAfee (1999), Amegashie (2002), Stein and Rapoport (2005), Moldovanu and Sela (2006), Matros (2006), Fu and Lu (2009), Groh et al. (2012)[1].). Though focusing on different strategic dimensions of elimination contests, all studies in this literature have captured three basic features, namely that (i) all contestants incur non-recoverable costs regardless of them winning or being selected through stages, (ii) the decisions in the different stages are interdependent and (iii) the contestants selected at each stage are determined by some variant of the contest function proposed by Tullock (1980)[2](Stein and Rapoport (2005)).
However, there is another as well critical feature of many multi-stage contests which has mostly been neglected in the literature, namely that contestants typically face a constraint on resources. This aspect comes easily in mind when thinking about the examples mentioned above: in sport races and tournaments players need to optimally allocate their energy between preliminary and final rounds; in R&D contests firms need to carefully allocate resources over the different stages of the R&D process; and candidates in political competitions need to distribute their budget between primary elections and final campaigns.
From all these examples it clearly emerges that the presence of a cap on resources implies that contestants face a fundamental trade-off when allocating scarce resources between early and later stages of a contest: a player that spends little resources on early stages in order to save resources for later stages, is likely to have a low performance at the beginning of the competition, thereby risking to be eliminated early, but if she manages to get to later stages, she has a high probability of winning in the end. On the other hand, a contestant that spends most resources in early stages, is more likely to get successfully to later stages, but may risk to run out of resources before the completion of the contest and consequently to lose in the end. Decisions that players make in early stages will impact, drive and constraint decisions taken at later stages.
Also, the presence of this trade-off implies that the winner of the contest may not be the player who has the largest budget at the beginning, but rather the player who is the ablest in allocating scarce resources between stages.
However, despite its relevance, just a few contributions have explicitly considered the crucial role of a constraint on resources in elimination contests (see Amegashie (1999), Amegashie (2002), Stein and Rapoport (2005), Matros (2006)). Indeed, this limited literature well emphasized the

[1]Refer to Section 2 for a thorough review of the literature.
[2]Tullock (1980) assumed that a contestant's probability of winning the contest equals the ratio between her own effort and the sum of other contestants' efforts, in order to capture the notion, common to many contests in practice, that random factors, or "noise", can play a role in determining the outcome of a contest.

presence of the basic trade-off that we have just illustrated. Strangely enough, however, none of these contributions realized that the interdependance between choices at different stages implies a critical fact, namely that *in each stage where a player is allowed to play, the total of the bids she has spent until that stage cannot be considered as sunk, since they are not strategically irrelevant in the decision of how much to bid in the following stages: past expenditures cap future possible expenditures.* Practically, this means that when solving a two-stage contest by backward induction, one would need to embed in the player's payoff function at the second stage also the bid that the player made in the first stage. Instead, to the best of our knowledge, all contributions in the relevant literature consider the bids paid in earlier stages as sunk, so that they do not appear in the expression of the final payoff function.

Also, we believe that one does not need to explicitly embed in the model a budget constraint to argue that players do face a cap on resources. Indeed, if contestants are to behave *rationally*, as it is conventionally assumed in economic theory, they will never spend more than their valuation, i.e., they will consider the value they assign to winning the prize as a "natural"cap on the total effort that they are willing to exert in the contest.

Inspired by these considerations, in this paper we analyze an elimination contest where players are *rationally* budget-constrained (in the sense explained above) and consider past bids as non-sunk when determining how much to spend in each stage they will be allowed to play. In particular, we model the competition in each stage as an all-pay auction where (i) all players who have survived from the previous stage bid against all others, (ii) a given number of highest bidders is selected to go to the subsequent stage, and (iii) all players forfeit their own bid [3].

Assuming non-sunk bids make the analysis much more complex. However, we think that our work allows to improve the quality of the modelization for many real scenarios. For example, our multi-stage all-pay auction with non-sunk bids provide a good modelization of R&D contests, capturing in particular the strategic interaction between firms when they need to optimally allocate scarce resources in a dynamic setting [4].

In the paper we analyze a simple two-stage model with complete information and asymmetric players, and we find that notwithstanding the presence of the trade-off between stages, the relative strengths of players are more important than their relative strategic abilities, in determining the equilibrium outcome of the game: due to the information structure, the ex-ante strongest player

[3] The all-pay auction is a fully discriminatory contest, that is a limit case of the Tullock contest when there is no noise at all and the outcome is completely determined by the effort exerted by players. All-pay auctions have been studied both under complete and incomplete information. Basic references in the former strand are Hillman and Samet (1987), Hillman and Riley (1989), Baye et al. (1993),Baye et al. (1996), Che and Gale (2003) and Siegel (2009). Basic references to the latter are Hillman and Riley (1989), Amann and Leininger (1996), Krishna and Morgan (1997), Moldovanu and Sela (2001), Moldovanu and Sela (2006) and Moldovanu et al. (2012).

[4] An example of such kind of R&D contests is an innovative practice of public procurement of R&D called Pre-commercial Procurement (PCP), which has been recently introduced at the EU level. In PCP a number of firms enter a R&D stepwise contest where competing projects are evaluated phase-by-phase and potential suppliers are gradually eliminated (for a detailed account of PCP see European Commission (2007)).

is always able to deter the other players from bidding positive in the first stage, so that she can guarantee herself shortlisting with a very small outlay and save most resources for the second stage. This implies that the first stage basically yields no revenue and that shortlisting to the second stage is inefficient, so that the expected revenue from the two-stage all-pay auction is lower than the one-stage one.

On the basis of these results, an elimination contest does not seem to be a very advantageous practice for the contest sponsor. However, the current version of our model is simple so that more work is needed to fully assess the performance of this mechanism.

Nevertheless, we believe that our work is indeed valuable insofar, by characterizing the equilibria of an elimination contest with non-sunk bids, we have not only filled a gap in the literature, but we have also hopefully inspired new reflection about the need to both exploit wisely the assumption of rationality of economic agents, and to assess carefully and thoroughly whether costs in a given dynamic decision problem are to be considered sunk or non-sunk, which is a relevant issue in many fields of economics.

The rest of the paper is structured as follows: Section 2 discusses the relation to the literature; Section 3 presents the model; Section 4 characterizes the equilibria; Section 5 concludes. All proofs are in the Appendix.

2. Relation to the literature

The body of literature on elimination contests, to which this work contributes, is large and various. The aim of this section is to review this literature, with a particular attention to assess how the various sub-strands or single contributions behave with respect to three basic features, i.e., (i) the way of modelling the shortlisting process, (ii) the objective of the analysis, whether normative or positive, and (iii) the inclusion in the model of a constraint on resources. This will help us to frame our work in the literature itself.

The bulk of the literature models the shortlisting process as follows: at each stage the remaining contestants are divided into groups where subcontests or "battles" are run, and then the battle winners compete again against each other in later stages (e.g., this is how shortlisting works in most team sport tournaments). Also, most contributions study contests from a normative point of view, adopting an *optimal contest design* perspective, that is they are aimed at identifying the set of "rules" which lead to the most-favorable outcomes for the contest designer, i.e., most importantly, total effort maximization. As stated in Gradstein and Konrad (1999) "contest design, that is the set of rules that define the victorious contestant(s), clearly has incentive effects as far as the amount of effort expended by the contestant goes". In particular, studies belonging to this research agenda address three main questions, i.e., (i) which is the optimal prize structure in contests, (ii) which is the optimal seeding of players, i.e., the best way to match players in subcontests on the basis of ability rankings - typically to avoid that the strongest competitors eliminate each other in early rounds, and (iii) which is the optimal structure of contests, i.e., the number of stages and the number of contestants remaining at each stage.

The seminal paper to this literature, Rosen (1986), addresses the first two questions, studying an elimination tournament with symmetric players where the probability of winning a match is a stochastic function of effort, and finds that prizes should be increasing in survival in order for players to exert a non-decreasing effort along the tournament, and that a random seeding yields higher total effort than the seeding where strong players meet weak players in the semifinals.

Groh et al. (2012) consider an elimination tournament with asymmetric players where each pair-wise match is modelled as an all-pay auction, and investigate the optimal seedings which are needed to achieve the maximization of, respectively, the total tournament effort, the probability of a final between the two strongest players, and the winning probability of the strongest player.

On the other hand, Gradstein and Konrad (1999) focus on the optimal structure issue and, studying a Tullock contest with symmetric players, ask under which conditions it is better for the designer to choose a multi-stage format rather than a single-stage one, thereby endogenizing the choice of the contest structure. They find that a single-stage contest is preferable only when the contest rules are discriminatory enough, i.e., when the effort exerted by contestants is relatively more important than random factors in determining the outcome of the contest (like in an all-pay auction). Also in this agenda, Amegashie (1999) models a two-stage Tullock contest with symmetric players, and finds that the result that an increase in the sensitivity of the contest sponsor yields an increase of exterted efforts, which was given as granted for one-stage contests, does not necessarily hold in a multi-stage context. Other papers belonging to this line of research are Moldovanu and Sela (2006) - who implement an analysis similar to Gradstein and Konrad (1999) but in an incomplete information setting - and Gradstein (1998) and Amegashie (2000).

A recent development from this strand of literature analyzes the role of information revelation in settings with incomplete information where players can signal and strategically misrepresent their preferences. When important information is revealed in the interim stages of a game, the incentives of players in earlier stages are altered, so that whether and to what extent it is optimal to reveal information about players' ability at a given stage of the game, is a fundamental aspect of contest design, which is also relevant to many contests in practice. Contributions to this agenda are Lai and Matros (2007), Zhang (2008) and Zhang and Wang (2009). These works typically investigate symmetric separating Perfect Bayesian Equilibria under different revelation information policies, and find that under revelation policies that allow for signalling effects, such kind of equilibria might fail to exist, and that in fact less information revelation may lead to more efficient outcomes.

Typically, sub-strands and contributions we have reviewed so far do not consider the role of the presence of a constraint on players' bidding capacity.

Our work departs from the bulk of the literature along a number of dimensions [5].

