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May 2014

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May 2014

Abstract

An allocation of indivisible items among $n \geq 2$ players is *proportional* if and only if each player receives a *proportional* subset—one that it thinks is worth at least $1/n$ of the total value of all the items. We show that a proportional allocation exists if and only if there is an allocation in which each player receives one of its *minimal bundles*, from which the subtraction of any item would make the bundle worth less than $1/n$.

We give a practicable algorithm, based on players' rankings of minimal bundles, that finds a proportional allocation if one exists; if not, it gives as many players as possible minimal bundles. The resulting allocation is maximin, but it may be neither envy-free nor Pareto-optimal. However, there always exists a Pareto-optimal maximin allocation which, when $n = 2$, is also envy-free. We compare our algorithm with two other 2-person algorithms, and we discuss its applicability to real-world disputes among two or more players.

An Algorithm for the Proportional Division of Indivisible Items

"The just is . . . proportional; the unjust is what violates the proportion."

—Aristotle, *Nichomachean Ethics*, Book V, Chapter 3 (trans. Sir David Ross, Oxford University Press, 1953).

1. Introduction

The problem of fairly dividing indivisible items among $n \geq 2$ players has proven nearly intractable. Even for $n = 2$ players, there may be no allocation in which each player receives at least ½ of its total value of all the items.

More generally, an allocation is *proportional* iff each player receives at least $1/n$ of its total value for all the items. In this paper, we give an algorithm that yields a proportional allocation to as many players as possible; it turns out to be maximin, but may not be Pareto-optimal or envy-free (more details are given below).

We begin by showing that a proportional allocation exists iff there is an allocation in which each player receives a *minimal bundle*, or a proportional subset of items with the property that the subtraction of any item would make it less than proportional.[1] Players rank their minimal bundles; these rankings determine the allocation.[2]

We analyze other properties that allocations may possess, including

Envy-freeness (EF): Each player values the subset of items it receives at least as much as the subset of items received by any other player, thereby precluding envy

[1] The concept of a "minimal bundle" was first used in Brams, Kilgour, and Klamler (2012), but there it was applied only to two players with exactly the same preferences for individual items. We will say more about this different usage in section 4 (ftn. 16).

[2] The best-known way of allocating items to two players is "divide-and-choose," in which the divider divides the items into two approximately equal bundles so he or she will be relatively indifferent whichever bundle the chooser selects. Here, by contrast, each player specifies all its minimal bundles; the problem is how to assign one to each player so that the resulting allocation is not only proportional but also, insofar as possible, envy-free, efficient, and maximin.

While a proportional allocation is envy-free in the case of two players, proportionality does not imply envy-freeness if $n > 2$. Thus, EF is a more demanding property than proportionality.

Two other properties that an allocation may satisfy are

- *Efficiency or Pareto-Optimality (PO):* There is no other allocation that is at least as good for all players and better for at least one.
- *Maximinality (MX)*: The allocation maximizes the minimum rank, over all players who receive a minimal bundle, of the most preferred minimal bundle.[3]

While PO is applicable to players receiving proportional allocations (i.e., minimal bundles), it can also be applied to players receiving less-than-proportional allocations if a proportional allocation to all the players is not possible. We call an allocation that assigns a minimal bundle to every player *complete*; any other allocation is *partial*. There may be more than one complete allocation, and even a complete allocation may not assign every item.

MX is related to a property of fairness championed by Rawls (1971), among others, who argued that one should try to help the disadvantaged to the extent possible. In the context of proportional fair division, it means ensuring that the lowest rank of a minimal bundle received by any player is as high as possible.

