Efficiency and Stability in a Process of Teams Formation

Boncinelli, Leonardo and Pin, Paolo

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Leonardo Boncinelli*   Paolo Pin†

Abstract

We analyze a *team formation process* that generalizes matching models and network formation models, allowing for overlapping teams of heterogeneous size. We apply different notions of stability: *myopic team-wise stability*, which extends to our setup the concept of pair-wise stability, *coalitional stability*, where agents are perfectly rational and able to coordinate, and *stochastic stability*, where agents are myopic and errors occur with vanishing probability. We find that, in many cases, coalitional stability in no way refines myopic team-wise stability, while stochastically stable states are feasible states that maximize the overall number of activities performed by teams.

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*Dipartimento di Economia e Management, Università degli Studi di Pisa, Italy.
Email: l.boncinelli@ec.unipi.it.

†Dipartimento di Economia Politica e Statistica, Università degli Studi di Siena, Italy.
Email: paolo.pin@unisi.it.

1 Introduction

1.1 Motivation

The *knapsack problem* is a classical problem in operation research (see Cabot, 1970): suppose that a rational agent has a finite set of objects – each with a given weight and a given value for her – and a rucksack with limited capacity (in weight), and she has to choose a selection of objects in discrete quantities to put in the sack. In its unbounded version (see Andonov et al., 2000), there are no restrictions on the number of copies of each kind of item that can be put in the sack. The above decision problem can also be interpreted as a choice over alternative time allocations: now the agent has a limited amount of time and several activities that she can do in discrete amounts, and each activity takes some time to be performed and gives a utility reward at its completion. The ensuing maximization problem involves a single agent who has to compare a finite number of possible choices; nevertheless, it is NP-complete, which roughly means that the number of operations required to find the optimal solution grows exponentially in the number of possible objects (or activities). The issue of time allocation by a single agent has recently been studied in Coviello et al. (2013). From their perspective, an agent has to carry on many different activities over time, and the resulting phenomenon of task juggling is investigated both theoretically and empirically.

However, in the real world, activities are often performed by people in *teams*, so that the constraints each agent has to take into consideration in her decision depend on the choices, and hence the constraints, of others. For instance, if Alice wants to allocate a couple of hours on Saturday morning to playing tennis, but all her friends have already fully allocated time on Saturday morning to other activities, then Alice's desire to play tennis will remain unsatisfied. This simple example shows the existence of indirect externalities that must be taken into account in every individual decision when activities are performed in teams, bringing an additional layer of difficulty in the knapsack problem presented above.

The following example is helpful to illustrate the generality of the setup that we want to deal with. Consider the question of co-authorships in academic research. People work simultaneously on different projects, and in order to do so they form teams of varying numerosity, or remain separate from one another. Each researcher can participate in more than one group, and every group can work on more than one project. Each project has a cost in terms of time and an expected reward in terms of career opportunities, and both time and reward are allowed to be heterogeneous across each member in the team. Moreover, every team that is formed may generate negative (or positive) direct externalities for non-members, due for instance to the reduction of effort that a researcher puts into each single project when she undertakes a new project.

Other applications where people are organized in teams to increase their production are given by a growing number of internet platforms that in recent times have started matching firms that post open problems with solvers (usually computer programmers) that apply to solve them for a monetary reward. This is the case of platforms like GitHub (https://github.com/), where programmers, who are connected in a Facebook-like social network, can self-organize in teams of any size and register to find a solution (this is the environment empirically analyzed by Majumder et al., 2012, and modeled theoretically, but under a different perspective from ours, by Fuchs and Garicano, 2012).

This team formation process can also be a representation for the production of new patents, or in the entrepreneurial activity related to the co-foundation of firms (two aspects surveyed, respectively, by Breschi and Lissoni, 2001 and by Stuart and Sorenson, 2008).

All the cases above are such that a social planner (or a research agency that wants to promote scientific research in the example of co-authorships) is interested in maximizing the number of successful teams. Consider an example of the latter kind of research agency; such an agency may well assume as primary goal the maximization of the total number of research papers that are published. Other information might well be relevant in principle, like journal rankings, but this is often difficult to take into account (e.g., see the list of reviews in the new Italian National Scientific Qualification, art. 16 of the law 30 December 2010, n. 240; it contains a single and very large class of reviews that are regarded as qualified).

Finally, there are also Internet platforms not devoted to professional activities, but simply to leisure, which allow consumers to match in groups in order to play together in sports games (as http://www.fubles.com/ for Futsal[1]) or to engage in consumption activities that need the coordination of a bunch of users, such as guided tourist tours (one of the services provided, e.g., by http://www.groupon.com). These examples have no clear objective functions from the point of view of a social planner, but in both cases there are platform organizers who would surely like to maximize the number of successful teams or users matched together. Therefore, maximization of the overall number of realized projects is in many setups, and for

[1]Futsal is indoor soccer played 5 vs. 5: http://en.wikipedia.org/wiki/Futsal.

different reasons, a focal objective of either the social planner or some other administrator.

1.2 Outline

In our model we try to encompass all the possible applications discussed above, generalizing matching models and network formation models. We imagine a finite set of agents, with heterogeneous constraints on time, who have the possibility of choosing between a finite set of *teams*. Not all teams are allowed to be formed. The description of which teams are allowed to be formed is called *technology*. Every team can be feasible or unfeasible, depending on many exogenous constraints that are summarized in the given technology. For instance, agents may form a team only if they are neighbors in an exogenously given social network, or if they match complementary exogenous skills, or if they have common communication tools.

Teams of different size are allowed, unless explicitly assumed otherwise (as in marriage, where only teams of size two can be formed). A configuration of feasible teams is itself said to be feasible if time constraints are satisfied for each agent. A feasible configuration is called a *state*, and each state provides a specific payoff to each agent. As we discuss in Section 2.4, this setup generalizes matching models and network formation models.

What we find is that, under the assumption of non-satiation with respect to teams for every agent, an extension of the simple notion of *pair-wise stability* (Jackson and Wolinsky, 1996) to this setup – which we call *myopic team-wise stability* – does not have a strong predictive power on the stable states of the model; in particular, feasible states that are maximal with respect to set inclusion are myopically team-wise stable. Therefore, we compare two possible refinements. The first is given by *stochastic stability*: in the presence of extremely rare errors that can create or dissolve teams, and with agents that adapt myopically to the current state of the system, we find that the stochastically stable states are those that actually maximize the overall number of teams. The second is a generalization of *coalitional stability* for cooperative games (Gillies, 1959): its predictive capability will turn out to be heavily dependent on the assumptions on payoff functions. Moreover, when all projects are equivalent for every agent in terms of costs (resources employed) and benefits (payoffs earned), we find that the states that satisfy this form of coalitional stability are exactly the same as those that are myopically team-wise stable. Therefore, this refinement – which is much more demanding in terms of agents' rationality – proves to have no greater predictive power with respect to myopic team-wise stability in a stark but significant case.

1.3 Relation to the theoretical literature

As far as matching models are concerned, the most recent papers that study multi-matching environments are Pycia (2012) and Hatfield et al. (2012), but their contributions are more focused and give more specific results. In the matching perspective, we provide a very general characterization of stochastically stable matchings under the assumptions of the marriage theorem (see Bose and Manvel, 1984 for a modern exposition, although this mathematical result is already in Hall, 1935).

With respect to network formation models, we generalize pair-wise stability (see Jackson and Wolinsky, 1996) and strong stability (Jackson and Van den Nouweland, 2005) to a more general setting of resource-constrained team formation. The constraint imposed on our agents by a fixed time resource has recently been modeled in network formation models, e.g., by Staudigl and Weidenholzer (2012). On the other hand, the constraints imposed by the technology can be related to many aspects recently introduced in the network formation literature: constraints may be due to homophily (see Currarini et al., 2009), because only similar agents may be able to form a team together; or they may be related to an exogenous network of opportunities, because only linked agents have the opportunity to match (on this, we are aware only of Franz et al., 2010); or they may be imposed by complementary exogenous skills that need to be matched together (see, e.g., Currarini and Vega-Redondo, 2010).

Stochastic stability (for which the references are discussed in Section 4) has been applied to networks (first by Jackson and Watts, 2002). More recently Klaus et al. (2010) used stochastic stability as a predicitive tool for roommate markets. In Boncinelli and Pin (2012), best shot games played in exogenous networks are analyzed, and stochastically stable states are proven to be the states with the maximum number of contributing agents if the error structure is such that contributing agents are much more likely to be hit by perturbations.

Our concept of coalitional stability stems from concepts of cooperative game theory, and particularly from the literature on coalition formation (see, e.g., Konishi and Ray, 2003, Gomes and Jehiel, 2005 and Hyndman and Ray, 2007) and clubs (see, e.g., Pauly, 1970 and Faias and Luque, 2012). We provide further references and undertake an in-depth discussion on this in Section 5.

1.4 Plan of the paper

The paper is structured as follows. In Section 2 we present all the aspects of the model, without any definition of stability. In Section 3 we introduce and discuss the weak notion of myopic team-wise stability, which is then refined with the tools of stochastic stability in Section 4, and with a concept of coalitional stability in Section 5. In Section 6 we combine our result on stochastic stability with the marriage theorem to provide a characterization of perfect matchings. Section 7 lists possible extensions of our study, and some additional discussion and results are in the Appendices.

2 Model

2.1 The team formation model

We take into consideration a finite set N of n agents. Each agent i has an endowment $w_i \in \mathbb{N}_1$ of a time resource. We denote by $\mathbf{w} \in \mathbb{N}_1^n$ the vector of endowments of all agents.[2] A *team* is a vector $\mathbf{t} \in \mathbb{N}^n$, $\mathbf{t} \leq \mathbf{w}$, with t_i indicating the amount of time employed by agent i in a joint task. We denote by T the set of teams.

Let A be a finite set of activities (or tasks). A project $p = (a, \mathbf{t})$ is an activity $a \in A$ carried out by a team $\mathbf{t} \in T$. We use set $P \subseteq A \times T$ to collect all projects $p = (a, \mathbf{t})$ such that team $\mathbf{t}$ is able to accomplish activity a. We can think of P as representing the *technology*, since it indicates, for every possible task, which combinations of inputs allow the task to be completed.[3] It will simplify the following exposition to introduce, with a slight abuse of notation, the auxiliary function $n(p) = n(a, \mathbf{t}) \equiv |\{i \in N : t_i > 0\}|$, which gives us the set of agents that put some positive amount of time (possibly different among agents) into

[2] We denote by $\mathbb{N}$ the set of non-negative integers, and by $\mathbb{N}_1$ the set of positive integers.

[3] A complementary interpretation of the technology P is based on the agents' preferences. From this point of view, P allows only for those teams in which no member would rather stay alone than participate in the project. In the literature on matching this condition is called *individual rationality* and it is also used in decentralized matching models (Roth and Vate, 1990). We observe, however, that interpreting P as individual rationality asks for a different model when combining mistakes with exit costs (see Appendix A): in such a case, a project that is formed by mistake persists over time due to the costs for exiting, even if some agent would prefer to stay alone. Finally, we point out that, when allowed, both interpretations for the technology can co-exist: a project is technologically feasible if such a team is able to perform the activity and, at the same time, every agent is willing to do so.

project p. Another notation we will use is $h(p) = h(a, \mathbf{t}) \equiv \sum_{i=1}^{N} t_i$, which indicates the total amount of time (e.g., *hours*) employed on aggregate by the agents in project p.

In the following discussion we will often use *teams* and *projects* as synonyms, but some clarification is necessary. A project $p = (a, t)$ characterizes an activity a performed by a team $\mathbf{t}$, where $\mathbf{t}$ specifies not only the members of the team (who are in the set $n(p)$) but also how much time each of them devote to the project. While every project p can occur only once in a state, because every activity a can be executed only once by the same team, the same team $\mathbf{t}$ can occur in different projects, if this is allowed by the technology P, i.e., if there are at least two projects $(a, \mathbf{t}), (b, \mathbf{t}) \in P$, with $a \neq b$.