[5] Another strand of literature that could be related to our work analyze multi-stage sequential all-pay auctions. In this research agenda, initiated by Leininger (1991), multi-stage all-pay auctions are modelled as dynamic games

First, we adopt a different modelization of shortlisting, namely that at each stage contestants do not meet just a subset of remaining competitors, but they rather confront all the other survivors. A few papers in the elimination contests literature adopt this kind of "all-against-all" shortlisting method. This works typically investigate two-stage contests where a *shortlisting stage* (or *entry stage*) is introduced by the designer before the proper contest stage, with the aim of inducing *efficient entry* in the contest, i.e., selecting the players with the highest valuations to participate in the contest. Higgins et al. (1985) study a two-stage contest where in the shortlisting stage players decide with a mixed strategy whether to enter (and pay an entry free) or not, so that first stage bids are exogenous, while the number of contestants to be shortlisted in the second stage is endogenous and determined such that contestants' expected profits are zero. Similarly, Baye et al. (1993) analyze a political lobbying two-stage game where in the first stage the politician decides to shortlist the potential lobbyists as to maximize rent-extraction. In another setting, Fullerton and McAfee (1999) analyze a research tournament where an auction is used to shortlist potential contestants for entry in the tournament, and show that for a large class of contests the optimal number of finalists is two, and that while neither a first-price nor second-price format can generally induce efficient entry, an all-pay auction amended to award a prize to all the entrants can. In the context of indicative bidding, Ye (2007) analyzes a very similar game and reaches the same conclusions. The "all-against-all" shortlisting method is also adopted by Fu and Lu (2009), who investigate the effort-maximizing structure and prize allocation in multi-stage contests where each stage is a multiple-winner multiple-loser nested Tullock contest - basically an extension of the Tullock contest where in each stage a set of prizes is awarded to multiple winners (see Clark and Riis (1996) and Clark and Riis (1998)).

A second point of divergence between our work and the bulk of literature is that we do not adopt a normative perspective but rather a positive one.

In some of the aforementioned papers both the multi-stage and the all-pay features are introduced to achieve efficient shortlisting, whereas in our model both features are a natural description of a stepwise competition where all players forfeit their outlay. However, by no means we think that a descriptive approach is superior to a normative one, which instead is extremely relevant in many contexts. We rather think that a normative analysis should be complemented with a positive one that detects the determinants of bidders' behavior when they are not constrained by the designer's maximization problem.

A third major distinct feature of our analysis is that we consider the role of a constraint on resources in elimination contests.

Despite the fact that in most real world contests players face a cap on resources, the literature has

where contestants enter the game sequentially (rather than simultaneously), as it is the case in many real contests where contestants perform one after the other. Other contributions to this strand are Konrad and Leininger (2007) and two very recent contributions by Segev and Sela (2011) and Segev and Sela (2012).

mostly neglected this important aspect[6]. Just a handful of papers embed it explicitly in the analysis. Amegashie (2002) analyzes a two-stage elimination contest and finds that when contestants face a symmetric cap on total effort between stages, full "burning out" can happen in equilibrium, i.e., all active players may find it optimal to spend all resources in the first-stage and be left with nothing in the second stage [7]. Notice that there are real-world phenomena where the contest designer may find it convenient to impose symmetry between players, typically to handicap an ex ante stronger player and ensure that the competition is balanced[8].

Stein and Rapoport (2005) introduce a budget constraint in a two-stage contest very similar to the one in Amegashie (1999) and find that as the rent increases, the ratio between second stage and first stage expenditures is constant if the budget constraint is not binding, while it decreases non linearly in the value of the rent if the constraint is binding. Last, Matros (2006) studies optimal seeding in a multiple-round elimination tournament similar to the one in Rosen (1986), where players face a fixed common budget constraint and the success function in each round is stochastic. He finds that players always spend (weakly) more resources in the initial than in the following rounds, since the expected payoffs are much higher in the first round than in subsequent rounds, and that in order to ensure equal resource allocation in all rounds and maximize total effort, the designer should implement the winner-takes-all prize scheme combined with an equal number of players in each group.

In our model, we do consider the fact that players face a constraint on resources, but, differently from the above contributions, we do not introduce explicitly a budget constraint. We rather consider that if players are to behave rationally, as is it customarily assumed in economic models, they will never let the sum of the outlays made through stages exceed their valuation, i.e. they will consider the value they assign to winning the good as a natural cap on their total spending.

The important point that we share with this limited litertaure, however, is that when contestants are (artificially or rationally) budget-constrained, a fundamental trade-off exists when allocating resources over stages: the more resources a player spends in a particular stage the higher the chance to get shortlisted to the next stage, but the lower the chance to get shortlisted in later stages and eventually win.

However, strangely enough, the aforementioned works did not realize that the very presence of this trade-off implies that at each given stage a player is allowed to bid, the total of the bids she has spent until that stage cannot be considered as sunk, since they are *not* strategically irrelevant in

[6]For the role of effort caps in single-stage contests see Che and Gale (2003), where it is shown that if contestants are asymmetric it is optimal to handicap the most efficient one, and Gavious et al. (2002) who endogenize the choice of imposing a bid cap.

[7]Burning out is peculiar given that the prize arrives only if a contestant is successful in all stages including the last stage. Contestants may expend all their effort in the first stage because if they do not peform well in the first-stage they cannot go to the second stage. Besides the presence of a symmetric cap on aggregate effort, the other necessary condition for burning out is that he contest success function is highy discriminatory.

[8]This is the rationale behind, for example, common salary caps in US professional sports.

her decision of how much to bid in the following stages. When there is a constraint on resources, expenditures made in earlier stages limit expenditures that can be done in later stages.
To the best of our knowledge the present paper is the first one in the literature to make this simple but relevant point.
The only paper that somehow considered that outlays in early stages may play a role in later stages, but with much different reasons with respect to ours, is Baik and Lee (2000). Motivated by those real world cases where efforts made in early stages affect the outcomes of later stages (e.g., sport games where scores in qualification stages are carried over to the final stage and added to the final-stage score), they study a two-stage Tullock contest with symmetric players, and consider the effects on rent dissipation of allowing first-stage efforts to be partially "carried over" to the second stage, i.e., first-stage efforts are (partially) taken into account in second-stage decision making.

3. The model

Consider that N risk-neutral players participate to a multi-stage contest organized as K consecutive all-pay auctions, where at each stage a given number of highest bidders is shortlisted to go to subsequent stages, and the players who are shortlisted in all stages are awarded a prize. Also, there is no minimum outlay required to enter the competition and the number of players to shortlist at each stage can vary from one stage to the other, since the contest designer has discretion over that.
Let us define the set of players $\mathcal{N} = \{1, 2, ..., i, ..., N\}$ and the set of stages $\mathcal{K} = \{a, b, ..., k, ..., K\}$, with $\mathcal{N}, \mathcal{K} \subset \mathbb{N}_+$ [9]. Players' valuations for the prize are $v_1 \geq v_2 \geq ... \geq v_N > 0$, where v_i is the valuation of the i-th player, with $v_i \in \mathbb{R}_+$, $\forall i \in \mathcal{N}$. We assume complete information, that is players know each other's valuations.
We indicate with q^k the number of players to be shortlisted at stage k, with $k \in \mathcal{K}$, $q^k \in \mathbb{N}_+$, so that q^{K-1} is the number of final winners, i.e., players that are awarded a prize. We assume that q^k is exogenous, i.e., is determined and announced ex-ante by the contest sponsor.
Technically, each stage-auction is a multi-unit all-pay auction where units are "shortlisting tickets" and each bidder demands one unit only [10]. Each bidder allowed to play at k offers a non-negative bid (effort) $x_i^k \in \mathbb{R}_+ \cup \{0\}$ (the vector $x^k \in \mathbb{R}_+^{q^k} \cup \{\mathbf{0}\}$ being the *profile* of actions taken by the players who can play at k) and the q^k highest bidders are awarded the q^k shortlisting tickets to stage $k+1$, but all players pay their own bid.
We further define as the *marginal bid at stage k*, and indicate it with $x^k_{(q^k)}$, the q^k-highest bid, i.e., the stage-k bid of the last - or "weakest" - player who manages to get a ticket to stage $k+1$, so that shortlisted players have bids: $x^k_{(1)} \geq x^k_{(2)} \geq ... \geq x^k_{(q^k)}$. The bidder i such that $x_i^k = x^k_{(q^k)}$

[9] In the following, the subscript indices will refer to players and the superscripts will refer to stages.
[10] We borrow the tickets analogy from previous works: Lai and Matros (2007) talk about "entry tickets" and Fu and Lu (2009) about "tickets to the next stage".

is the *marginal bidder at stage* k [11]. The bidders who bid below the marginal bid at stage k are eliminated, while if a player bid exactly the marginal bid, two cases can occur:

"More tickets than marginal bidders" case: either there are no ties at the marginal bid or the number of players tying at the marginal bid is lower or equal than the number of shortlisting tickets remaining after that players who have bid higher than the marginal bid have already been shortlisted. In this case no tie-breaking is needed: all the players who tie at the marginal bid will get shortlisted with certainty.

"More marginal bidders than tickets" case: the number of players tying at the marginal bid is higher than the number of remaining shortlisting tickets. In this case tie-breaking is needed and ties are broken uniformly at random so that player i has $\frac{\#\text{ rem. tickets}}{m^k}$ probability of getting shortlisted to stage $k+1$, where m^k is the number of players (included i) who tie at the marginal bid in stage k.

The shortlisting process is depicted in Figure 3, where for each player bidding at a given stage, it is indicated the bid which allowed her to get shortlisted to that stage.

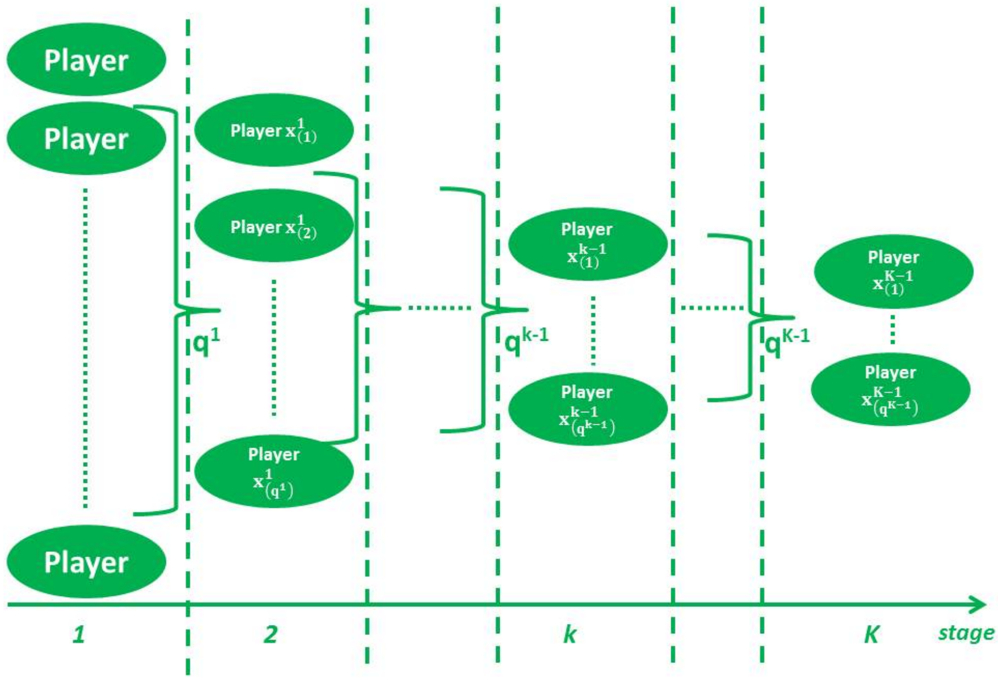

Figure 1: The shortlisting process in the elimination contest

[11] We borrow this definition from Fullerton and McAfee (1999), with a slight difference: in their work "marginal bidder" indicates the "first rejected bidder" rather than the "last accepted" one as in our case.