Proportional allocations, even if not complete, offer a good starting point for the fair division of indivisible items. In this paper, we give a practicable algorithm that

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[3] We show later, in Example 4, that an allocation may fail MX but satisfy PO if there is another allocation in which some players receive two or more lower-ranked minimal bundles, which give them greater value than a single higher-ranked minimal bundle, without degrading the minimal bundles that other players receive.

produces a *maximally proportional* allocation—whereby as many players as possible receive minimal bundles—that is also MX. Our algorithm uses a method we call *fallback allocation*, in which players "fall back" on their preferences from more preferred to less preferred minimal bundles.[4]

Our algorithm produces a maximally proportional allocation—proportional for as many players as possible—that is also MX, but this allocation may be neither PO nor EF. In fact, these properties are *independent*: Even when one property is satisfied, another may fail. Thus, one cannot guarantee that some proportional allocation satisfies all three properties, and sometimes not even two.[5] Indeed, a proportional allocation that is EF can be Pareto-dominated by a PO and MX allocation that is not EF.

Despite these limitations, players will often have an incentive to be truthful about their minimal bundles when using our algorithm. If they exaggerate by specifying a bundle to be minimal that is of greater-than-minimal size, then they may receive an allocation that is not even proportional. To be sure, a player can sometimes benefit by exaggerating its claim or by concealing information about its minimal bundles, but because such a strategy can backfire, it will presumably be eschewed by risk-averse players.

As an alternative to proportional allocations, Procaccia and Wang (2014) defined a "maximin share guarantee:" Each player's value for its allocation is at least the value of

[4] Analogous procedures have been applied to bargaining (Brams and Kilgour, 2001), voting (Brams and Sanver, 2008), and coalition formation (Brams, Jones, and Kilgour, 2005). The latter two articles, in somewhat modified form, are chapters 3 and 8 of Brams (2008). The concept of a "minimal bundle" was first used in Brams, Kilgour, and Klamler (2012), but there it was applied only to two players with exactly the same preferences for individual items. We will say more about this different usage in section 5 (ftn. 15).

[5] If each player receives at most one minimal bundle, there exists a maximally proportional PO-MX allocation, but it need not be EF. By the same token, if each player receives an EF allocation, it need not be either PO or MX. These incompatibilities among properties are illustrated later by Example 5.

its least preferred bundle after it divides the items into as many bundles as there are players. While they showed that allocations with the maximin-share guarantee for all players may not exist, they provide an algorithm that guarantees each player at least 2/3 of its maximin share. Possible connections between this algorithm and ours remain to be explored.

The paper proceeds as follows. In section 2 we prove a proportionality theorem, showing that the use of minimal bundles is necessary and sufficient to ascertain whether a proportional allocation is complete. We also give several examples illustrating partial and complete proportional allocations and indicate which properties they satisfy.

In section 3 we give our algorithm, showing that it yields maximally proportional allocations that are also MX, but these allocations need not be EF or PO. In section 4, we prove the independence of the three properties and illustrate the manipulability of our algorithm.

In section 5 we focus on the 2-player case, comparing proportional allocations using our algorithm with allocations given by two other fair-division procedures. One uses ordinal, and the other cardinal, information on players' preferences for individual items (not minimal bundles). In section 6, we summarize our results and discuss the relevance of our analysis to settling real-life disputes, such as dividing an inheritance among multiple heirs or settling the claims of multiple parties in a lawsuit, and we draw several conclusions.

2. The Proportionality Theorem and Examples

We make three assumptions:

1. There are $n \geq 2$ players, with $m \geq n$ items available to be allocated.

2. All items, including the players' least-preferred items, have positive utility for all the players.
3. The utility of a bundle of items to players is the sum of the utilities of the items that comprise it, so there are no synergies, positive or negative.[6]

Before stating the proportionality theorem, we next give formal definitions of the concepts to be used:

• A subset S is *proportional* for player i iff i regards S as worth at least $1/n$ of i's total value for all items.

• An *allocation* is an ordered collection of non-overlapping subsets, $A = (S_1, S_2, \ldots, S_n)$, with the interpretation that player i receives bundle S_i. Note that $S_i \cap S_j = \phi$ whenever $i \neq j$—that is, each item is assigned to at most one player.

• An allocation $A = (S_1, S_2, \ldots, S_n)$ is proportional for player i iff S_i is proportional for i.

• If an allocation A is proportional for all players, A is a *complete proportional* allocation. If A is proportional for some but not all players, A is a *partial proportional* allocation.

