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Chen, Yongmin and Zhang, Tianle

University of Colorado at Boulder, Lingnan University

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Yongmin Chen[†] Tianle Zhang[‡]

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Abstract. This paper studies a model of interpersonal bundling, in which a monopolist offers a good for sale under a regular price and a group purchase discount if the number of consumers in a group—the bundle size—belongs to some menu of intervals. We find that this is often a profitable selling strategy in response to demand uncertainty, and it can achieve the highest profit among all possible selling mechanisms. We explain how the profitability of interpersonal bundling with a minimum or maximum group size may depend on the nature of uncertainty and on parameters of the market environment, and discuss strategic issues related to the optimal design and implementation of these bundling schemes. Our analysis sheds light on popular marketing practices such as group purchase discounts, and offers insights on potential new marketing innovation.

Keywords: Interpersonal bundling, bundling, group purchase, group discount, demand uncertainty

[†]University of Colorado, Boulder, USA; yongmin.chen@colorado.edu

[‡]Lingnan University, Hong Kong; tianlezhang@ln.edu.hk

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1. INTRODUCTION

This paper studies a form of product bundling where a good is offered for sale under both a regular price and a group purchase discount if the group size—the bundle size—meets certain requirement. The defining characteristic of this selling format is that the purchase of the bundle is made by different consumers—and hence we term it *interpersonal bundling*—rather than by a single consumer as under traditional mixed bundling.[1]

Interpersonal bundling (denoted as IB) is a widely observed selling practice. In many markets and for many goods, multiple consumers may form a purchase group to qualify for a group discount, as, for example, when buying tickets for a concert, purchasing a tour, or dining at a restaurant.[2] In recent years, many Internet sites have emerged that allow sellers to offer IB, where consumers purchasing with group coupons receive substantial discounts when the minimum group size is reached. Launched in November 2008, Groupon was a pioneer in this selling format on the Internet, and it exceeded a billion dollars in revenue in just its third year of operation (Levin, 2012).[3] Despite its popularity, the pricing and profitability of interpersonal bundling have not been studied in a general framework that allows a menu of bundle sizes. How should a seller optimally choose prices and bundle sizes? When will IB be more profitable than separate selling?[4] What determine the magnitude of its profit advantage? We provide some answers to these questions in this study.

[1] Mixed bundling refers to offering goods for sale both as a package and as individual components.

[2] Miller Farms, a local family farm in Colorado, runs the Fall Harvest Festival each year. In 2012, a customer is charged \$15 to participate in the Festival and pick up vegetables to take home. For a group of 10 or more, the price per person is lowered to \$13.

[3] Many other group buying websites offer variants of interpersonal bundling, including Livingsocial, where a consumer receives a free deal if she gets three people buy the product. There are numerous interpersonal bundling sites around the global, such as uBuyiBuy, Gaopeng, and Lashou in Asia, MyCityDeal in Europe, Downtown Colombia in South America, and Spreets in Australia.

[4] Here, separate selling means offering a good for sale under a single unit price to all consumers, whereas a pure bundle would consist of multiple units of the same good under a unit price for group purchase. The recent economics literature has investigated product bundling that is different from traditional mixed bundling. See, for example, the study of bundle size pricing by Chu, Leslie, and Sorensen (2011), and of inter-firm bundling by Gans and King (2006) and Armstrong (2012).

The literature on product bundling has found that mixed bundling often is more profitable than separate selling through two main mechanisms: segmenting the consumer population to facilitate price discrimination and reducing the dispersion of consumer values to extract consumer surplus (e.g., Adams and Yellen 1976; Schmalensee, 1984; Long, 1984; McAfee, McMillan, and Whinston 1989; and Chen and Riordan, 2013).[5] This paper will explore an alternative motive for bundling: as a profitable strategy in response to demand uncertainty. While this motive can also arise when each bundle is purchased by an individual consumer,[6] it is especially relevant and important for interpersonal bundling.

We consider a stylized model where a monopolist sells to a population of low- and high-value consumers, with the numbers of these consumers being uncertain and following some joint probability distribution. Under separate selling, the seller would *ideally* pursue a high-price strategy if high-value consumers is numerous, or a low-price strategy that will also attract low-value consumers if their number is sufficiently large. However, because price is set before the uncertainty is resolved, a single price is generally not optimal. By offering the good for sale under IB, it is possible that a high or low price will become effective only when that price is optimal under the demand realization. Thus, interpersonal bundling potentially enables the seller to use optimal option pricing under uncertain demand, leading to higher profit than separate selling.

Our analysis of this model, in a general setting where the seller can commit to a menu that specifies multiple bundle size intervals to which the group discount applies, leads to two results. First, we show that a bundle menu with at most two (disjoint) intervals is more profitable than separate selling, provided that demand uncertainty is relevant for the choice of optimal prices under separate selling. Second, under a plausible sufficient condition, IB

[5] In a standard model of two goods, some consumers may value one good highly but another very little, while others may value two goods together relatively highly, and values for the bundle may be less dispersed than values for individual goods. By charging the former (who purchase only a single unit) a higher price and the latter a bundle discount, mixed bundling generally leads to higher profit than separate selling.

[6] Under standard mixed bundling with two goods, there can be uncertainties on each individual consumer's valuation for the two goods, and mixed bundling can thus be viewed as a form of option pricing, where a consumer will obtain the bundle discount only if she has sufficiently high demand for both goods.

achieves the highest profit among all possible selling mechanisms. Both results are obtained without assuming functional forms on the joint distribution of consumer numbers, and they provide a general and elegant characterization of the properties of interpersonal bundling.

To gain insights on when the general conditions on the profitability of IB are satisfied and how to implement IB in various market environments, we further study two especially simple forms of IB: interpersonal bundling with a minimum or maximum group size, denoted respectively as IBmin or IBmax. For each of them, we derive a sufficient condition for its superiority over separate selling. Interestingly, these two conditions, both invariant to the functional form of the consumer distributions, reveal contrasting patterns of demand uncertainty. Specifically, relative to separate selling, IBmin tends to be more profitable when the number of *low-value* consumers is more dispersed, whereas IBmax tends to be more profitable when the number of *high-value* consumers is more dispersed. On the other hand, their profitability is also affected by some other aspects of the market environment in similar ways. We illuminate the intuition behind these findings, relate them to observed marketing practices, and suggest that IBmax, as a potentially profitable marketing innovation, can be implemented similarly as IBmin on the Internet and through intermediaries such as Groupon and Amazon.

We further explore how a seller may incorporate additional strategic considerations in the design of IBmin, by explicitly modeling the decision process of individual consumers in two variants of the main model. In the first variant, we allow the possibility that some consumers are initially uninformed about the existence of the seller's product. Then, in order to qualify for the group discount, informed consumers may take (costly) actions to transmit product information to the uninformed, and the seller can exploit this incentive in setting the bundle size. IBmin can thus increase the seller's profit by facilitating the dissemination of product information.[7]

[7] This informational role of group buying has also been identified and explored in Jing and Xie (2011), but their model focuses on exogenously fixed group size and known demand. By contrast, bundle size is a key decision variable in our analysis of IBmin, and demand uncertainty is a central feature of our model that interacts with the consideration for information transmission.

In the second variant, we consider the possibility that high-value consumers need to incur transaction costs to sign up for group purchase. The seller may then partially segment the consumer population, charging the regular price to high-value consumers with high sign-up costs while attracting low-value consumers with the bundle discount. To allow for a richer modeling of consumers' decision process, we consider two forms of IBmin in a two-period setting: a simultaneous format where the seller does not inform period-2 consumers how many buyers signed up in period 1, and a sequential format where the seller does. Hu, Shi and Wu (2013) find in a parallel setting that the seller prefers the sequential mechanism, because it encourages consumer participation by removing their uncertainty in period 2, which leads to higher group formation rates. Interestingly, in our case the seller, who aims to maximize profit, may instead prefer the simultaneous format. This is because the simultaneous format does *not* remove uncertainty to the high-value consumers, which facilitates price discrimination by discouraging them from obtaining the bundle discount.

In addition to offering a new perspective on product bundling, this paper is also closely related to the literature on pricing under demand uncertainty. In Dana (1999, 2001), for example, demand can be *either* high or low. He finds that a monopolist optimally offers two prices, with only a limited quantity offered under a low price, which is set for the low demand state. A high price then allows the firm to extract additional consumer surplus when demand turns out to be high, in which case the limited quantity available at the low price will sell out so that some high-priced units will be purchased. Anand and Aron (2003), in an early study of web-based group buying, also consider a model with either a high or a low demand regime, represented by two linear demand functions. They demonstrate that group buying may enable the seller to set price-quantity schedules that optimize revenue under each demand regime, and that the profitability of group buying relative to posted pricing depends on whether the two linear demand functions are parallel or intersecting.[8] Our paper departs from this literature by adopting a different analytical approach, capturing the group

[8] Also related are Gale and Holmes (1992, 1993), who study how a monopolist may use advance purchase discounts to allocate capacity more efficiently in the presence of demand uncertainty. See also Dana (1998) for a related analysis in competitive markets.

buying problem in a general bundling framework. One clear advantage of this approach is that it enables us to analyze group-discount schemes with minimum or maximum group sizes in a unified model and to uncover the interesting relations between them. Additionally, we are concerned with the uncertainty of a different nature: there are *both* high- and low-value consumers, and the uncertainty is over their respective numbers. We believe that this captures plausible market environments faced by many firms, complementary to the settings studied in other papers in this literature. Furthermore, our analysis leads to interesting new results on the profitability and optimal design of interpersonal bundling. Our paper thus contributes to the literatures on product bundling, on pricing under demand uncertainty, and, more generally, on the economics and management of marketing.

In the rest of the paper, we establish the two general properties of interpersonal bundling in section 2, and analyze in more detail its two simple forms, IBmin and IBmax, in section 3. Section 4 explores the optimal design of IBmin incorporating additional strategic considerations of information dissemination and price discrimination. Section 5 concludes.

2. DEMAND UNCERTAINTY AND INTERPERSONAL BUNDLING

2.1 The Model

A monopolist offers a product for sale. There are two types of consumers, high-value and low-value, whose product valuations are respectively H and L, with $H > L > 0$. A consumer's type is her private information, and each consumer desires to purchase at most one unit. The numbers of low- and high-value consumers (denoted as L-consumers and H-consumers) are respectively x and y, which are realizations of random variables X and Y that have joint distribution function $G(x, y)$ on support $[a_x, b_x] \times [a_y, b_y]$, where $0 \leq a_x < b_x$ and $0 \leq a_y \leq b_y$. The marginal distribution functions of X and Y are $F_x(x)$ and $F_y(y)$, respectively. Production cost is normalized to zero, and the firm maximizes expected profit.