For the sake of clarity, consider the following simple example. Suppose that there are 6 players allowed to bid at a given stage and that a total of 4 shortlisting tickets are available for the next stage. In this case the marginal bid is the fourth highest bid. Suppose that two players bid higher than the marginal bid, one player bids lower and 3 players tie at the marginal bid. The high-bid players get with certainty the ticket to the next stage, and the low-bid player is eliminated. The number of tickets left is 2, but there are 3 players tying at the marginal bid, so that we are in the "more marginal bidders than tickets" case. Tie-breaking is needed: the 3 players at the tie have $\frac{2}{3}$ probability each to get a ticket. Suppose instead that there are just two players tying at the marginal bid, and two players bidding lower. In this case we are in the "more tickets than marginal bidders" case and there is no need of tie-breaking: both players at the tie get the ticket with certainty.

We define as the *net valuation* of player i who is shortlisted at stage k, and indicate it with NV_i^k, player i's valuation net of the sum of the bids she has made until stage k, i.e.,

$$NV_i^k \triangleq v_i - \sum_{l=1}^{k} x_i^l \quad \text{with} \quad NV_i^k \in \mathbb{R} \quad \forall i \in \mathcal{N}, k \in \mathcal{K} \tag{1}$$

Importantly, we assume that the total value each player *rationally* bids over stages cannot exceed her valuation. As discussed in the previous sections, a player's valuation, which represents the maximum amount of resources the player is willing to spend for winning the prize, represents a "natural" cap on the total effort each player can exert as a whole in the competition.

Therefore, the i-th player faces the following *implicit budget (or effort) constraint*:

$$\sum_{k=1}^{K} x_i^k \leq v_i \tag{2}$$

Notice that the maximum amount a player can spend at a given stage k is her net valuation at stage $k-1$, so that, relative to each stage k, we could rewrite the budget constraint of Equation 2 as:

$$x_i^k \leq NV_i^{k-1} \tag{3}$$

Obviously, if player i spends exactly her net valuation, $x_i^k = NV_i^{k-1}$, she will have no resources left to spend from stage $k+1$ onwords. Therefore it is clear that in each stage where a player is allowed to play, the total of the bids she has spent until that stage are not sunk, since they are not strategically irrelevant in her decision of how much to bid in following stages.

In game-theoretic terms, our multi-stage all-pay auction can be defined as an extensive game with complete information and simultaneous moves [12]. The game is described by the following elements:

[12]We use the same definition in Chapter 6 of Osborne and Rubinstein (1994).

1. A set of players: $\mathcal{N} = \{1, 2, ..., N\}$.
2. A set of histories: $\mathcal{H} = \left\{h^k, 0 \leq k \leq K\right\}$, where h^0 is the empty sequence (i.e., $\emptyset$) and $h^r = (x^k)_{k=1,...,r}$ with $r \leq K$.
   Each member of $\mathcal{H}$ is a *history*, i.e., a sequence of profiles of actions taken by players. A history h^r is terminal if $r = K$. The set of terminal histories is called $\mathcal{Z}$. The set of actions available for players after history h^r is $\mathcal{A}(h^r) = \{x^{r+1} : (h^r, x^{r+1}) \in \mathcal{H}\}$.
3. A function P that assigns to each non terminal history (each member of $\mathcal{H} \backslash \mathcal{Z}$) a set of members of $\mathcal{N}$. P is called the *player function*, $\mathcal{P}(h^r)$ being the set of players who take an action after history h^r:

$$\mathcal{P}(h^0) = \{1, 2, ..., N\}, \mathcal{P}(h^r) = \left\{i : x_i^r \geq x_{(q^r)}^r\right\} \tag{4}$$

   $x_{(1)}^r > x_{(2)}^r > ... > x_{(q^r)}^r$
4. For each player i a *preference relation* $\succeq_i$ on $\mathcal{Z}$. These preferences are represented by the payoff function of player i at the final stage K, i.e.,

$$\Pi_i(h^K) = \begin{cases} v_i - \sum_{k=1}^K x_i^k, & \text{if } \{x_i^K > x_{(q^K)}^K\} or \{x_i^K = x_{(q^K)}^K and \text{ rem. tickets} > \text{ties}\} \\ \frac{v_i}{m^K} - \sum_{k=1}^K x_i^k, & \text{if } \{i \text{ ties at } x_{(q^K)}^K and \text{ ties} > \text{rem. tickets}\} \\ -\sum_{k=1}^r x_i^k \quad (r \leq K), & \text{otherwise} \end{cases} \tag{5}$$

The payoff function above tells us that player i can end up in three situations: (i) if she makes it to the final stage and offers a bid higher than the marginal bidder (which, at this stage, is the last player who is awarded a prize), she "wins" (i.e., she is awarded a prize) with certainty. This also happens if player i is the marginal bidder but either she does not tie with anyone, or the number of ties is lower than the number of tickets left after the shortlisting of highest bidders ("more tickets than marginal bidders" case). In both cases she gets a payoff equal to her valuation net of the sum of all the bids she has made until the final stage; (ii) if player i ties with some other player(s) at the marginal bid in the final stage and the number of ties is higher than the number of tickets left ("more marginal bidders than tickets" case), she is awarded a prize only if she is selected by the uniformly random tie-breaking mechanism, but the sum of the bids she made is a certain outlay; (iii) if player i is eliminated before the final stage, or if she makes it but bids below the marginal bid, she incurs a loss equal to the sum of the bids she has made until the last stage she has been shortlisted in. Notice that the losses are greater the further the player is shortlisted through the stages.

It is plausible to think that if player i expects to get either a negative or a zero payoff in the game, she has no incentive to bid positive in any of the stages she will have a chance to bid in. In particular, we impose the two following assumptions to hold in the model:

Assumption 1 Players who bid zero in a given stage are allowed to be shortlisted to the next stage. In fact, it can be the case that the marginal bid is zero so that players bidding (with or without ties) at zero have a positive probability to get shortlisted. Therefore, unlike similar models, bidding zero is not equivalent to stay out of the contest.

Assumption 2 Players who expect to get either a negative or a zero expected payoff always enter the game and bid zero in all stages, as if they were not indifferent between staying out from the contest and entering, even if expected payoffs are the same.

The first assumption could seem strong, however other works adopt similar ones. For example Fu and Lu (2009) assume that if all contestants who participate in a round make zero effort, the winner of that round is selected at random. We have used Assumption 1 in the current version of the model, but we are currently working to amend the model so that zero bidders are no longer allowed to get shortlisted. However, we do not expect our results to change significantly.
At this stage of work we consider the basic case were $\mathcal{N} = \{1, 2, 3\}$, $\mathcal{K} = \{a, b\}$, $q^a = 2$ and $q^b = 1$. That is, there are three players and two stages, at the first stage two players are shortlisted and at the end only one player wins (i.e., one prize only is awarded). The model can be extended to the general case described above, and we plan to do that.
We analyze the case where players are asymmetric, i.e., have different valuations for the prize, $v_1 > v_2 > v_3$.

4. Characterization of equilibria

We solve the game by backward induction, looking for the complete set of Subgame Perfect Nash Equilibria (SPNE), i.e., of triples of strategies $(x_i^{a*}, x_i^{b*}) \quad \forall i \in \{1, 2, 3\}$. Therefore, we first look for the Nash Equilibria of the generic subgame after any possible (length-1) history h^a, namely the Stage-b all pay auction, and then, on the basis of continuation payoffs, we analyze the bidding in Stage-a [13].

4.1. Stage-b all-pay auction

In Stage-b the two players who got shortlisted from Stage-a, that is players $i, j : x_i^a, x_j^a \in \{x_{(1)}^a, x_{(2)}^a\}$, bid again. Notice that since we are working backwards, we do not know the identity of these players, so that we indicate them generically with i, j. The Stage-b (i.e., final) payoff

[13] For each $h^a \in \mathbb{R}_+^3$ there is a subgame, so that there are infinite possible subgames.

function of player i (analogously for j) is as follows:

$$\Pi_i(h^b) = \begin{cases} v_i - x_i^a - x_i^b, & \text{if} \quad x_i^b > x_j^b \\ \frac{v_i}{2} - x_i^a - x_i^b, & \text{if} \quad x_i^b = x_j^b \\ -(x_i^a + x_i^b), & \text{if} \quad x_i^b < x_j^b \end{cases} \tag{6}$$

The payoff function above tells that: (i) if shortlisted player i bids higher than the other shortlisted player, she wins the prize with certainty and gets a payoff equal to her valuation net of the sum of the bids she has made in the two stages; (ii) if i ties with the other player, the winner of the prize is selected randomly but the the sum of the bids is a certain outlay for both; (iii) if i bids lower than her opponent, she incurs a loss equal to the sum of the bids.

Notice that the Stage-a bid must be present in the Stage-b payoff since, as said, we do not consider them as sunk. The Stage-a bid caps the maximum amount player i can spend in Stage-b, i.e., $x_i^b \leq v_i - x_i^a$.

Aside from the indeterminacy of players who enter the subgame and the presence of the net valuations in place of the standard valuations, the Stage-b subgame is analogous to a standard one-stage two-player all-pay auction with complete information. It is well known that this kind of auction does not have a Nash equilibrium in pure strategies, neither in the case where players are asymmetric nor in the case with symmetric players, but, in both cases, it does have a Nash equilibrium in mixed strategies (e.g., see Hillman and Samet (1987) for the symmetric case and Hillman and Riley (1989) and Baye et al. (1996) for the asymmetric case.)