• If an allocation A does not assign all items to the players, we call the unassigned items *excess items* for A. Even a complete proportional allocation may have excess items.

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[6] This assumption of "additive separability" is not necessary, but without it two different bundles of a player, if valued the same, might not be minimal with respect to the subtraction of a particular item. For example, because of synergy with other items in a bundle, an item's subtraction from a bundle might cause a loss that makes the bundle less than minimal, whereas this item's subtraction from an equally valued other bundle would not have this effect.

Theorem 1 (Proportionality). *Let Q be any subset of players. An allocation that is proportional for all players in Q exists iff there is an allocation in which every player in Q receives one of its minimal bundles. In particular, a complete proportional allocation exists iff there is a complete allocation in which every player receives one of its minimal bundles.*

Proof. An allocation in which each player receives one of its minimal bundles is clearly a proportional allocation. To show the converse, fix a proportional allocation and consider the subset T assigned to player $i \in Q$. Because the allocation is proportional, T must be worth at least $1/n$ to i.

If T is not already a minimal bundle for i, remove items from T, beginning with any item that is least preferred by i, stopping as soon as removing the next item would make the remaining subset less than proportional. The current subset must be a minimal bundle for i. Moreover, because it is contained in the original allocation to i, it cannot overlap any other player's allocation. Proceeding in this way demonstrates that every player's proportional allocation contains a minimal bundle for that player. Therefore, the existence of the original allocation implies the existence of an allocation of minimal bundles to the players in Q. ■

Theorem 1 implies that if no complete allocation of minimal bundles exists, then no complete allocation exists. This is not to say, however, that there is always a complete allocation, even if there are at least as many items as players. A partial allocation of minimal bundles may be the best that one can do, as we will illustrate with an example in which one or more players receive either no items or items of insufficient value to constitute a minimal bundle.

Example 1. $n = 3$ players, {A, B, C}; 3 items, {1, 2, 3}, to be allocated. Assume the players' utilities for the items are $u_A(1, 2, 3) = (40, 35, 25)$, $u_B(1, 2, 3) = (35, 40, 25)$, and $u_C(1, 2, 3) = (40, 25, 35)$, respectively. Then their minimal bundles are all single items, ranked as follows:[7]

A: $1 \succ 2$

B: $2 \succ 1$

C: $1 \succ 3$

There are two complete proportional allocations, (1, 2, 3) and (2, 1, 3) to (A, B, C). (Henceforth, we indicate allocations in the alphabetical order of the players.) In both allocations, each player receives at least 1/3 of its total value. Note that only (1, 2, 3), which Pareto-dominates (2, 1, 3), is PO.

In allocation (2, 1, 3), A and B will envy each other, because each player prefers the assignment of the other player. But even the efficient proportional allocation, (1, 2, 3), causes envy, because C envies A for obtaining item 1, which C prefers to the item (item 3) that it receives. The inefficient proportional allocation also makes C envious, but of B (rather than A) for obtaining item 1.

Example 2. $n = 4$ players, {A, B, C, D}; 4 items, {1, 2, 3, 4}, to be allocated. Assume the players rank their minimal bundles as follows:

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[7] Thus, A's minimal bundles are {1} and {2}, and A prefers item 1 to item 2. In other examples, minimal bundles may contain more than one item. In this and the next three examples (Examples 1-4), it can be shown easily that the players' rankings of minimal bundles are consistent with some set of utilities. Thus in Example 1, the utilities must be at least 1/3 of the total for each of the player's two most-preferred items and, therefore, less than 1/3 for its least-preferred item. Barring ties, when a player's minimal bundles do not overlap, there must be fewer than n of them in total, because each minimal bundle must be worth at least $1/n$ of the total.

A: $1 \succ 2$

B: $2 \succ 1$

C: $1 \succ 3$

D: $1 \succ 34$

where 34 denotes the subset {3, 4}.

Because there are 4 players and 4 items, and item 4 is not a minimal bundle for any player, no complete proportional allocation can exist, as some player would have to receive item 4 alone. But there are four partial proportional allocations, in which item 4 is an excess item, namely (1, 2, 3, -), (2, 1, 3, -), (2, -, 3, 1), and (-, 2, 3, 1). Three of these are PO, but (2, 1, 3, -) is not.