Let $\bar{x}$ and $\bar{y}$ be the expected number of L- and H-consumers, respectively. Then

$$\bar{x} = \int_{a_x}^{b_x} x dF_x(x); \qquad \bar{y} = \int_{a_y}^{b_y} y dF_y(y). \tag{1}$$

We allow the possibility that either $y = \bar{y}$ is a constant or $x = \bar{x}$ is a constant, in which case $G(x, y)$ degenerates to $F_x(x)$ or $F_y(y)$.[9]

As a benchmark, consider the case of separate selling where the firm posts a single unit price to all consumers. Then, profit is higher under $p = H$ if $H\bar{y} > L(\bar{x} + \bar{y})$ and under $p = L$ if $H\bar{y} < L(\bar{x} + \bar{y})$. It follows that the optimal price and the corresponding profit are, respectively:[10]

$$p^s = \begin{cases} H & if \quad \bar{x} \leq \left(\frac{H}{L} - 1\right)\bar{y} \\ L & if \quad \bar{x} > \left(\frac{H}{L} - 1\right)\bar{y} \end{cases}, \qquad \pi^s = \begin{cases} H\bar{y} & if \quad \bar{x} \leq \left(\frac{H}{L} - 1\right)\bar{y} \\ L(\bar{x} + \bar{y}) & if \quad \bar{x} > \left(\frac{H}{L} - 1\right)\bar{y} \end{cases}. \tag{2}$$

Therefore, if the expected number of L-consumers ($\bar{x}$) is small, the firm will only sell to the H-consumers at $p^s = H$; otherwise, it will sell to all consumers at $p^s = L$.

Under interpersonal bundling (IB), the firm sets a stand-alone unit price p, a discounted unit price under group purchase $q \leq p$, and a condition that the group discount becomes effective if and only if the number of consumers belongs to the set $B \equiv \{[m_i, M_i] \ : i = 1, ..., n\}$ for some integer n, with $0 \leq m_i \leq M_i < \infty$.[11] An example of B with two intervals $(n = 2)$ is $B = \{[0, 1], \ [2, 3]\}$. IB may also contain a bundle with a single interval, $B = [m, M]$, which further nests two special cases: (1) IB with a minimum group size, or IBmin: (p, q, m), where each consumer can separately purchase the good at price p, but consumers who sign up for group purchase can buy at the discounted price q if and only if there are at least m consumers in the group. (2) IB with a maximum group size, or IBmax: (p, q, M), where consumers who sign up for group purchase can buy at the discounted price q if and only if the group size does not exceed M.

Except for Subsection 4.2, we assume that there is no transaction cost to join group

[9] At least one of X and Y is not a constant. We also allow X or Y to be discrete random variables, in which case a_x and b_x or a_y and b_y correspond respectively to the smallest and largest values that can be realized.

[10] For ease of exposition, when profit is the same under $p = H$ and $p = L$, we assume $p^s = H$.

[11] When $G(x, y)$ is not continuous, some inequalities in the bundle size conditions $m_i \leq B_i \leq M_i$ may be strict. More generally, IB can take the form of a general bundling menu $\{(p_i, B_i)\}$, where consumers in bundle size B_i are charged with price p_i. In our simple context, we use our equivalent formulation for notational convenience.

purchase, which implies that if $q < p$, all consumers will attempt to purchase at q.

2.2 Profitability of Interpersonal Bundling

We first demonstrate that a simple form of IB, with $p = H$, $q = L$, and some B containing at most two intervals, is generally more profitable than separate selling under demand uncertainty.

Given (p, q, B), all consumers will purchase at price q if $x + y \in B$ and $q \leq L$, whereas when $x + y \in \bar{B}$ only H-consumers will purchase at price p if $L < p \leq H$, where $\bar{B}$ is the complement of set B. The firm's problem is to maximize (expected) profit:

$$\max_{q \leq L < p \leq H, B} \pi(p, q, B) = q \int\int_B (x + y)\, dG(x, y) + p \int\int_{\bar{B}} y dG(x, y). \tag{3}$$

Since $\pi(p, q, B)$ weakly increases in p and q for any B, the optimal p and q that maximize $\pi(p, q, B)$ are $p^* = H$ and $q^* = L$. Hence the firm's maximum profit under IB and the optimal B are:

$$\pi^* \equiv \max_B \pi(H, L, B); \quad B^* = \arg\max_B \pi(H, L, B). \tag{4}$$

Since $\pi(H, L, B) = L(\bar{x} + \bar{y})$ if $B = [a_x + a_y, b_x + b_y]$ and $\pi(H, L, B) = H\bar{y}$ if $B = (b_x + b_y, \infty)$, we have $\pi^* \geq \pi^s$. Thus, same as mixed bundling, IB will always be at least as profitable as separate selling.

The seller's problem can be written as maximizing

$$\begin{aligned}
\pi(H, L, B) &= L \int\int_{x+y \in B} (x + y)\, dG(x, y) + H \int\int_{x+y \in \bar{B}} y dG(x, y) \\
&= \int\int_{x+y \in B} [L(x + y) - Hy]\, dG(x, y) + H\bar{y} \\
&= L(\bar{x} + \bar{y}) + \int\int_{x+y \in \bar{B}} [Hy - L(x + y)]\, dG(x, y).
\end{aligned}$$

Our result below will assume a regularity condition on uncertainty: there exist (small) intervals δ_1 and δ_2 on $[a_x + a_y, b_x + b_y]$, where δ_i can be a single number if $G(x, y)$ is not continuous, such that

$$\Pr\left(x + y > \frac{H}{L} y \,\middle|\, x + y \in \delta_1\right) = 1; \quad \Pr\left(x + y < \frac{H}{L} y \,\middle|\, x + y \in \delta_2\right) = 1. \tag{5}$$

That is, conditional on $x+y$ belonging to δ_1, $x+y > \frac{H}{L}y$; and conditional on $x+y$ belonging to δ_2, $x+y < \frac{H}{L}y$. This assumption rules out trivial cases where $x+y$ is always higher or lower than $\frac{H}{L}y$ for (almost) all possible realizations of (x, y), in which case under separate selling the optimal price will be independent of the realization of (x, y). As it will become clear later, condition (5) holds quite generally; in fact, if it is not satisfied, then separate selling will always be an optimal selling scheme (see the argument immediately following (14) in the next subsection), and the resolution of demand uncertainty will not affect the choice of p^s.

Proposition 1 *Under (5), interpersonal bundling (H, L, B'), with B' containing at most two intervals, is more profitable than separate selling.*

Proof. Since under separate selling $\pi^s = \max\{H\bar{y}, L(\bar{x}+\bar{y})\}$, we show that there is some B' such that $\pi(H, L, B') > \pi^s$ whether $\pi^s = H\bar{y}$ or $\pi^s = L(\bar{x}+\bar{y})$, and B' contains at most two intervals.

Suppose that $\pi^s = H\bar{y}$. Then, let $B' = \delta_1$, we have,

$$\pi^* \geq \pi\left(H, L, B'\right) = \int\int_{x+y\in\delta_1} \left[L(x+y) - Hy\right] dG(x,y) + H\bar{y} > H\bar{y}.$$

(The first inequality above is due to revealed preference, and the second to the definition of δ_1, which is a single interval and could be a single number if it is a mass point.)

Next, suppose instead that $\pi^s = L(\bar{x}+\bar{y})$. Then, let $B' = \bar{\delta}_2$, which contains two intervals, with $\bar{B}' = \delta_2$. We then have

$$\begin{aligned}\pi^* &\geq \pi\left(H, L, B'\right) = L(\bar{x}+\bar{y}) + \int\int_{x+y\in\bar{B}'} \left[Hy - L(x+y)\right] dG(x,y) \\ &= L(\bar{x}+\bar{y}) + \int\int_{x+y\in\delta_2} \left[Hy - L(x+y)\right] dG(x,y) > L(\bar{x}+\bar{y}).\end{aligned}$$

■

The reason for the profitability of IB is simple. With uncertain demand, the optimal uniform price depends on the realization of the numbers of H- and L-consumers. Under separate selling, the firm chooses the price that is only optimal *on average*. When $p^s = H$, the L-consumers are not served, and adding a discounted price L that becomes effective only if the realized consumer group size corresponds to a region where $p^s = L$, which is ensured by the construction of bundle B', leads to a higher expected profit than separate selling under $p^s = H$. Similarly, when $p^s = L$, profit can be increased by also offering a higher regular price H that becomes effective only if the realized consumer group size corresponds to a region where profit is higher under $p^s = H$. Thus, IB implements profitable option pricing under demand uncertainty, boosting profit.

Notice that for IB to dominate separate selling with $p^s = H$, the bundle size is only required to satisfy $x+y \in \delta_1$, i.e., $m \leq x+y \leq M$ for some $m \leq M$; and for IB to dominate $p^s = L$, the bundle size is only required to satisfy $x+y \in \delta_2$, i.e., $x+y \leq m$ or $x+y \geq M$ for some $m \leq M$. Therefore a profitable bundle B' contains at most two intervals. However, B' may not be the optimal bundle. IB with a more general B can potentially achieve higher profit. In fact, as we show next, IB with a general bundle menu B is an optimal selling scheme if a plausible sufficient condition is satisfied.

2.3 Interpersonal Bundling as an Optimal Selling Mechanism

This subsection demonstrates that IB (H, L, B^*), with $B^* = \{[m_i^*, M_i^*] \ : i = 1, ..., n\}$, is an optimal selling mechanism under the following sufficient condition (explained shortly):

$$\left\{(x,y) : x+y > \frac{H}{L}y\right\} = \Omega; \qquad \left\{(x,y) : x+y \leq \frac{H}{L}y\right\} = \bar{\Omega}, \tag{6}$$

where

$$\Omega \equiv \cup_{i=1}^{n} \{(x,y) : \ m_i^* \leq x+y \leq M_i^*\}.$$

We prove this by first characterizing the seller's highest possible profit under its information constraint, using a general mechanism design approach. We then show that optimal IB achieves this (constrained) first best under (6).