Notice that even if players are ex-ante asymmetric, i.e., they have different valuations, after bidding in Stage-a they can become symmetric, in the sense that they can get the same net valuation. Therefore, we should consider two different cases:

ex-ante asymmetric - ex-post asymmetric players ("asym-asym" case): shortlisted players have different net valuations: in this case we will denote by H the player who is shortlisted to the Stage-b subgame with the *higher net valuation*, $NV_H = v_H - x_H^a$, and by L the player who is shortlisted to Stage-b with the *lower net valuation*, $NV_L = v_L - x_L^a$, with $H, L \in \{i, j\}$ and $NV_H > NV_L$.

ex-ante asymmetric - ex-post symmetric players ("asym-sym" case): shortlisted players have the *same net valuations*, $NV_i = NV_j$: in this case we will denote them by S and use the notation $NV_S = v_S - x_S^a$, with $S \in \{i, j\}$. Notice that this case can only happen if the ex-ante stronger between the shortlisted players made in Stage-a a higher bid than the ex-ante weaker player. However, as we will see, the "asym-sym" case will never occur in the SPNE.

The two following propositions characterize the equilibrium of the Stage-b game:

Proposition 1 (*Existence of mixed strategy Nash equilibrium in the Stage-b all-pay auction*)

1.1 No pure strategy equilibrium can exist in the subgame, neither in the "asym-asym" case nor in the "asym-sym" case.

1.2 The equilibrium bid of each shortlisted player i in the Stage-b subgame is a random variable with cumulative distribution function (CDF) $F_i(x_i^b)$ which is continuous over $(0, \infty)$.

1.3 The support of the equilibrium CDF is the same $\forall i$ and is $[0, NV_L]$ in the "asym-asym" case and $[0, NV_S]$ in the "asym-sym" case.

1.4 In equilibrium at most one agent bids zero with strictly positive probability.

Proof: In the Appendix 6.1.

Proposition 2 (*Nash equilibrium of the Stage-b all-pay auction*)

The Stage-b all-pay auction has one of two possible different asymmetric Nash equilibria in mixed strategies, depending on which case occurs:

2.1 "asym-asym" case: If players got shortlisted with different net valuations, there is an unique equilibrium where the player who got shortlisted with the higher net valuation (player H) randomizes continuously over $(0, NV_L]$ according to the mixed strategy $F_H(x_H^b) = \frac{x_L^a + x_H^b}{v_L}$, and the player who got shortlisted with the lower net valuation (player L) randomizes continuously over $(0, NV_L]$ according to the mixed strategy $F_L(x_L^b) = \frac{NV_H - NV_L}{v_H} + \frac{x_H^a + x_L^b}{v_H}$. Players' equilibrium payoffs are respectively $u_H^* = NV_H - NV_L$ and $u_L^* = 0$.

2.2 "asym-sym" case: If players got shortlisted with the same net valuation, there is an unique equilibrium where both players i and j randomize continuously over $(0, NV_S]$ according, respectively, to mixed strategies $F_i(x_i^b) = \frac{x_j^a + x_i^b}{v_j}$ and $F_j(x_j^b) = \frac{x_i^a + x_j^b}{v_i}$, and both get an equilibrium payoff of zero, $u_S^* = 0$ with $S \in \{i, j\}$.

Proof: In the Appendix 6.2.

Notice that each player's equilibrium strategy, in both the "asym-asym" and the "asym-sym" case, contains the Stage-a bid of the opponent: as we expected, given that in our model Stage-a bids are not sunk, Stage-b equilibrium strategies take into account what happened in Stage-a. Except for the presence of the Stage-a bids, the equilibrium strategies of the "asym-asym" case are identical to the ones of the one-stage all-pay auction with asymmetric players (see Theorem 3 in Baye et al. (1996)). On the other hand, the equilibrium of the "asym-sym" case resembles somehow the one of the one-stage all-pay auction with symmetric players (see Proposition 3 in Hillman and Samet (1987)), insofar both players get a zero expected payoff; however, differently from the one-stage game, equilibrium strategies embed the Stage-a bids and also display player-specific valuations at the denominator (obviously, given the ex-ante asymmetry between players). Notice also that equilibrium payoffs are expressed in terms of net valuations, so that they take into account Stage-a bids. Interpretation is that player L (in the "asym-asym" case) and player S (in the "asym-sym" case), who get a zero expected continuation payoff, sometimes win and sometimes lose so that, on average, they are able to exactly cover the sum of their outlays, whereas player H (in the "asym-asym" case) is winning more often than losing, so that, on average, she is able to more than cover the sum of her outlays. Also, notice that we cannot check yet whether the equilibrium strategies respect requirement [1.4] in Proposition 1, since at this stage we do not know the equilibrium values of Stage-a bids. As we shall see, such requirement is met in the SPNE strategies.

Proposition 2 tells us something interesting: when players are ex-post asymmetric ("asym-asym" case) *the winner is, on average, the player who got shortlisted to Stage-b with the higher net valuation*, rather than the player with the original higher valuation (i.e., the ex-ante strongest player), as it was the case in the equilibrium of the one-stage game. The intuition behind this result is that *when players have limited resources to allocate between stages, winning depends crucially not only on the relative strengths of players, but also on how players allocate resources between the two stages.* However, we will see in the following that, due to the asymmetry between players, the strongest player (i.e., player 1) has an advantage over the other players, so that in equilibrium she will always be the player who gets shortlisted with the higher net valuation, and, consequently, will have better chances to be the final winner. Instead, when players are ex-post symmetric ("asym-sym" case), no player has an advantage, so that they have symmetric chances to be the final winner.

4.2. Stage-a all-pay auction

Now we go backward: given the continuation payoffs in Proposition 2, we ask what are the optimal choices of players at Stage-a, i.e., after history h^0.

Notice that $x^a_{(2)}$ is the marginal bid at Stage-a, so that player i (with $i = \{1, 2, 3\}$) will get shortlisted with certainty only if her Stage-a bid is greater than the marginal bid, or if she bids the marginal bid but we are in the "more tickets than marginal bidders" case, which in this case means that either $x^a_i = x^a_j > x^a_k$ or $x^a_j > x^a_i > x^a_k$. Instead, if i ties with other players at the marginal bid but we are in the "more marginal bidders than tickets" case, which in this case means that either $x^a_j > x^a_i = x^a_k$ (i.e., $m^a = 2$) or $x^a_i = x^a_j = x^a_k$ (i.e., $m^a = 3$), ties will be uniformly broken at random.

Formally, the Stage-a payoff function of player i is as follows [14]:

$$\Pi_i(h^a) = \begin{cases} u_i^*, & \text{if } \{x_i^a > x_{(2)}^a\} \ or \ \{x_i^a = x_j^a > x_k^a\} or \{x_j^a > x_i^a > x_k^a\} \\ \frac{\#\text{ rem. tickets}}{m^a} u_i^*, & \text{if } \{x_j^a > x_i^a = x_k^a\} \ or \ \{x_i^a = x_j^a = x_k^a\} \\ -x_i^a, & \text{otherwise} \end{cases} \tag{7}$$

where u_i^* is the continuation payoff of player i in case she is shortlisted, such that:

$$u_i^* = \begin{cases} 0, & \text{if } \quad i = \{L, S\} \\ NV_H - NV_L, & \text{if } \quad i = H \end{cases} \tag{8}$$

Therefore, player i knows that: (1) if she is not shortlisted, she will incur a certain loss equal to her Stage-a bid; (2) if she is shortlisted with NV_L or NV_S, she will get a zero expected payoff, i.e., on average she will not be the final winner, while (3) if she is shortlisted with NV_H, she will get a positive expected payoff, i.e, on average she will be the final winner.
From Assumption 2 in Section 3 we have that players who expect to end up either in (1) or in (2) always enter the game and bid zero in Stage-a, and from Assumption 1 in Section 3 we have that players who bid zero in Stage-a have a chance to be shortlisted to Stage-b. On the other hand, if player i expects to end up in (3), she will bid in Stage-a the lowest possible amount that allows her to get shortlisted. From the discussion above it is clear that each player i's optimal choice in Stage-a is to make such a bid that allows her to get shortlisted with NV_H. However, the presence of an ex-ante asymmetry between bidders implies the following:

Proposition 3 *(Advantage for ex-ante stronger players)*
A player with a higher valuation has an "advantage" over a weaker player: if she underbids (at limit, overlaps to) a bidder with a lower valuation, she will get shortlisted with NV_H with certainty, provided that her bid allows her to get shortlisted. Formally:

$$\forall x_i^a, x_j^a \in \{x_{(1)}^a, x_{(2)}^a\} \quad \text{with} \quad x_i^a \leq x_j^a, \quad \text{if} \quad v_i > v_j \quad \text{then} \quad v_i - x_i^a > v_j - x_j^a \tag{9}$$

On the basis of this result, notice that for any couple of *positive* bids she expects from her rivals, $x_i^a > x_j^a > 0$ with $i, j \in \{2, 3\}$, player 1's optimal choice is to "bid in between" i.e., such that $x_i^a > x_1^a > x_j^a$, since this always ensures that she is shortlisted with NV_H. Notice that the case $x_2^a = x_3^a > 0$ will not occur in equilibrium insofar player 3 has no incentive to overlap with a strictly positive bid to stronger players, since in case she manages to get shortlisted, she will always get NV_L and hence a zero continuation payoff.

[14]Notice that each possible history of length-1 is a profile of actions, so that $h^a = x^a$, with $x^a \in \mathbb{R}_+^3$

But anticipating that this way they will never be able to get shortlisted with NV_H, players 2 and 3 have no incentive to bid positive in Stage-a. One could think therefore that player 1's optimal response, anticipating that player 2 and 3 will bid zero, might be to overlap and bid zero as well. However, notice that this way she would risk not to get shortlisted at all. For this reason she might rather prefer to bid an arbitrarily small positive amount, $\epsilon > 0$.
We prove that in fact this is exactly the case and that the following result holds:

Proposition 4 *(Equilibrium of the Stage-a all-pay auction)*
Given continuation payoffs in Equation 8, the Stage-a all-pay auction has a unique equilibrium in pure strategies, which is $x^{a*} = \{x_1^{a*} = \epsilon, x_2^{a*} = 0, x_3^{a*} = 0\}$. In this equilibrium only player 1 bids a positive amount, which is indetermined but very close to zero. Also, only the "asym-asym" case occurs in equilibrium, so that player 1 is always shortlisted with NV_H.