Giving the excess item to the player who receives no minimal bundle would maximize the minimum value that any player receives, which would extend the idea of MX to nonminimal bundles. But our algorithm in the next section, which we restrict to the allocation of minimal bundles, does not make any such extension.[8]

There are four more partial proportional allocations, (1, 2, -, 34), (2, 1, -, 34), (2, -, 1, 34), and (-, 2, 1, 34). Only the second is not PO. None of the aforementioned partial proportional allocations is EF.

Example 3. n = 4 players, {A, B, C, D}; 4 items, {1, 2, 3, 4}, to be allocated. Assume the players rank their minimal bundles as follows:

A: 1

[8] To apply this algorithm to the allocation of individual items would require that players who do not receive minimal bundles rank items in a manner analogous to their ranking of minimal bundles.

B: $1 \succ 2$

C: $1 \succ 2 \succ 3$

D: $1 \succ 23 \succ 24 \succ 34$

Note that all four players' minimal bundles are consistent with the ranking $1 \succ 2 \succ 3 \succ 4$ of individual items.

Six partial proportional allocations are PO: (1, 2, 3, -), (-, 2, 3, 1), (1, 2, -, 34), (1, -, 2, 34), (-, 1, 2, 34), and (-, 2, 1, 34). All are maximal, assigning items to three players, but none is envy-free—the other three players will envy whichever player receives item 1. In addition, C will envy B in the first three allocations for obtaining item 2.

As in Example 2, in the first two allocations, the excess item, 4, could be assigned to the player who received nothing (A or D) to maximize the minimum value of any player. By comparison, the allocation of (1, 2, -, 34) makes all players except A envious. C, who receives nothing, envies every other player.

Example 1 showed that a complete proportional allocation may not be EF. Our final example in this section shows that a proportional allocation may fail PO as well as EF, even when all players receive their most preferred minimal bundles.

Example 4. $n = 4$ players, {A, B, C, D}; 6 items, {1, 2, 3, 4, 5, 6}, to be allocated. Assume the players rank their minimal bundles as follows:

A: $12 \succ 3 \succ 4$

B: $34 \succ 1 \succ 2$

C: 5

D: 6

There is a complete proportional allocation of the players' most-preferred minimal bundles: (12, 34, 5, 6). But this allocation is Pareto-dominated by the complete proportional allocation, (34, 12, 5, 6), in which A and B receive both their 2nd and 3rd most-preferred minimal bundles.

To see why (34, 12, 5, 6) dominates, note that, for A, each of items 3 and 4 must be worth at least ¼ of the total value, so together they must be worth at least ½. On the other hand, bundle 12 must be worth less than ½, because item 1 is not a minimal bundle and must, therefore, be worth less than ¼, and similarly for item 2. Thus, bundle 12 must be worth at least ¼ but less than ½ of the total value. Therefore, A prefers 34 to 12. Similarly, B prefers 12 to 34.

Also note that C and D receive the same assignments in both of the complete proportional allocations, (12, 34, 5, 6) and (34, 12, 5, 6). Although the latter allocation is PO, it does not assign minimal bundles to A and B. By contrast, our algorithm, described in section 3, cannot return (34, 12, 5, 6), because it assigns each player either a minimal bundle or nothing.[9] We will comment further on this exclusion later.

The preceding examples show that proportional allocations may be complete or partial (the allocations in Example 1 and 4 are complete; those in Examples 2 and 3 are partial). Even a complete allocation may cause envy, whether it is PO (Example 1) or not (Example 4). We next describe our algorithm and assess what it does, and does not, do.

3. The Proportional Algorithm (PR) and Its Properties

[9] Note that we do not regard return (34, 12, 5, 6) as an allocation of minimal bundles, because it assigns A not one minimal bundle but two—and similarly for B. There are additional proportional allocations of (3, 1, 5, 6), (3, 2, 5, 6), (4, 1, 5, 6), and (4, 2, 5, 6). The first Pareto-dominates the other three, and it is the one that is most preferred by A and B if (34, 12, 5, 6) is not available. But it is not MX, and because it is Pareto-dominated by (12, 34, 5, 6), it is also not PO.

