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30 April 2014

Designing Matching Mechanisms under Constraints: An Approach from Discrete Convex Analysis

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Abstract. In this paper, we consider two-sided, many-to-one matching problems where agents in one side of the market (hospitals) impose some distributional constraints (e.g., a minimum quota for each hospital). We show that when the preference of the hospitals is represented as an M$^\natural$-concave function, the following desirable properties hold: (i) the time complexity of the generalized GS mechanism is $O(|X|^3)$, where $|X|$ is the number of possible contracts, (ii) the generalized Gale & Shapley (GS) mechanism is strategyproof, (iii) the obtained matching is stable, and (iv) the obtained matching is optimal for the agents in the other side (doctors) within all stable matchings.
Furthermore, we clarify sufficient conditions where the preference becomes an M$^\natural$-concave function. These sufficient conditions are general enough so that they can cover most of existing works on strategyproof mechanisms that can handle distributional constraints in many-to-one matching problems. These conditions provide a recipe for non-experts in matching theory or discrete convex analysis to develop desirable mechanisms in such settings.

1 Introduction

The theory of two-sided matching has been extensively developed, and it has been applied to design clearinghouse mechanisms in various markets in practice.[4] As the theory has been applied to increasingly diverse types of environments, however, researchers and practitioners have encountered various forms of distributional constraints. As these features have been precluded from consideration until recently, they pose new challenges for market designers.

The *regional maximum quotas* provide one such example. Under the regional maximum quotas, each agent on one side of the market (which we call a hospital) belongs to a region, and each region has an upper bound on the number of agents

[4] See Roth and Sotomayor [15] for a comprehensive survey of many results in this literature.

on the other side (who we call doctors) who can be matched in each region. Regional maximum quotas exist in many markets in practice. A case in point is Japan Residency Matching Program (JRMP), which organizes matching between medical residents and hospitals in Japan. Although JRMP initially employed the standard deferred acceptance algorithm [6], it was criticized as placing too many doctors in urban areas and causing doctor shortage in rural areas. To address this criticism, Japanese government now imposes a regional maximum quota to each region of the country. Regulations that are mathematically isomorphic to regional maximum quotas are utilized in various contexts, such as Chinese graduate admission, Ukrainian college admission, Scottish probationary teacher matching, among others [10].

Furthermore, there are many matching problems in which *minimum quotas* are imposed. School districts may need at least a certain number of students in each school in order for the school to operate, as in college admissions in Hungary [1]. The cadet-branch matching program organized by United States Military Academy (USMA) imposes minimum quotas on the number of cadets who can be assigned to each branch [16]. Yet another type of constraints takes the form of *diversity constraints*. Public schools are often required to satisfy balance on the composition of students, typically in terms of socioeconomic status [2]. Several mechanisms have been proposed [2, 3, 7, 8, 10] for each of these various constraints, but previous studies have focused on tailoring mechanisms to specific settings, rather than providing a general framework.

This paper develops a general framework for handling various distributional constraints, in the setting of 'matching-with-contracts' [9]. We begin by aggregating hospital preferences and distributional constraints into a preference of a representative agent, "the hospitals," as in [10]. The key of our analysis is to associate the preference of the hospitals to a mathematical concept called M$^\natural$-concavity [11]. M$^\natural$-concavity is an adaptation of concavity to functions on discrete domains, and has been studied extensively in discrete convex analysis, which is a branch of discrete mathematics. We show that if the hospitals' aggregated preferences can be represented by an M$^\natural$-concave function, then the following key properties of two-sided matching hold: (i) the time complexity of the generalized GS mechanism is $O(|X|^3)$, where $|X|$ is the number of possible contracts, (ii) the generalized GS mechanism is strategyproof for doctors, (iii) the resulting matching is stable (in the sense of Hatfield and Milgrom [9]), and (iv) the obtained matching is optimal for each doctor among all stable matchings.

Although these properties can be obtained by combining already known results in matching theory and discrete convex analysis, these two research areas have not been well-connected so far. While general frameworks on two-sided matchings are proposed by utilizing discrete convex analysis in [4, 5], strategyproof mechanisms are not considered in these models. One contribution of this paper is to establish a link between matching theory and discrete convex analysis.

Equipped with this general result, we study conditions under which the hospitals' preferences can be represented by an M$^\natural$-concave function. We start by separating the preference of hospitals into two parts. More specifically, we divide hospital preferences to hard distributional constraints for the contracts to be feasible, and soft preferences over a family of feasible contracts. Drawing upon techniques from discrete convex analysis, we show that if a family of hospital-feasible contracts forms a matroid [11], and the soft preferences satisfy certain easy-to-verify conditions (e.g., it can be represented as a weighted sum of contracts), then hospital preferences can be represented by an $\mathrm{M}^\natural$-concave function.

One of the main motivations of our work is to provide an easy-to-use recipe, or a toolkit, for organizing matching mechanisms under constraints. Although our general result is stated in terms of the abstract $\mathrm{M}^\natural$-concavity condition, market designers do not need advanced knowledge on discrete convex analysis or matching theory. On the contrary, our sufficient conditions in the preceding sections suffice for most practical applications. To use our tool, all one needs to show is that the given hard distributional constraints produce a matroid (note that requirements over soft preferences are elementary, e.g., weighted sum of contracts). Fortunately, there exists a vast literature on matroid theory, and what kinds of constraints produce a matroid is well-understood. Therefore, it is usually sufficient to show that the hard distributional constraints can be mapped into existing results in matroid theory. We confirm this fact by demonstrating that most distributional constraints can be represented using our method. The list of applications includes matching markets with the regional maximum quotas [8, 10], individual minimum quotas [3], regional minimum quotas [7], and diversity requirements in school choice [2]. As such, we believe that this study contributes to the advance of practical market design (or "economic engineering") as emphasized in the recent literature (see Roth [14] for instance), by providing tools for organizing matching clearinghouses in practice.