Since all consumers are *ex ante* the same, we can consider mechanisms for a representative consumer. From the revelation principle, we can limit our search for an optimal selling scheme to direct mechanisms where the consumer is asked to report her type $\theta \in \{H, \ L\}$, who will receive a unit of the good with probability $\lambda(\cdot)$ by paying $p(\cdot)$,[12] and truth reporting is optimal for the consumer. Given that there is a continuum of consumers, $\lambda(\cdot)$ and $p(\cdot)$ will depend on θ and on some aggregate measure(s) of consumers. For the first best, we assume that a mechanism may depend on the realized total demand from each of the two types of consumers, x and y.[13] Then, a mechanism specifies $\{\lambda(\theta; x, y),\ p(\theta; x, y)\}$.

The seller chooses $\{\lambda(\theta; x, y),\ p(\theta; x, y)\}$ to maximize

$$\pi = \int\int [xp(L; x, y)\lambda(L; x, y) + yp(H; x, y)\lambda(H; x, y)]\, dG(x, y), \tag{7}$$

subject to individual rationality constraints

$$\begin{aligned} (L - p(L; x, y))\lambda(L; x, y) &\geq 0, &(8)\\ (H - p(H; x, y))\lambda(H; x, y) &\geq 0; &(9) \end{aligned}$$

and incentive compatibility constraints

$$\begin{aligned} (L - p(L; x, y))\lambda(L; x, y) &\geq (L - p(H; x, y))\lambda(H; x, y), &(10)\\ (H - p(H; x, y))\lambda(H; x, y) &\geq (H - p(L; x, y))\lambda(L; x, y). &(11) \end{aligned}$$

From standard arguments, $p(L; x, y) = L$ so that the L-type receives no information rents, and (10) holds with $p(H; x, y) \geq L$. From (11), which holds in equality at the optimum, and with $p(L; x, y) = L$, we have

$$p(H; x, y)\lambda(H; x, y) = H\lambda(H; x, y) - (H - L)\lambda(L; x, y). \tag{12}$$

Thus (9) and (12) are the two remaining constraints. Substituting (12) into (7), with $p(L; x, y) = L$, we obtain

$$\pi = \int\int \{[xL - y(H - L)]\lambda(L; x, y) + yH\lambda(H; x, y)\}\, dG(x, y),$$

[12] We can also allow a transfer payment when the consumer does not receive the good, but it would be optimal for the seller to set this payment to zero.

[13] In reality, the seller may only be able to commit to prices based on aggregate demand $x + y$, but not on individual values of x and y, because x and y may not be separately verifiable while $x + y$ potentially is. Thus the first-best profit is only a benchmark. Remarkably, as we show next, IB can achieve this first best if (6) holds.

which increases in $\lambda(H;x,y)$. Since constraint (9) is not less likely satisfied with an increase in λ, it follows that $\lambda(H;x,y) = 1$ at the optimum. Then, (9) and (12) become $H \geq p(H;x,y) = H - (H-L)\lambda(L;x,y)$, and the seller chooses $\lambda(L;x,y)$ to maximize

$$\pi = \int\int \{(x+y)L\lambda(L;x,y) + yH[1-\lambda(L;x,y)]\}\,dG(x,y), \tag{13}$$

which can be written as

$$\pi = \int\int [(x+y)L - yH]\,\lambda(L;x,y)\,dG(x,y) + H\bar{y}.$$

Therefore, letting $\lambda(L;x,y) = 1$ whenever $x + y > \frac{H}{L}y$ and $\lambda(L;x,y) = 0$ whenever $x + y \leq \frac{H}{L}y$, the highest possible profit that the seller can achieve is

$$\pi^* = \int\int_{x+y>\frac{H}{L}y} [(x+y)L - yH]\,dG(x,y) + H\bar{y}. \tag{14}$$

There are two cases to consider in (14): (i) if $x+y < \frac{H}{L}y$ for (almost) all (x,y), or if $x+y > \frac{H}{L}y$ for (almost) all (x,y), we have $\pi^* = H\bar{y}$ or $L(\bar{x}+\bar{y})$ respectively, and hence separate selling as a special case of IB would achieve the first best. And (ii), $\left\{(x,y) : x+y > \frac{H}{L}y\right\}$ has a positive measure and (5) holds. Then $\pi^* > H\bar{y}$. In either case, under condition (6), (H, L, B^*) with $B^* = \{[m_i^*, M_i^*] \ : i = 1, ..., n\}$ will achieve the first-best π:

$$\pi^* = \int\int_{x+y\in B^*} [(x+y)L - yH]\,dG(x,y) + H\bar{y}. \tag{15}$$

We summarize the above with the following:

Proposition 2 *Under (6), interpersonal bundling* (H, L, B^*) *with* $B^* = \{[m_i^*, M_i^*] \ : i = 1, ..., n\}$ *is an optimal selling scheme.*

As (14) suggests, the highest possible profit for the seller is equal to $H\bar{y}$, the expected profit under uniform price H, plus an additional term that reflects the expected increase in profit if the price could be lowered to sell also to L-consumers when the demand realization is such that doing so would raise profit; i.e., when $(x,y) \in \left\{(x,y) : x+y > \frac{H}{L}y\right\}$. Under a general interpersonal bundling scheme, the idea is to divide this set into regions corresponding to intervals of $x + y$. With bundle sizes designed to match these intervals, IB can achieve the first-best profit, as in (15), and it is therefore an optimal selling scheme.[14]

Propositions 1 and 2 are complementary to each other. Proposition 2 shows that IB is an optimal selling method under (6), but it may not dominate separate selling, which is a special case of IB, and the optimal bundle might be rather complicated. By contrast, Proposition 1 shows that a simple form of IB is more profitable than separate selling, provided that the regularity condition is satisfied, but it does not address the issue of whether IB is an optimal selling scheme. Together, they imply that IB dominates separate selling *and* achieves the highest possible profit when (5) and (6) are both satisfied.

While Propositions 1 and 2 shed light on the properties of IB in general, it is not explicit how they relate to the parameter values of the model and to observed marketing practices. We next turn to simpler forms of IB that have been or can potentially be used relatively easily in practice, to gain insights on when the conditions for these results are satisfied and how to design IB in different market environments.

3. IB WITH A MINIMUM OR MAXIMUM GROUP SIZE

In this section, we discuss two especially simple forms of interpersonal bundling, IBmin and IBmax. One motivation for the study of IBmin is that it is a widely observed selling practice, popularized especially on the Internet by Groupon. After studying IBmin, we also examine IBmax As it turns out, the profitability condition for IBmax is interestingly connected to that for IBmin.

[14] Condition (6) is needed, essentially because the first-best mechanism can depend separately on x and y, while IB can only condition prices on $x + y$. With (6), information about $x + y$ is sufficient for the implementation of the first best. Notice that if $\left\{(x, y) : x + y > \frac{H}{L}y\right\}$ is empty, (6) obviously holds with $i = 1$ and $m_1 = M_1$. In general, while (6) holds naturally in many situations (as we illustrate later), it need not always be true.

3.1 Profit Advantages of IBmin

Notice that IBmin (p, q, m) is equivalent to $p^s = q$ if $m \leq a_x + a_y$, and to $p^s = p$ if $m > b_x + b_y$. Thus IBmin is always at least as profitable as separate selling. To investigate when IBmin is more profitable than separate selling and how large its profit advantage is, we note that, from (3) and (4), under IBmin the seller maximizes:

$$\max_m \pi(H, L, m) = L \int\int_{x+y \geq m} (x+y)\, dG(x,y) + H \int\int_{x+y<m} y dG(x,y). \tag{16}$$

Let π^* now be the highest profit under IBmin. Condition (A1) below provides a sufficient condition for $\pi^* > \pi^s$:

$$\left(1 + \frac{a_x}{a_y}\right) < \frac{H}{L} < \left(1 + \frac{b_x}{b_y}\right). \tag{A1}$$

Proposition 3 *IBmin (H, L, m) is more profitable than separate selling if (A1) holds.*

Proof. From (A1), $Hb_y < L(b_x + b_y)$. Thus there exists $\varepsilon_1 \equiv \frac{1}{2}\left(b_x + b_y - \frac{H}{L} b_y\right) > 0$ such that $b_x + b_y - \varepsilon_1 > \frac{H}{L} b_y$. Hence, if $x + y \in \delta_1 = [b_x + b_y - \varepsilon_1, b_x + b_y]$, $x + y > \frac{H}{L} b_y \geq \frac{H}{L} y$. It follows that

$$\Pr\left(x + y > \frac{H}{L} y \middle| x + y \in \delta_1\right) = 1.$$

Also from (A1), $Ha_y > L(a_x + a_y)$. Thus there exist $\varepsilon_2 \equiv \frac{1}{2}\left[\frac{H}{L} a_y - (a_x + a_y)\right] > 0$ such that $a_x + a_y + \varepsilon_2 < \frac{H}{L} a_y$. Hence, if $x + y \in \delta_2 = [a_x + a_y, a_x + a_y + \varepsilon_2]$, $x + y < \frac{H}{L} a_y \leq \frac{H}{L} y$. It follows that

$$\Pr\left(x + y < \frac{H}{L} y \middle| x + y \in \delta_2\right) = 1.$$

Therefore, condition (5) is satisfied, and hence IBmin is more profitable than separate selling from Proposition 1. ■

Intuitively, under (A1), if $p^s = H$, profit can be increased by keeping the regular price but adding a group bundle with unit price L and a minimum size that is slightly lower than $b_x + b_y$ (the maximum possible total number of consumers); if $p^s = L$, profit can be increased by rasing the regular price to H and adding a bundle with unit price L and a minimum size that is slightly higher than $a_x + a_y$ (the minimum possible total number of consumers). (A1), which is sufficient but not necessary for (5), ensures that these changes will indeed strictly increase profit. Condition (A1) is thus invariant to the functional form of the joint distribution of X and Y, depending only on the upper and lower limits of the support for the distribution. It holds if the H/L ratio is relatively large compared to a_x/a_y but small compared to b_x/b_y. IBmin allows the firm to sell at the low price only if profit is higher under the low price—otherwise the high price will prevail—thereby assuring a higher profit than separate selling.[15]

In many situations where group coupons are issued by sellers such as restaurants and hair salons, H could be considered as the regular price at which the seller has less uncertainty about the number of consumers. Thus the difference between a_y and b_y tends to be relatively small. On the other hand, there might be more uncertainty about the number of consumers who will purchase at the sale price L, so the difference between a_x and b_x tends to be relatively large. In such situations, condition (A1) is likely satisfied.[16]