We use $\mathbf{e}(x) = \sum_{(a,\mathbf{t}) \in x} \mathbf{t}$ to indicate the vector collecting the overall amount of resources employed in state x, agent by agent. We say that x is *feasible* if $\mathbf{e}(x) \leq \mathbf{w}$, and we denote by X the set of feasible states. We also introduce function $\ell(x) = |x|$ that simply counts the number of projects that are completed in state x.

Finally, we introduce *utilities* that agents earn depending on the state they are in. For every $i \in N$, and for every $x \in X$, we denote by $u_i(x)$ the utility gained by agent i in state x.

Given these elements, it is possible to define a *team formation model* with the quintuple $(N, \mathbf{w}, P, X, \mathbf{u})$. The primitives are the set N of agents involved, their constraints $\mathbf{w}$, the set P of projects allowed by technology, and agents' utilities $\mathbf{u}$. Given N, $\mathbf{w}$ and P, it is possible to derive the partially ordered set X of all feasible states.

2.2 Assumptions

In deriving our results, we employ the following restrictions on the possible structure of teams (first three) and on utilities (second group of three). We explicitly refer to each of these assumption whenever used. We note that some of them are a refinement of one another, while others are incompatible.

Assumption **t1.** *In every $(a, \mathbf{t}) \in P$, we have for every $i \in N$ that either $t_i = 0$ or $t_i = 1$.*

Assumption t1 states that the time allocated to each feasible project by every agent is always 0 or 1, or simply (up to a normalization of time) that the time allocated to each feasible project by its participants is a constant of the model which is homogeneous across projects for every agent.

In contrast, the next two are assumptions that exogenously fix the number of members in each team. We will discuss them in more detail in Section 2.4 where we will see how our model is a generalization of other common theoretical setups.

Assumption **s1.** *There is a $k \in \mathbb{N}_1$, such that for every $p \in P$, we have that $|n(p)| = k$.*

The following Assumption s2 is a refinement of Assumption s1, where k is fixed to be equal to 2.

Assumption **s2.** *For every $p \in P$, we have that $|n(p)| = 2$.*

We now present some assumptions that specify how agents gain utilities by performing activities in teams.

Assumption **v0.** *For every $x, x' \in X$, with $x' \neq x$ and $x' = x \cup \{p\}$, and for every $i \in N$ such that $i \in n(p)$, we have that $u_i(x') > u_i(x)$.*

Assumption v0 is the only one that is needed for our main result. It states that the marginal utility in forming a team, for each of its members, is always positive, independently of all other teams in place. We note that this assumption allows for a large variety of externalities that a project may have on the utility of non-members of that team, or on the fact that the same team could bring different marginal effects to its members, depending on the state.

An additional possible restriction is to impose that the aggregate utility of each project is constant across projects (normalized to 1).

Assumption **v1.** *For each $x \in X$, $\sum_{i \in N} u_i(x) = |x|$.*

Finally, we will consider also a more restrictive assumption that asks for linearity in teams membership, so making the marginal value of each team, for each of its members, independent on states.

Assumption **v2.** *For each state $x \in X$, and any agent $i \in N$, we have that $u_i(x) = v \cdot |\{p \in x : i \in n(p)\}|$, with $v \in \mathbb{R}^+$.*

The last two assumptions convey different ideas on the assignment of utilities: while Assumption v1 imposes that the aggregate marginal value of each team is 1, Assumption v2 says that the payoff earned by each agent i is merely given by the number of projects in which i participates. We note that the two assumptions are compatible only if Assumption s1 holds as well, in which case we have $v = \frac{1}{k}$.

2.3 Maximal states

We observe that X is a partially ordered set under inclusion. This is because, for any two states x and x' belonging to X, we can have that either x is included in x', or x' is included in x, or no set inclusion relationship can be established between them. However, as the empty state x_0 is included in any other state, it is the only *minimal* state (or the *least* state) and, given two states x and x', the set of those states that are included in both is always non-empty. On the other hand, as there is a threshold $\mathbf{w}$ on the overall available resources, there may not always be a common superset for any two states. In general there will be many *maximal* states, i.e., states above which it is not possible to include other teams, because otherwise the threshold would be exceeded.

We denote by $\mathcal{M}$ the set of maximal states, $\mathcal{M} = \{x \in X : x \subseteq x' \text{ and } x \neq x' \Rightarrow x' \notin X\}$. We denote by $\mathcal{L}$ the set of states with maximum number of completed projects, $\mathcal{L} = \{x \in X : |x| \geq |x'|, \text{ for all } x' \in X\}$. We observe that $\mathcal{L} \subseteq \mathcal{M}$. In fact, if $x \in X$ and $x \notin \mathcal{M}$, then there exists a feasible state that can be obtained from x by adding some project, and x cannot maximize the number of projects. In contrast, there exist in general maximal states that do not maximize the number of projects, as the following example shows.

Example 1 (Maximal states and maximum number of projects). Consider the case in which $N = \{i, j, k, m\}$, $\mathbf{w} = (2, 2, 2, 2)$, $A = \{a, b\}$, and $P = \{(a, (1, 1, 0, 0)), (b, (1, 1, 0, 0)), (a, (0, 1, 1, 0)), (b, (0, 1, 1, 0)), (a, (0, 0, 1, 1)), (b, (0, 0, 1, 1))\}$. This is a situation in which there are four agents with two units of time each, there are two activities to be performed, and each activity requires that either $\{i, j\}$, or $\{j, k\}$, or $\{k, m\}$ must be involved, with one unit of time each. We note that Assumptions t1 and s2 hold. Figure 1 illustrates the partial order on set X resulting from the above assumptions: an arrow from a state x to another state y indicates that we can pass from x to y by adding a single project.[4] There are three maximal states, but only one of them maximizes the number of projects. □

2.4 Why this generalization?

The theoretical setup we have introduced encompasses different matching models with non-transferable utility that have been developed in the literature. We acknowledge that some

[4]In order to provide a simplified figure, we have summarized in a single node the states that are the same in any respect apart from the labels of activities.

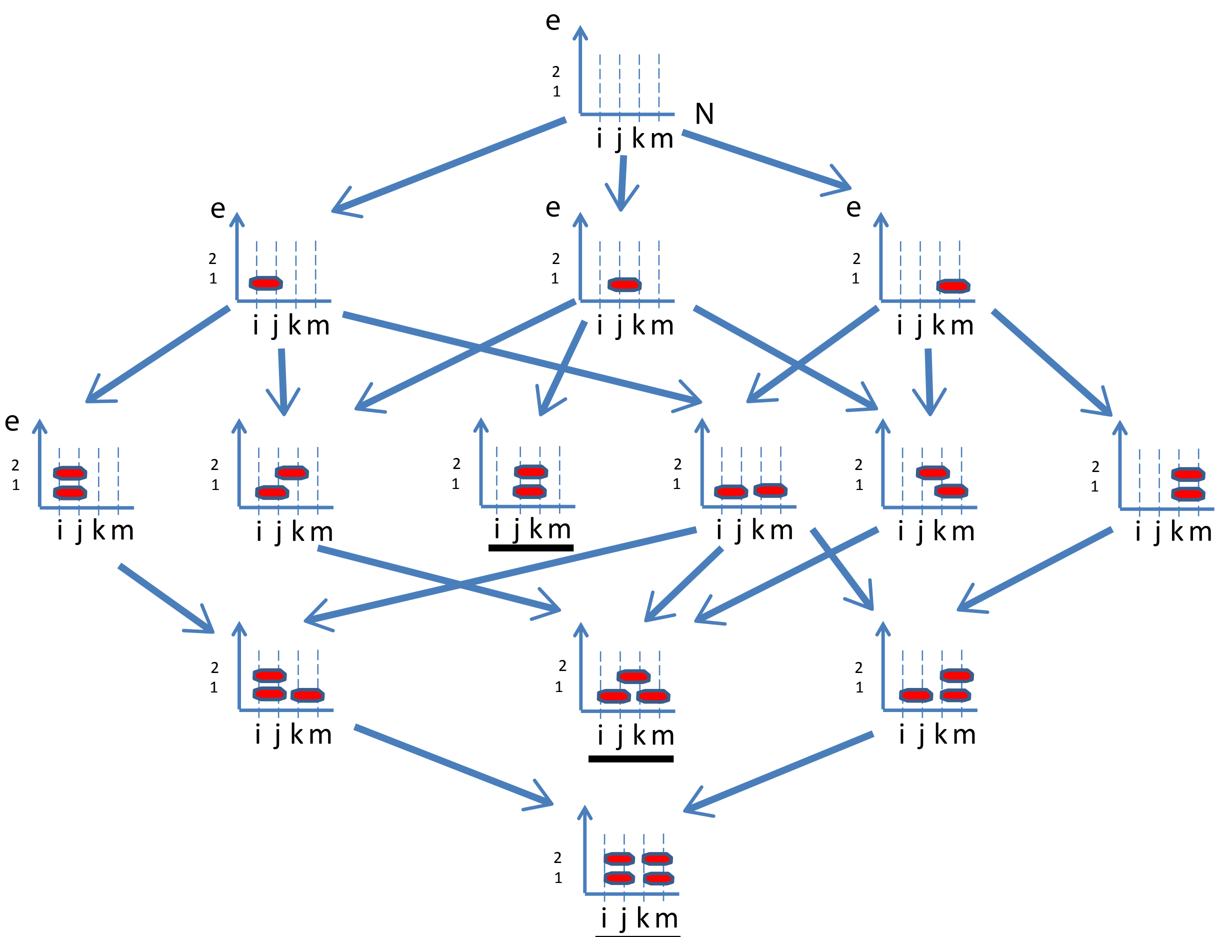

Figure 1: The partially ordered set X of all feasible states for the team formation model of Example 1. In this graphical representation – where we have not considered differences in activities, focussing only on their number – there are three maximal states, which are those underlined in black.

of the existing models might be stretched to deal with most of the cases analyzed within our setup. However, we claim that our model is a natural and simple container for all these models, and we find a value in its capability to adapt easily so as to take into consideration specific cases. In the following we will illustrate this capability.

The knapsack problem (presented in Subsection 1.1) is obtained if we have only one agent who can carry an overall weight of w (the resource endowment) in her knapsack, and she can choose objects (or activities) in set A, where every object $a \in A$ has value v_a and weight w_a. A project is here an object with its weight, i.e., (a, w_a). A state is a set of objects within the knapsack, such that the sum of weights does not exceed w. The agent's utility in state x is the sum of values of the objects in the knapsack.

Cooperative games with non-transferable utility are obtained in our setup if we specify that each agent can belong to one coalition only, and that no externalities are allowed. In order to deal with matching, we simply need to add Assumption s2, so that only teams of size two are allowed to be formed. Marriage – that is bipartite matching – can be obtained through adequate constraints on the technology; after dividing the set of agents between males and females, only heterosexual pairs are allowed in P, and additional constraints can also be considered. Figure 2 provides an example. We will discuss in detail this example and a general application of our setup to marriage in Section 6.

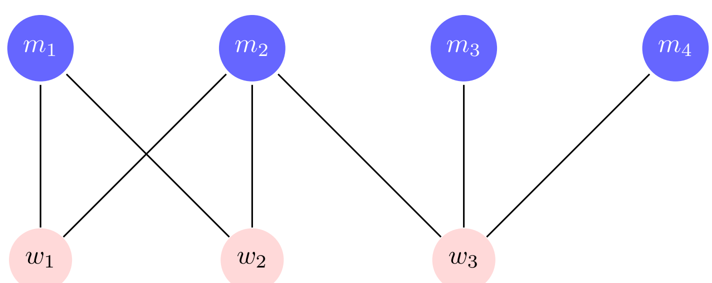

Figure 2: A marriage problem. In this example there are three women ($w1$, $w2$ and $w3$) and four men (from $m1$ to $m4$). Feasible pairs are joined by a link.

If, instead, we relax the upper bound on $\mathbf{w}$, still following Assumption s2, then any state can be considered as a network that satisfies the constraints imposed by $\mathbf{w}$ (concerning the maximum degree of nodes), and the connections made available by technology P (representing the exogenous network of opportunities). This is illustrated in the following example.