2 Model

A market is a tuple $(D, H, X, \succ_D, f)$. D is a finite set of doctors and H is a finite set of hospitals. X is a finite set of contracts. Each contract $x \in X$ is bilateral, in the sense that x is associated with exactly one doctor $x_D \in D$ and exactly one hospital $x_H \in H$. Each contract can also contain some terms of contracts such as working time and wages. Let $\succ_D = (\succ_d)_{d \in D}$, where each $\succ_d$ represents the preference of each doctor d over the contracts that are related to her (which are denoted as X_d).

We assume some distributional constraints are enforced on feasible contracts. We assume such distributional constraints and hospital preferences are aggregated into a preference of a representative agent, which we call "hospitals". The preference of the hospitals is represented as a function f, i.e., for two sets of contracts $X', X'' \subseteq X$, the hospitals prefer X' over X'' if $f(X') > f(X'')$ holds. If $X' \subseteq X$ violates some distributional constraint, $f(X') = -\infty$. We assume f is normalized by $f(\emptyset) = 0$. Also, we assume f is unique-selecting, i.e., $\forall X' \subseteq X, \arg\max_{X'' \subseteq X'} f(X'')$ is a singleton.

For notation simplicity, we assume each contract $x \in X$ is acceptable both for doctors and hospitals. If a doctor (or hospitals) consider a contract unacceptable, it is not included in X.

Now, we define several concepts used in this paper.

Definition 1 (feasibility). *For a subset of contracts $X' \subseteq X$, we say X' is hospital-feasible iff $f(X') \neq -\infty$. Also, we say X' is doctor-feasible iff it contains at most one contract related to each doctor, i.e., $|X'_d| \leq 1$ holds for all $d \in D$. Then, we say X' is feasible iff it is doctor- and hospital-feasible. We also call a feasible set of contracts* matching*.*

Definition 2 (choice functions). *For each doctor d, its choice function $Ch_d(X')$ chooses $\{x\}$, i.e., a set that contains exactly one contract $x \in X'_d$, where x is the most preferred contract in X'_d, or $Ch_d(X') = \emptyset$ if $X'_d = \emptyset$. Then, the choice function of all doctors are defined as $Ch_D(X') := \bigcup_{d \in D} Ch_d(X')$.*

For the hospitals, its choice function $Ch_H(X')$ is defined as: $\arg\max_{X'' \subseteq X'} f(X'')$*. Since we assume f is unique-selecting, Ch_H is uniquely determined by f.*

Definition 3 (stability [9]). *We say a matching X' is stable iff there exists no $x \in X \setminus X'$ such that $x \in Ch_H(X' \cup \{x\})$ and $x \in Ch_D(X' \cup \{x\})$ hold.*

For notation simplicity, we write $X' + x$ and $X' - x$ to represent $X' \cup \{x\}$ and $X' \setminus \{x\}$, respectively. Also, when $x = \emptyset$, they mean that nothing is added to X' and nothing is removed from X', respectively.

Definition 4 (M$^\natural$-concavity). *We say that f is M$^\natural$-concave when $\forall Y, Z \subseteq X, \forall y \in Y \setminus Z$, there exists $z \in Z \setminus Y \cup \{\emptyset\}$ such that $f(Y) + f(Z) \leq f(Y - y + z) + f(Z - z + y)$ holds.*

The generalized Gale & Shapley (GS) mechanism [9] is a generalized version of the well-known deferred acceptance algorithm [6], which is adapted for the 'matching-with-contracts' model.

Definition 5 (generalized Gale & Shapley (GS) mechanism [9]). *The generalized GS mechanism gives the following matching:*

1. *$R \leftarrow \emptyset$.*
2. *$X' \leftarrow Ch_D(X \setminus R), X'' \leftarrow Ch_H(X')$.*
3. *If $X' = X''$ then return X', otherwise, $R \leftarrow R \cup (X' \setminus X'')$, goto 2.*

3 Properties of generalized GS mechanism

In this section, we show properties of the generalized GS mechanism, assuming f is M$^\natural$-concave and unique-selecting. Due to space limitations, most of proofs are deferred to Appendix.

Theorem 1. *The time complexity of the generalized GS mechanism is $O(|X|^3)$.*

The following theorem immediately follows from existing results in discrete convex analysis.

Theorem 2. *The generalized GS mechanism is strategyproof for doctors, i.e., no doctor has any incentive to misreport her preference, no matter what the other doctors report. Also, it always produces a stable matching, and the obtained matching is doctor-optimal among all stable matchings, i.e., all doctors weakly prefer the obtained matching to any other stable matching.*

4 Sufficient Conditions for M$^\natural$-concavity

In this section, we present several sufficient conditions under which function f becomes M$^\natural$-concave. Without loss of generality, we can assume f is represented by the summation of two parts, i.e., $f(X') = \tilde{f}(X') + \hat{f}(X')$, where $\tilde{f}$ represents hard distributional constraints for hospital-feasibility and $\hat{f}$ represents soft preference over hospital-feasible contracts. More specifically, $\tilde{f}(X')$ returns 0 when X' is hospital-feasible, and otherwise, $-\infty$. $\hat{f}(X')$ returns a bounded non-negative value. Let us first introduce a structure called *matroid*. [11].

Definition 6 (matroid). *Let X be a finite set, and F be a family of subsets of X. A pair (X, F) is a matroid iff it satisfies the following conditions.*

1. *$\emptyset \in F$.*
2. *If $X' \in F$ and $X'' \subset X'$, then $X'' \in F$ holds.*
3. *If $X', X'' \in F$ and $|X'| > |X''|$, then there exists $x \in X' \setminus X''$ such that $X'' \cup \{x\} \in F$.*

Let us denote a family of hospital-feasible contracts as $\mathrm{dom}\, f = \mathrm{dom}\, \tilde{f} = \{X' \mid X' \subseteq X, \tilde{f}(X') \neq -\infty\}$. The following theorem holds.