To illustrate and to make explicit profit comparisons, consider the example below:

Example 1 *Suppose that X and Y are independently and uniformly distributed on $[0,3]$ and $[1,2]$, respectively. Then, $\bar{x} = \bar{y} = \frac{3}{2}$, $p^s = H$ if $H \geq 2L$, $p^s = L$ if $H < 2L$, and (A1) holds if $H < \frac{5}{2}L$. Under IBmin,*

$$\pi(H, L, m) \equiv L \int_{\max\{1,m-3\}}^{2} \int_{\max\{m-y,0\}}^{3} (x+y) \frac{1}{3} dxdy + H \int_{1}^{\min\{m,2\}} \int_{0}^{\min\{m-y,3\}} y \frac{1}{3} dxdy.$$

Setting $\partial\pi(H, L, m)/\partial m = 0$, we find the optimal minimum bundle size as

$$m^* = \begin{cases} \frac{H}{2L-H}, \text{ with } \pi^* > \pi^s & if \quad H \leq \frac{4}{3}L \\ \frac{3}{2}\frac{H}{L}, \text{ with } \pi^* > \pi^s & if \quad \frac{4}{3}L < H < 2.6L \\ \geq 5, \text{ with } \pi^* = \pi^s & if \quad 2.6L \leq H \end{cases}.$$

For instance, if $L = 1$ and $H = 2$, then $m^* = 3$ and $\pi^* = 3.3333 > \pi^s = 3$, so IBmin increases (expected) profit by about 11%.

[15] If H/L is too small, it may be optimal always to sell at $p^s = L$, so the option to sell at alternative prices has no value. Likewise, if H/L is too large, it could be optimal always to sell at $p^s = H$, which would then achieve the same profit as interpersonal bundling.

[16] We may view IBmin as allowing the seller to experiment with a lower price that will prevail only when the number of purchasing consumers reaches a minimum size, or only when it is more profitable than the regular price.

Several observations can be made in Example 1. First, condition (A1) is sufficient, but not necessary, for the profitability of IBmin. In Example 1, while (A1) holds for $H < 2.5L$, IBmin is also profitable when $H \in [2.5L, 2.6L)$.

Second, (A1) is fairly tight as a sufficient condition. When $H \geq 2.6L$, IBmin is no longer profitable. In this case, $\frac{3}{2}\frac{H}{L} \geq \frac{3}{2}(2.6) = 3.9$. However, for any $m \in [3.9, 5)$, the expected profit under $x + y \geq m$, in which case all sales will occur at the discounted price L, is lower than the expected profit under separate selling. Therefore, it is optimal for the seller not to offer the bundle, which is equivalent to setting a sufficiently large bundle size ($m^* \geq 5$).

Third, when IBmin is profitable, m^* increases in H but decreases in L. A marginal increase in m reduces the probability that the sale will occur at the low price (with a large volume) and raises the probability that the sale will occur at the high price. Thus, m^*, which balances these two effects, increases with the high price and decreases with the low price.

We now turn to the question of how the advantage of IBmin, relative to separate selling, may vary with the market environment. We first consider how the ratio H/L, or the difference between the reservation prices of the high- and low-value consumers, affects the relative profitability of bundling.

Corollary 1 *Suppose that (A1) holds and L is fixed. Then, $\pi^* - \pi^s$ exhibits an inverted-U shape with respect to changes in H, first increasing and then decreasing, reaching maximum at $H = \left(1 + \frac{\bar{x}}{\bar{y}}\right) L$.*

Proof. When $H < \left(1 + \frac{\bar{x}}{\bar{y}}\right) L$, $\pi^* - \pi^s = \max_m \pi(H, L, m) - L(\bar{x} + \bar{y})$. From (16), $\pi(H, L, m)$ increases in H for all interior m. Thus, if (A1) holds so that $\pi^* > \pi^s$, $\max_m \pi(H, L, m)$ is also increasing in H, and so is $\pi^* - \pi^s$. Similarly, when $H \geq \left(1 + \frac{\bar{x}}{\bar{y}}\right) L$, $\pi^* - \pi^s = \max_m \int\int_{x+y\geq m} [L(x+y) - Hy]\, dG(x, y)$, which decreases in H. ■

When H/L is low (or high), the profit advantage of IBmin is low relative to separate selling, because selling at price L (or H) is often more profitable than at price H (or L), which implies that the option to sell at one of the two prices contingent on the realizations of $X + Y$ under IBmin has very limited value. This option becomes more valuable when

H/L is at some intermediate level, implying more profound profit advantage of IBmin.[17]

We next consider how the dispersion of X affects the profits under IBmin. Intuitively, when X is more dispersed, demand is more uncertain and the advantage of IBmin is larger. The result below shows that this is indeed the case under some conditions, assuming that X and Y are independent with the (marginal) distribution of Y being $F_y(y)$, and comparing profits under two different distributions of X.

Following Johnson and Myatt (2006), we say that distribution $\hat{F}_x(x)$ is more dispersed than $F_x(x)$ if $\hat{F}_x(x)$ is a rotation of $F_x(x)$ such that $x \gtreqless \hat{x} \Longleftrightarrow \hat{F}_x(x) \lesseqgtr F_x(x)$ for some rotation point $\hat{x}$. Under $\hat{F}_x(x)$ and $F_x(x)$, respectively, let $\bar{x}_{\hat{F}}$ and $\bar{x}_F$ be the expected values of X, $\hat{b}_x$ and b_x the upper limits of $\hat{F}_x$ and F_x, and $\hat{m}^*$ and m^* the optimal bundle sizes, where $\hat{b}_x \geq b_x$ and $\bar{x}_{\hat{F}} \geq \bar{x}_F$. Let the corresponding profits be $\hat{\pi}^*$ and π^* under bundling, and $\hat{\pi}^s$ and π^s under separate selling.

Corollary 2 *Suppose (A1) holds and $\hat{F}_x$ is a rotation of F_x such that: (i) $H\bar{y} \geq L\left(\bar{y} + \bar{x}_{\hat{F}}\right)$, (ii) $\hat{x} \leq m^* - b_y$, and (iii) $\int_{a_y}^{b_y} [Lm^* - Hy] \left[\hat{F}_x(m^* - y) - F_x(m^* - y)\right] dF_y(y) \leq 0$. Then, $\hat{\pi}^* - \hat{\pi}^s > \pi^* - \pi^s$; that is, the profit advantage of IBmin relative to separate selling is larger if X is more dispersed.*

Proof. See the appendix. ■

Although the result seems intuitive, the comparison of profits under $\hat{F}_x(x)$ and $F_x(x)$ turns out to be subtle. Condition (i) ensures that $p^s = H$ under separate selling for both $\hat{F}_x(x)$ and $F_x(x)$. Under (ii), $\hat{F}_x(x) < F_x(x)$ for $x \geq m^* - y$, so that more dispersion under $\hat{F}_x$ leads to higher probabilities for higher realizations of x; and under (iii) this similarly holds on average weighted by the density of Y. All three conditions can be easy to verify. For instance, in Example 1, where $F_x(x) = \frac{x}{3}$, these conditions are satisfied for any rotation $\hat{F}_x(x) = \frac{x}{\alpha}$ with $\alpha > 3$ and $H/L \in [(3+\alpha)/3,\ 2.6)$, where $m^* = \frac{3}{2}\frac{H}{L} > 3 > b_y = 2$, and $\hat{x} = 0$.

[17] We have also considered a variant of our model in which the reservation prices of the two types of consumers are random variables instead of constant values, with similar insights on the profitability of IBmin. The assumption that the reservation prices take constant values H and L allows us to illustrate our ideas most transparently.

3.2 Profitability of IBmax and the Nature of Uncertainty

We now study IBmax, (p, q, M), which also weakly dominates separate selling, since it is equivalent to $p^s = q$ if $M > b_x + b_y$ and to $p^s = p$ if $M \leq a_x + a_y$. We first establish the profitability of IBmax and its profit advantage relative to separate selling, in parallel to and in comparison with the analysis for IBmin. We then discuss limited-quantity discount (LQD), which is closely related to IBmax, and comment on how to implement IBmax.

From (3) and (4), under IBmax the seller maximizes:

$$\max_M \pi(H, L, M) = L \int\int_{x+y\leq M} (x+y)\, dG(x,y) + H \int\int_{x+y>M} y dG(x,y). \tag{17}$$

Let π^* now be the highest profit under IBmax. As another application of Proposition 1, Condition (A1') below provides a sufficient condition for $\pi^* > \pi^s$:

$$\left(1 + \frac{b_x}{b_y}\right) < \frac{H}{L} < \left(1 + \frac{a_x}{a_y}\right). \tag{A1'}$$

Proposition 4 *IBmax is more profitable than separate selling if (A1') holds.*

Proof. From (A1'), $Ha_y < L(a_x + a_y)$. Thus there exists $\varepsilon_1 \equiv \frac{1}{2}\left(a_x + a_y - \frac{H}{L}a_y\right) > 0$ such that $a_x + a_y - \varepsilon_1 > \frac{H}{L}a_y$. Hence, if $x+y \in \delta_1 = \left[a_x + a_y, a_x + a_y + \frac{L}{H}\varepsilon_1\right]$, then $y \leq a_y + \frac{L}{H}\varepsilon_1$ and

$$x + y \geq a_x + a_y > \frac{H}{L}a_y + \varepsilon_1 \geq \frac{H}{L}\left(y - \frac{L}{H}\varepsilon_1\right) + \varepsilon_1 = \frac{H}{L}y.$$

It follows that

$$\Pr\left(x + y > \frac{H}{L}y \middle| x + y \in \delta_1\right) = 1.$$

Also from (A1'), $Hb_y > L(b_x + b_y)$. Thus there exists $\varepsilon_2 \equiv \frac{1}{2}\left[\frac{H}{L}b_y - (b_x + b_y)\right] > 0$ such that $b_x + b_y + \varepsilon_2 < \frac{H}{L}b_y$. Hence, if $x + y \in \delta_2 = \left[b_x + b_y - \frac{L}{H}\varepsilon_2, b_x + b_y\right]$, then $y \geq b_y - \frac{L}{H}\varepsilon_2$ and

$$x + y \leq b_x + b_y < \frac{H}{L}b_y - \varepsilon_2 \leq \frac{H}{L}\left(y + \frac{L}{H}\varepsilon_2\right) - \varepsilon_2 = \frac{H}{L}y.$$

It follows that

$$\Pr\left(x + y < \frac{H}{L}y \middle| x + y \in \delta_2\right) = 1.$$

Therefore, condition (5) is satisfied, and hence IBmax dominates separate selling from Proposition 1. ■

Note that (A1') is more likely to hold if a_y is small relative to a_x and b_y is large relative to b_x; that is, if Y is more dispersed than X. When the number of high-value consumers (y) tends to be either much lower or much higher than the number of low-value consumers (x), the seller wishes to set a low price to sell to all consumers if y turns to be low, but a high price to sell only to the high-value consumers if y turns out to be high. By specifying that the low price becomes effective only when the consumer group does not exceed a certain size, the seller can implement profitable option pricing, charging a low price when demand is low and a high price when demand is high.