Proof: In the Appendix 6.3.

Proposition 5 *(Equilibrium of the Stage-a all-pay auction - continued)*
The Stage-a all-pay auction has no equilibria in mixed strategies, so that the pure-strategy equilibrium is the unique equilibrium of Stage-a.

Proof: In the Appendix 6.4.

4.3. SPNE of the two-stage all-pay auction

From Propositions 1-5 we get the following result:

Proposition 6 *(SPNE of the two-stage all-pay auction)*
When players have valuations $v_1 > v_2 > v_3$, the two-stage all-pay auction has a unique SPNE which is as follows:

$$\begin{cases} \left(x_1^{a*} = \epsilon \simeq 0, \quad F_1(x_1^b) = \frac{x_1^b}{v_j} \quad \forall x_1^b \in [0, v_j]\right) \\ \left(x_j^{a*} = 0, \quad F_j(x_j^b) = \frac{v_1 - v_j}{v_1} + \frac{x_j^b}{v_1} \quad \forall x_j^b \in [0, v_j]\right) \end{cases} \tag{10}$$

with $j \in \{2, 3\}$. Equilibrium payoffs are $u_1^* = NV_1 - NV_j = v_1 - \epsilon - v_j$ and $u_j^* = 0$.

In this equilibrium only player 1, i.e., the ex-ante strongest player, bids positive in Staga-a. Consequently, she gets always shortlisted and always with NV_H. Player 2 and 3 bid zero, and one of

them gets shortlisted at random, and always with NV_L[15]. In Stage-b shortlisted players randomize over a common support, whose upper bound is ex-ante indetermined and equal to $NV_j = v_j$ (since $x_j^{a*} = 0$). Player 1 gets a positive expected payoff, i.e., on average she wins the game, while the other shortlisted player gets a zero expected payoff, i.e., on average she makes no losses, and the player who is not shortlisted gets an actual zero payoff.
Therefore, even if the subgame result tells us that the player who will win on average the game is not the ex-ante strongest player, but rather the player who is able to allocate the resources such that she manages to get shortlisted with the higher net valuation, however the SPNE result tells us that *in fact, this player is always player 1, so that it seems that players' relative ex-ante strengths are more important, in determining the outcome of the game, than their relative abilities to allocate optimally limited resources over stages.*
The intuition for this result is that since there is complete information and the game is dynamic, players can use information about continuation payoffs and rivals' valuations to bid optimally in Stage-a. By Proposition 3, a player who anticipates that in the second-stage she will meet a stronger player, does not want to bid positive in the first stage. Because of Assumption 1, she can still be shortlisted but on average she will make no loss in the end. The information structure allows the strongest player to deter other players from bidding positive in the first stage, so that she can ensure to get shortlisted with a very small outlay and save most resources for the second stage. Therefore, even if the information structure in our model is different, we get a result similar to the literature about the effect of signalling in elimination contests with information revelation (e.g.,Lai and Matros (2007)): too much information does not necessarily lead to "good" outcomes. In both cases efficient shortlisting of players is prevented: in their case this is due to misrepresentation of preferences, whereas in our case it is due to the predatory behavior of the strongest player. Unsurprisingly, this is detrimental for total revenue (effort) extraction, as illustrated by the following Proposition.

Proposition 7 *(Expected Revenue)*
The two-stage all-pay auction yields a *lower expected revenue* than the one-stage all-pay. The first stage yields virtually no revenue, whereas the second stage yields a lower expected revenue than the two-player all-pay due to inefficient shortlisting, that is the fact that the weakest player (i.e., player 3), has a positive chance to get shortlisted and to win eventually.

Proof: In the Appendix 6.5

This result is consistent with the prediction in Gradstein and Konrad (1999) that when the contest rules are discriminatory enough, a one-stage contest yields a higher total effort than its

[15] Notice that equilibrium strategies meet the requirement 1.4 of Proposition 1.

multi-stage counterpart. In our case this is due to the fact that the multi-stage contest implies a positive probability that the second strongest player does not reach the final stage, so that shortlisting is inefficient.

Therefore, what emerges from our analysis is that a multi-stage all-pay auction does not seem to have very appealing features. However, it is reasonable to think that our results may depend on the simplicity of our analysis. In the next section we discuss some possible work developments aimed at enhancing the descriptive power of the model.

5. Conclusion

In this paper we have studied a multi-stage elimination all-pay contest with budget-constrained players and non-sunk bids.

Differently from related literature, we realize that when players face a cap on resources, they do not regard past bids as strategically irrelevant in their decision of how much to bid in following stages. This happens because they face a basic trade-off when deciding how to allocate scarce resources between earlier and later stages of the contest: the more resources a player spends in a particular stage the higher the chance to get shortlisted to the next stage, but the lower the chance to get shortlisted to later stages and eventually win. Choices made in earlier stages do influence and bind choices to be made later. Further, the presence of the trade-off implies that the winner may not be the player with the biggest ex-ante budget - as it is on average the case for one-stage all-pay contests - but rather the player who is the most able in allocating limited resources over stages.

Also, although we did not explicitly embed a budget constraint in the model, we assumed that players *do* face a constraint on resources. Indeed, if contestants are to behave *rationally*, as it is conventionally assumed in economic theory, they never spend more than their valuation, i.e., they consider the value they assign to winning the prize as a "natural" cap on the total effort that they are willing to exert in the contest.

On the basis of these original considerations, we focused on a simple two-stage contest with complete information and asymmetric players, and we found that the relative strengths of players are more important than their relative trading-off abilities, in determining the outcome of the game. This result stemmed from the fact that since the game is dynamic and there is complete information, players can use information about continuation payoffs and rivals' valuations to bid optimally in the first stage: a player who anticipates that in the second-stage will meet a stronger player, does not want to spend resources in the first stage. The information structure allows the strongest player to always deter other players from bidding positive in the first stage, so that she can ensure to get shortlisted with a very small outlay and save most resources for the second stage (where mixed strategies are played). This leads to the result - consistent with Gradstein and Konrad (1999)'s prediction - that the two-stage all-pay auction yields a lower expected revenue than the one-stage all-pay, since the first stage yields basically no revenue (due to the strongest player

exerting a minimal effort in the first stage) and shortlisting is inefficient (since the weakest player has a positive chance to get shortlisted and eventually win).
On the basis of these results, our elimination contest does not seem to be an advantageous allocation mechanism for the contest sponsor. However, we are aware that the current version of our model is simple so that more work is needed to fully assess the performance of this mechanism. Therefore, we are currently working at the implemention of a number of steps aimed at enhancing the descriptive power of the model. In particular, we are modifying the model so to no longer allow that players who bid zero in the first stage have a chance to get shortlisted, and to allow for a more realistic information structure. These steps will enable us to check whether our current results are in fact to some extent underestimating the potential performance of the all-pay elimination contest. We also plan to extend the analysis to the general case with N players and K stages [16].
We believe that our work is valuable along at least two dimensions. First, our paper provided a contribution to the elimination contest literature: to the best of our knowledge this is the first attempt to characterize the equilibria of a multi-stage elimination all-pay contest with non-sunk bids and where players's valuations are regarded as a natural budget cap. Although non-sunk bids made the analysis more complex, they allowed to improve the quality of the modelization for many real competitive scenarios, like R&D contests and sport tournaments. Second, we hopefully inspired new reflection about the general issue of how the assumption of agents'rationality should be exploited, and more in particular, about the need to be careful and thorough when assessing whether costs in a given dynamic decision problem are to be considered sunk or non-sunk, which is a relevant issue in many fields of economics.

6. Appendix

6.1. Proof of Proposition 1

Arguments needed for the characterization of the equilibrium in this case are totally analogous to those used by Hillman and Riley (1989) and Hillman and Samet (1987) for characterizing the equilibria of the one-stage all-pay auction, respectively for the asymmetric and the symmetric case. However, for the sake of clarity, we will reformulate all the main arguments so that they fit our case.

Lemma 1: No pure-strategy equilibrium can exist in the Stage-b subgame, neither in the "asym-asym" case nor in the "asym-sym" case.
Proof.
"Asym-asym" case. For any bid of the other player which is below the lower net valuation NV_L, each player has an incentive to slightly overbid the other player, so that there is a race to the top

[16] The only step left to complete the characterization of the two-stage all-pay auction is to analyze the case with ex-ante symmetric players.

until NV_L (there is no equilibrium below NV_L). There is no equilibrium above NV_L either, since player L will never bid more than her net valuation, and consequently neither player H would bid above. Also, there is no equilibrium for ties at NV_L, since player L would be better off bidding zero, and the race to the top would start again. Therefore, the "asym-asym" case cannot have an equilibrium in pure strategies.

"Asym-sym" case. For any bid of the other player which is below the common net valuation NV_S, each player has an incentive to slightly overbid the other player, so that there is a race to the top until NV_S (there is no equilibrium below NV_S). There is no equilibrium above NV_S either, since no player will bid more than her net valuation. Also, players will never tie at NV_S, since they are better off by bidding zero. Therefore the " asym-sym" case neither can have an equilibrium in pure strategies. Q.E.D

Lemma 2 No player will, in equilibrium, ever spend a *positive* amount with strictly positive probability, i.e., equilibrium strategies are *continuous* mixed strategies.

Proof. Suppose to the contrary that player i spends some $x_i^b = \beta > 0$ with strictly positive probability. Then player j will always beat that bid with a marginally greater bid (the probability that j beats i rises discontinuously as a function of x_j^b at $x_j^b = \beta$). Therefore, there is some $\epsilon > 0$ such that j will bid in the interval $[(\beta - \epsilon), \beta]$ with zero probability. But then, agent i would be better off by bidding $\beta - \epsilon$ rather than β, since her probability of winning would be the same, contradicting the hypothesis that $x_i^b = \beta$ is an equilibrium strategy in the subgame.

Lemma 3 In equilibrium the two players must have the same maximum spending level.

Proof. From Lemma 2 it follows that, if $\bar{x}_i^b$ is player i's maximum spending level, player j wins with probability 1 by spending $\bar{x}_i^b$ and viceversa. Hence, the upper bound of the support is the same for both players and it is equal to NV_L in the "asym-asym" case and NV_S in the "asym-sym" case.

Lemma 4 In equilibrium the minimum outlay is zero for each player.