The proportional algorithm we propose, which we call PR, yields an allocation that is maximally proportional—if not complete, then proportional for as many players as possible—that is also MX but may be neither PO nor EF. PR's rules are as follows:

1. Each player specifies all its minimal bundles and ranks them from best to worst. Ties are allowed – i.e., rankings need not be strict.

2. Starting with the players' top-ranked minimal bundles (level 1), descend the players' rankings, one rank at a time, and find an allocation of minimal bundles for as many players as possible.[10]

3. If the allocation is complete, go to step (4). Otherwise, repeat step (2) until there are no lower-ranked minimal bundles.

4. If there are two or more complete or maximizing partial allocations, discard those that are inefficient (Pareto-dominated); if more than one remains, choose one at random at the highest minimum level.[11]

We emphasize that PR allocates only minimal bundles so as many players as possible receive them.[12] In Example 4, this occurs at level 1, when all players receive a

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[10] This is the manner in which players fall back on their preferences. If there are ties in a player's ranking, then, at the k^{th} iteration, any of its minimal bundles may be assigned to it, provided the minimal bundle assigned is strictly less preferred than at most $k - 1$ of its other minimal bundles.

[11] Instead of a random choice, other tie-breaking criteria could be invoked, such as choosing the allocation with the highest Borda score, based on the ranks of the minimal bundles, or choosing an envy-free allocation if one exists.

[12] We also considered an algorithm in which any minimal bundle that is a superset of a minimal bundle of another player is removed prior to steps 2—4 in order to allocate subsets that are as small as possible. But this variant may in fact preclude a complete allocation, as the following example demonstrates:

A: 1
B: $2 \succ 3$
C: $34 \succ 1$

At level 1, PR gives a complete allocation of (1, 2, 34). But if we eliminate 34 because it is a superset of 3, PR gives only a partial allocation of (1, 2, -) or (-, 2, 1).

minimal bundle, making the allocation complete. (But this allocation is not PO, as we showed in section 2.) There is also a complete allocation in Example 1, which occurs at level 2.

In Examples 2 and 3, PR does not produce a complete allocation, even when the descent reaches the bottom level (level 2 in Example 2 and level 4 in Example 3), but as noted earlier, no complete proportional allocation exists. Hence, maximally proportional allocations in these examples are partial.

In Example 3, in particular, there are two maximally proportional allocations of (1, 2, 3, -) and (-, 2, 3, 1) at level 3. The remaining maximally proportional allocations, (1, 2, -, 34), (1, -, 2, 34), (-, 1, 2, 34), and (-, 2, 1, 34), occur at level 4. Only the former two are MX.

PR searches for a complete allocation at level 4. Finding none, it selects a partial allocation "at the highest minimum level" (rule 4)—that is, at level 3 in Example 3, which ensures that the allocation is MX.

This leaves item 4 "on the table" in Example 3; it is also on the table in Example 2 if C receives item 3 at level 2 when A and B receive, respectively, items 1 and 2 (by rule 4). Thus, PR may produce excess items, which it makes no effort to allocate.[13]

Examples 1, 2, and 3 show that PR may not produce an EF allocation, and Example 4 that PR may not produce a PO allocation. However, we do have

Theorem 2. A *PR allocation is maximally proportional and MX*.

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[13] For it to do so would require information of the kind described in ftn. 8. Also, to ensure that the resulting allocation is PO would require that PR descend to the bottom-ranked minimal bundles of all the players in order to determine whether a complete proportional allocation that occurs earlier is Pareto-dominated (see Example 4, in which this is the case).

Proof. If there is a complete allocation, it is MX when the descent stops, because any other complete allocation not at this level must include a minimal bundle at a lower level.