Example 2 (Co-authorships). A popular and seminal model in the economic literature on networks is the "Co-author model" of Jackson and Wolinsky (1996): agents are researchers and links are pair-wise collaborations on scientific projects, which are costly but provide payoffs that depend endogenously on the negative externalities given by each collaboration to the other co-authors of an author.

We can include in our setup a payoff function with costs and negative externalities of a project $p = (a, \mathbf{t})$ on the members of other teams that are formed by the members of $\mathbf{t}$, as in the original model. However, our model allows for more generality and also for more realistic time constraints[5] that can be imposed on the available (multi-)matchings. First of all, (i) some agents may work alone, but even three or more agents can set up a team together and produce a paper, as happens in the profession. Then, (ii) with regard to constraints, there could be an exogenous network G of acquaintances, so that a group of co-authors is possible only if they are mutually connected in G. Or, (iii) the researchers could have exogenous complementary skills, and only projects involving agents with enough diversity could be successful. Aspects such as the three listed above, and even others, could all be modeled by some technology P. □

So, what is the added value of our setup with respect to existing ones, in terms of representation of real world phenomena? To provide an answer through an example, let us stick to the co-authorship model of Example 2. Imagine that agents i, j and k set up a project p together, so that $|n(p)| = 3$. This could be represented in the original co-authorship model of Jackson and Wolinsky (1996), that allows only for couples, by saying that i is linked to j and k, and j and k are also linked together. We observe that the link between i and j would have a negative externality on each neighbor of these agents, including k. However, in the general setup and in reality, the fact that i and j are in a three-agent collaboration has a positive externality on the third agent k, and a negative externality on the others. Formally, this could be done in the original network formation model by specifying, for each link between two agents, and for any other agent, the sign of the externality of that link on

[5]It can be reasonable to assume (at least in certain contexts) that everyone likes to be in as many projects as possible, and hence that Assumption v0 is satisfied. With this interpretation, costs arise indirectly as opportunity costs due to the constraint given by available time.

the third agent. It is clear that this would seriously complicate the notation, and that only a more general framework such as the one we use can overcome such difficulties.

We finish this section with two stylized but fairly general applications that show how the possibilities and the competing incentives of some environments cannot be dealt with using the standard models of matching and network formation.

2.5 Application One: The publishing model

We present here an extended example providing the general idea of the model. We continue to adopt an intuition related to the daily experience of everyone in the academic profession, but it is clear that it can easily be extended to R&D between firms that are competing in a market, as in the model of Goyal and Joshi (2003). Consider a world where there are n homogeneous scientific authors, each trying to form teams of collaborators and each with a common time constraint w. They all have two goals: a good output in terms of publications (on which they compete with colleagues), but also the objective of doing good research that can provide advancements in the field. Each author maximizes in each project both the probability of being published and the probability of authoring a good idea. We assume no constraint on the multi-matching technology P, except for the fact that agents cannot work alone: $p \in P$ if and only if $|n(p)| \geq 2$.

Here we assume that a project p has a strictly positive *divulgative fitness* (i.e., an expected popularity) that we call $\phi(p)$. The divulgative fitness of a paper may depend on the amount of work that is put into the paper by its members. This fitness can clearly also be related to heterogeneous exogenous factors.

Accordingly, a paper's *probability of being published* has the multinomial form:[6]

$$P_{pub}(p) = \frac{\phi(p)}{\sum_{q \in x} \phi(q)}.$$

When a new team is formed there are clear negative externalities (increasing in the divulgative fitness of the new project) for all the agents that are not members of the new team, because their probabilities of being published decrease.

[6] We set up the model as if only one paper is *published*, but this can be easily relaxed as long as the number of published papers is fixed and independent of the aggregate fitness.

In a related but not necessarily collinear way, we assume that each project has a strictly positive probability of providing a good idea which is $P_{good}(p)$. This probability is reasonably increasing in effort, but there may be communication and coordination costs which make it decrease in the size, in terms of members, of the team. Or, there could be positive externalities from the aggregate quality of all the scientific production as a whole. In general, the whole environment of x can provide both positive and negative externalities, with network effects like those described in the *connection model* and in the *co-authorship model* of Jackson and Wolinsky (1996).

To provide a simplified functional form, which maintains the general idea, we assume that each author i receives a payoff in a generic state x that is:

$$u_i(x) = \sum_{p:i\in n(p)} \left(U P_{pub}(p) + \frac{V}{|n(p)|} \cdot P_{good}(p) \right),$$

where U and V are positive numbers, homogeneous for all agents,[7] and $P_{good}(p)$ does not depend on other existing projects in x. We observe that, while the utility U coming from a publication is not affected by the number of authors (what matters is to have a publication in the curriculum), the benefits V deriving from a good idea must be shared among the participants (consider, for instance, the earnings that come from a patented idea).

For this simplified model it is not difficult to prove that it satisfies Assumption v0.[8] That is because, for an agent i, if we call $\Phi \equiv \sum_{q\in x:i\in n(q)} \phi(q)$, the marginal utility for being member of a new project p' is:

$$u_i(x \cup p') - u_i(x) = U \left(\phi(p') \frac{\sum_{q\in x:i\notin n(q)} \phi(q)}{(\Phi + \phi(p'))\,\Phi} \right) + V \cdot P_{good}(p').$$

[7] In a context of industrial organization, with R&D between firms, U could be the aggregate value of a fixed market, on which firms compete for shares, while V could be the expected value of further markets that new products could open.

[8] It is possible to provide more complicated payoff functions, which are non-linear or for which $P_{good}(p)$ is state dependent, and which also satisfy Assumption v0. A simple but reasonable first step of generalization is by assuming the existence of non-negative net externalities on $P_{good}(p)$ that come from a member of $n(p)$ that participates in other projects. This would bring our model closer to the *connection model* than to the *co-authorship model*. Indeed, the negative externalities of the *co-authorship model* are an indirect way to take into account the scarcity of time that researchers face in their activity; the reason for such externalities is removed in our setting, where agents are explicitly given time endowments.

The first term is non-negative, and it is null only if that agent was already a member of each existing team. The second term is strictly positive by definition.

2.6 Application Two: The connection model with heterogeneous skills

Finally, we conclude the section with a model that is again related to the connection model of Jackson and Wolinsky (1996), but agents also have heterogeneous skills and in equilibrium we have the features of assortative matching (Becker, 1973). We consider n agents (n is even), where each agent i has an exogenous and individual specific skill $\sigma(i)$. We have also ℓ activities, where agents can participate in couples, so that Assumption s2 holds and a team is always composed of two agents, but in contrast to a standard network formation model the agents can form at most ℓ couples (one for each activity) and overlapping couples (two agents working on more than one project) are possible. In this sense it can be seen as an overlapping matching model (see Example 7 on the possibility of excluding overlaps) and to maintain this relation we will call the teams matchings as well.

The utility accruing to agents in a generic state has two separable (to maintain things easy) components. First, agent i has a payoff from each matching that she makes with any agent j, and this payoff is a function $g(\sigma(i), \sigma(j))$ that is increasing in the partner's skill. For the case of a single activity ($\ell = 1$), it is well known that this leads to positive assortative matching. But there is also a second component to the utility function: if we consider all the matchings over all the ℓ activities we obtain a network, where a link between agents i and j is present if and only if they are matched in at least one activity. On this network we consider the geodesic distance $d(i, j)$ between any two nodes (i.e., the length of the minimum path between them), and the standard connection payoff (see again Jackson and Wolinsky (1996)), that assigns to agent i a payoff $\beta \cdot \delta^{d(i,j)}$ for each other agent (node) j to which she is directly or indirectly connected. As these payoffs are non negative and have non negative externalities on the payoffs of other teams, it is easy to check that property v_0 holds.

Depending on the parameter β and on the functional form of g, this model can exhibit some externalities that are not internalized by the agents. In particular, if β is too small, each activity will have independent incentives, so that the matchings on each of them will tend to be exact replicas, and the resulting network will be made only of $n/2$ disconnected couples. This may lead to an inefficient network, because it is poorly connected.

In Examples 6 and 7, when comparing stability notions, we will consider a specific analytical example for this model.

3 Myopic team-wise stability

The first equilibrium notion that we provide – called *myopic team-wise stability* – is a direct generalization of the concept of *pair-wise stabiliy*, from Jackson and Wolinsky (1996), which is used in network formation games.

3.1 Myopically team-wise stable states

The following definition formalizes the notion:

Definition 1. *A state x is* myopically team-wise stable **[MTS]** *if*

(i) for any project $p \in x$, and for any agent $i \in n(p)$, we have that $u_i(x) \geq u_i(x \backslash \{p\})$;

(ii) there exists no project $p \in P$ such that $x \cup \{p\} \in X$ and, for any agent $i \in n(p)$, $u_i(x) \cup \{p\} > u_i(x)$. □

In words, a state x is *myopically team-wise stable*, if (i) there is no agent that would be better off by deleting a project she belongs to; and (ii) there is no project that could be added to state x, without hitting the constraints of its members, and which would make them all strictly better off.[9] With some abuse of notation, we also denote by MTS the set of states that are myopically team-wise stable.

The reason why we call it myopic is that it considers only deviations of one single step in the partially ordered set X.[10] Note also that, even if a dynamic is implicit in the definition, this concept of equilibrium is a static one. Again, for comparison with and discussion on the network formation case see Jackson and Watts (2002).

The following Lemma is the first building block of our results.

[9] We require a strict Pareto improvement for the members of a project that is going to be formed in order to conclude that a state is not myopically team-wise stable, while a weak Pareto improvement is usually considered sufficient. We remark that our choice – which in principle yields a weaker equilibrium concept – makes no actual difference if Assumption v0 is satisfied.

[10] The network concept of pair-wise stability is likewise *myopic*. On this see Page et al. (2005) and discussion in Mantovani et al. (2011).

Lemma 1. *Take a team formation model satisfying Assumption v0. We have that $MTS = \mathcal{M}$.*

Proof. If a state is maximal it is not possible to add a project, and any deletion would damage the members of the removed project (by Assumption v0). Therefore, that state is MTS.

Suppose that a state is not maximal, then it would be possible to add a project, which would add a positive marginal amount to the utility of all its members (by Assumption v0), and thus that state is not myopically team-wise stable. □

3.2 Direct and indirect externalities

Our specification allows for externalities between the agents, or for a non-trivial structure of preferences of the agents toward the other team members. All the inefficiencies arising because positive and negative externalities are not endogenized by the agents would give rise to a comparison between stable and efficient outcomes that would be very similar to the one extensively analyzed in network formation models (see Jackson, 2005).

However, even if utilities from states had a simple structure, e.g., as in the case of the one imposed by Assumption v2, numerous indirect effects would arise from the constraints imposed by the technology P, and by the vector $\mathbf{w}$ of endowments, as will be clear from the following examples.

First of all, consider again the case illustrated in Example 1 and Figure 1. Because of the constraints imposed by the technology, agents j and k clearly have a negative externality on the other two agents when they form a team together: by forming a team on an activity they reduce the available teams for agents i and m.

Example 3. Consider a team formation model with four agents: i, j, k and m, all with an endowment of 3 units of time. Agent i can form a team with j only, if one of the two puts in 1 unit and the other 2 units of time. The same holds for the couple formed by k and m. Agents j and k, on the other hand, can form a team together by investing only one unit of time each. As illustrated graphically in Figure 3, this team formation model has three myopically team-wise stable sets: I, II and III.[11] The payoff of a project for an agent is

[11] We use here the same simplifying representation that we have employed in Figure 1 and briefly discussed in footnote 4.

always $\frac{1}{2}$. Therefore, this example does not satisfy Assumption t1, but satisfies Assumptions s2 and v2.

In this case, even if all the teams have the same payoff for each of their members, there are indirect preferences for some of the agents. In particular, if agents j and k could *choose ex ante* their teams they *would rather* form teams together, because this would allow them to form up to 3 teams, while forming a team with i or m (respectively) would bind them to myopically team-wise stable sets where they can form at most two teams. We will formalize in Section 5 the idea of '*choosing ex ante*' and '*would rather*'.