By contrast, the sufficient condition for IBmin to dominate separate selling, (A1), is more likely to hold if a_y is large relative to a_x and b_y is small relative to b_x; that is, if X is more dispersed than Y. Thus IBmax and IBmin are profitable selling schemes in response to demand uncertainties of a different nature. IBmin tends to be profitable when the total number of consumers and their (average) valuation are likely negatively correlated: a higher number of consumers is likely associated with more low-value consumers; IBmax tends to be profitable when the total number of consumers and their (average) valuation are likely positively correlated: a higher number of consumers is likely associated with more high-value consumers.[18]

Moreover, for fixed L, it is straightforward to show that the profit advantage of IBmax (relative to separate selling) is an inverted-U function of H, reaching maximum at $H = \left(1 + \frac{\bar{x}}{\bar{y}}\right) L$, similar to that of IBmin. It can also be verified that this profit advantage is more pronounced if F_y is more dispersed, similar to the larger profit advantage of IBmin when F_x is more dispersed. Hence, there is a version of Corollary 1 and of Corollary 2 for IBmax, and we omit their formal statements to avoid repetition.

We next further show that IBmin or IBmax is in fact an optimal selling scheme, if there

[18] IB may not be profitable if neither (A1) nor (A1') is satisfied. For example, if $L = 1$, $H = 2$, and $(X, Y) = (1, 10)$ or $(2, 12)$ with equal probability, then $p^s = 2$ and $\pi^s = 22$, which cannot be improved through IB.

exists either m^* or M^* with one of the following monotonic properties on the entire support of (x, y):

$$\begin{aligned} &\text{(C1)} \quad x + y \geq m^* \text{ if and only if } x + y \geq \frac{H}{L} y; \text{ or} \\ &\text{(C2)} \quad x + y \leq M^* \text{ if and only if } x + y \geq \frac{H}{L} y, \end{aligned}$$

where the inequalities may be strict if m^* or M^* is a mass point, and m^* (or M^*) is in the interior of $[a_x + b_x,\ a_y + b_y]$ if it is not a mass point.

Condition (C1) holds, for example, if y is a constant, in which case $m^* = x^* + y$ for $x^* \equiv y\left(\frac{H}{L} - 1\right)$. Obviously, (C1) can also hold if y is not a constant. For example, suppose $L = 10$, $H = 12$, and (X, Y) takes either (0, 50) or (100, 0) with equal probability. Then (C1) holds with strict inequality and $m^* = 50$.

Condition (C2) holds, for example, if x is a constant, in which case $M^* = x + y^*$ for $y^* \equiv x/\left(\frac{H}{L} - 1\right)$. For an example where x is not a constant, suppose $L = 10$, $H = 12$, and (X, Y) takes either (0,100) or (50, 0) with equal probability. Then (C2) holds with $M^* = 50$.

Note that condition (5), under which IB dominates separate selling, and condition (6), under which IB is an optimal selling mechanism, are both satisfied when either (C1) or (C2) holds. Therefore, from Propositions 1 and 2, we have:

Corollary 3 *(1) Under (C1), IBmin dominates separate selling and also achieves the first-best profit. (2) Under (C2), IBmax dominates separate selling and also achieves the first-best profit.*

In general, however, neither (C1) nor (C2) is necessarily satisfied. For example, suppose (X, Y) can take three possible pairs of values with equal probability: $(2, 0)$, $(3, 3)$, and $(10, 4)$, with $L = 1$ and $H = 2$. Then, on the support of (x, y): $x + y \geq \frac{H}{L} y$ if $x + y \leq 2$ or if $x + y \geq 14$, but $x + y < \frac{H}{L} y$ if $2 < x + y < 14$. Nevertheless, Propositions 1 and 2 both still apply here: bundle $(2, 1, B')$, with $B' = \{[0, 2], [14, 15]\}$, dominates separate selling and also achieves the first-best profit.

While IBmin has been used in the sale of many goods, IBmax does not appear common. But a related format of IBmax, limited-quantity discount (LQD), has been used in many markets. LQD sets a low price for a limited quantity and raises the price for the additional quantity exceeding the limit. This is a popular selling strategy by airlines, hotels, stadiums, theaters, and even department stores, as discussed and analyzed in Dana (1999, 2001).[19] Dana (2001; page 650) gives the following example:

Suppose that demand will be either high or low, each with probability 1/2. Low demand consists of 50 consumers with a reservation value of \$10, and high demand consists of 100 consumers with a reservation value of \$12. The seller must set its prices in advance. Within his context, the following is an optimal selling strategy under zero marginal cost: print 50 tickets at a price of \$10 and 50 tickets at a price of \$12. This yields an expected profit of \$800, higher than \$750, the highest expected profit under a uniform price.

With IBmax, however, the seller can do even better in this example. Reformulating the example with the notations of our model, we have $L = 10$, $H = 12$ and (X, Y) takes either (50,0) or (0,100) with equal probability. IBmax with regular price 12, discounted price 10, and maximum group size $M^* = 50$ leads to an expected profit of \$850: The seller offers IBmax $(12, 10, 50)$ to all consumers for simultaneous sign up. If the demand state is low, at most 50 consumers will sign up for group purchase and will all receive the discount price \$10. If the demand state turns out to be high, more than 50 consumers will sign up, in which case the price becomes 12, and all 100 consumers pay the higher price.

Intuitively, when the group size exceeds the maximum limit M, under LQD the low price is still effective for units up to M. If many of those who would purchase M units at the low price, when $x + y > M$, are high-value consumers (in the above example all of them are), the seller can do better with IBmax by making the low price unavailable if the group size exceeds M. On the other hand, if those who would purchase M units at the low price, when $x + y > M$, are mostly low-value consumers, LQD can potentially be more profitable.

[19] Denote the profit under LQD by $\tilde{\pi}(p, q, M)$, where M is the limited quantity to which the lower price q applies. In our context, since $\tilde{\pi}(H, L, 0) = H\bar{y}$ and $\tilde{\pi}(H, L, b_x + b_y) = L(\bar{x} + \bar{y})$, LQD is at least as profitable as separate selling, and can often be more profitable.

One possible reason why LQD has seen wide applications but IBmax has not is that it is more difficult for a seller to commit to IBmax and to communicate it to potential buyers. However, IBmax does not seem much more difficult to implement than IBmin. For instance, a seller could announce in advance a sale price L that is effective only if the number of orders it receives does not exceed M^* for a certain time period, and if it exceeds M^*, all of those who still wish to purchase the good will need to pay the regular price H. The announcement can be made through some intermediary such as Groupon or Amazon, and the number of orders received will be kept confidential until the time period expires (so that consumers essentially submit orders simultaneously). The goods could be theater or sports tickets, vacation packages, restaurant meals, consumer electronics, and so on. Thus, we believe that IBmax, like many other marketing innovations, will potentially find its profitable applications in the marketplace after it is conceived by researchers and understood by practitioners.

4. INFORMATION DISSEMINATION AND PRICE DISCRIMINATION

Our main model has focused on the role of demand uncertainty in the profitability of interpersonal bundling. Demand uncertainty is a common phenomenon in many markets, and our analysis demonstrates how firms can use this selling strategy to increase profit in such market environments. In this section, we discuss how a seller may incorporate two additional strategic considerations in the design of IB to enhance its profitability, in two variants of the main model. We shall devote our attention to IBmin, due to its high relevance to the applications we have in mind, and also for the sake of keeping the discussions concise.

4.1 Dissemination of Product Information

The existence of a seller's product may be known to some consumers but unknown to others. In order to achieve the group size to qualify for the low bundle price under IBmin, an informed potential buyer may have the incentive to transmit the sale information to other consumers. A seller should take this incentive into account in its bundle design.

To formalize this idea in a simple setting, we consider a variant of the main model by assuming that the number of H-consumers is initially a given number $n \geq 1$, and each of them ($i = 1, ..., n$) can make an effort in order to inform a set of $k > 0$ H-consumers who are initially unaware of the seller's product and prices.[20] Define set $N \equiv \{i : i = 1, ..., n\}$. Each $i \in N$ succeeds in transmitting the information to the k uninformed consumers with probability β_i at a personal cost $C(\beta_i)$, where $C'(\cdot) > 0$ with $C'(0) \to 0$, $C''(\cdot) \geq 0$, and the k uninformed consumers become informed if at least one $i \in N$ succeeds. Thus, the number of H-consumers is potentially

$$y = \begin{cases} n + k & \text{with probability} \quad 1 - \Pi_{i=1}^{n}(1 - \beta_i) \\ n & \text{with probability} \quad \Pi_{i=1}^{n}(1 - \beta_i) \end{cases}.$$

Other aspects of the model are the same as in Section 2. In particular, all L-consumers are informed about the seller's product and price(s), and their number, x, is the realization of random variable X that has distribution $F(x)$. (We drop the subscript x in $F(x)$ for this section.) Under separate selling, informed consumers have no incentive to incur the cost to transmit product information. Hence $p^s = L$ and $\pi^s = L(n + \bar{x})$ if $L(n + \bar{x}) > Hn$, whereas $p^s = H$ and $\pi^s = Hn$ if $L(n + \bar{x}) \leq Hn$.

Under IBmin, the seller first posts (p, q, m), after which all $i \in N$ simultaneously choose β_i. Both x and y are then realized, and possible purchases are made. For convenience, we again treat m as a continuous number, and without loss of generality, we can confine our search for the optimal (p, q, m) to $q \leq L < p \leq H$.