Theorem 3. *If f is M$^\natural$-concave and $f(\emptyset) = 0$, then $(X, \mathrm{dom}\, \tilde{f})$ is a matroid.*

Theorem 3 means that, in order to utilize the theory of M$^\natural$-concavity, it is necessary for the set of hospital-feasible contracts to form a matroid.

We utilize the following properties [11].

Property 1 (simultaneous exchange property). Let (X, F) be a matroid. $\forall Y, Z \in F, \forall y \in Y \setminus Z$, there exists $z \in Z \setminus Y \cup \{\emptyset\}$ such that $Y - y + z \in F$ and $Z - z + y \in F$ hold.

Property 2 (summation with weights). If $\tilde{f}(X')$ is M$^\natural$-concave, then $f(X') = \tilde{f}(X') + \sum_{x \in X'} w(x)$ is also M$^\natural$-concave, where $w(x)$ is a weight associated to x.

Property 3 (laminar concave function). Assume $\mathcal{T}$ is a *laminar* family of subsets of X, i.e., for any $Y, Z \in \mathcal{T}$, either one of the following conditions holds: (i) $Y \cap Z = \emptyset$, (ii) $Y \subset Z$, or (iii) $Z \subset Y$. Also assume for each $Y \in \mathcal{T}$, a univariate concave function f_Y is associated. Then, $f(X') = \sum_{Y \in \mathcal{T}} f_Y(|X' \cap Y|)$ is M$^\natural$-concave. Such an f is called a laminar concave function on $\mathcal{T}$.

First, let us consider one simple but very general method for defining $\hat{f}$.

Definition 7 (preference based on total order on X). *Assume there exists a total preference ordering $\succ_H$ over X, i.e., $x_1 \succ_H x_2 \succ_H x_3 \succ_H \ldots$. Furthermore, we assume for each x, a positive weight $w(x)$ is defined so that $w(x) > w(x')$ holds when $x \succ_H x'$. Then, we assume $\hat{f}(X')$ is given as $\sum_{x \in X'} w(x)$.*

Theorem 4. *If (X, F), where $F = \mathrm{dom}\, \tilde{f}$, is a matroid and $\hat{f}(X')$ is given as $\sum_{x \in X'} w(x)$, then f is M$^\natural$-concave.*

Proof. From Property 1, it is clear that $\tilde{f}$ is M$^\natural$-concave. Then, from Property 2, $f(X') = \tilde{f}(X') + \sum_{x \in X'} w(x)$ is M$^\natural$-concave.

It must be noted that although we assume a weight value $w(x)$ is given for each contract and f is defined by the sum of these weight values, $Ch_H(X')$ is determined only by the relative ordering of these weight values. Thus, the specific cardinal choice of these weight values is not important.

Let us introduce a useful matroid and an M$^\natural$-concave function based on it.

Theorem 5. *Assume $\mathcal{T}$ is a laminar family of subsets of X, and positive integers q_T for $T \in \mathcal{T}$ are given. Then, (X, F), where $F = \{X' \subseteq X \mid |X' \cap T| \leq q_T (\forall T \in \mathcal{T})\}$, is a matroid. Furthermore, assume $\tilde{f}$ is defined so that $\mathrm{dom}\, \tilde{f} = F$, and $\hat{f}$ is a laminar concave function on $\mathcal{T}$, then f is M$^\natural$-concave.*

Proof. The proof is straightforward from Property 3.

Let us introduce another type of useful M$^\natural$-concave functions/matroids. We first introduce a concept called a group of contracts.

Definition 8 (group of contracts). *Let $G = \{g_1, \ldots, g_n\}$ be a partition of X, i.e., $g \cap g' = \emptyset$ for any $g \neq g'$ and $\bigcup_{g \in G} g = X$. We call each element g as a group in G.*

A natural way of dividing contracts into groups is based on hospitals, i.e., each g_i represents the set of contracts related to hospital h_i.

For contracts within each group g, we assume a priority ordering $\succ_g$ is defined. Furthermore, we assume a positive weight $w(x)$ is defined for each x, so that $w(x) > w(x')$ when both x and x' belong to g and $x \succ_g x'$.

Furthermore, we assume a finite sequence of groups is defined, in which a group can appear repeatedly. This sequence determines a preference over the numbers of accepted contracts of each group. Such a preference is called an *order-respecting* preference, which can model a wide variety of preferences of hospitals [10]. For example, a sequence $g_1, g_1, g_2, g_3, g_1, g_1, g_2, g_3, \ldots$ means that, g_1 can accept two contracts, then g_2 can take one, and g_3 can take one, and so on, as long as there exists a contract related to each group. If a sequence is defined based on a round-robin ordering, e.g., it is given as $g_1, g_2, \ldots, g_n, g_1, g_2, \ldots, g_n, g_1, g_2, \ldots$, then it means that hospitals prefer that the numbers of accepted contracts of each group become as equal as possible.

Then, let $w_g(i)$ denote the weight associated with the i-th selection/turn of group g. We assume $\forall i, g, w_g(i) > w_g(i+1)$ holds. Also, if the i-th appearance of group g is earlier than the j-th appearance of group g' in the sequence, we assume $w_g(i) > w_{g'}(j)$ holds. Thus, these weights represent the sequence. We call these weights *group weights*. Let us define $W_g(k) := \sum_{i=1}^{k} w_g(i)$.

Definition 9 (order-respecting preference). *For groups G, an order-respecting preference $\hat{f}$ is given as follows:*

$$\hat{f}(X') = \sum_{g \in G} W_g(|X' \cap g|) + \sum_{x \in X'} w(x), \tag{1}$$

where $w_g(i) >> w(x)$ holds for any $g, i,$ and x.

Now, we clarify the condition on $\tilde{f}$, such that $f(X') = \tilde{f}(X') + \hat{f}(X')$ becomes M$^\natural$-concave, assuming $\hat{f}(X')$ is order-respecting.

Definition 10 (symmetry of groups). *We say G is symmetric in (X, F), if for a matroid (X, F), $\forall g \in G, \forall x, x' \in g, \forall X' \subset X$ such that $\{x, x'\} \cap X' = \emptyset$, $X' + x \in F$ iff $X' + x' \in F$ holds.*

The following theorem holds.