Proof. Suppose to the contrary that player i picks $x_i^b = \beta > 0$ as her minimum bid (i.e., she spends less than $x_i^b = \beta > 0$ with zero probability). Then, any bid in the interval $(0, \beta)$ would yield a negative payoff to player j, since the probability of winning is zero in that interval. Since player j can always bid zero, it follows that she neither will bid in the interval $(0, \beta)$. But then player i could reduce her bid below β without changing her probability of winning, contradicting the hypothesis that agent i's optimal minimum spending level was some $\beta > 0$. Hence, the lower bound of the support is the same for both players and it is equal to zero.

Given these results, if we define $1 - F_i(x_i^b)$ to be the probability that player i spends more than x_i^b, then $F_i(x_i^b)$ is continuous over $(0, \infty)$. If $0 < F_i(0) < 1$ then player i spends a strictly positive amount with probability less than 1 and her alternative is to spend zero.

Lemma 5 At most one agent bids zero with strictly positive probability.

Proof: If both players bid zero with positive probability then each has a chance of winning. However, this will not occur in equilibrium, for if one player spends zero with positive probability, the other can with an arbitrarily small positive bid increase her probability of winning and hence her expected payoff.

6.2. Proof of Proposition 2

Player i's expected payoff (with $i = \{H, L, S\}$) is:

$$E\Pi_i(h^b) = (v_i - x_i^a - x_i^b)F_j(x_i^b) + (-x_i^a - x_i^b)[1 - F_j(x_i^b)] = v_iF_j(x_i^b) - x_i^a - x_i^b \tag{11}$$

Equilibrium requires that, for any bid in her support, each player earns a constant expected payoff, given the mixed strategy of the other player.

The equilibrium condition we need to impose for player i is therefore as follows:

$$E\Pi_i(h^b) = v_iF_j(x_i^b) - x_i^a - x_i^b = u_i^* \quad \forall x_i^b \in [0, NV_n] \tag{12}$$

where u_i^* is a constant and $n \in \{L, S\}$. Setting $x_i^b = NV_n$, we are able to derive the expression for player i's equilibrium expected payoff:

$$u_i^* = NV_i - NV_n \tag{13}$$

Therefore for the "asym-asym" case (Proposition 2.1) we have the two following cases:

Case1. If player i is shortlisted with the *lower net valuation*, i.e., $i = L$ and $NV_i = NV_L$, then she will get on average a *zero* equilibrium payoff, $u_L^* = 0$.

Case2. If player i is shortlisted with the *higher net valuation*, i.e., $i = H$ and $NV_i = NV_H$, then she will get on average a *positive* equilibrium payoff, $u_H^* = NV_H - NV_L > 0$.

whereas for the "asym-sym" case (Proposition 2.2) we have $NV_i = NV_j = NV_S$, so that both of them will on average get a zero equilibrium payoff $u_S^* = 0$.

Consequently, the equilibrium conditions for the "asym-asym" case will be as follows:

$$E\Pi_H(h^b) = v_HF_L(x_H^b) - x_H^a - x_H^b = NV_H - NV_L \quad \forall x_H^b \in [0, NV_L] \tag{14}$$

$$E\Pi_L(h^b) = v_LF_H(x_L^b) - x_L^a - x_L^b = 0 \quad \forall x_L^b \in [0, NV_L] \tag{15}$$

The equilibrium mixed strategies are uniquely determined as the solutions of the system of the two above equations, and are as follows:

$$F_L(x_L^b) = \begin{cases} \frac{NV_H - NV_L}{v_H} + \frac{x_H^a + x_L^b}{v_H}, & \forall x_L^b \in [0, NV_L) \\ 1, & \forall x_L^b \geq NV_L \end{cases} \tag{16}$$

$$F_H(x_H^b) = \begin{cases} \frac{x_L^a + x_H^b}{v_L}, & \forall x_H^b \in [0, NV_L) \\ 1, & \forall x_H^b \geq NV_L \end{cases} \tag{17}$$

On the other hand, the equilibrium conditions for the "asym-sym" case are as follows:

$$E\Pi_i(h^b) = v_i F_j(x_i^b) - x_i^a - x_i^b = 0 \quad \forall x_i^b \in [0, NV_S] \tag{18}$$

$$E\Pi_j(h^b) = v_j F_i(x_j^b) - x_j^a - x_j^b = 0 \quad \forall x_j^b \in [0, NV_S] \tag{19}$$

from which the following equilibrium strategies are uniquely determined:

$$F_i(x_i^b) = \begin{cases} \frac{x_j^a + x_i^b}{v_j}, & \forall x_i^b \in [0, NV_S) \\ 1, & \forall x_i^b \geq NV_S \end{cases} \tag{20}$$

$$F_j(x_j^b) = \begin{cases} \frac{x_i^a + x_j^b}{v_i}, & \forall x_j^b \in [0, NV_S) \\ 1, & \forall x_j^b \geq NV_S \end{cases} \tag{21}$$

Therefore in both the "asym-asym" case and in the "asym-sym" case, the Stage-b subgame has a unique asymmetric equilibrium in mixed strategies Q.E.D.

6.3. Proof of Proposition 4

The proof is articulated in two parts. We first prove that the triple ($x_1^a = \epsilon, x_2^a = 0, x_3^a = 0$) is an equilibrium of the Stage-a all-pay auction, and then that it is the unique pure-strategy equilibrium.

6.3.1. Proof that the triple ($x_1^a = \epsilon, x_2^a = 0, x_3^a = 0$) is an equilibrium of Stage-a all-pay auction

We need to check whether any player has any incentive to deviate.
Consider player 1 first. Obviously, she would never deviate upward, but we need to check that in fact she does not find it profitable to bid zero rather than a positive amount. Player 1 will profitably

deviate *iff* the expected payoff of deviating is higher than the expected payoff of not-deviating, namely [17]:

$$E\Pi_1(x_1^a = 0, x_{-1}^{a*}) > E\Pi_1(x^{a*}) \tag{22}$$

Notice that if player 1 deviates, so that all players bid zero, the marginal bid is zero and we are in the "more marginal bidders than tickets" case and ties are broken randomly. In this case two possible events may occur:

- with probability 2/3 player 1 is shortlisted. In that case she will meet player 2 with probability 1/3 and player 3 with probability 1/3. By Proposition 3 we have that, given equality between bids, player 1 will always have NV_H whoever the other shortlisted player among player 2 and 3 will be. Her expected continuation payoff will be $u_1^* = NV_1 - NV_i = v_1 - v_i$, with $i \in \{2, 3\}$;
- with probability 1/3 player 1 is not shortlisted. However she makes no loss since her bid is zero.

The expected payoff from deviating is hence:

$$E\Pi_1(x_1^a = 0, x_{-1}^{a*}) = \frac{1}{3}(v_1 - v_2) + \frac{1}{3}(v_1 - v_3) \tag{23}$$

On the other hand, when player 1 does not deviate and bids $\epsilon > 0$, she is shortlisted and pays her bid with certainty. With probability 1/2 player 1 will meet player 2 and with probability 1/2 she will meet player 3, but whether she will have NV_H, NV_L or NV_S depends on ϵ:

if $\epsilon < v_1 - v_i$ (with $i \in \{2, 3\}$): then $NV_1 > NV_i$, i.e., $v_1 - \epsilon > v_i$, so that $1 = H$ and $u_1^* = NV_1 - NV_i = (v_1 - \epsilon) - v_i$.

if $\epsilon > v_1 - v_i$: then $NV_1 < NV_i$, so that $1 = L$ and $u_1^* = 0$

if $\epsilon = v_1 - v_i$: then $NV_1 = NV_S$, so that $1 = S$ and $u_1^* = 0$

Therefore, for the expected payoff of not deviating one should make three cases [18]:

$$E\Pi_1(x^{a*}) = \begin{cases} \frac{1}{2}(v_1 - v_2) + \frac{1}{2}(v_1 - v_3) - \epsilon, & \text{if} \quad 0 < \epsilon < v_1 - v_2 \\ \frac{1}{2}(v_1 - \epsilon - v_3), & \text{if} \quad v_1 - v_2 \leq \epsilon < v_1 - v_3 \\ 0, & \text{if} \quad \epsilon \geq v_1 - v_3 \end{cases} \tag{24}$$

[17]Remember that each possible history of length-1 is a profile of actions, so that $h^a = x^a$, with $x^a \in \mathbb{R}_+^3$.

[18]Notice that since $v_1 > v_2 > v_3$, when the condition $\epsilon \leq v_1 - v_2$ holds, then it also holds that $\epsilon < v_1 - v_3$. Specularly, when $\epsilon \geq v_1 - v_3$, then $\epsilon > v_1 - v_2$.

Remember that continuation payoffs are expressed in terms of net valuations, so that they take into account Stage-a bids. Interpretation is that player L (who gets a zero continuation payoff) is able on average to exactly cover the sum of her outlays, whereas player H on average is able to more than cover the sum of her outlays.

It is clear that the payoff function in Equation 24 is maximized when ϵ is as closest as possible to zero (of course it must be strictly positive, otherwise she would be deviating) so that player 1 will optimally choose the smallest ϵ above zero [19]. Therefore, to see when it is profitable to deviate for player 1 (Equation 22) we need to compare the payoff of deviating (Equation 23) with only the first line of the payoff of not deviating (Equation 24). We easily get that the condition in Equation 22 holds *iff*:

$$\epsilon > \frac{1}{3}v_1 - \frac{1}{6}(v_2 + v_3) \tag{25}$$

Since the ϵ that player 1 optimally chooses is as close as possible to zero, she will never find it profitable to deviate.

Consider now player 2. From Proposition 3 we know that she does not find it convenient neither to overlap nor to overbid player 1, since this way she would be shortlisted with NV_L with certainty. Hence, the only possibly profitable deviation would be to underbid player 1 by an amount δ. Therefore, player 2 will deviate *iff*:

$$E\Pi_2(x_2^a = (\epsilon - \delta), x_{-2}^{a*}) > E\Pi_2(x^{a*}) \tag{26}$$

If player 2 does not deviate and bids zero, she has $\frac{1}{2}$ probability to get shortlisted. In that case she meets player 1 and always get shortlisted with the NV_L, since by the maximization problem of player 1 (Equation 24) we have that player 1's optimal bid ϵ is such that $NV_1 > NV_2$, so that player 2's expected payoff from shortlisting is 0. With $\frac{1}{2}$ probability she is not shortlisted and she gets an actual payoff of zero, since she bid zero. Therefore:

$$E\Pi_2(x^{a*}) = 0 \tag{27}$$

On the other hand, if player 2 deviates and underbids player 1, she gets shortlisted with certainty. She happens to have NV_H (and hence get a positive payoff $u_2^* = NV_2 - NV_1$) *iff*

$$v_2 - (\epsilon - \delta) > v_1 - \epsilon \rightarrow v_1 - v_2 < \delta \quad \forall \epsilon > 0 \tag{28}$$

Otherwise she has NV_L (and gets $u_2^* = 0$).