We consider a symmetric equilibrium where each $i \in N$ chooses the same β. Given (p, q, m), and all other H-consumers' choice β, i chooses her β_i to maximize her expected surplus:

$$U(\beta_i | m, \beta) = (H - q)\Pr(X + Y \geq m) + (H - p)\Pr(X + Y < m) - C(\beta_i),$$

[20] Unlike in Section 2, the number of initial H-consumers is now an integer. This avoids the problem that no consumer is willing to incur the information transmission cost when the number is a continuum. For convenience, we assume that the initially uninformed consumers also are all of the H-type.

where $\Pr(X+Y \geq m) =$

$$\left[1-F\left(m-n-k\right)\right]\left[1-\left(1-\beta\right)^{n-1}\left(1-\beta_i\right)\right]+\left(1-F\left(m-n\right)\right)\left(1-\beta\right)^{n-1}\left(1-\beta_i\right).$$

To see the trade-off for i in choosing optimal β_i, notice that the optimal β_i satisfies $\partial U\left(\beta_i \middle| m,\beta\right)/\partial\beta_i|_{\beta_i=\beta}=0$ in a symmetric equilibrium, which, denoting the equilibrium β by $\beta \equiv \beta\left(p,q,m\right)$ for any given (p,q,m), becomes:

$$(p-q)\left[F\left(m-n\right)-F\left(m-n-k\right)\right]\left(1-\beta\right)^{n-1}-C'\left(\beta\right)=0, \tag{18}$$

where $\left[F\left(m-n\right)-F\left(m-n-k\right)\right]$ is the increased probability of meeting m due to the addition of k uninformed consumers and $\left(1-\beta\right)^{n-1}$ is the probability that other $(n-1)$ H-type informed consumers fail to reach the uninformed. Thus, as n goes to infinitive, β approaches zero because of the free riding problem. If n is finite, however, rearranging the above equation gives

$$(p-q)\left[F\left(m-n\right)-F\left(m-n-k\right)\right]=\frac{C'\left(\beta\right)}{\left(1-\beta\right)^{n-1}}.$$

Thus, for a finite fixed n, the equilibrium $\beta=\beta\left(p,q,m\right)$ is positive and increases in $(p-q)$ because $C''>0$. The choice of β balances the marginal benefit of increasing the probability of meeting the bundle size m and the marginal effort cost of disseminating information. Holding other things constant, the bundle discount $(p-q)$ has to increase with n if the seller wishes to induce the same amount of effort from the informed consumers. Hence, the firm's ability to use IBmin as an information dissemination device will be more limited when n becomes larger.

Anticipating $\beta\left(p,q,m\right)$ in equilibrium, the seller will choose the equilibrium bundle $\left(p^*,q^*,m^*\right)$, and the equilibrium β is then $\beta^*=\beta\left(p^*,q^*,m^*\right)$. In setting $\left(p^*,q^*,m^*\right)$, the seller knows that IBmin now can increase profit for two distinct reasons. First, as a profitable pricing strategy under uncertainty, it increases profit even if $\beta_i=0$ for all i (in which case uninformed consumers do not learn about the product information). From Proposition 3 and (A1), this is ensured if

$$\left(1+\frac{a_x}{n}\right)<\frac{H}{L}<\left(1+\frac{b_x}{n}\right). \tag{A2}$$

Second, IBmin can motivate consumers to transmit product information to the uninformed, or to choose $\beta_i > 0$ at a personal cost. In choosing m^* to provide this incentive, the seller balances two opposing effects: while a higher m motivates the informed consumers to disseminate information, it may also diminish this incentive if the threshold is set too high. Our next result, which provides a sufficient condition for higher profit under IBmin with the additional channel of encouraging information transmission to expand demand (i.e., in equilibrium $\beta_i = \beta^* > 0$), refers to the following condition:

$$\left(1 + \frac{a_x}{n}\right) < \frac{H}{L} \leq \left(1 + \frac{b_x}{n+k}\right). \tag{A2'}$$

Note that (A2'), which implies the weaker condition (A2), similarly holds if H/L is in an intermediate range.

Proposition 5 *Suppose that (A2') holds. Then, IBmin has higher profit than separate selling with* $\beta^* > 0$, $p^* = H$, *and* $m^* \in (n + a_x, n + k + b_x)$.

Proof. See the appendix. ■

Since the discount price can be valid only if m^* is reached, the informed consumers have the incentive to transmit costly product information to the uninformed, hoping that more consumers will join the group purchase. As is shown in the proof for the symmetric equilibrium contained in the appendix, it is indeed optimal for each informed consumer to choose β^*, given that other informed consumers will do the same, and there exists an interior minimum size m^*. It is worth emphasizing that the optimal m^* is now chosen also to provide the incentive for β^*, in addition to responding optimally to demand uncertainty. In other words, IBmin also provides a mechanism to expand market demand.

To illustrate, consider the next example:

Example 2 *Suppose that* $n = 2$, $k = 1$, $C(\beta_i) = \frac{1}{2}\beta_i^2$, $F(x) = \frac{x}{3}$ *for* $x \in [0, 3]$, *and* $L < H < \frac{5}{2}L$. *Then, condition (A2) is satisfied, which is sufficient for IBmin to increase profit. With* $L = 1$, *Table 1 below lists the equilibrium bundle and the profit comparisons with separate selling.*

Table 1

	p^*	q^*	m^*	β^*	π^*	π^s	$\frac{\pi^*-\pi^s}{\pi^s}$
$H=2$	2	1	4.875	0.25	4.805	4	10%
$H=1.8$	1.8	1	4.278	0.21	4.377	3.6	22%
$H=1.5$	1.5	1	3.398	0.143	3.914	3.5	12%

As in Example 1, given L, m^* is higher for higher H. Furthermore, β^* is also higher for higher H, directly because of the larger bundle discount $(H-L)$, and indirectly because of the higher bundle size (m^*).

For tractability, our model of information dissemination has made some simplifying assumptions. In particular, our assumption that each informed consumer can transmit sale information to all uninformed consumers with some probability is restrictive. While it is possible that an informed consumer can publicize the group coupon information to all uninformed consumers through media such as facebook or an online forum, our assumption is made mainly for the tractability of analysis. We expect that the basic insights will still be valid in a more realistic setting where various numbers of uninformed consumers may become informed with different probabilities.

Another of our simplifying assumptions is that the uninformed are all H-consumers. One may wonder what would happen if some L-consumers were also in the uninformed pool. To see this most strikingly, suppose that the uninformed are all L-consumers. Then, the incentive for the informed H-consumers to disseminate information remains unchanged, because both types of consumers are willing to buy at the discounted price L. However, the firm's problem would need a slight modification: if m^* is reached, the firm would earn profit $L(x+n+k)$, same as when all the uninformed have the high valuation; but if m^* is not reached, the firm would earn profit Hn while the profit is $H(n+k)$ when all the uninformed have the high valuation. Thus, the difference when the uninformed all have the low valuation is that the firm earns less profit if m^* is not reached. In response, the firm would set a lower m^* compared to the case where all the uninformed are H-consumers. Nevertheless, the firm will still find it profitable to impose a minimum bundle size to motivate consumers to disseminate information.[21]

4.2 Price Discrimination

To obtain the group discount under IB, a consumer may need to incur transaction costs to sign up for group purchase. If H-consumers have higher time costs, they are less likely to participate. Interpersonal bundling can thus be a device for price discrimination, as in the textbook example of price discrimination through coupons. With IBmin, however, there is an additional instrument to screen the buyers: Through the choice of the minimum bundle size that may not be reached due to uncertainty, the seller can further discourage H-consumers from attempting to receive the group discount.

To illustrate, consider another variant of the main model, where the L-consumers have no cost to participate in group purchase, but the H-consumers incur a transaction cost t to do so. Assume that t is distributed on $[\underline{t}, \bar{t}]$ with p.d.f. $\phi(t) > 0$, c.d.f. $\Phi(t)$, and $0 \leq \underline{t} < \bar{t}$. The number of L-consumers is again x with cumulative distribution function $F(x)$, while the mass of H-consumers is normalized to 1. Under separate selling, $p^s = H = \pi^s$ if $H \geq L(\bar{x}+1)$, whereas $p^s = L$ and $\pi^s = L(\bar{x}+1)$ if $H < L(\bar{x}+1)$.

As in the main model, the game under IBmin proceeds as follows: First, the seller offers (p, q, m). Second, the number of L-consumers and the private t for each H-consumer are realized. Third, consumers choose whether to sign up for group purchase. Fourth, the total number of consumers who sign up becomes known. If this number exceeds m, each group member pays q while consumers who have not signed up will pay p; otherwise, all consumers are charged regular price p.

In order to analyze price discrimination under alternative forms of IBmin, we further assume that consumers can sign up for group purchase possibly in two periods, 1, or 2. (Neither the seller nor consumers discount time.) Under the *simultaneous* format, at the beginning of period 2 the seller does not reveal how many consumers signed up in the first

[21] The analysis can also be properly modified to deal with the more general case where the uniformed were a mix of the two types of consumers, and $\gamma \in [0,1]$ were the portion of the H-consumers in the uninformed pool. It can be shown that the result (Proposition 5) would be qualitatively similar.

period, whereas under the *sequential* format the firm does. Hence, with the former all consumers effectively make sign-up decisions simultaneously, whereas with the latter they make sign-up decisions sequentially.

Simultaneous Format

In this case, an H-consumer, if she wishes to participate, needs to incur t before it becomes known how many L-consumers have joined group purchase, or what the realization of x is (it is optimal for all L-consumers to sign up for group coupon since they incur no sign-up cost). Suppose that there is some $t^* \in [0, \bar{t}]$ that solves

$$H - p = \int_{x+\Phi(t^*)\geq m} (H - q) f(x)\, dx + \int_{x+\Phi(t^*)<m} (H - p) f(x)\, dx - t^*. \tag{19}$$

Then, there will be an equilibrium where all L-consumers sign up for group purchase, and an H-consumer will sign up if and only if $t \leq t^*$.[22] We shall focus on this equilibrium.[23] Rearranging (19), we obtain

$$t^* = (p - q)\left[1 - F\left(m - \Phi\left(t^*\right)\right)\right]. \tag{20}$$

The seller's problem is, with $t^* = t^*(p, q, m)$, to maximize

$$\pi(p, q, m) = \int_{m-\Phi(t^*)}^{b_x} \left[q\left(x + \Phi\left(t^*\right)\right) + p\left(1 - \Phi\left(t^*\right)\right)\right] f(x)\, dx + p \int_{a_x}^{m-\Phi(t^*)} f(x)\, dx \tag{21}$$

subject to $q \leq L$, $L \leq p \leq H$, $a_x \leq m - \Phi(t^*) \leq b_x$. The solution to (21) defines the equilibrium (p^*, q^*, m^*).