If there is no complete allocation, the algorithm descends to the lowest-ranked minimal bundle. Then, if there is an allocation to the same number of players at a higher minimum level (as in Example 3 above), PR chooses this allocation, according to rule 4, making it MX. ■

We showed earlier that allocations produced by PR, whether complete or partial, need not be EF. The interrelations of the properties of proportional allocations are surprisingly complex—for example, a non-EF allocation can Pareto-dominate an EF allocation, as we prove in the next section.

4. Incompatibilities and Manipulability

In this section we show that none of our three properties implies any other, so there is no guarantee that a complete allocation satisfying one property will satisfy another. The situation is different when $n = 2$, which we consider later.

Theorem 3 (Independence). If $n > 2$, *a complete proportional allocation that satisfies one of EF, PO, or MX does not necessarily satisfy either of the other properties. Moreover, a non-PO proportional allocation can be EF when a PO proportional allocation that Pareto-dominates it is not.*

Proof. Consider the following example:

Example 5. $n = 4$ players, {A, B, C, D}; 6 items, {1, 2, 3, 4, 5, 6}, to be allocated. Assume the players rank their minimal bundles as follows:

A: $1 \succ 2 \succ 3$

B: $1 \succ 4$

C: $2 \succ 5$

D: 6

For complete proportional allocations, we show that none of the three properties implies any other:

• *EF vs. PO*. Proportional allocations $(1, 4, 2, 6)$, $(2, 1, 5, 6)$, and $(3, 1, 2, 6)$ are PO but not EF, whereas $(3, 4, 5, 6)$ is proportional and EF but not PO. Thus, for proportional allocations, neither PO nor EF implies the other. Moreover, all three PO allocations Pareto-dominate the EF allocation, proving the second claim of the theorem.

• *MX vs. EF*. Allocations $(1, 4, 2, 6)$, $(2, 1, 5, 6)$, and $(2, 4, 5, 6)$ are MX but not EF, whereas $(3, 4, 5, 6)$ is EF but not MX. Thus, for proportional allocations, neither MX nor EF implies the other.

• *PO vs. MX*. Allocation $(2, 4, 5, 6)$ is MX but not PO, whereas $(3, 1, 2, 6)$ is PO but not MX. Thus, for proportional allocations, neither MX nor PO implies the other.

To complete the proof, we show that a proportional allocation that satisfies any of the three properties may fail the other two. In Example 5, the proportional allocation $(3, 1, 2, 6)$ is PO but neither EF nor MX; the proportional allocation $(3, 4, 5, 6)$ is EF but neither PO nor MX; and the proportional allocation $(2, 4, 5, 6)$ is MX but neither PO nor EF. Thus, a complete proportional allocation that satisfies one of the three properties may fail both of the other two. ■

Interestingly enough, however, one pair of properties can always be satisfied:

Theorem 4. *If a complete proportional allocation of minimal bundles exists, then there is a complete proportional allocation that is both PO and MX.*

Proof. Clearly, at least one complete proportional allocation produced by PR must be MX. If it is not PO, then it must be Pareto-dominated by a PO allocation. That PO allocation must necessarily be MX. ■

In Example 5, proportional allocations (1, 4, 2, 6) and (2, 1, 5, 6) are both PO and MX, but neither is EF. Nonetheless, there does exist a proportional EF allocation, namely (3, 4, 5, 6).

This conflict between PO and EF, in particular, has been noted elsewhere, though not in the context of proportional allocations (Brams, Edelman, and Fishburn, 2001; Brams and King, 2005). In the concluding section, we explore circumstances under which one might prefer an EF to a PO allocation, or vice versa, when there is no allocation that simultaneously satisfies both properties (as in Example 5).

We next consider the special case when $n = 2$.

Theorem 5. *Let $n = 2$. If a complete proportional allocation of minimal bundles exists, then there exists a complete proportional allocation that is EF, PO, and MX.*

Proof. In any complete proportional allocation, each player must receive a minimal bundle, so it must be assigned at least ½ of the total value of all the items. Such an allocation is EF. By Theorem 4, at least one complete proportional allocation must be PO and MX, so there is at least one allocation that satisfies all three properties. ■

This is not to say that there may not be other complete 2-player allocations that do not satisfy all three properties. Consider the following example:

Example 6. $n = 2$ players, {A, B}; 4 items, {1, 2, 3, 4}, to be allocated. Assume the players rank their minimal bundles as follows:

A: $12 \succ 14 \succ 24$

B: $124 \succ 13 \succ 23 \succ 34$

[These minimal bundles are consistent with the following utilities: $u_A(1, 2, 3, 4) = (35, 30, 10, 25)$; $u_B(1, 2, 3, 4) = (25, 21, 34, 20)$.]