Notice that since $\delta < \epsilon$ by definition, then we have that the condition from player 1's maximization

[19] The fact that the optimal ϵ is undetermined is due to the tie-breaking rule. If tie-breaking were in favour of player 1 (i.e., player 1 wins in all ties), player 1 would optimally bid exactly zero.

problem (Equation 24) i.e. $\epsilon < v_1 - v_2$, implies $\delta < v_1 - v_2$, so that the condititION above on δ is never met, and:

$$E\Pi_2(x_2^a = (\epsilon - \delta), x_{-2}^{a*}) = 0 \tag{29}$$

Therefore player 2 never finds it profitable to deviate.
Also, notice that since $v_2 > v_3$, then $\delta < v_1 - v_2$ implies $\delta < v_1 - v_3$, so that player 3 neither has any incentive to deviate.
Therefore, since no player has any incentive to deviate, we can conclude that $(x_1^a = \epsilon(\simeq 0), x_2^a = x_3^a = 0)$ is an equilibrium of Stage-a all-pay auction Q.E.D.

6.3.2. Proof that the triple $(x_1^a = \epsilon, x_2^a = 0, x_3^a = 0)$ is the unique equilibrium of Stage-a all-pay auction

The proof is by contradiction and articulated in lemmas.

Lemma 1. No triple of the form $(x_i^a > x_j^a > x_k^a)$, with $x_k^a \geq 0$, can be an equilibrium.
Proof. Suppose it is. Then no player has any profitable deviation. Consider players i and j. Either $v_i > v_j$ or $v_j > v_i$. If $v_i > v_j$, then by Proposition 3 i makes a profitable deviation by underbidding (at limit overlapping) to x_j^a. On the other hand, if $v_j > v_i$, then i makes a profitable deviation by bidding zero. Therefore $(x_i^a > x_j^a > x_k^a)$ is never an equilibrium Q.E.D.

Lemma 2. No triple of the form $(x_i^a = x_j^a > x_k^a)$, with $x_k^a \geq 0$, can be an equilibrium.
Proof. Suppose it is. Then no player has any profitable deviation. Again, consider players i and j. Either $v_i > v_j$ or $v_j > v_i$. If $v_i > v_j$, then j makes a profitable deviation by bidding zero. On the other hand, if $v_j > v_i$, then i makes a profitable deviation by bidding zero. Therefore $(x_i^a = x_j^a > x_k^a)$ is never an equilibrium Q.E.D.

Lemma 3. No triple of the form $(x_i^a = x_j^a = x_k^a)$, with $x_k^a \geq 0$, can be an equilibrium.
Proof. Suppose it is. Then no player has any profitable deviation. There are two possible cases:

3.1 $(x_i^a = x_j^a = x_k^a = 0)$
We have just proved that player 1 finds it always profitable to bid a positive amount rather than zero. Therefore 1 makes a profitable deviation by bidding a positive amount rather than zero.

3.2 $(x_i^a = x_j^a = x_k^a > 0)$
Player 3 makes a profitable deviation by bidding zero rather than a positive amount.

In both cases we reach a contradiction with the initial assumption, so that $(x_i^a = x_j^a = x_k^a)$ can never be an equilibrium Q.E.D.

Corollary. From lemmas 1-3 it follows that all plausible equilibria must be of the form $(x_i^a > x_j^a = x_k^a)$, with $x_k^a \geq 0$.

Lemma 4. No triple of the form $(x_3^a > x_1^a = x_2^a)$, with $x_1^a = x_2^a \geq 0$ can be an equilibrium (i.e., player 3 cannot be the highest bidder).

Proof. Suppose it is. Then no player has any profitable deviation. But player 3 does a profitable deviation by bidding zero rather than a positive amount. Therefore $(x_3^a > x_1^a = x_2^a)$ cannot be an equilibrium Q.E.D.

Lemma 5. No triple of the form $(x_2^a > x_1^a = x_3^a)$, with $x_1^a = x_3^a \geq 0$ can be an equilibrium (i.e., player 2 cannot be the highest bidder).
Proof. Suppose it is. Then no player has any profitable deviation.
Consider player 1. We need to check whether she has any incentive to slightly overbid player 3, such that $x_1^{a\prime} = x_3^a + \epsilon \leq x_2^a$, with $\epsilon > 0$.
If she does not deviate she will get shortlisted with probability 1/2. In this case she will meet player 2 and by Proposition 3 she will always have the NV_H, and get a positive continuation payoff $u_1^* = NV_1 - NV_2 = v_1 - x_1^a - (v_2 - x_2^a)$. With probability 1/2 she will not get shortlisted and will incur a loss equal to her bid, $u_1^* = -x_1^a$, with $x_1^a \geq 0$. Therefore, the expected payoff of non deviating is:

$$E\Pi_1(x^{a*}) = \frac{1}{2}(v_1 - v_2 + x_2^a) - x_1^a \tag{30}$$

Now suppose that she deviates and bids $x_1^{a\prime} = x_3^a + \epsilon = x_1^a + \epsilon$, with $x_1^a = x_3^a \geq 0$. In this case she will get shortlisted with certainty, and always with NV_H, so that the expected payoff from deviating is:

$$E\Pi_1(x_1^{a\prime} = (x_1^a + \epsilon), x_{-1}^{a*}) = v_1 - v_2 + x_2^a - x_1^a - \epsilon \tag{31}$$

from which we easily get that

$$\forall \epsilon : 0 < \epsilon < \frac{1}{2}(v_1 - v_2 + x_2^a) \tag{32}$$

it holds that

$$E\Pi_1(x_1^{a\prime} = (x_1^a + \epsilon), x_{-1}^{a*}) > E\Pi_1(x^{a*}) \tag{33}$$

i.e., player 1 will find it convenient to deviate. Notice that player 1 will optimally choose an ϵ

which is closest as possible to zero, so that Equation 33 will always hold and player 1 will always deviate. Therefore the triple $(x_2^a > x_1^a = x_3^a)$, with $x_1^a = x_3^a \geq 0$ cannot be an equilibrium Q.E.D.

Corollary. From the previous steps it follows readily that the only possible equilibrium of the Stage-a all-pay auction is $(x_1^a = \epsilon \cong 0, x_2^a = x_3^a = 0)$. Moreover since the optimal x_1^a is very close to zero, it will always be the case that $NV_1 > NV_2$ if player 2 is shortlisted, and $NV_1 > NV_3$ if player 3 is shortlisted, so that only the "asym-asym" case occurs in equilibrium. Q.E.D.

6.4. Proof of Proposition 5

We know from Equations 7 and 8 that, in the event a player gets shortlisted (which we will refer to as *SH* in the following for notational convenience), she will get a positive continuation payoff *iff* her net valuation will be higher than the net valuation of the other shortlisted player, and a zero continuation payoff if her net valuation will be lower or equal than her opponent's; if instead she does not get shortlisted (*NO-SH* in the following), she will incur a loss equal to her Stage-a bid. Therefore, each player i's expected payoff from playing mixed strategies in Stage-a is as follows:

$$\begin{aligned} E\Pi_i(h^a) &= (NV_i - NV_j)P(NV_i > NV_j | i,j\ SH)P(i,j\ SH) + \\ &+ (NV_i - NV_k)P(NV_i > NV_k | i,k\ SH)P(i,k\ SH) - x_i^a P(i\ NO\text{-}SH) \end{aligned} \tag{34}$$

with $i, j, k \in \{1, 2, 3\}$.

Remember that a couple of players i, j get shortlisted with certainty if their Stage-a bids are the two highest i.e., we have either $(x_i^a > x_j^a > x_k^a)$ or $(x_j^a > x_i^a > x_k^a)$ or $(x_i^a = x_j^a > x_k^a)$. On the other hand, when either $(x_i^a > x_j^a = x_k^a)$ or $(x_j^a > x_i^a = x_k^a)$ occur, there is $\frac{1}{2}$ probability that both i and j are shortlisted, whereas in case $(x_i^a = x_j^a = x_k^a)$, there is $\frac{1}{3}$ probability that both i and j are shortlisted. Since events are mutually esclusive, the probability that the couple of players i, j are shortlisted is as follows:

$$\begin{aligned} P(i,j\ SH) &= P(x_i^a > x_j^a > x_k^a) + P(x_j^a > x_i^a > x_k^a) + P(x_i^a = x_j^a > x_k^a) \\ &+ \frac{1}{2}P(x_i^a > x_j^a = x_k^a) + \frac{1}{2}P(x_j^a > x_i^a = x_k^a) + \frac{1}{3}P(x_i^a = x_j^a = x_k^a) \end{aligned} \tag{35}$$

On the other hand, in the events $(x_i^a < x_j^a < x_k^a)$, $(x_i^a < x_k^a < x_j^a)$ and $(x_i^a < x_j^a = x_k^a)$ player i never gets shortlisted, whereas in the events $(x_i^a = x_k^a < x_j^a)$, $(x_i^a = x_j^a < x_k^a)$ and $(x_i^a = x_j^a = x_k^a)$ there is a positive probability that player i does not get shortlisted (respectively, $\frac{1}{2}$ in the first two events and $\frac{1}{3}$ in the third). Therefore the probability of no shortlisting for player i (with

$i \in \{1,2,3\}$) is:

$$P(i\ NO\text{-}SH.) = P(x_i^a < x_j^a < x_k^a) + P(x_i^a < x_k^a < x_j^a) + P(x_i^a < x_j^a = x_k^a) \\ + \frac{1}{2}P(x_j^a > x_i^a = x_k^a) + \frac{1}{2}P(x_k^a > x_i^a = x_j^a) + \frac{1}{3}P(x_i^a = x_j^a = x_k^a) \tag{36}$$