With regular price p and discounted bundle price q, an H-consumer may nevertheless prefer to purchase at p, because she incurs t for group purchase and she may lose t without receiving the bundle discount if m^* is not reached. Hence, a higher m will reduce the incentive of an H-consumer to engage in group purchase. IBmin may thus price discriminate more effectively both than traditional coupons and than traditional mixed bundling. A

[22] Equation (19) says that the marginal H-consumer with t^* will just be willing to sign up, given (p, q, m) and given the equilibrium behavior of all other consumers.

[23] There can also be a trivial equilibrium where no one signs up for the group coupon, due to there being a continuum of consumers.

higher m, however, can hurt the seller if the sales to the L-consumers do not materialize. Notice that any q below L will lower the seller's profit when the good is sold at a discount and will also make participating in group purchase more attractive to the H-consumers. Thus it is optimal for $q^* = L$. On the other hand, a higher p may increase the profit from the H-consumers paying p but makes the bundle discount more attractive. Consequently, the optimal p is determined jointly with m.

Again denote the seller's equilibrium profit under IBmin by π^*. To derive a sufficient condition under which $\pi^* > \pi^s$, we utilize the condition below

$$\text{(i) } \bar{t} > H - L; \quad \text{(ii) } \frac{H}{L} < 1 + \frac{b_x}{\Phi(H-L)}. \tag{A3}$$

Since $p^* \leq H$, part (i) in (A3) ensures that some H-consumers will not incur t for the bundle discount, and, from (21),

$$\pi^* \geq \pi(H, L, a_x) = L[\bar{x} + \Phi(H-L)] + H[1 - \Phi(H-L)] > L(\bar{x}+1) = \pi^s|_{p^s=L},$$

so that $(p, q, m) = (H, L, a_x)$ is always more profitable than $p^s = L$. Moreover, since $t^* \leq H - L$ and part (ii) in (A3) implies $H\Phi(t^*) < L[\Phi(t^*) + b_x]$, if $m = b_x + \Phi(t^*) - \varepsilon$ for small enough $\varepsilon > 0$ (i.e., m is slightly below $b_x + \Phi(t^*)$, we have, from (21):

$$\begin{aligned}\pi^* &\geq \pi(H, L, b_x + \Phi(t^*) - \varepsilon) = \int_{b_x-\varepsilon}^{b_x} [L(x + \Phi(t^*)) - H\Phi(t^*)] f(x)\,dx + H \\ &\geq \int_{b_x-\varepsilon}^{b_x} [L(b_x - \varepsilon + \Phi(t^*)) - H\Phi(t^*)] f(x)\,dx + H > H = \pi^s|_{p^s=H},\end{aligned}$$

where the last inequality above holds because $H\Phi(t^*) < L[b_x + \Phi(t^*) - \varepsilon]$ for sufficiently small ε. Hence $(p, q, m) = (H, L, b_x + \Phi(t^*) - \varepsilon)$ is always more profitable than $p^s = H$. Therefore, since $p^s = L$ or H, under condition (A3) it must be true that $\pi^* > \pi^s$ and $p^* > L = q^*$. We have thus established:

Proposition 6 *Suppose that condition (A3) is satisfied. Then, the seller's profit is higher under IBmin than under separate selling with $p^* > L = q^*$.*

Notice that since y is normalized to 1 in this variant of the model, (A1) becomes $(1 + a_x) < H/L < (1 + b_x)$. Hence, (A3) is less stringent than (A1), because $\Phi(H - L) < 1$ for

$\bar{t} > H - L$. IBmin dominates separate selling under broader conditions here than in the main model, because it now may increase profit also through price discrimination. To illustrate, suppose $H/L < 1 + a_x$. In this case, if, as in the main model, no consumer has sign-up costs, IBmin is not profitable and the firm will optimally choose $p^s = L$. However, with positive sign-up costs for H-consumers, IBmin becomes profitable through price discrimination. In fact, bundle (H, L, a_x), under which $m = a_x$ is always reached but H-consumers with $t > H - L$ will choose not to join the group and will hence pay price H, yields a higher profit than separate selling. (The seller may do even better by optimally choosing some m^* that is different from a_x.)

Sequential Format

Now consider the sequential format. Since an L-consumer has no cost to sign up, it is optimal for her to do so in the first period. Therefore in equilibrium all L-consumers sign up in period 1 and their number is then publicly known.

Next consider the sign-up decision of H-consumers, for whom it is optimal to wait until the beginning of period 2 to make the choice.[24] Suppose for a moment that, in equilibrium, depending on the realization of x, there exists a cutoff value $t^{**}(x)$ such that only H-consumers with $t \leq t^{**}$ will sign up for group purchase. Given such a strategy by other consumers, an H-consumer with sign-up cost t chooses to sign up only if this leads to a (weakly) higher surplus for her and if a group discount is expected to be offered:

$$H - q - t \geq H - p \qquad \text{and} \qquad x + \Phi(t^{**}) \geq m.$$

Hence the marginal H-consumer has $t = p - q$. It follows that, if $x \geq \hat{x}$, it is optimal for any H-consumer with $t \leq t^{**}$ to sign up given that the others will do the same, where

$$t^{**} = p - q \qquad \text{and} \qquad \hat{x} = m - \Phi(p - q), \tag{22}$$

[24] In reality, it might also be costly for an H-consumer to learn how many consumers have already joined the group, possibly because of the cost to visit the sign-up website. For convenience, we assume that t is incurred when the consumer actually signs up for group purchase, such as transaction costs to open an account or to place an order.

and the group size will be reached. Therefore, under the sequential format, there is indeed an equilibrium, where the seller chooses (p, q, m) optimally, L-consumers sign up in the first period, and: (i) if $x \geq \hat{x}$, then H-consumers with $t \leq t^{**}$ will sign up in the second period and m will be reached, so that group participants will pay discounted price q while non-participants (H-consumers with $t > t^{**}$) will pay regular price p; (ii) if $x < \hat{x}$, no H-consumers will sign up and only regular price p is available.[25]

Comparing (22) with (20), we have $t^{**} > t^{*}$. That is, more H-consumers will sign up for group purchase under the sequential than under the simultaneous format of IBmin. This implies that, for the same bundle, group purchases will occur more often under the sequential format. The intuition behind this finding, as in Hu, Shi, and Wu (2013), is that the sequential format removes the uncertainty faced by period-2 consumers about the number of participating consumers in period 1, which makes period-2 consumers more willing to sign up. Although our model and analysis differ from those in Hu, Shi, and Wu (2013),[26] our finding supports their conclusion that the sequential group-buying mechanism will lead to higher deal success rates. While this implies that a seller would prefer the sequential format if, as they assume, it aims to maximize the deal success rates, in our model the seller, whose objective is to maximize profit, may actually prefer the simultaneous format.

To see that profit can be higher under simultaneous than under sequential IBmin, we notice that the seller's profit function for the sequential format can be obtained by using the profit expression for the simultaneous format in (21) but replacing t^{*} with t^{**}:

$$\pi(p, q, m) = \int_{m-\Phi(t^{**})}^{b_x} \left[q\left(x + \Phi(t^{**})\right) + p\left(1 - \Phi(t^{**})\right) \right] f(x)\, dx + p \int_{a_x}^{m-\Phi(t^{**})} f(x)\, dx. \tag{23}$$

While a complete comparison of profits under the two formats is rather complicated and

[25] Potentially there can also be an equilibrium in which some of the H-consumers with low t sign up in the first period, which may enhance the probability of a discrete benefit of the group discount. In the appendix, we argue that this equilibrium, when it exists, has qualitatively similar properties as the equilibrium here.

[26] Among other differences, in their group-buying mechanisms consumers have heterogenous valuations but identical participation costs, whereas in our model high-value consumers differ in participation costs but have identical valuation.

beyond the scope of our paper, we demonstrate that profit can be higher in the simultaneous format with the following example:

Example 3 *Assume that* $\phi(t) = 1$ *on* $[0, 1]$, $f(x) = \frac{1}{2}$ *on* $[0, 2]$, *and* $L = 1$. *For different values of* H, *Table 2 compares equilibrium simultaneous and sequential IBmin, denoted with superscripts* * *and* **, *respectively.*

Table 2

	Simultaneous Format						Sequential Format				
H	p^*	q^*	m^*	t^*	π^*		p^{**}	q^{**}	m^{**}	t^{**}	π^{**}
1.6	1.6	1	0.97	0.44	2.335		1.6	1	0.96	0.6	2. 272
1.5	1.5	1	0.80	0.40	2. 300		1.5	1	0.75	0.5	2. 266
1.4	1.4	1	0.62	0.35	2. 262		1.4	1	0.56	0.4	2. 246

Example 3 makes it clear that a profit-maximizing seller may prefer the simultaneous over the sequential format. This is because the seller wishes to price discriminate when using IBmin, and, unlike the sequential format, the simultaneous format does not remove uncertainty for the H-consumers, thereby discouraging them from signing up to obtain the group discount.

5. CONCLUDING REMARKS

This paper has conducted a strategic analysis of interpersonal bundling. As a mechanism for option pricing under demand uncertainty, interpersonal bundling will often dominate separate selling with just one or two bundle size intervals, and it is optimal among all selling mechanisms under a plausible sufficient condition. The profitability conditions of interpersonal bundling with a minimum or maximum group size exhibit interesting similarities and differences: each is likely profitable when the ratio of reservation prices of the two consumer types (H/L) is within some intermediate range, and each's profit advantage (relative to separate selling) tends to be an inverted-U function of H/L; but IBmin (respectively, IBmax) tends to be more profitable when the number of low-value (respectively, high-value) consumers is more dispersed. Furthermore, the profitability of IBmin will be enhanced if the incentive to qualify for group purchase motivates buyers to disseminate product information, and if more high-value consumers can be induced to pay the regular instead of the discounted price.