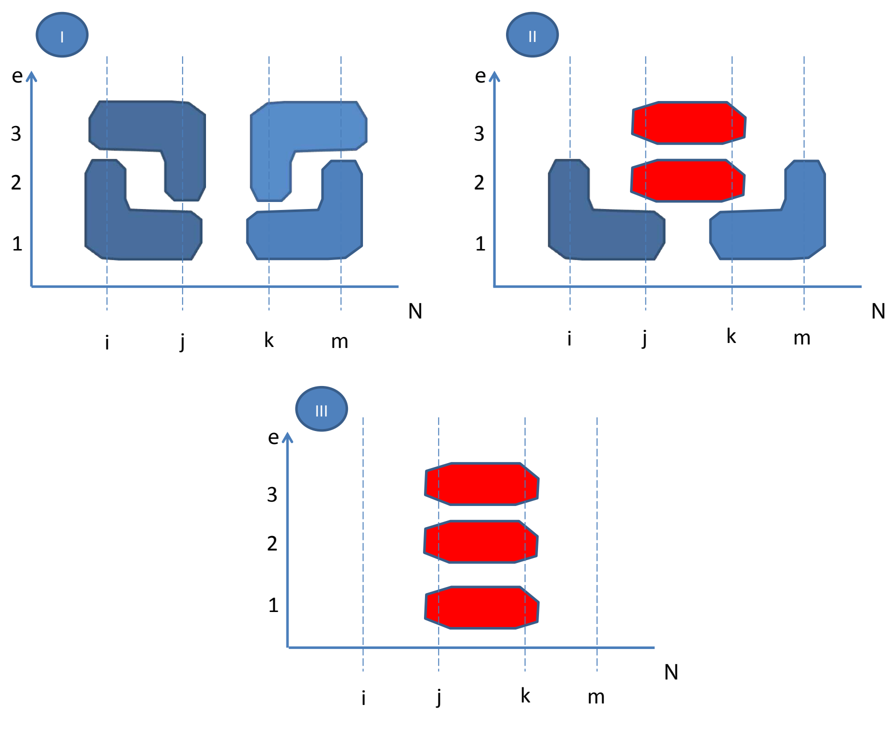

Figure 3: MTS states of Example 3.

Example 4. As the model is specified, it is not possible to specify indirect externalities of the following form: a team is feasible if and only if another team is not, and possibly also the other way round. As an example consider Case I in Figure 4, where we may want to express a condition by which the team (i, j) can be active only if the team (k, m) is not active. Another example could be one in which the same subset of agents cannot simultaneously work on more than one project, even if many are independently feasible (and this is a case we will discuss in Example 7). It is possible however to model a similar situation including fictitious agents, with a dummy utility function, whereby the agents have limited resources of time and must be included as members in the teams that we want to be mutually exclusive. We will not develop all the formal definitions of this approach, but we maintain the simple example given in Figure 4: as illustrated in Case II it is possible to add a fictitious agent h with $w_h = 1$, and such that the possible teams are now (i, j, h) and (k, m, h). In this way the original two teams become mutually exclusive.

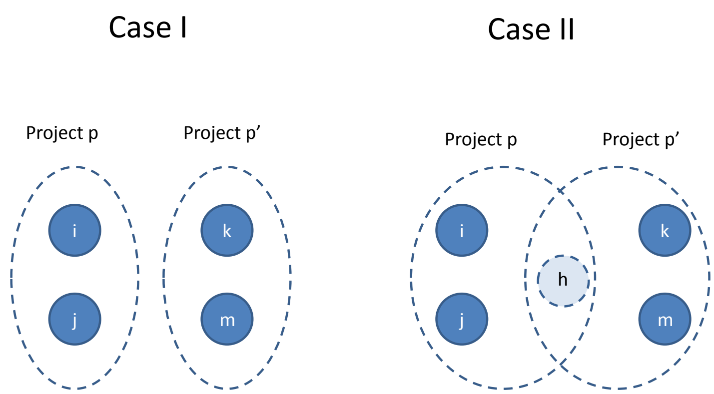

Figure 4: The two cases discussed in Example 4.

4 Stochastic stability

To refine MTS we consider an unperturbed dynamics where absorbing states are myopically team-wise stable states, and we then insert vanishing perturbations with the aim of refining our prediction by means of *stochastic stability*.

4.1 Unperturbed dynamics and preliminary results

In order to deal with stochastic stability, we need to introduce an underlying dynamics which describes the probabilistic passage from state to state, and then to add perturbations. In particular, we work with discrete time and we indicate it with $s = 0, 1, \ldots$. We denote the state of the system at time s with x^s. At time $s + 1$ a single project $p \in P$ is drawn, with every project in P having positive probability of being drawn.

One remark is worth making at this point. Extensive heterogeneity is allowed between the probabilities of different projects being drawn; for instance, it might be reasonable to assume that better projects (in some sense) are more likely to be selected and then implemented. As long as every feasible project has a positive, even tiny, probability of being drawn all our results remain valid.

The extracted project p will actually be formed if $x \cup \{p\}$ is feasible, i.e., $\mathbf{e}(x \cup \{p\}) \leq \mathbf{w}$. Otherwise, such a team is not formed, since its creation is not possible due to resource constraints. In any case, state x^{s+1} is reached. We refer to this dynamic process as *myopic team-wise dynamics*.

A Markov chain (X, D) turns out to be defined, where X is the state space and D the transition matrix, with $D_{xx'}$ denoting the probability of moving from state x to state x'. We recall some concepts and results in Markov chain theory, following Young (1998). Given any two states $x, x' \in X$, state x' is said to be *accessible* from state x if there exists a sequence of states starting from x and reaching x' such that the system can move with positive probability from each state to the next state. A set $\mathcal{E}$ of states is called an *ergodic set* (or *recurrent class*) when each state in $\mathcal{E}$ is accessible from any other state in $\mathcal{E}$, and no state outside of $\mathcal{E}$ is accessible from any state in $\mathcal{E}$. If $\mathcal{E}$ is an ergodic set and $\mathbf{x} \in \mathcal{E}$, then $\mathbf{x}$ is called *recurrent*. Let $\mathcal{R}$ denote the set of all recurrent states of (X, D). If an ergodic set is made of a single element, such a state is called *absorbing*. Equivalently, x is absorbing when $D_{\mathbf{xx}} = 1$. Let $\mathcal{A}$ denote the set of all absorbing states of (X, D). Clearly, an absorbing state is recurrent, hence $\mathcal{A} \subseteq \mathcal{R}$.

The following Proposition proves that there are no recurrent states other than absorbing states, and provides a characterization of absorbing states as maximal states, and thus as myopically team-wise stable states.

Proposition 2. *Take a team formation model satisfying Assumption v0 and a myopic team-wise dynamics. We have that* $\mathcal{A} = \mathcal{R} = \mathcal{M} = MTS$.

Proof. From the definitions of $\mathcal{R}$ and $\mathcal{A}$, we know that $\mathcal{A} \subseteq \mathcal{R}$. Moreover, from Lemma 1, we have that $\mathcal{M} = MTS$.

Take $x \notin \mathcal{M}$. An additional project can be formed, and once formed, state x will never be visited again in the future. This shows, by contraposition, that $\mathcal{R} \subseteq \mathcal{M}$ and $\mathcal{A} \subseteq \mathcal{M}$.

Finally, consider any state $x \in \mathcal{M}$. Since existing projects never disappear, and no new project is feasible starting from x, we can conclude that x is absorbing. Therefore, $\mathcal{M} \subseteq \mathcal{A}$. □

4.2 Perturbed dynamics and stochastically stable states

We are ready to introduce perturbations in the unperturbed dynamics considered in the previous subsection, and then use the techniques developed by Foster and Young (1990), Young (1993), Kandori et al. (1993). Basically, we suppose that with a tiny amount of probability active projects may accidentally dissolve, and possibly (but not necessarily) non-existing projects can be formed if feasible. By so doing, the dynamic system under consideration becomes ergodic, and from known results it follows that there exists a unique probability distribution among states that is stationary and describes the limiting behavior of the Markov chain as time goes to infinity, irrespectively of the initial state. Then we consider the limit of this stationary distribution for the amount of perturbation decreasing to zero. Those states that are visited with positive probability in this limiting stationary distribution are called stochastically stable.

We invite the reader who is interested in a formal exposition of perturbed Markov chain theory to consult Young (1993) and Ellison (2000), while in the following we simply make use of the *resistance* function $r : X \times X \to \mathbb{R}^+ \cup \{\infty\}$, where $r(x, x')$ indicates the minimal amount of perturbations required to move the system from x to x' in one unit of time. If $r(x, x') = 0$ then the system moves from x to x' with positive probability in the unperturbed dynamics, i.e., $T_{xx'} > 0$, while $r(x, x') = \infty$ is interpreted as impossibility of moving from x to x' in one unit of time even when perturbations are allowed.

We rely on the techniques and results illustrated in Foster and Young (1990), Young (1993) and Young (1998), as they provide a relatively easy way to identify which states are stochastically stable. More precisely, we restrict attention to absorbing states (since there are no other recurrent states by virtue of Proposition 2), and for any pair (x, x') of absorbing states we define $r^*(x, x')$ as the minimum sum of the resistances between states over any path starting in x and ending in x'. Then, for any absorbing state x, we define an x-tree as a tree having root at x and all absorbing states as nodes. The resistance of an x-tree is defined as the sum of the r^* resistances of its edges. Finally, the *stochastic potential* of x is said to be the minimum resistance over all trees rooted at x.

A state x is proven to be stochastically stable (Foster and Young, 1990) if and only if x has minimum stochastic potential in the set of absorbing states. Intuitively, stochastic stability selects those states that are easiest to reach from other states, with "easiest" interpreted as requiring the fewest mutations (as measured by the stochastic potential).

We now introduce two alternative perturbation schemes in the unperturbed dynamics considered in the previous subsection. The two perturbation schemes lead us to the same results. The first one is called a *uniform perturbation scheme*, and it is such that at every time s each project $p \in P$ has an i.i.d. probability ϵ subject to an error; such an error makes the project disappear if existing, and be formed if non-existing and $x^s \cup \{p\} \in X$. The second perturbation scheme is such that only existing projects can be hit by a perturbation, so that each existing project disappears with an i.i.d. probability ϵ. We refer to this modeling of errors as a *uniform destructive perturbation scheme*.[12] It is easy to check that the perturbed system is irreducible and aperiodic: from any state $x \in X$, every existing project may dissolve by means of perturbations, and then one project per period may form, leading the system to any state x'. Every project can form in the unperturbed dynamics, thanks to Assumption v0, and hence perturbations creating new projects (that are allowed only in the uniform perturbation scheme) do not play any significant role. Aperiodicity is ensured since there are no recurrent states other than absorbing states (see Proposition 2).

We denote by SS the set of stochastically stable states. The following proposition identifies stochastically stable states as the states with the maximum number of existing projects.

Proposition **3.** *Take a team formation model satisfying Assumption v0, a team-wise dynamics and either a uniform destructive perturbation scheme or a uniform perturbation scheme. Then,* $SS = \mathcal{L}$.