Similarly to what noticed for the pure-strategy analysis, we have here that relatively stronger players have an advantage over weaker ones, since they can restrict optimally their support so to make sure to get a positive continuation payoff, *conditional on shortlisting*.
In fact, one can see that by choosing the interval $[0, v_1 - v_2)$ as her support, player 1 makes it sure that for any bid she may plausibly expect from player $j \in \{2,3\}$, i.e., for all $x_j^a \in [0, v_j)$[20], it will always be true that $x_1^a < (v_1 - v_j) + x_j^a$, so that $P(NV_1 > NV_j|1, j\ SH) = 1$ and $P(NV_j > NV_1|1, j\ SH) = 0$. Therefore, player 1 is able to optimally choose the support to make it sure that, in case she gets shortlisted, she will have the higher net valuation, and hence get a positive continuation payoff, regardless of whom the other shortlisted player is.
On her hand, player 2 can do a similar reasoning and pick $[0, v_2 - v_3)$ as her support, which ensures that in case she gets shortlisted with player 3, she always has the higher net valuation and gets a positive continuation payoff: for any plausible bid from player 3 - i.e., for all $x_3^a \in [0, v_3)$ - it will always hold that $x_2^a < (v_2 - v_3) + x_3^a$, so that $P(NV_2 > NV_3|2{,}3\ SH) = 1$ and $P(NV_3 > NV_2|2{,}3\ SH) = 0$, whereas she knows that if she gets shortlisted with Player 1 she will get a zero continuation payoff.
Consequently, player 3 knows that in case she gets shortlisted, she cannot do nothing to prevent her opponent to have the higher net valuation. So she expects a zero continuation payoff from shortlisting regardless of whom the other shortlisted player is.
Note that the presence of an upper bound on players' rational bidding has an impact on the probability of shortlisting, but players do not have an interest in getting shortlisted if they expect not to take a positive continuation payoff: their goal is to maximize their expected payoff, rather than getting shortlisted per se.
Given the considerations above, we have that players' expected payoffs from randomizing in Stage-a are as follows:

$$E\Pi_1(h^a) = (NV_1 - NV_2)P(1{,}2\ SH) + (NV_1 - NV_3)P(1{,}3\ SH) - x_1^a P(1\ NO\text{-}SH) \tag{37}$$

$$E\Pi_2(h^a) = (NV_2 - NV_3)P(2{,}3\ SH) - x_2^a P(2\ NO\text{-}SH) \tag{38}$$

[20]Notice that no player would bid her own entire valuation in Stage-a, since in case she gets shorlisted she will have no resources left to bid in Stage-b, so that the other shortlisted player would be able to win with an infinitesimal amount, making her losing the entire budget.

$$E\Pi_3(h^a) = -x_3^a P(3\ NO\text{-}SH) \tag{39}$$

Since $E\Pi_3(h^a) \leq 0$, player 3 never finds it convenient to randomize and, due to Assumption 2 in the model (Section 3), we conclude that player 3 will bid zero with probability 1, i.e., $P(x_3^a = 0) = 1$. Considering that (i) players randomize independently, (ii) given $x_3^a = 0$ then $P(x_i < x_3) = 0$ (with $i \in \{1,2\}$) and (iii) $P(x_i = x_j) = 1 - P(x_i > x_j) - P(x_i < x_j) \quad \forall i, j \in \{1,2,3\}$, we can explicit the expected payoff of Player 1 (Equation 37) as follows:

$$\begin{aligned}
&E\Pi_1(h^a) = (NV_1 - NV_2)[P(x_1^a > x_2^a)P(x_2^a > x_3^a) + P(x_2^a > x_1^a)P(x_1^a > x_3^a) + \\
&+ (1 - P(x_1^a > x_2^a) - P(x_1^a < x_2^a))P(x_2^a > x_3^a) + \frac{1}{2}P(x_1^a > x_2^a)(1 - P(x_2^a > x_3^a)) + \\
&+ \frac{1}{2}P(x_2^a > x_1^a)(1 - P(x_1^a > x_3^a)) + \frac{1}{3}(1 - P(x_1^a > x_2^a) - P(x_1^a < x_2^a))(1 - P(x_2^a > x_3^a))] + \\
&+ (NV_1 - NV_3)[\frac{1}{2}P(x_1^a > x_2^a)(1 - P(x_2^a > x_3^a)) + \\
&+ \frac{1}{3}(1 - P(x_1^a > x_2^a) - P(x_1^a < x_2^a))(1 - P(x_2^a > x_3^a))] - x_1^a[P(x_1^a < x_2^a)(1 - P(x_2^a > x_3^a)) + \\
&+ \frac{1}{2}P(x_2^a > x_1^a)(1 - P(x_1^a > x_3^a)) + \frac{1}{3}(1 - P(x_1^a > x_2^a) - P(x_1^a < x_2^a))(1 - P(x_2^a > x_3^a))]
\end{aligned} \tag{40}$$

Notice that in Equation 40, we assumed that Player 1's support is $[0, v_1 - v_2)$, so that she might play zero with a positive probability. In the following we show that Player 1's expected payoff from keeping zero in the mix, i.e., randomizing over $[0, v_1 - v_2)$ is lower than the expected payoff she could get by dropping the bid on 0 from the mix, i.e., randomizing over $(0, v_1 - v_2)$.
By imposing $P(x_1 > 0) = 1$ in Equation 40, we can calculate

$$\begin{aligned}
&E\Pi_1(\text{mix on } (0, v_1 - v_2)) = (NV_1 - NV_2)[P(x_1 > x_2)P(x_2 > x_3) + \\
&+ (1 - P(x_1 > x_2) - P(x_1 < x_2))P(x_2 > x_3) + P(x_2 > x_1) + \\
&+ \frac{1}{2}P(x_1 > x_2)(1 - P(x_2 > x_3)) + \frac{1}{3}(1 - P(x_1 > x_2) - P(x_1 < x_2))(1 - P(x_2 > x_3))] + \\
&+ (NV_1 - NV_3)[\frac{1}{2}P(x_1 > x_2)(1 - P(x_2 > x_3)) + \\
&+ \frac{1}{3}(1 - P(x_1 > x_2) - P(x_1 < x_2))(1 - P(x_2 > x_3))] + \\
&- x_1[P(x_1 < x_2)(1 - P(x_2 > x_3)) + \frac{1}{3}(1 - P(x_1 > x_2) - P(x_1 < x_2))(1 - P(x_2 > x_3))]
\end{aligned} \tag{41}$$

By imposing $E\Pi_1(\text{mix on } (0, v_1 - v_2)) \geq E\Pi_1(\text{mix on } [0, v_1 - v_2)) = E\Pi_1(h^a)$ we get the condition $(v_1 - v_2)(1 - P(x_1 > 0)) + x_2(1 - P(x_1 > 0)) \geq 0$ which is always true. Therefore player 1 will optimally never put mass on zero. From Equation 38 we know that the only chance for

player 2 to get a positive expected payoff from randomizing is to get shortlisted with 3. But the probability that both player 2 and player 3 will be shortlisted is 0, since we know that player 1 will get shortlisted with certainty. Therefore, player 2 neither has any advantage from randomizing, since if she gets shortlisted she will meet player 1 for sure, and hence get a zero continuation payoff. By Assumption 2 in the model we conclude that also player 2 prefers to bid zero with probability 1 rather than randomizing. Given that players 2 and 3 play zero with probability 1, player 1 will optimally bid a infinitesimal positive amount ϵ, and we are back to the pure-strategy case.
Therefore there is no mixed strategy equilibrium for the Stage-a all-pay auction Q.E.D.

6.5. Proof of Proposition 7

Notice that since Stage-a yields basically no revenue, the relevant comparison is between the Stage-b all-pay auction and the standard all-pay auction with two asymmetric players. Recalling from Baye et al. (1996) the equilibrium of the standard asymmetric all-pay auction (with $v_1 > v_2$),

$$\begin{cases} F_1(x_1) = \frac{x_1}{v_2} \quad \forall x_1 \in [0, v_2] \\ F_2(x_2) = \frac{v_1 - v_2 + x_2}{v_1} \quad \forall x_2 \in [0, v_2] \end{cases} \tag{42}$$

we have that player 1's spending is distributed uniformly on the interval $[0, v_2]$, and so her expected outlay is $E[x_1] = \frac{v_2}{2}$.
Conditional upon bidding positive, player 2's outlay also is distributed uniformly on $[0, v_2]$, so that her expected outlay is $E[x_2] = \frac{v_2}{2}\left(\frac{v_2}{v_1}\right)$[21]. Therefore, the total expected revenue from a standard two-player all-pay auction is:

$$E[x_1 + x_2] = \frac{v_2}{2} + \frac{v_2}{2}\left(\frac{v_2}{v_1}\right) = \frac{v_2}{2}\left(1 + \frac{v_2}{v_1}\right) \tag{43}$$

Turning to the Stage-b all-pay auction, we have from Proposition 5 that the upper bound of the equilibrium support is *ex-ante indetermined*, since player L will be player 2 with probability $\frac{1}{2}$ and player 3 with probability $\frac{1}{2}$. Therefore, player 1's outlay will be distributed uniformly on $[0, v_2]$ in half of the cases and on $[0, v_3]$ in the other half, so that her expected outlay will be:

$$E[x_1^b] = \left(\frac{1}{2}\right)\frac{v_2}{2} + \left(\frac{1}{2}\right)\frac{v_3}{2} \tag{44}$$

As for player 2, in case she is shortlisted, she bids according to an uniform distribution on the

[21] Decomposing player 2's equilibrium distribution in its continuos and discrete parts, we can write:
$F_2(x_2) = 1 - \frac{v_2}{v_1} + \left(\frac{v_2}{v_1}\right)\frac{x_2}{v_2}$
where $\frac{v_2}{v_1}$ is the probability of bidding positive and $[1 - \frac{v_2}{v_1}]$ is the probability of bidding zero.

interval $[0, v_2]$ conditional upon spending positive [22]. Therefore her expected outlay will be:

$$E[x_2^b] = \left(\frac{1}{2}\frac{v_2}{v_1}\right)\frac{v_2}{2} \tag{45}$$

Analogously, player 3's expected spending will be:

$$E[x_3^b] = \left(\frac{1}{2}\frac{v_3}{v_1}\right)\frac{v_3}{2} \tag{46}$$

Therefore, neglecting player 1's first-stage bid which is very close to zero, we have that the total expected revenue from the two-stage all-pay auction is as follows:

$$E[x_1^b + x_2^b + x_3^b] = \frac{v_2}{4}\left(1 + \frac{v_2}{v_1}\right) + \frac{v_3}{4}\left(1 + \frac{v_3}{v_1}\right) \tag{47}$$

which can be easily seen to be smaller than $E[x_1 + x_2]$ in the standard all-pay auction (Equation 43) Q.E.D.

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[22] Notice that we need conditioning on both the event of shortlisting and the event of spending positive.

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