There are three complete proportional allocations that are both EF and PO, (12, 34), (24, 13), and (14, 23). The latter two are MX—giving a player, at worst, its 3rd choice minimal bundle—whereas (12, 34) assigns B its 4th-choice minimal bundle.

We next consider some strategic properties of proportional allocations. As we have shown, there may be several proportional allocations, which can satisfy different properties. Independent of how these are prioritized, a natural question is whether a player can do better by not being truthful about its minimal bundles, either in specifying them or ranking them.

Theorem 6. *By falsely designating a bundle to be minimal when it is greater-than-minimal, a player can hurt itself under PR compared with being truthful.*

Proof. In Example 3, assume that C indicates its ranking of minimal bundles to be

C′: $1 \succ 2 \succ 34$ instead of C: $1 \succ 2 \succ 3$,

so it claims, falsely, that bundle 34 is minimal when, in fact, item 3 alone is a minimal bundle (the rankings of the other players remain the same). Then PR gives two partial proportional allocations, (1, 2, 34, -) and (1, 2, -, 34).

If one is chosen randomly, C, instead of obtaining item 3 with certainty, would obtain bundle 34 with probability ½, which, on an expected-value basis, is worse than being truthful because C—as well as the other players—ranks item 4 below item 3. Moreover, half the time C will get nothing. ■

But truthfulness is not always optimal:

Theorem 7. *By falsifying its preferences, a player can do better under PR than when it is truthful.*

Proof. Consider the following example:

Example 7. $n = 3$ players {A, B, C}; 4 items {1, 2, 3, 4}, to be allocated. Assume the players rank their minimal bundles as follows:

A: $1 \succ 2$

B: $1 \succ 2$

C: $1 \succ 2 \succ 34$

PR yields two complete allocations of (1, 2, 34) or (2, 1, 34), one of which is to be chosen randomly. But if A indicated that only item 1 was a minimal bundle,

A′: 1

the unique complete allocation would be (1, 2, 34), so A would obtain item 1 with certainty. Thus, A does better by truncating its preferences. ■

Taken together, Theorems 6 and 7 show that false claims may either help (Theorem 7) or hurt (Theorem 6) a player, at least in terms of expected utility, reflecting random choices in the last step of PR. Thus, manipulative strategies are a double-edged sword, risky to employ unless one has complete information about the preference rankings of the other players and can exploit it, either when specifying one's own minimal bundles or ranking them.

In the next section, we briefly describe and illustrate two 2-person procedures that elicit different information from the players than their ranking of minimal bundles. We compare the allocations they give with PR allocations, and more generally with allocations based on ranking minimal bundles.

5. Comparison with 2-Person Algorithms

It is not always possible, even with only two players, to give each an EF allocation. As a trivial example, assume $m = n = 2$, and that both players strictly rank the two items the same, so the only minimal bundle for both is their preferred item. An allocation is proportional for a player iff it receives the preferred item; the other player will be envious because it receives less than ½ the value of all the items.

If there are more than two items, however, and the rankings of minimal bundles by the two players are sufficiently different, there may be multiple proportional allocations, as we illustrate next.

Example 8. $n = 2$ players {A, B}; 6 items {1, 2, 3, 4, 5, 6}, to be allocated.