Theorem 6. *$f(X') = \tilde{f}(X') + \hat{f}(X')$ is M$^\natural$-concave if (X, F), where $F = \mathrm{dom}\,\tilde{f}$, is a matroid, G is symmetric in (X, F), and $\hat{f}$ is order-respecting.*

Let us introduce a few matroids and a method to create a new matroid from given matroids [13], which are used in our case studies.

Definition 11 (uniform matroid). *(X, F) is a uniform matroid if $F = \{X' | X' \subseteq X, |X'| \leq k\}$ for some non-negative integer k.*

Definition 12 (gammoid). *A gammoid (X, F) is defined based on a directed graph (V, E, S, T). Here, V is a set of vertexes, and E is a set of directed edges. We assume $S \subset V$, $T \subset V$, and $S \cap T = \emptyset$. Here, S is a set of start vertexes and T is a set of terminal vertexes. We assume $S = X$, and $X' \subseteq X$ is an element of F, iff there exist vertex-disjoint paths for each $x \in X'$ to some element in T.*

Definition 13 (union). *Assume $(X, F_1), \ldots, (X, F_k)$ are matroids. Then (X, F), where $F = \{X' | X' = \bigcup_{1 \leq i \leq k} X'_i,$ where $X'_i \in F_i\}$ is also a matroid.*

5 Case Studies

In this section, we examine existing works on constrained matching and show that the sufficient conditions described in Section 4 hold in these cases.

5.1 Standard Model

In the standard model of two-sided matching [6], a contract $x \in X$ is a pair (d, h), which represents a matching between doctor d and hospital h. Each hospital h has its maximum quota q_h, i.e., X' is hospital-feasible iff $|X'_h| \leq q_h$ for all h, where $X'_h = \{(d, h) \in X' \mid d \in D\}$. (X, F), where $F = \mathrm{dom}\, \tilde{f}$, is a matroid, since it is a union of uniform matroids.

Each hospital h has a priority ordering $\succ_h$ among contracts X_h. Let us assume a positive weight $w(x)$ is defined for each $x = (d, h)$, so that $w((d, h)) > w((d', h))$ when $(d, h) \succ_h (d', h)$ holds. Assuming $\hat{f}(X') = \sum_{x \in X'} w(x)$, f is M$^\natural$-concave by Theorem 4. It must be noted that although we assume a weight value $w(x)$ is given for each contract and f is defined by the sum of these weight values, $Ch_H(X')$ is determined only by the relative ordering of these weight values among the contracts that belongs to the same hospital. Thus, the specific cardinal choice of these weight values, or the relative ordering among contracts for different hospitals, is not important.

In the standard model, a matching is stable iff there exists no blocking pair [6]. Here, we assume each hospital has its own choice function Ch_h, and $Ch_H(X')$ is given as $\bigcup_{h \in H} Ch_h(X')$. X' is stable if there exists no pair of d and h, where $x \in X \setminus X'$ is a contract related to d and h, $x \in Ch_h(X' \cup \{x\})$ and $x \in Ch_d(X' \cup \{x\})$ hold. It is easy to see that our stability is equivalent to standard stability, and the standard deferred acceptance mechanism is identical to the generalized GS-mechanism where Ch_H is defined as the maximization of this f.

However, if some additional distributional constraints are imposed, Ch_H cannot be defined in this way. Thus, our stability can be different from standard stability. It is common that no matching satisfies standard stability when additional distributional constraints are imposed.

5.2 Regional Maximum Quotas [8, 10]

The model used in [8, 10] is almost identical to the standard model. The only difference is that hospitals are grouped into regions $R = \{r_1, \ldots, r_l\}$, where each region r is a subset of hospitals, and has its regional maximum quota q_r. We assume R is a laminar family of H, i.e., these regions have a hierarchical structure. Let X'_r denote $\bigcup_{h \in r} X'_h$. X' is hospital-feasible iff $|X'_h| \leq q_h$ for all $h \in H$, and $|X'_r| \leq q_r$ for all $r \in R$. From Theorem 5, (X, F), where $F = \mathrm{dom}\, \tilde{f}$, is a matroid, since $\mathcal{T} = \{X_{r_1}, X_{r_2}, \ldots, X_{h_1}, X_{h_2}, \ldots\}$ is a laminar family of X.

Goto et al. [8] assume there exists a total preference ordering $\succ_H$ over X, i.e., $x_1 \succ_H x_2 \succ_H x_3 \succ_H \ldots$. We assume a positive weight $w(x)$ for each x is defined so that $w(x) > w(x')$ when $x \succ_H x'$. Let us assume $\hat{f}(X')$ is given as $\sum_{x \in X} w(x)$. Then, f is M$^\natural$-concave by Theorem 4. In Goto et al. [8], a mechanism called Priority-List based Deferred Acceptance mechanism (PLDA) is presented. PLDA is identical to the generalized GS mechanism where Ch_H is defined as the maximization of this f.

In Kamada and Kojima [10], a round-robin ordering among hospitals is defined, e.g., $h_1 \rightarrow h_2 \rightarrow \ldots$, and each hospital can sequentially accept one contract according to this round-robin ordering.[5] Let $w_{h_i}(j)$ denote the weight associated with the j-th choice of hospital h_i. Then, we can define $w_{h_i}(j)$ as $C - |H| \cdot j - i$, where C is a large positive constant. $W_h(k)$ denotes $\sum_{j=1}^{k} w_h(j)$. Then, $\hat{f}(X')$ can be defined as $\sum_{h \in H} W_h(|X_h|) + \sum_{x \in X'} w(x)$, where $C >> w(x)$ for all $x \in X$. Assuming $G = \{X_{h_1}, X_{h_2}, \ldots\}$, it is clear that G is symmetric in (X, F). Thus, from Theorem 6, $f(X') = \tilde{f}(X') + \hat{f}(X')$ is M$^\natural$-concave. In Kamada and Kojima [10], a mechanism called Flexible Deferred Acceptance mechanism (FDA) is presented. FDA is identical to the generalized GS mechanism where Ch_H is defined as the maximization of this f.