Proof. Consider any two absorbing states $x, x' \in X$. In order to move from x to x' it is necessary for all projects that exist at x and do not exist at x' to be stopped, and this

[12] For our result we do not require the probabilities to be equal, and every team $\mathbf{t}$ could have any state and time dependent utility $\eta(p, x, s, \epsilon)$, depending also on some positive real number ϵ. Thus we merely require that all such probabilities converge to zero with the same order as ϵ goes to 0.

can only occur by means of $\ell(x) - \ell(x \cap x')$ perturbations, both in the uniform destructive perturbation scheme and in the uniform perturbation scheme. In contrast, projects that exist at x' and do not exist at x can form in the unperturbed dynamics, thanks to Assumption v0. Therefore:

$$r^*(x, x') = \ell(x) - \ell(x \cap x'). \tag{1}$$

We are ready to prove $SS \subseteq \mathcal{L}$. We proceed by contradiction. Suppose $x \notin \mathcal{L}$. Then we can find an x' such that $\ell(x') > \ell(x)$. Take any x-tree and consider the path from x' to x, say $(x', x_1, \ldots, x_i, \ldots, x_k, x)$. By (1), the sum of resistances over this path is $\ell(x') - \ell(x' \cap x_1) + \ell(x_1) - \ell(x_1 \cap x_2) + \ldots + \ell(x_{k-1}) - \ell(x_{k-1} \cap x_k) + \ell(x_k) - \ell(x_k \cap x)$. We now consider the x'-tree obtained from the x-tree by reversing the path from x' to x. Again by (1), the sum of resistances over this reversed path is $\ell(x) - \ell(x \cap x_k) + \ell(x_k) - \ell(x_k \cap x_{k-1}) + \ldots + \ell(x_2) - \ell(x_2 \cap x_1) + \ell(x_1) - \ell(x_1 \cap x')$. Taking the difference between the above sums of resistances over the two paths, we obtain that the x'-tree has a resistance which is equal to the resistance of the x-tree $+\ell(x) - \ell(x')$. Since $\ell(x') > \ell(x)$, we can conclude that for any x-tree we can find an x'-tree with a lower overall resistance, and hence the stochastic potential of x' is lower than the stochastic potential of x. Therefore, x cannot be stochastically stable.

We now prove $\mathcal{L} \subseteq SS$. Since at least one stochastically stable state must exist, and we have just seen that no state outside of $\mathcal{L}$ is stochastically stable, we can therefore conclude that there exists an $x \in \mathcal{L}$ that is stochastically stable. Consider any other $x' \in \mathcal{L}$. Following exactly the same reasoning as above we obtain that the stochastic potential of x' must be the same as the stochastic potential of x. Therefore, x' is stochastically stable as well. □

Proposition 3 is our main result. It provides a very precise characterization of stochastically stable states for every team formation model that satisfies the general assumption v0, and under the perturbation schemes that we have considered.

To aid intuitive comprehension, we provide the following discussion. In the representation of all the states in X as a partially ordered set, with the empty state at the top and maximal states at the bottom, an error can be seen as a step upwards, while the adapting best response of agents (under Assumption v0) can be seen as a step downwards. Consider Example 1 as represented in Figure 1. In this case, the bottom left state is SS, because it would need at least two errors before the system can move in the unperturbed dynamics to another MTS state. To move away from the other two MTS states, on the other hand, only one error is required.

Clearly, the fact that SS states maximize the number of teams in the state, does not tell us anything about efficiency, because the utility function could have a structure that highly rewards agents in states with few teams. However, under an additional assumption we can prove the following corollary.

Corollary **4.** *Given a team formation model, under Assumption v1, every stochastically stable state is Pareto efficient.*

Proof. By Proposition 3 a stochastically stable state x maximizes the number of projects $\ell(x)$. By Assumption v1 we have $\ell(x) = \sum_{i \in N} u_i(x)$, so that x also maximizes the aggregate utility to agents. Thus, there is no other state that can provide a higher utility for some agent, without damaging any other agent. □

4.3 Descriptive value of stochastic stability

In this section we have shown that in a team formation model stochastically stable states coincide with maximal states. This result rests not only on Assumption v0 and a large class of perturbation schemes, but also on individual behavior that is boundedly rational. In particular, no agent will ever exit from existing projects in order to free up time and start other projects, despite the fact that this may increase her utility. Given this circumstance, one may then query the descriptive value of stochastic stability. An answer clearly depends on the specific case under consideration. We limit ourselves to the following observations. Even if agents have sufficient cognitive skills to recognize the possibility of an increase in utility, there are at least two kinds of reasons that might prevent them from doing so. First, coordination issues: in order to carry out a utility enhancement, an agent has to quit projects with some teams and contextually start other projects with new teams, and this involves coordinating the actions of several agents. Second, switching costs: these costs may be due to legal obligations – consider for instance divorce costs in marriage – or to learning how to operate in new teams.

Nevertheless, in the following section we analyze a refinement of myopic team-wise stability based on strong rationality and absence of coordination and switching costs.

In Appendix A we consider a variant of the team formation model where agents are endowed with unlimited cognitive and coordination skills, but face switching costs when leaving an existing project. We show that when switching costs are high enough, stochastically stable states are exactly those states having the maximum number of existing projects.

5 Coalitional stability

In this section we introduce a concept of stability – *coalitional stability* – which is strongly based on coordination opportunities and rationality of agents. Then we compare it with myopic team-wise stability and stochastic stability, providing a class of situations where coalitional stability has no refining power.

5.1 Coalitionally stable states

We consider the following definition:

Definition **2.** *A state x is* coalitionally stable **[CS]** *(or* coalition proof*) if there exists no subset $C \subseteq N$ such that*

(i) there exists a set of projects $y \subseteq x$ such that $\forall\, p \in y$, $\exists\, i \in C$ such that $i \in n(p)$;

(ii) there exists a set of projects z such that $z \cap x = \emptyset$ and, $\forall\, p \in z$, if $j \in n(p)$ then $j \in C$;

(iii) $(x \backslash y) \cup z \in X$ and for any agent $i \in C$ we have that $u_i((x \backslash y) \cup z) > u_i(x)$. □

In words, a state x is *coalitionally stable* if there is no coalition that can (i) erase a set of projects, each of which contains at least one agent of the coalition, (ii) form a set of other projects, where all the members of each are also members of the coalition, and (iii) make all the members of the coalition strictly better off in the resulting state.[13] We denote by CS the set of states that are coalitionally stable.

It is evident that this definition allows for a profound rationality by the agents: they can identify and coordinate a deviation through a long path in the partially ordered set X. This definition allows the agents to maintain some of their existing teams with other agents outside of the coalition, and in this sense it is a generalization of *bilateral deviations* defined in a network formation setting by Goyal and Vega-Redondo (2007). The literature on clubs (e.g., Pauly, 1970 and Faias and Luque, 2012) focusses on deviations where all *clubs* are deleted when members are both outside and inside the coalition. Note finally that we

[13] In contrast to what happens for myopic team-wise stability (see footnote 9), the choice to require a strict Pareto improvement for the agents of a blocking coalition – instead of asking for a weak Pareto improvement – can enlarge the set of coalitionally stable states even when Assumption v0 is satisfied. However, this difference ceases to exist when we introduce costs to exit from existing projects (see Appendix A).

are not providing a general result of existence of CS states, for which we may need a more general concept of stability such as the one provided in more specific settings by Herings et al. (2009, 2010). However, we will focus on situations in which we can compare CS states with SS states, so that a simpler definition suffices.[14] In Appendix A we provide more general definitions, which integrate this approach with that of stochastic stability.

5.2 A comparison between notions of stability

In the representation of all the states in X as a partially ordered set, the deviation of a coalition can be seen as a path that moves first upwards and then downwards. Consider as an example the whole states of Example 1 and Figure 1. Suppose that in this case the utility that agents j and k receive from being together is always greater than the utility they receive from being respectively with i and m. Then, the bottom left state that maximizes the number of projects would not be CS: j and k could coordinate to delete all the existing projects and start two projects together, moving to the state on the extreme right. Therefore, in general, SS and CS states are not necessarily related concepts and can have empty intersections. This can also clearly be seen if we look back at Example 3 and the related Figure 3. I and II are stochastically stable, because they maximize the number of teams at four, while II and III are coalitionally stable, because I can be broken by the coalition $\{j, k\}$.

There are cases in which SS and CS states can be a subset of one another, and given the flexibility of the utility function $\mathbf{u}$, a setup can always be provided, even under Assumption v0, such that any desired subset of states would always be chosen by the grand coalition N.

On the other hand, there are many other cases, based on simple and general utility functions, where the CS states do not provide a clear or apparently improving refinement upon the CTS states. As an example, consider that in general CS states may not even be

[14] Some clarification is needed. The definition of *farsighted coalitionally stable* states, as proposed by Herings et al. (2009, 2010) in a context where agents are farsighted players who evaluate the desirability of a deviation in terms of its future consequences (see also Dutta et al., 2005 and Navarro, 2013), can easily be generalized to our context, and this is what we achieve in Appendix B. The good thing about the above definition is that there always exists a farsighted coalitionally stable state, while existence is not guaranteed for simple coalitionally stable states, as we define them. Accordingly, they are a super-set of the *coalitionally stable* states. However, our point is that in many contexts coalitional stability has too little predictive power, so that the coalitionally stable states are too numerous or can be anything. We are not concerned here with the point that in other contexts coalitional stability can be a concept so restrictive that no state satisfies it.

Pareto efficient, because agents adhere to deviating coalitions only if their marginal profit from doing so is strictly positive: thus, it could be that a Pareto improving deviation is feasible through a coalition whose members will not all be strictly better off. An instance of this sort is in the case provided by Example 1, under Assumption v2: the top right maximal state of Figure 1, with only agents b and c forming teams, is a CS state, even if it is Pareto dominated by the bottom left state (with all agents taking part in two teams each).

Here below we present a result for a case (which actually encompasses the previous example) where CS states coincide with MTS states, so that they provide no refinement at all with respect to the myopic and boundedly rational concept of myopic team-wise stability.

PROPOSITION **5.** *Given a team formation model, under Assumptions t1 and v2, we have that* $CS = MTS$.

Proof. Since it is trivially true that $CS \subseteq MTS$, we focus on showing that $MTS \subseteq CS$. Let us suppose this is not the case; then there is a team-wise stable state x which is not coalitionally stable. From the definition, this means that in x there is a coalition C that can erase a set y of projects and start a set z of new projects.

As agents in C need to strictly increase their utility (i.e., the number of teams to which they belong, from Assumption v2), for each of the agents there is at least one unit of free time in x; formally we have that $e_i(x) < w_i$ for each $i \in C$.

As all the projects give strictly positive payoffs, then z is nonempty.

But then, as each team in z is formed by all and only agents from C, and by Assumption t1 it would cost one unit of time each, then each project $p \in w$ could be started already in x, and $x \cup \{p\} \in X$, i.e., it would be feasible. We have reached a contradiction with the hypothesis that x is a team-wise stable state. □

5.3 Some examples

We conclude this section with three examples, one on the publishing model and two on the connection model with heterogeneous skills.

EXAMPLE 5. Consider the publishing model presented in Section 2.5. Assume that the divulgative fitness of a project is simply given by the amount of work put into it:

$$\phi(p) = h(p).$$

In a related way, the probability that a project is good is equal to:

$$P_{good}(p) = (\beta \cdot h(p))\, \gamma^{|n(p)|-2},$$

where β is a small positive number, such that $1/\beta > \max_{m \in \{2,\ldots n\}} \{wm\gamma^{m-2}\}$ (and hence $P_{good}(p)$ can correctly be interpreted as a probability), and $\gamma \in (0, 1)$.[15]

We consider the case in which there are only 4 agents, each with an endowment of two hours (i.e., $w = 2$). We recall that the only technological constraint in the publishing model is represented by a minimum size of two agents for a team to be able to write a paper.

Our characterization of SS allows us to conclude that SS states are those with four projects, where every project is made up of a team of two co-authors working one hour each. The utility earned by each agent in this case is $U/2 + 2\beta V$. Furthermore, every agent earns the same level of utility even in the states with only three or two projects, which are all made up of two co-authors, possibly investing two hours each in a project. Consider now the case in which three agents choose to exclude the fourth agent, who remains alone and is unable to write a paper by herself. The three agents can either invest two hours each on a single project, or one hour each in two distinct projects. In any case their utility is $U + 2\beta V\gamma$. If, instead, all four agents choose to form one big project, or two distinct projects, the individual utility turns out to be $U + 2\beta V\gamma^2$. By solving the inequalities arising when we compare the above individual utilities,[16] we obtain that for $\gamma \leq 1 - U/(4\beta V)$, coalitionally stable states are states are states where all hours are invested and all projects are carried out by two co-authors, while for $\gamma \geq 1 - U/(4\beta V)$ they are states where three co-authors invest all their time in one or two projects.

We assume that it is possible to make welfare considerations by aggregating utilities across agents. It is easy to check that if $\gamma \leq \sqrt{1 - U/(4\beta V)}$ then efficient states are those where all time is invested and all projects have two co-authors (and hence stochastic stability leads to efficiency), while if $\gamma \geq \sqrt{1 - U/(4\beta V)}$ then efficient states are those where projects have four authors and all time is invested. Figure 5 summarizes all the results.