Like other selling formats, interpersonal bundling can achieve its potential benefits for the seller only if it is properly implemented. In particular, losses may occur if the bundle discount under group purchase is too big. For example, when a restaurant offers a group coupon for 70% off its regular price, it could be unwisely pricing below marginal cost.[27] While many businesses have profited from offering IBmin on the Internet, there have also been media reports about how a merchant is hurt by its deep group discount through Groupon and other "social buying" intermediaries.[28] Part of the problem is a potential conflict in incentives: even though the seller should use the advertised deal to maximize its profit, an intermediary like Groupon benefits from a higher deal success rate. However, it need not be in the best interests of the sellers (and, in the long run, also their Internet intermediaries such as Groupon) to focus only on deal success rates. As our theory suggests, the seller's profit is sometimes higher when the deal is off—if the realized number of low-value consumers is not high.[29] And, it would be even worse for sellers if below-cost group sale prices are used to boost deal success rates.

We have studied monopoly interpersonal bundling in this paper. It would be desirable

[27] The restaurant may want to attract repeat customers by taking a one-time loss, but is the loss necessary? Our analysis suggests that interpersonal bundling can be profitable without the repeat-business effect, and a seller need not incur losses in order to generate repeat businesses.

[28] See, for example, "Groupon demand almost finishes cupcake-maker" (November 22, 2011, *The Telegraph),* which tells the story of a British cakemaker who offered her product at 75% off its regular price through Groupon and had to produce at costs substantially above price in order to meet a huge demand increase. See also Byers, Mitzenmacher and Zervas (2012) for discussions about negative side effects for merchants using Groupon.

[29] As a form of advertising, IBmin on the Internet can also serve as a promotional device that encourages consumers to try the product and become repeat customers. While we do not model such roles, they can also be important. Indeed, some sellers may have used Groupon as an advertising platform to attract repeat customers, or to fill up their off-peak capacity.

for future research to analyze interpersonal bundling by competing firms. The profitability of this selling strategy, and its potential adoption by a firm, may then depend on competitive conditions, possibly also including considerations such as product differentiation. For tractability, our model has made some restrictive assumptions, such as that there are only two types of consumers. It would be desirable for future research to extend the analysis to more general settings. While our analysis has demonstrated the profitability of general bundle menus, it remains to be seen when they will be implemented by innovative firms. The simple form of a maximum bundle size, though, can perhaps easily find its profitable applications in the Internet market.

APPENDIX

Proof of Corollary 2. From (i), $H\bar{y} \geq L\left(\bar{y} + \bar{x}_{\hat{F}}\right)$. Hence under separate selling the optimal price is H for either $\hat{F}_x$ or F_x. It follows that

$$\begin{aligned}
\hat{\pi}^* - \hat{\pi}^s &= \int\int_{x+y\geq\hat{m}^*} [L(x+y) - Hy]\, d\hat{F}_x(x)\, dF_y(y) + H\bar{y} - H\bar{y} \\
&\geq \int_{a_y}^{b_y} \left\{ \int_{m^*-y}^{\hat{b}_x} [Lx - (H-L)y]\, d\hat{F}_x(x) \right\} dF_y(y),
\end{aligned}$$

where the inequality is due to revealed preference. Since $\hat{F}_x(x) < F_x(x)$ for $x \geq m^* - y$ from (ii), we have

$$\begin{aligned}
& \int_{m^*-y}^{\hat{b}_x} [Lx - (H-L)y]\, d\hat{F}_x(x) \\
= \;& \left[L\hat{b}_x - (H-L)y\right] - [Lm^* - Hy]\,\hat{F}_x(m^*-y) - \int_{m^*-y}^{b_x} L\hat{F}_x(x)\, dx - \int_{b_x}^{\hat{b}_x} L\hat{F}_x(x)\, dx \\
> \;& [Lb_x - (H-L)y] - [Lm^* - Hy]\,\hat{F}_x(m^*-y) - \int_{m^*-y}^{b_x} LF_x(x)\, dx.
\end{aligned}$$

Thus

$$\begin{aligned}
\hat{\pi}^* - \hat{\pi}^s &> \int_{a_y}^{b_y} [Lb_x - (H-L)y]\, dF_y(y) - \int_{a_y}^{b_y} [Lm^* - Hy]\,\tilde{F}_x(m^*-y)\, dF_y(y) \\
&\quad - \int_{a_y}^{b_y}\int_{m^*-y}^{b_x} LF_x(x)\, dx dF_y(y). \\
\text{(from (iii))} &\geq \int_{a_y}^{b_y} [Lb_x - (H-L)y]\, dF_y(y) - \int_{a_y}^{b_y} [Lm^* - Hy]\, F_x(m^*-y)\, dF_y(y) \\
&\quad - \int_{a_y}^{b_y}\int_{m^*-y}^{b_x} LF_x(x)\, dx dF_y(y) \\
&= \int_{m^*-y}^{b_x} [Lx - (H-L)y]\, dF_x(x) = \pi^* - \pi^s.
\end{aligned}$$

■

Proof of Proposition 5. First, in equilibrium, β_i satisfies (18). The firm's problem is:

$$\begin{aligned}
& \max_{q \leq L < p \leq H, m} \pi(p,q,m) \qquad (24) \\
= \; & q\left[[1-(1-\beta)^n]\int_{x\geq m-n-k}(x+n+k)\,dF(x) + (1-\beta)^n\int_{x\geq m-n}(x+n)\,dF(x)\right] \\
& +p\left[[1-(1-\beta)^n](n+k)F(m-n-k)+(1-\beta)^n nF(m-n)\right].
\end{aligned}$$

Next, from (18) and with $C'' \geq 0$, we have $\beta \equiv \beta(p,q,m)$ increasing in p and decreasing in q; and furthermore

$$\frac{\partial \beta(p,q,m)}{\partial m} = \frac{(p-q)[f(m-n)-f(m-n-k)](1-\beta)^{n-1}}{(n-1)(p-q)[F(m-n)-F(m-n-k)](1-\beta)^{n-2}+C''}.$$

Thus $\beta(p,q,m)$ is increasing in m at $m = n + a_x$ but decreasing in m at $m = n + k + b_x$. At the optimum, $\pi(p,q,m)$ must increase in β. Thus, since $\pi(p,q,m)$ and $\beta(p,q,m)$ both increase in p, the solution to problem (24) must have $p = H$, so that problem (24) becomes $\max_{q\leq L,m} \pi(H,q,m)$. Next,

$$\begin{aligned}
\frac{\partial \pi(H,q,m)}{\partial \beta} = \; & qn(1-\beta)^{n-1}\left[\int_{x\geq m-n-k}(x+n+k)\,dF(x) - \int_{x\geq m-n}(x+n)\,dF(x)\right] \\
& +Hn(1-\beta)^{n-1}\left[(n+k)F(m-n-k)-nF(m-n)\right],
\end{aligned}$$

with

$$\left.\frac{\partial \pi}{\partial \beta}\right|_{m=n+a_x} = qn(1-\beta)^{n-1}k > 0, \quad \left.\frac{\partial \pi}{\partial \beta}\right|_{m=n+k+b_x} = Hn(1-\beta)^{n-1}k > 0.$$

Next, since $Hn \geq L(n+a_x)$ by assumption (A2'),

$$\begin{aligned}
& \left.\frac{\partial \pi(H,q,m)}{\partial m}\right|_{m=n+a_x} \\
= \; & [1-(1-\beta)^n][H(n+k)-qm]f(m-n-k)|_{m=n+a_x} \\
& + (1-\beta)^n(Hn-qm)f(m-n)|_{m=n+a_x} + \left.\frac{\partial \pi(p,q,m)}{\partial \beta}\frac{\partial \beta(p,q,m)}{\partial m}\right|_{m=n+a_x} \\
\geq \; & \left.\frac{\partial \pi(p,q,m)}{\partial \beta}\right|_{m=n+a_x} \left.\frac{\partial \pi(p,q,m)}{\partial \beta}\right|_{m=n+k+b_x} > 0.
\end{aligned}$$

On the other hand, at $m = n+k+b_x$, $\frac{\partial \pi(p,q,m)}{\partial \beta}\frac{\partial \beta(p,q,m)}{\partial m} < 0$, $f(m-n) = 0$, $f(m-n-k) > 0$, β is not affected by q from (18), but $\pi(H,q,m)$ increases in q, which implies that $q^* = L$

at $m = n + k + b_x$. And since $H(n+k) \leq L(n+k+b_x)$ by assumption (A2'), we have $\left.\frac{\partial \pi(H,q,m)}{\partial m}\right|_{m=n+k+b_x} < 0$. Therefore, the equilibrium m is interior: $m^* \in (n + a_x, n + k + b_x)$. It follows from (18) that $\beta^* > 0$. ■

Alternative Equilibrium under Sequential IBmin in Section 4.2

We argue below that at the alternative equilibrium (see footnote 25), the H-consumers who join group purchase and the seller's optimal choice of (p, q, m) are the same as those in the equilibrium in Section 4.2 under the sequential format, even though some of the group-buying H-consumers sign up in period 1 here.

Consider a potential equilibrium where all L- and some H-consumers with cost $t \leq t^0$ sign up in the first period and the H-consumers with cost $t \in (t^0, t^{***}(x)]$ sign up in the second period. Given an equilibrium cutoff value t^{***}, a consumer with $t > t^0$ will choose to sign up in the second period if

$$H - q - t \geq H - p \qquad \text{and} \qquad x + \Phi\left(t^0\right) + \left[\Phi\left(t^{***}\right) - \Phi\left(t^0\right)\right] \geq m.$$

The cutoff values of t and x are thus identical to those in condition (22) on p. 29. Moreover, there exists $t^0 > 0$ such that H-type consumers with $t \leq t^0$ optimally sign up in the first period. This is because the expected benefit of an early sign-up, which is discount $(H - L)$ multiplied by the expected increase in the probability of reaching the minimum bundle size, is a positive constant and thus, for a sufficiently low $\underline{t}$, there exists a cutoff $t^0 \in (\underline{t}, \bar{t})$ such that the consumer with $t = t^0$ is indifferent between signing up in first period or not doing so. Accordingly, the firm will offer the optimal (p, q, m) that maximizes (23) on p.30. Therefore, if we compare the equilibrium discussed here with the one in Section 4.2 under sequential format, the cutoff value t for the marginal H-consumer who join group purchase (t^{**} and t^{***}, respectively) and the optimal (p, q, m) are characterized by the same set of equations, even though some of the group-buying H-consumers sign up in period 1 here.

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