5.3 Regional Minimum Quotas [7]

The model used in [7] is almost identical to the regional maximum quotas model described in Section 5.2, but each individual hospital h has its minimum quota p_h. Furthermore, each region r has its minimum quota p_r rather than its maximum quota. The model presented in [3] is a special case of this model in which no regional minimum quota is imposed.

X' is (original) hospital-feasible iff $p_h \leq |X'_h| \leq q_h$ for all h, and $p_r \leq |X'_r|$ for all r. Here, we assume each doctor can accept any hospital, and each hospital can accept any doctor, i.e., $\forall d \in D, \forall h \in H, (d, h) \in X$. For each region r, let q_r denote $\sum_{h \in r} q_h$. Without loss of generality, we assume there exists a root region H, whose minimum quota is set as $p_H = n$ (and $q_H = \sum_{h \in H} q_h$). From these assumptions, regions R and individual hospitals form a tree, in which H is the root node (as shown in Figure 1 (a)). We assume for each region r, $\sum_{h \in r} p_h \leq p_r \leq q_r$ holds. From these assumptions, we can guarantee that a feasible matching always exists.

If we use the original definition of hospital-feasibility, the family of hospital-feasible contracts cannot be a matroid, since $\emptyset$ is not hospital-feasible. Here, we relax the original hospital-feasibility as follows. X' is hospital-feasible iff there exists $X'' \supseteq X'$ such that X'' is original hospital-feasible.

We create a gammoid that represents these regional constraints as follows.

- We set the start vertexes S as X.
- For each hospital h, we create p_h terminal vertexes and $q_h - p_h$ intermediate vertexes. There exist links from each (d, h) to these vertexes.
- For each region r, we create $p_r - \sum_{h \in r} p_h$ terminal vertexes. There exist links from intermediate vertexes created for $h \in r$ to these vertexes.
- We assume $\tilde{f}(X') = 0$ iff there exist vertex-disjoint paths from each $x \in X'$ to some terminal vertex, and otherwise, $-\infty$.

Assume there are four hospitals $h_1, \ldots, h_4$. Their maximum and minimum quotas are 3 and 1, respectively. They are divided into two regions r_1, r_2. Their minimum quotas are 3 (Figure 1 (a)). Thus, we require at least one doctor is assigned to

[5] To be more precise, Kamada and Kojima [10] assume *target quotas* for each hospital can be introduced. Here, we consider a case where these target quotas are the same for all hospitals that belongs to the same region.

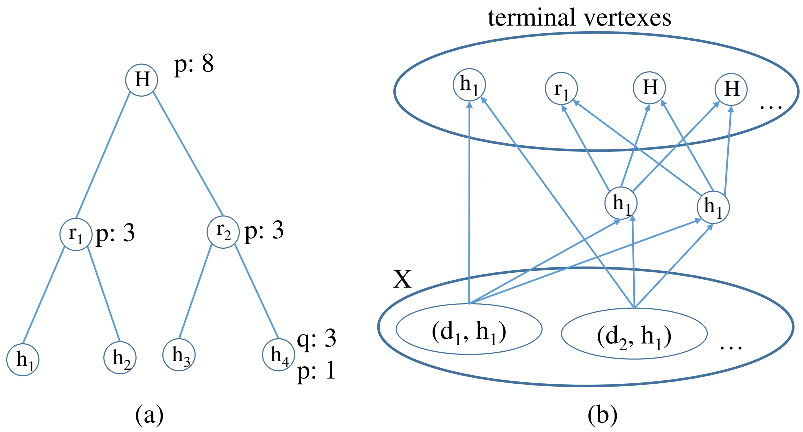

Fig. 1. Example of Gammoid

both h_1 and h_2, and additional one doctor is assigned to either h_1 or h_2. Then, for h_1, we create one terminal vertex, and $3-1=2$ intermediate vertexes. There exist links from each contract related to h_1 to these vertexes. Also, for r_1, we create $3-(1+1)=1$ terminal vertex. There exist links from intermediate vertexes for h_1 (as well as h_2) to these vertexes. For group H, we create $8-(3+3)=2$ terminal vertexes. There exist links from intermediate vertexes for h_1 (as well as h_2, h_3, and h_4) to these vertexes (Figure 1 (b)). The following theorem holds.

Theorem 7. *X' is hospital-feasible, i.e., there exists $X'' \supseteq X'$ such that X'' is original hospital-feasible, iff $\tilde{f}(X')=0$.*

$\hat{f}(X')$ is defined in a similar way as in Section 5.2. Then, f becomes M$^\natural$-concave from Theorem 6.

Goto et al. [7] present a mechanism based on the deferred acceptance mechanism called Round-robin Selection Deferred Acceptance mechanism for Regional Minimum quotas (RSDA-RQ). RSDA-RQ is identical to the generalized GS mechanism where Ch_H is defined as the maximization of f described above. Fragiadakis et al. [3] present a mechanism based on the deferred acceptance mechanism called Extended Seat Deferred Acceptance mechanism (ESDA). ESDA is a special case of RSDA-RQ, in which no regional minimum quota is imposed.

5.4 Controlled School Choice [2]

We assume each doctor d has its type $\tau(d) \in T = \{t_1, \ldots, t_k\}$. A type of doctor may represent race, income, gender, or any socioeconomic status. Each hospital h has a priority ordering $\succ_h$ among contracts X_h. Furthermore, each hospital sets minimum and maximum quotas for each type t, which are denoted as $\underline{q}_{h,t}$ and $\overline{q}_{h,t}$, respectively. These quotas are *soft-bounds* and does not affect feasibility.