[15]For this example we could also have chosen $\phi(p) = P_{good}(p)$. We did not, because in our view it is more realistic to assume that the number of authors has a negative externality only on the quality of the project, and not on its *divulgative fitness*. Consider also that the following part of this example would work even when there is no competition for publishing: i.e., when $U = 0$.

[16]In actual fact, we should check that all other states are not relevant for the analysis. This amounts merely to a list of simple but boring inequalities to be developed, which we prefer to skip.

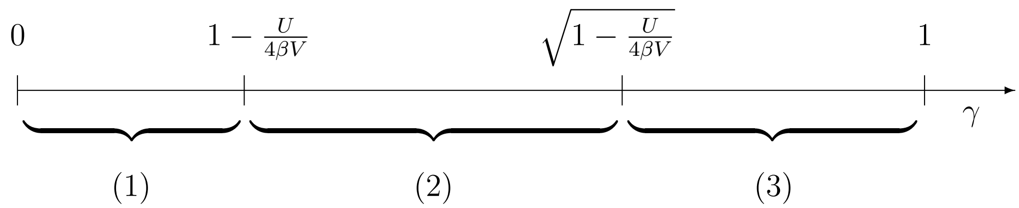

Figure 5: (1) SS coincides with CS, and is efficient; (2) SS is efficient but differs from CS, which are states with 3 authors per project; (3) SS differs from CS, which are states with 3 authors per project, and from efficient states, which are those with 4 agents per project.

We conclude by remarking that the relationship between SS, CS and efficiency can be far from obvious even in relatively simple examples.

Example 6. Consider the framework of the connection model with heterogeneous skills (Section 2.6). Assume that agents are indexed by the n (n even) natural numbers, and that the skill of each agent is precisely the corresponding number, so that $\sigma(i) = i \in \{1, \ldots, n\}$. Utilities are additive over projects for their members: $g(i,j) = g(j,i) = \alpha(i+j)$, with $\alpha > 0$. And finally, there is no decay in the connection part of the utility function (i.e., $\delta = 1$) and β is so *small* that $\beta < \frac{\alpha}{n-1}$. In this model, a state in which all agents make ℓ matchings and the resulting network is connected provides an aggregate payoff to agents equal to $2\alpha \left(\sum_{i=1}^{n} i\right) + \beta \cdot n(n-1) = (\alpha + \beta)n(n-1)$.

It can be seen that with these assumptions there is a unique coalitionally stable state in which agents 1 and 2 match together in all the ℓ activities (they make a profitable deviation by connecting together even if they lose all the connections in the network), agents 3 and 4 match together in all activities, and so on up to agents $n-1$ and n (who are left alone and have only this matching possibility). This single coalitionally stable state reaches a non-optimal aggregate utility for the agents of $2\alpha \left(\sum_{i=1}^{n} i\right) + \beta \cdot n = (\alpha(n-1) + \beta)n$.

On the other hand, any myopically team-wise stable state in which all agents make ℓ matchings is stochastically stable. In Appendix C we show that for any $\ell \geq 3$ most of the networks generated by these overlapping matchings are connected, and that as n grows the ratio of the connected networks approaches 100%. Hence, the stochastically stable states are efficient *most of the times*, and as $n \to \infty$ they are almost surely efficient, in terms of aggregate utility.

EXAMPLE 7. The *trick* in the previous example is that agents could seem to have the possibility of overlapping their match in multiple activities, and that the resulting network is therefore not connected. However, it is possible to exclude overlapping matchings by introducing fictitious agents, as discussed in Example 4. Even if we assume the same structure of the utility function as in the previous Example 6, and the limitation that two agents cannot work on more than one activity together, then the coalitionally stable state would still not result in a connected network.

However, if the number of activities is ℓ, and n is a multiple on $\ell+1$, the first $\ell+1$ agents will form a fully connected cluster together, the agents indexed from $\ell+2$ to $2\ell+2$ will do likewise, and so on up to agents from $n-\ell$ to n (who are left alone and have only these matching possibilities).

An example along the above lines is shown in Figure 6, for the case $\ell = 3$ and $n = 4$. In this case the first matching is the standard assortative one. In the second matching agent 1, who is pivotal, chooses her second choice, namely agent 3, and the other 6 agents adapt accordingly. Finally, in the third matching agent 1 chooses her third choice. The resulting network is made up of two separate fully connected clusters of 4 agents each.

6 An application to marriage

In this section we apply our results to a specific and typical group formation problem, namely marriage.

We take into consideration two sets, W and M, that we refer to as a set of women and a set of men respectively. Each man is interested in being matched with a woman he likes, and each woman is interested in being matched with a man she likes. However, not all matches are feasible (because of preferences and/or because of other constraints). We indicate by $P \subseteq W \times M$ the set of feasible pairs. In graph theory language, (W, M, P) is equivalent to a bipartite graph. Figure 2 provides an example.

It is useful to define for every woman $w \in W$ the set of her feasible mates, $M(w) = \{m \in M : (w, m) \in P\}$. Analogously, we define for every man $m \in M$ the set of his feasible mates,

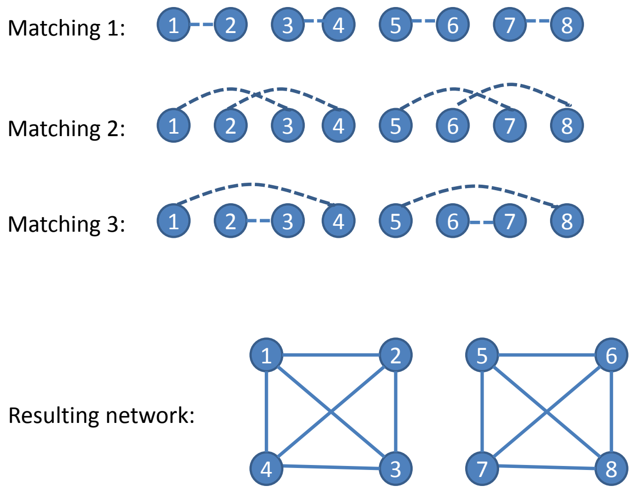

Figure 6: The three matchings and the resulting network discussed in Example 7.

$W(m) = \{w \in W : (w, m) \in P\}$. For the sake of an easy notation, we also define for every $W' \subseteq W$, $M(W') = \bigcup_{w \in W'} M(w)$, and for all $M' \subseteq M$, $W(M') = \bigcup_{m \in M'} W(m)$.

A matching is a feasible set of pairs, i.e., a set of feasible pairs such that every woman is not matched with more than one man and every man is not matched with more than one woman. Formally, a set of pairs x is said to be a *matching* if and only if $x \subseteq T$ and for any two distinct pairs $(j, w) \in x$ and $(j', w') \in x$ we have that $j \neq j'$ and $w \neq w'$. We denote by X the set of all matchings.

Given a matching x we can define the set of unmarried women, $W^u(x) = \{w' \in W : w' \neq w \text{ for all } (w, m) \in x\}$. Analogously, we can define the set of unmarried men, $M^u(x) = \{m' \in M : m' \neq m \text{ for all } (w, m) \in x\}$. Finally we define the set of pairs that can still feasibly be formed, $F(x) = \{(w, m) \in P : w \in W^u(x), m \in M^u(x)\}$.

In the above setting, the Marriage Theorem (see for example Bose and Manvel, 1984, pag. 205-209) states a necessary and sufficient condition for the existence of a matching where no woman and no man remain unmarried: basically, for any subset of women, and of men, the corresponding set of feasible mates contains an at least equal number of mates. Preliminarily, we say that a matching x is *perfect* if and only if $W^u(x) = \emptyset$ and $M^u(x) = \emptyset$.

THEOREM **6** (the Marriage Theorem)**.** *Given* (W, M, T)*, the following two conditions are* *equivalent:*

1. *there exists a perfect matching;*

2. *for every* $W' \subseteq W$, $|M(W')| \geq |W'|$, *and for every* $M' \subseteq M$, $|W(M')| \geq |M'|$.

Figure 7 provides an instance where a perfect matching exists, and another instance where a perfect matching does not exist.

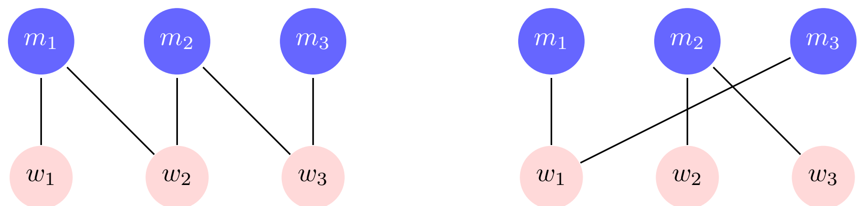

Figure 7: On the left, an example is drawn where a perfect matching exists. In the example on the right, instead, no perfect matching exists; in fact, $W(\{m_1, m_3\}) = \{w_2\}$, and hence $|W(\{m_1, m_3\})| = 1 < 2 = |\{m_1, m_3\}|$.

6.1 Myopic team-wise stability and coalitional stability

We now endow agents with preferences over mates. In particular, we use $u(m, w)$ to denote the utility that man m earns if matched with woman w. Similarly, we use $u(w, m)$ to denote the utility that woman w earns if matched with woman m. Let $u(m, \emptyset)$ and $u(w, \emptyset)$ indicate the utility of remaining unmatched for man m and woman w respectively. The set of feasible pairs P is such that $u(m, w) < u(m, \emptyset)$ or $u(w, m) < u(w, \emptyset)$ imply that $(m, w) \notin P$. Therefore, Assumption v0 turns out to be satisfied. Apart from preferences, pairs can be unfeasible for *technological* reasons (e.g., they cannot meet since they live too far away from each other).

We observe that marriage is a specific instance of the general model presented in Section 2. In this context there is only one activity: marriage. So there is a one-to-one correspondence between teams and projects. Moreover, $w_i = 1$ – i.e., each agent's endowment is 1 – and every $\mathbf{t} \in P$ is such that there exist $i, j \in N$, $i \neq j$, with $t_i = t_j = 1$ and $t_k = 0$ for every $k \neq i, j$ – i.e, only pairs can be feasible teams. As an application of Lemma 1, we can state that the set of myopically team-wise stable states coincides with the set of matchings where no additional pair can be formed, i.e., matchings x with $F(X) = \emptyset$.

In marriage problems, a standard notion of stability is employed. A collection of pairs $x \subseteq P$ is said to be a *stable matching* if for all $(w, m) \in x$, we have that if m' is such that $(w, m') \in P$ and $u(w, m') > u(w, m)$, then $u(m', w) > u(m', w')$ with $(w', m') \in x$. It has been known since Gale and Shapley (1962) that a stable matching always exists in marriage problems. It is also a known result that the set of stable matchings in a marriage problem coincides with the core. Moreover, the notion of coalitional stability (see Section 5) becomes equivalent to the core when each agent can belong to only one group. Therefore, we have that $CS \neq \emptyset$. As we already know, we also have $CS \subseteq MTS$.

We may wonder if coalitional stability maximizes the number of pairs that are formed. In particular, the Marriage Theorem tells us under what conditions a perfect matching is feasible; we may now hope that coalitional stability ensures that agents are actually able to achieve a perfect matching in such cases. Example 8 shows that such hope is vain.

EXAMPLE 8. Consider the left part of Figure 7. Assume the following preferences. For every woman, men are ranked in this way: m_1 is better than m_2, who is better than m_3. For every man, women are ranked in this way: w_3 is better than w_2, who is better than w_1. The only stable matching is $x = \{(w_2, m_1), (w_3, m_2)\}$. We observe that a perfect matching exists: $x' = \{(w_1, m_1), (w_2, m_2), (w_3, m_3)\}$.

6.2 Stochastic stability

Following the discrete dynamic discussed in Section 4.1, suppose that x^s is the actual matching at time s. At every time-step a pair of agents (w, m) is randomly selected, and such a pair is formed if and only if it can feasibly be formed given x^s, i.e., $(w, m) \in F(x^s)$. As an application of Proposition 2, in the resulting Markov chain there are no recurrent states other than absorbing states, and a state x is absorbing if and only if $F(x) = \emptyset$.

We can then analyze the perturbed Markov chain where, at every time s, each existing pair has an i.i.d. probability ϵ of dissolving. This makes the dynamic system irreducible and aperiodic, and allows us to apply the theory of regularly perturbed Markov chains. In particular, we can use Proposition 3 to conclude that the stochastically stable states are matchings where the number of pairs is maximized. This result can be put at work together with the Marriage Theorem to obtain the following corollary.

COROLLARY **7.** *Given* (W, M, P)*, if for every* $W' \subseteq W$*,* $|M(W')| \geq |W'|$*, and for every* $M' \subseteq M$*,* $|W(M')| \geq |M'|$*, then the set of perfect matchings is non-empty and coincides with the set of stochastically stable states.*

To illustrate an application of the definitions, we can consider the left-hand panel of Figure 7 (Example 8), where we have that:

- the following matchings are myopically team-wise stable states: $\{(w_1, m_1), (w_3, m_2)\}$, $\{(w_2, m_1), (w_3, m_2)\}$, $\{(w_2, m_1), (w_3, m_3)\}$, $\{(w_1, m_1), (w_3, m_2)\}$, $\{(w_1, m_1), (w_2, m_2), (w_3, m_3)\}$;
- the only matching that is a coalitionally stable state is: $\{(w_2, m_1), (w_3, m_2)\}$;
- the only matching that is a stochastically stable state is: $\{(w_1, m_1), (w_2, m_2), (w_3, m_3)\}$.

In conclusion, even when a perfect matching is feasible, rational behavior may prevent agents from achieving it. In contrast, myopic best-reply behavior coupled with tiny perturbations is capable of maintaining the system almost always with no uncoupled agent. This kind of result can be of particular interest in a firm/worker interpretation, in particular where preferences are not strong on either side, as could be the case in a market for non-skilled workers with homogeneous wages.[17] Moreover, our general framework of Section 2 allows us to easily manage extensions where firms can hire multiple workers, or the same workers can perform more than one job simultaneously.

7 Conclusions and future research

In this paper we have provided a model which describes how teams of individuals arise in order to perform activities, investing amounts of a scarce resource (typically time) to conduct the activities. The kinds of interaction that can be modeled in the proposed framework are many and widespread in economic and social spheres. Unfortunately, in a context like this the complexity of analysis can increase very rapidly, so that predictions become very hard to make. Nevertheless, we introduce and discuss alternative notions of stability – myopic

[17] The latter assumption on preferences helps to consider stochastic stability as a more prominent concept than coalitional stability, decreasing the benefits of exiting from a project to form a new one. Alternatively, a variant of the model with switching costs can be considered, along the lines of Appendix A.

team-wise stability, stochastic stability and coalitional stability – and we are able to provide results that are rather clear-cut (especially for stochastic stability).

Future work can highlight the relevance of the model for specific applications. The setting that we have provided is sufficiently general and flexible to accommodate many different sets of assumptions, and this allows a proper fine-tuning of the model. The examples throughout the paper illustrate its applicative potential. In Appendix A, a variant of the model is presented where we add switching costs, which can be considered as a realistic feature for several applications. A promising direction for research could explore its potential applicability to the job market, where the result that stochastic stability selects states with the highest number of projects has an interesting interpretation in terms of unemployment reduction.

On a purely theoretical ground, we provide here below three possible lines along which research may lead to interesting advancements.

The first question is related to the concept of coalitional stability that we define in Section 5. We are able to provide examples of non-existence of coalitionally stable states, but also (as in Proposition 5) to prove their existence under specific assumptions. We partly address this question in Appendix B, where we define a wider set of *farsightedly stable states*, whose existence is always granted. However, we conjecture that the existence of simple coalitionally stable states holds also for more general assumptions than those of Proposition 5.

The second question is related to welfare issues, and follows from the discussion in Sections 4 and 5: under which conditions on the utility function are SS states and CS states Pareto efficient, or do they maximize the objective function of some social planner? Clearly a simple case is the one in which the objective function is monotonic in the number of teams, so that SS states are *optimal*; or the case in which the utility function is monotonic for all agents in the objective function, so that CS states would be *optimal*, because even the grand coalition made of all the N agents is better off in those states. But how much of the two simple statements above can be generalized in order to have non-trivial results?

The third question follows from the previous one but has a mechanism design approach. Suppose that, through incentives, a planner can slightly change the utility function, not to obtain the trivial forms discussed above but something approaching that. Alternatively, the planner could have the possibility of modifying the technology, at least to the point at which feasibility depends on the structure of connections among agents: only agents that are close (in some sense) can work together, and the planner can adopt policies to affect who is close to whom. What are the sufficient conditions that would allow the planner to make SS or CS states Pareto efficient, or make them approach maxima of some objective function?

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Appendix A Robustness of stochastic stability with respect to switching costs

[FOR ONLINE PUBLICATION]

In this appendix we endow agents with the same degree of rationality and coordinating abilities that we have used in Section 5 for coalitional stability. However, we introduce switching costs to exit from existing projects, showing that if such costs are sufficiently high then coalitional stability coincides with myopic team-wise stability. Moreover, we consider an unperturbed dynamics based on coalitional stability, and we reproduce the results of Proposition 2. Finally, we show that the introduction of uniform perturbations like in Section 4 yields the same predictions as Proposition 3, i.e., stochastically stable states are maximal states.

The following definition provides an adjusted notion of coalitional stability.

Definition A. *A state x is* coalitionally stable **[CS]** *(or* coalition proof*) if there exists no subset $C \subseteq N$ such that*

(i) there exists a set of projects $y \subseteq x$ such that $\forall\, p \in y$, $\exists\, i \in C$ such that $i \in n(p)$;

(ii) there exists a set of projects z such that $z \cap x = \emptyset$ and, $\forall\, p \in z$, if $j \in n(p)$ then $j \in C$;

(iii) $(x \backslash y) \cup z \in X$ and for any agent $i \in C$ we have that $u_i((x \setminus y) \cup z) - c \sum_{p \in y} 1_i(n(p)) > u_i(x)$. □

The only difference with respect to the definition provided in Section 5 concerns point (iii), in particular the presence of switching costs that affect the advantage of exiting from projects and forming new ones. More precisely, each agent is aware that she will pay a cost equal to $c \geq 0$ for each project she wants to leave. Function $1_i(n(p))$ denotes the indicator function, giving 1 if agent i belongs to project p, and 0 otherwise. We denote by $CS(c)$ the set of coalitionally stable states when switching costs are equal to c.

For ease of presentation of our argument, we define a blocking operation as a quintuple (x, C, y, z, c) such that for every member i of coalition C we have that $u_i((x \setminus y) \cup z) - c \sum_{p \in y} 1_i(n(p)) > u_i(x)$. We denote by $BO(c)$ the set containing all blocking operations when switching costs are c.

The following proposition shows a kind of continuity of coalitional stability with respect to c. In the presence of a tiny amount of switching costs, the predictions given by coalitional stability do not change compared to the case without switching costs. By contrast, when c is large enough, then coalitional stability gives the same predictions as myopic team-wise stability. In the proposition we use MTS and CS as defined in Sections 3 and 5.

PROPOSITION **A.** *Take a team formation model satisfying Assumption v0. Then:*

(i) if c is low enough, then $CS(c) = CS$;

(ii) if c is high enough, then $CS(c) = MTS$.

Proof. Preliminarily, we provide the following thresholds for switching costs:

$$\begin{aligned} \underline{c} &= \min_{(x,C,y,z,0)\in BO(0)} \min_{i\in C} u_i((x \setminus y) \cup z) - u_i(x) \\ \overline{c} &= \max_{(x,C,y,z,0)\in BO(0)} \min_{i\in C} u_i((x \setminus y) \cup z) - u_i(x) \end{aligned}$$

We observe that $\underline{c}$ and $\overline{c}$ are well-defined, since a maximum and a minimum always exist in finite sets, and N, X and P are all finite.

To prove point (i), we show the double implication. We start from $CS \subseteq CS(c)$ for every $c \geq 0$. Take a state $x \in CS$. We know that no blocking coalition exists in the absence of switching costs, and no blocking coalition can arise when switching costs are added, since the advantage for a coalition to exit from projects and form new ones cannot increase. Hence, $x \in CS(c)$ for every $c \geq 0$.

We now show that $CS \subseteq CS(c)$ if c is sufficiently low. We fix $c < \underline{c}$ and we consider a state $x \in CS(c)$. For such a state no blocking operation exists when switching costs are c. Suppose ad absurdum that a blocking operation exists in the absence of switching costs, i.e, there exists $(x, C, y, z, 0) \in BO(0)$. This means that there must exist $i \in C$ such that $u_i((x \setminus y) \cup z) > u_i(x)$ and $u_i((x \setminus y)c \sum_{p\in y} 1_i(n(p)) \cup z) > u_i(x)$, but this is in contradiction with $c < \underline{c}$. Hence, $x \in CS$ if $c < \underline{c}$.

To prove point (ii), we exploit Lemma 1, and we show the double implication between $CS(c)$ and $\mathcal{M}$. Clearly, $CS(c) \subseteq \mathcal{M}$ for every $c \geq 0$. By contraposition, if $x \notin \mathcal{M}$, then there exists a project $p \in P$ such that $p \notin x$ and $x \cup \{p\} \in X$. Due to Assumption v0, $(x, n(p), \emptyset, \{p\}, 0) \in BO(0)$, and hence $x \notin CS$.

Finally, we show that $\mathcal{M} \subseteq CS(c)$ if c is sufficiently high. We fix $c \geq \overline{c}$. Then, no blocking operation exists such that $y \neq \emptyset$. By contraposition, if $x \notin CS(c)$, then there exists a blocking operation $(x, C, \emptyset, z, 0)$. This means that $x \cup z \in X$, and hence $x \notin \mathcal{M}$. $\square$

As a corollary of Proposition A, we observe that the set of coalitionally stable states is always non-empty if switching costs are sufficiently high.

We now introduce an adjusted unperturbed dynamics which is based on coalitional stability, and we refer to it as *coalition-wise dynamics*. Basically, everything is as in the myopic team-wise dynamics used in Section 4, with the following difference. At each time two sets of projects, y and z, are randomly selected from the set of all subsets of P.[18] The state of the system changes from the current state x to $(x \setminus y) \cup z$ if there exists a coalition C that has the power to destroy all projects in y and to form all projects in z, and has advantage in doing so, i.e., each of its members strictly increases her utility if she behaves in such a manner. Clearly, this coalition is exactly of the type which is excluded by the definition of coalitional stability.

In the coalition-wise dynamics, we denote by $\mathcal{R}(c)$ and $\mathcal{A}(c)$, respectively, the set of recurrent states and the set of absorbing states when the level of switching costs is c. A characterization analogous to that in Proposition 2 can be provided for the current setup.

Proposition **B.** *Take a team formation model satisfying Assumption v0 and a coalition-wise dynamics. If c is high enough, then $\mathcal{A}(c) = \mathcal{R}(c) = \mathcal{M} = MTS$.*

Proof. We fix $c \geq \bar{c}$. We already know by definition that $\mathcal{A}(c) \subseteq \mathcal{R}(c)$.

Suppose now that $x \notin \mathcal{M}$. Then, there exists a project $p \in P$ such that $x \cup \{p\} \in X$. Such a project can be selected in the coalition-wise dynamics and, given Assumption v0, it will be formed by agents belonging to $n(p)$. Since no project will ever cease to exist, due to $c \geq \bar{c}$, we can conclude that $x \notin \mathcal{R}(c)$. Hence, $\mathcal{R}(c) \subseteq \mathcal{M}$.

Moreover, $\mathcal{M} \subseteq \mathcal{A}$, since by starting from a state $x \in \mathcal{M}$ no new project can be formed and existing projects are too costly to be destroyed.