Let us assume a contract is represented as (d, h, t, s), where $t \in T$ and $s \in \{0, 1, 2\}$. $s = 0, 1, 2$ mean that doctor d is accepted for hospital h for type t's priority seat, normal seat, and extended seat, respectively. Let $X'_{h,t,s}$ denote $\{(d, h, t, s) \in X' \mid d \in D\}$. Let us assume for each x, its weight $w(x)$ is defined. We assume $w((d, h, t, 0)) > w((d', h', t', 1))$ and $w((d, h, t, 1)) > w((d', h', t', 2))$ hold for any d, d', t, and t', i.e., hospitals first try to fill their priority seats, then normal seats, and finally extended seats. Also, we assume $w((d, h, t, s)) > w((d', h, t, s))$ if $d \succ_h d'$, i.e., the preference of an individual hospital over doctors is respected, as long as doctors have the same type.

Let us define $\tilde{f}(X')$ as 0 when $|X'_h| \leq q_h$, $|X'_{h,t,0}| \leq \underline{q}_{h,t}$, and $|X'_{h,t,1}| \leq \overline{q}_{h,t} - \underline{q}_{h,t}$ hold $\forall h \in H$, $\forall t \in T$, and $-\infty$ otherwise. Also, let us define $\hat{f}(X')$ as $\sum_{x \in X'} w(x)$. From Theorem 5, $\tilde{f}(X')$ forms a matroid, since $\mathcal{T} = \{X_{h,t,s} | h \in H, t \in T, s \in \{0, 1, 2\}\} \cup \{X_h | h \in H\}$ is a laminar family of X. Thus, f is M$^\natural$-concave from Theorem 4.

To run the generalized GS mechanism, we modify the preference of each doctor d so that $(d, h, t, s) \succ_d (d, h', t, s')$ holds for any $h \neq h'$, s, and s' if $h \succ_d h'$, and $(d, h, t, 0) \succ_d (d, h, t, 1) \succ_d (d, h, t, 2)$ holds for any h. Ehlers et al. [2] present a mechanism based on the deferred acceptance mechanism called Deferred Acceptance Algorithm with Soft Bounds (DAASB). DAASB is identical to the generalized GS mechanism where Ch_H is defined as the maximization of f. Ehlers et al. [2] also introduce a new stability concept, in which doctor d and hospital h can form a blocking pair, where d is currently assigned to hospital h', $h \succ_d h'$, and $\tau(d) = t$, if either one of the following conditions holds:

1. $|X'_h| < q_h$ or $|X'_{h,t,0}| < \underline{q}_{h,t}$, or
2. another doctor d', where $\tau(d') = t'$ is assigned to h, and either
   (a) $t = t'$ and $d \succ_h d'$,
   (b) $t \neq t'$, $\underline{q}_{h,t} \leq |X'_{h,t}| < \overline{q}_{h,t}$, $\underline{q}_{h,t'} < |X'_{h,t'}| \leq \overline{q}_{h,t'}$, and $d \succ_h d'$,
   (c) $t \neq t'$. $\underline{q}_{h,t} \leq |X'_{h,t}| < \overline{q}_{h,t}$, and $|X'_{h,t'}| > \overline{q}_{h,t'}$, or
   (d) $t \neq t'$, $|X'_{h,t}| \geq \overline{q}_{h,t}$, $|X'_{h,t'}| > \overline{q}_{h,t'}$, and $d \succ_h d'$ holds.

This stability concept is identical to our stability.

6 Conclusion

We proved that in two-sided, many-to-one matching problem, in which some distributional constraints are imposed on feasible matchings, several desirable properties hold when the preference of hospitals is represented as an M$^\natural$-concave function. Furthermore, we derived sufficient conditions under which the preference becomes an M$^\natural$-concave function. These conditions provide a recipe for non-experts of matching theory and discrete convex analysis to develop desirable mechanisms that handle many-to-one matching problems with distributional constraints.

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A Proof of Theorem 1

In Definition 5, if $X' = X''$, i.e., if no contract is rejected by hospitals, the procedure terminates immediately. Thus, at least one contract must be rejected in each execution of line 2. Thus, the generalized GS mechanism executes line 2 at most $|X'|$ times. The calculation of Ch_D is $O(|X'|)$. By Lemma 1, $Ch_H(X')$ can be calculated in $O(|X'|^2)$. Thus, the time complexity of the generalized GS mechanism is $O(|X|^3)$.

Lemma 1. *$Ch_H(X')$ can be calculated in $O(|X'|^2)$.*

Proof. The following greedy algorithm does this computation.

1. $S \leftarrow \emptyset$, $U \leftarrow X'$.
2. Repeat the following procedure.
   Choose $x \in U$ so that $f(S+x)$ is maximized and $f(S+x) > f(S)$, and set S to $S+x$ and U to $U \setminus \{x\}$. If no such x exists, return S.

We show the correctness of the algorithm. Assume that $S = \{s_1, s_2, \ldots, s_k\}$ denotes the output of the algorithm and its elements are added in ascending order of index. We will use M-optimality theorem and M-minimizer cut theorem (see [11] for details). In our context, M-minimizer cut theorem guarantees that if $x^* \in X'$ maximizes f among $x \in X'$ and $f(\emptyset) < f(x^*)$ then the maximizer $X^* \in \arg\max_{X'' \subseteq X'} f(X'')$ must contain x^*. Thus, s_1 must be contained in X^*. By iteratively using M-minimizer cut theorem for $\{s_1, \ldots, s_i\}$ with $i < k$, we can show that $\{s_1, \ldots, s_i, s_{i+1}\} \subseteq X^*$, and finally, $S \subseteq X^*$. On the other hand, $f(S) \geq f(S+x)$ for all $x \in X' \setminus S$ holds. These facts together with M-optimality theorem which says

$$f(\tilde{X}) \geq f(X'') \quad (\forall X'' \subseteq X') \Longleftrightarrow \begin{cases} f(\tilde{X}) \geq f(\tilde{X} + x - y) & (\forall x, y \in X') \\ f(\tilde{X}) \geq f(\tilde{X} \pm x) & (\forall x, y \in X'), \end{cases}$$

imply $S = X^*$.

B Proof of Theorem 2

By Lemma 2, assuming f is M$^\natural$-concave and unique-selecting, Ch_H satisfies the irrelevance of rejected contracts, the substitute condition, and the law of aggregate demand. Hatfield and Milgrom [9] show that when Ch_H satisfies these three conditions, the generalized GS mechanism is strategyproof for doctors, and it obtains the doctor-optimal matching among all stable matchings.

Lemma 2. *$Ch_H(X')$ satisfies the following three properties.*

Irrelevance of rejected contracts: for any $X' \subseteq X$ and any $x \in X \setminus X'$, $Ch_H(X') = Ch_H(X' \cup \{x\})$ whenever $x \notin Ch_H(X' \cup \{x\})$.

Substitutes condition: for any $X', X'' \subseteq X$ with $X' \subseteq X''$, $Re_H(X') \subseteq Re_H(X'')$ holds, where $Re_H(X') = (X' \setminus Ch_H(X'))$.

Law of aggregate demand: for any $X', X'' \subseteq X$ with $X' \subseteq X''$, $|Ch_H(X')| \leq |Ch_H(X'')|$.

Proof. Since $Ch_H(X')$ is defined as $\arg\max_{X'' \subseteq X} f(X'')$ and f is unique-selecting, it is clear that irrelevance of rejected contracts holds. Also, Fujishige and Tamura [4] show that the substitutes condition holds if f is M$^\natural$-concave and unique-selecting. Furthermore, Murota and Yokoi [12] show that the law of aggregate demand holds if f is M$^\natural$-concave and unique-selecting.

C Proof of Theorem 3

Let $\eta : \boldsymbol{Z}^N \to \boldsymbol{R} \cup \{-\infty\}$ be an M$^\natural$-concave function such that $\mathrm{dom}\,\eta$ is bounded and has 0 as the minimum point. For each $i \in N$, let $c_i = \max\{z(i) \mid z \in \mathrm{dom}\,\eta\}$ for all $i \in N$. Let us consider a finite set X and a partition $G = \{g_1, g_2, \ldots, g_n\}$ of X with $|g_i| \geq c_i$ for all $i \in N$. Let us define $\zeta(X')$ as $(|X' \cap g_1|, \ldots, |X' \cap g_n|)$. Each element $\zeta_i(X')$, where $1 \leq i \leq n$, is a non-negative integer. We define the family F of subsets of X defined by

$$F = \{X' \subseteq X \mid \zeta(X') \in \mathrm{dom}\,\eta\}.$$

Then, the following lemma holds. The definition of the symmetry is given in Definition 10. This lemma is the converse of Lemma 4.

Lemma 3. *(X, F) is a matroid and G is symmetric in (X, F).*

Proof. The fact that G is symmetric in (X, F) is obvious by the definition of F. Since $0 \in \mathrm{dom}\,\eta$, we have $\emptyset \in F$.

Let $X', X'' \in F$ with $|X'| > |X''|$. We denote $\zeta(X')$ and $\zeta(X'')$ by z_1 and z_2, respectively. It follows from $|X'| > |X''|$ that there exists $i \in N$ with $z_1(i) > z_2(i)$. The M$^\natural$-concavity of η guarantees that (a) $z_1 - \chi_i, z_2 + \chi_i \in \mathrm{dom}\,\eta$ or (b) there exists $j \in N$ such that $z_2(j) > z_1(j)$ and $z_1 - \chi_i + \chi_j, z_2 + \chi_i - \chi_j \in \mathrm{dom}\,\eta$, where χ_i is a unit vector such that its i-th element is 1 and other elements are 0.

In the case (a), we have $X'' \cup \{x\} \in F$ for some $x \in g_i \cap (X' \setminus X'')$. In the case (b), there exist $x \in g_i \cap (X' \setminus X'')$ and $y \in g_j \cap (X'' \setminus X')$ with $\hat{X}' := X' - \{x\} + \{y\} \in F$. We note that $|\hat{X}'| = |X'|$ and $X' \cap X''$ is a proper subset of $\hat{X}' \cap X''$. We replace X' by $\hat{X}'$, and continue the above discussion. After a finite number of iterations, the above (a) must occur by $|X'| > |X''|$.

From the above discussion, for $X'' = \emptyset$ and $x \in X'$, we have $X' - \{x\} \in F$. Hence, if $X'' \subset X' \in F$ then $X'' \in F$.

We finally prove Theorem 3. Suppose that $X = \{x_1, x_2, \ldots, x_m\}$. Since f is an M$^\natural$-concave function on X and $f(\emptyset) = 0$, from Lemma 3 for the partition $G = \{\{x_1\}, \{x_2\}, \ldots, \{x_m\}\}$ and X, we have $F = \{X' \mid X' \subseteq X, f(X') \neq -\infty\}$ is a matroid.

D Proof of Theorem 6

Since $G = \{g_1, \ldots, g_n\}$ is symmetric in a matroid (X, F), when we check whether $X' \in F$, only the number of members for each group matters. Then, we can assume $\tilde{f}(X')$ is equal to $\tilde{\eta}(\zeta(X'))$, where $\tilde{\eta}(z)$ is 0 if $\exists X' \in F$ such that $z = \zeta(X')$, and otherwise $-\infty$. Here, $\zeta(X')$ is defined as $(|X' \cap g_1|, \ldots, |X' \cap g_n|)$.

It is enough to show that the function $\acute{f}$ defined by $\acute{f}(X') = \acute{\eta}(\zeta(X')) = \tilde{\eta}(\zeta(X')) + \sum_{1 \le i \le n} W_{g_i}(\zeta_i(X'))$ is M$^\natural$-concave, because f is equal to the sum of $\acute{f}$ and a linear function.