Finally, we observe that $\mathcal{M} = MTS$ by Lemma 1. □

To provide results on stochastic stability, we have to introduce a perturbed dynamics. In particular, we can adopt each of the two perturbation schemes of Section 4. More precisely, in the uniform perturbation scheme, at every time s each project $p \in P$ is hit by an error with an i.i.d. probability ϵ: if p is an existing project then it disappears, while if p is a non-existing project than it is formed unless $x^s \cup \{p\} \notin X$. In the uniform destructive

[18]The details of this probabilistic selection are not important for the results as long as every pair y, z is chosen with positive probability.

perturbation scheme, on the other hand, only existing projects can be hit by perturbations. In the coalition-wise dynamics with uniform (destructive) perturbation scheme, we denote by $SS(c)$ the set of stochastically stable states.

PROPOSITION **C.** *Take a team formation model satisfying Assumption v0, a coalition-wise dynamics and a uniform perturbation scheme. If c is high enough, then $SS(c) = \mathcal{L}$.*

Proof. We fix $c \geq \bar{c}$. By Propositions 1 and B, we know that the set of absorbing states is the same as in the model of Section 4.

We now consider resistances. Since switching costs are so high that no agent will ever exit from an existing project, we have that a perturbation is required for every project to be destroyed. New projects will instead be formed in the unperturbed dynamics, since they are certainly advantageous due to Assumption v0. Therefore, $r^*(x, x') = \ell(x) - \ell(x \cap x')$. Hence, even the resistances between absorbing states are the same as in the model of Section 4.

Having the same set of absorbing states and the same resistances between them, we can invoke Proposition 3 to conclude that $SS(c) = \mathcal{L}$. □

Appendix B Farsightedly stable sets of states

[FOR ONLINE PUBLICATION]

The set of coalitionally stable states, as defined in Section 5, may be empty. In this appendix we apply results from Herings et al. (2009, 2010) to show that it is possible, on the other hand, to define a wider set of *farsightedly stable states*, whose existence is always granted. The two papers are applied respectively to network formation games and cooperative games, and as our setup generalizes both, the two papers also offer the possibility of finding examples of the non-existence of CS states for our model.

First of all we need to define *improving paths*, with a simple rephrasing of Definition 2.

DEFINITION **B** (Improving paths). *Given a state x there is an improving path from x to another state x' if there exists a subset $C \subseteq N$ such that*

(i) there exists a set of projects $y \subseteq x$ such that $\forall\ p \in y$, $\exists\ i \in C$ such that $i \in n(p)$;

(ii) there exists a set of projects z such that $z \cap x = \emptyset$ and, $\forall\ p \in z$, if $j \in n(p)$ then $j \in C$;

(iii) $x' = (x \backslash y) \cup z \in X$ and for any agent $i \in C$ we have that $u_i(x') > u_i(x)$. □

Definition B generalizes *farsighted improving paths* from Definition 3 in Herings et al. (2009) and Definition 1 in Herings et al. (2010). For a state x we can define as $F(x)$ the set of all the states x' that can be reached from x along improving paths, and as $C_{x \to x'}$ the coalition that is profitably moving from state x to state x'. It is clear that a state x is CS if and only if $F(x) = \emptyset$.

Now, the improving path promoted by a coalition is moving on the partially ordered set X of all possible states to a new state. What if another coalition were to start a new path from this new state? This could harm the members of the original coalition, who would not have been *farsighted* enough. To increase the rationality of the agents the following definition is used.

Definition **C** (Farsightedly stable sets of states)**.** *A set of states $S \subseteq X$ is* farsightedly stable *if:*

(i) for any $x \in S$, $x' \notin S$, such that $x' \in F(x)$, there is an $x'' \in F(x')$ such that there is an agent $i \in C_{x \to x'}$, for which $u_i(x'') < u_i(x)$;

(ii) for any $x' \in X \backslash S$, $F(x') \cap S \neq \emptyset$;

(iii) there is no $S' \subsetneq S$ such that S' satisfies conditions (i) and (ii) above.

Definition C generalizes *farsightedly stable sets* from Definition 4 in Herings et al. (2009) and Definition 4 in Herings et al. (2010). Condition (i) says that if there is an improving path from a state x belonging to S, to another state x' outside it, then this is due to the fact that the improving path in question could possibly harm one of the members of the coalition in $C_{x \to x'}$, because of a new improving path, promoted by another coalition, from x'. Condition (ii) says that from every state outside X there is an improving path into X. Finally, since the previous two conditions are trivially satisfied by X itself, condition (iii) says that a set of states S is farsightedly stable if no proper subset of states of S is also farsightedly stable. A farsightedly stable set S of states always exists, and this is proven by the following proposition, which generalizes Theorem 2 in Herings et al. (2009) and Proposition 1 in Herings et al. (2010).

Proposition **D** (Existence of a farsightedly stable sets of states)**.** *For every team formation model characterized by a quintuple $(N, \mathbf{w}, P, X, \mathbf{u})$, there exists a farsightedly stable sets of states $S \subseteq X$.*

Proof. We proceed by contradiction. The set X of states, whose cardinality is given by a finite natural number ω_0, satisfies conditions (i) and (ii) in Definition C. Therefore, X may not be farsightedly stable because it does not satisfy condition (iii), i.e., there is a set $X_1 \subsetneq X$, whose cardinality is given by some finite natural number $\omega_1 < \omega_0$, which also satisfies those two conditions. If X_1 does also not satisfy condition (iii), we can iterate the reasoning. If we never find a set that satisfies all conditions from Definition C, it means that we have found an infinite series $\{\omega_i\}_{i\in\mathbb{N}}$ such that $\omega_{i+1} < \omega_i$ for every $i \in \mathbb{N}$. Such an infinite series is impossible, and this proves the statement. □

Appendix C Combinatorics for the connection model with heterogeneous skills

[FOR ONLINE PUBLICATION]

In this appendix we prove formally some results previewed in Example 6, where we introduced the network corresponding to ℓ overlapping matchings between n agents. First, we obtain that if the overlapping networks are randomly generated with uniform and independent probabilities, then, for any $\ell \geq 3$, as $n \to \infty$, the resulting network will almost surely be connected. This result is based on Molloy and Reed (1995), which generalizes Erdös and Rényi (1960) (see Chapter 2 of Vega-Redondo, 2007 for a less technical presentation of this approach).

Proposition **E.** *Consider ℓ independent and uniformly random matchings between $n \geq \ell$ agents, and the resulting network where two agents i and j are connected if and only if they are matched in at least one of those matchings. For any $\ell \geq 3$, the probability that the resulting network is connected goes to 1 as $n \to \infty$.*

Proof. Call $p(k)$ the probability that an agent i (note that the agents are ex-ante identical) has degree k in the network. We have that

$$p(\ell) = \frac{(n-1)(n-2)\dots(n-\ell)}{(n-1)^\ell}, \tag{a}$$

where, above, we have the number of all the ℓ matchings that do not repeat the identity of the agents i is matched to, and, below, all possible matchings between i and the other $n-1$ agents.

Theorem 1 in Molloy and Reed (1995) proves that if the following strict inequality holds

$$\lim_{n\to\infty}\sum_{k=1}^{n-1} k(k-2)p(k) > 0, \tag{b}$$

then the network is almost surely connected as n grows to infinity.

As $n \to \infty$, it follows from (a) that $p(\ell) \to 1$. Then $\sum_{k=1}^{n-1} k(k-2)p(k) \to \ell(\ell-2)$, and this latter quantity is strictly positive for any $\ell \geq 3$. □

Even if formally correct, the above result is asymptotic, while the intuition for, and the formal definition of, stochastic stability are based on a finite set of states. For this reason, we consider finite ℓ and n, and the ratio of the resulting networks that are not connected, on all the possible resulting networks. We give the proof here below, finding an upper bound for this ratio, and finding that, even for limited values of $\ell \geq 3$ and n, this ratio is bounded above by very low values.

Proposition F. *Consider ℓ independent and uniformly random matchings between $n = 2m \geq \ell$ agents, and the resulting network where two agents i and j are connected if and only if they are matched in at least one of those matchings. Among all the possible matchings, the ratio of those that result in a corresponding disconnected network is bounded above by the quantity*

$$\sum_{k=1}^{m-1} \frac{\binom{m}{k}^{\ell}}{\binom{2m}{2k}^{\ell-1}}, \tag{c}$$

where $\binom{h}{j} = \frac{h!}{j!(h-j)!}$ is the standard notation for all the possible ways to extract j elements from a set of cardinality h.

Proof. Consider all the possible matchings between $2m$ agents, without partition as in a single roommate matching, with $m \in \mathbb{N}_+$. Call $N(m)$ the number of all those matchings. The combinatorial problem is equivalent to choosing, without replacement, two agents among the first $2m$, then two more among the remaining $2m-2$, and so on. In formulas we have

$$\begin{aligned} N(m) &= \frac{1}{m!}\binom{2m}{2}\binom{2m-2}{2}\cdots\binom{2}{2} \\ &= \frac{(2m)!}{m!\,2^m}, \end{aligned}$$

where the first term $\frac{1}{m!}$ divides by all possible permutations among the matchings.
Consider now ℓ separate matchings between the same $2m$ agents. Their number is $N(m)^{\ell}$.

We study the corresponding network, where a node between two agents is present if they are matched in at least one of the matchings. The number of ways in which ℓ distinct matchings result in a disconnected network is certainly bounded above by the following formula:

$$\#\ \textit{disconnected} \ \leq \ \sum_{k=1}^{m-1} \binom{2m}{2k} (N(k))^{\ell} (N(m-k))^{\ell}, \tag{d}$$

where the term $\binom{2m}{2k}$ considers all possible ways to make a partition of the $2m$ agents into two nonempty subsets, after which we consider all the possible matchings between them in the ℓ matchings. This number is an upper bound because there are disconnected resulting networks, with more than two components, that are counted several times.[19] [20] Finally, expressing the terms we have

$$\begin{aligned} \frac{\#\ \textit{disconnected}}{\#\ \textit{all}} \ &\leq \ \sum_{k=1}^{m-1} \frac{\binom{2m}{2k} (N(k))^{\ell} (N(m-k))^{\ell}}{N(m)^{\ell}} \\ &= \ \sum_{k=1}^{m-1} \frac{(m!)^{\ell}\ 2^{m\ell}\ (2k!)^{\ell-1}\ ((2m-2k)!)^{\ell-1}}{(2m!)^{\ell-1}\ (k!)^{\ell}\ 2^{k\ell}\ ((m-k)!)^{\ell}\ 2^{(m-k)\ell}}, \end{aligned}$$

and simplifying, the last expression is equivalent to (c). □

The term in (c) may be difficult to interpret, but Table 1 shows that it approaches very small quantities even for limited values of n and ℓ.[21]

[19] As an example, if $m = 4$, consider the case in which all the matchings overlap and the resulting network is made up of 4 disconnected pairs of nodes. This case is counted both when we consider two components of 2 and 6 elements respectively, and also when we consider two components of 4 elements each.

[20] There is also another reason why the right-hand part of inequality (d) is an upper bound for the left hand part: when the size of the two subsets of agents are asymmetric (e.g. 2 and 6) all the possible disconnected networks are counted in the reverse order as well (6 and 2). Therefore, the right-hand part is almost twice as extensive as the left-hand part. We do not consider this in the proof in order to avoid a distinction between even and odd m.

[21] It is clear that the result of Proposition F could be used to prove Proposition E, because the limit of (c), for $n \to \infty$, is 0. However, we find the present proof for Proposition E more direct and elegant.

Table 1: Computation of expression (c), for varying ℓ and $n = 2m$.

	n=10	20	30	40	50	100
ℓ= 3	0.16881	0.067102	0.039556	0.028198	0.021957	0.010454
4	0.015877	0.002998	0.001242	0.000678	0.000426	0.000103
5	0.001627	0.000154	$4.25 \cdot 10^{-5}$	$1.73 \cdot 10^{-5}$	$8.68 \cdot 10^{-6}$	$1.04 \cdot 10^{-6}$
6	0.000174	$8.08 \cdot 10^{-6}$	$1.46 \cdot 10^{-6}$	$4.43 \cdot 10^{-7}$	$1.77 \cdot 10^{-7}$	$1.05 \cdot 10^{-8}$