We first show that $\acute{\eta}$ is M$^\natural$-concave. $\acute{\eta}$ is M$^\natural$-concave when its effective domain $\mathrm{dom}\,\acute{\eta} = \{z | \acute{\eta}(z) \neq -\infty\}$ is not $\emptyset$ and for all $z, z' \in \mathrm{dom}\,\acute{\eta}$ and $i \in N$ with $z_i > z'_i$, either (a) $\acute{\eta}(z) + \acute{\eta}(z') \le \acute{\eta}(z - \chi_i) + \acute{\eta}(z' + \chi_i)$ or (b) there exists $j \in N$ such that $z'_j > z_j$ and $\acute{\eta}(z) + \acute{\eta}(z') \le \acute{\eta}(z - \chi_i + \chi_j) + \acute{\eta}(z' + \chi_i - \chi_j)$, holds, where χ_i is a unit vector such that its i-th element is 1 and other elements are 0.

Lemma 4. *$\acute{\eta}$ is M$^\natural$-concave, and* $\mathrm{dom}\,\acute{\eta}$ *has* 0 *as the minimum point.*

Proof. Since $\tilde{f}$ gives a matroid, $\mathrm{dom}\,\acute{\eta}$ has 0 as the minimum point. It is known that the sum of an M$^\natural$-concave function and a separable concave function is also M$^\natural$-concave. Since $\acute{\eta}$ is the sum of $\tilde{\eta}$ and the separable concave function $\sum_{1 \le i \le n} W_{g_i}(\zeta_i(X'))$, it is enough to show that $\tilde{\eta}$ is M$^\natural$-concave. Furthermore, since the value of $\tilde{\eta}$ in its effective domain is always 0, to show the M$^\natural$-concavity of $\tilde{\eta}$, it is sufficient to show that $\mathrm{dom}\,\tilde{\eta}$ is M$^\natural$-convex, i.e., for all $z, z' \in \mathrm{dom}\,\tilde{\eta}$ and $i \in N$ with $z_i > z'_i$, either (a) $z - \chi_i, z' + \chi_i \in \mathrm{dom}\,\tilde{\eta}$ or (b) there exists $j \in N$ such that $z'_j > z_j$ and $z - \chi_i + \chi_j, z' + \chi_i - \chi_j \in \mathrm{dom}\,\tilde{\eta}$, holds. Let X_1, X_2 be elements of matroid (X, F) such that $z = \zeta(X_1)$ and $z' = \zeta(X_2)$. By the symmetry of G in (X, F), we can assume that either $X_1 \cap g_k \subseteq X_2 \cap g_k$ or $X_2 \cap g_k \subseteq X_1 \cap g_k$ for each $k \in N$. By $z_i > z'_i$, there exists $x \in g_i \cap (X_1 \setminus X_2)$. Simultaneous exchange property for X_1, X_2 and x guarantees that (a') $X_1 - x, X_2 + x \in F$ or (b') there exists $y \in (X_2 \setminus X_1)$ with $X_1 - x + y, X_2 + x - y \in F$. In the case (a'), we have $z - \chi_i = \zeta(X_1 - x)$ and $z' + \chi_i = \zeta(X_2 + x)$, that is, (a) holds. In the case (b'), $y \notin g_i$ must hold, and therefore, there exists $j \in N$ with $y \in g_j$. By our assumption, $|X_1 \cap g_j| < |X_2 \cap g_j|$ must be satisfied. Thus, in this case, (b) holds.

Lemma 5. *$\acute{f}$ is M$^\natural$-concave.*

Proof. Let $X', X'' \in \mathrm{dom}\,\acute{f}$ and $x \in X' \setminus X''$. We assume that $x \in g_i$. We denote $\zeta(X')$ and $\zeta(X'')$ by z' and z'', respectively. If $|X' \cap g_i| \le |X'' \cap g_i|$ then there exists $y \in g_i \cap (X'' \setminus X')$. By the symmetry between (X, F) and G, $f(X') = f(X' - x + y)$ and $f(X'') = f(X'' + x - y)$.

In the sequel, we suppose that $|X' \cap g_i| > |X'' \cap g_i|$, i.e., $z'_i > z''_i$. The M$^\natural$-concavity of $\acute{\eta}$ guarantees that:

(i) $\acute{\eta}(z') + \acute{\eta}(z'') \le \acute{\eta}(z' - \chi_i) + \acute{\eta}(z'' + \chi_i)$

or

(ii) there exists $j \in N$ such that $z''_j > z'_j$ and

$$\acute{\eta}(z') + \acute{\eta}(z'') \leq \acute{\eta}(z' - \chi_i + \chi_j) + \acute{\eta}(z'' + \chi_i - \chi_j).$$

In the case (i), we have $\acute{f}(X') + \acute{f}(X'') \leq \acute{f}(X' - x) + \acute{f}(X'' + x)$. In the case (ii), there exist $y \in g_j \cap (X'' \setminus X')$ such that

$$\acute{f}(X') + \acute{f}(X'') \leq \acute{f}(X' - x + y) + \acute{f}(X'' + x - y).$$

Hence $\acute{f}$ is M$^\natural$-concave.

E Proof of Theorem 7

The number of terminal vertexes is $(n - \sum_{r \in R} p_r) + \sum_{r \in R}(p_r - \sum_{h \in r} p_h) + \sum_{h \in H} p_h = n$. Thus, at most n contracts are accepted. Assume X' is hospital-feasible. For each hospital h, the total number of contracts are at most $p_h + (q_h - p_h) = q_h$. Thus, accepted contracts satisfy all individual maximum quotas. Also, if one terminal vertex t is not connected to any contract in X', we can find $x' \in X \setminus X'$ so that there exists a vertex disjoint path from x' to t. By adding new contracts, X' is extended to X'' so that n contracts are accepted. Then, each hospital accepts at least p_h contracts. Also, each region accepts at least p_r contracts (i.e., p_h contracts for each $h \in r$, and $p_r - \sum_{h \in r} p_h$ additional contracts for the terminal vertexes for region r). Thus, X'' satisfies all minimum quotas and X' is a subset of (original) hospital-feasible matching.