Assume that the players' utilities are $u_A(1, 2, 3, 4, 5, 6) = (25, 24, 23, 21, 4, 3)$, $u_B(1, 2, 3, 4, 5, 6) = (4, 25, 24, 21, 23, 3)$. It follows that their rankings of the individual items are

A: $1 \succ 2 \succ 3 \succ 4 \succ 5 \succ 6$

B: $2 \succ 3 \succ 5 \succ 4 \succ 1 \succ 6$

and that each has 13 minimal bundles, comprising either three or four individual items as follows (slashes between adjacent bundles indicate ties):[14]

A: $123 \succ 124 \succ 134 \succ 234 \succ 125 \succ \underline{126}/135/2456 \mid \succ 136/235/3456 \succ 145/236$

B: $235 \succ 234 \succ 254 \succ \underline{354} \succ 231 \succ 236/251/3416 \mid \succ 256/351/5416 \succ 241/356$

Each player's minimal bundles are worth, in order, 72, 70, 69, 68, 53, 52/52/52, 51/51/51, 50/50 utiles to the player.

As shown by the vertical bars in the players' rankings of minimal bundles, there is a complete proportional allocation, (126, 354), at level 6 that is also MX, which we have underscored. In addition, there are three complete proportional allocations that are EF and PO but not MX: (124, 356), (134, 256), and (136, 254). A prefers (124, 356) and (134, 256) to the MX allocation, whereas B prefers (136, 254) to the MX allocation.

Notice that 11 of the 13 minimal bundles of each player contain 3 items. The two that contain 4 items cannot be part of a complete proportional allocation, because each player must receive at least three items to obtain at least 50 utiles.

In real-life fair division, it is reasonable to expect that some complete allocations will involve an unequal split of the items. If there are, say, 10 items to be allocated, a player's top 5 items are almost surely not minimal; more likely, players will find their top

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[14] The maximum number of minimal bundles that a player can have occurs when the players value all items the same, and hence all minimal bundles are the same size. If this were the case when $m = 6$, as in Example 8, each player's minimal bundles would contain 3 items, and there would be $\binom{6}{3} = 20$ of them.

For a variety of combinatorial results on the fair division of indivisible items, see Brams and Fishburn (2000), Edelman and Fishburn (2001), Brams, Edelman, and Fishburn (2003), Budish (2011), and Brams, Kilgour, and Klamler (2012). Results on the computational complexity of determining allocations that satisfy different levels of proportionality and envy-freeness are given in Aziz, Gaspers, Mackenzie, and Walsh (2013).

2 or 3 items to be worth at least ½ the total value of all the items. By the same token, it may take 6 or 7 of their bottom-ranked items to constitute a minimal bundle.

We next compare the EF, PO, and MX allocation of (126, 354) with the results provided by two 2-person fair-division algorithms. The first uses ordinal information based on player rankings of individual items, whereas the second uses cardinal information on player valuations of the items.

The AL algorithm requires that the players rank individual items—not minimal bundles—from best to worst (Brams, Kilgour, and Klamler, 2014). It then effects an equal division of as many items as possible that are *pairwise envy-free*: Each player can match its items to the other player's items such that each player pairwise prefers each of its items to the item with which it is matched.[15]

AL, based just on A's and B's ranking of individual items, does *not* give a complete allocation that is pairwise EF but, instead, a partial allocation of (13, 25): A pairwise prefers item 1 to item 2 and item 3 to 5, and B pairwise prefers item 2 to item 3 and item 5 to item 1. Hence, neither player is envious of the other player for the allocation of these four items.

As for items 4 and 6, AL finds no pairwise matching, because both A and B prefer item 4 to item 6. Consequently, AL puts these items into a "contested pile," precluding them from being allocated to either player.[16]

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[15] What we call pairwise envy-freeness is called "necessary envy-freeness" in Bouveret, Endriss, and Lang (2010). For a simple necessary and sufficient condition for there to be a complete allocation based on pairwise envy-freeness ("Condition D"), see Brams, Kilgour, and Klamler (2014).

[16] See Brams, Kilgour, and Klamler (2012) for an algorithm, the "undercut procedure," that allocates items in the contested pile when, as here, both players agree on the ranking of the items in the contested pile. (Note that their preferences on subsets of items may be very different.) In the undercut procedure, a minimal bundle refers to a subset that is minimal because the subtraction of one item causes it to be worth less than ½ the total value; or to a subset in which the substitution of a less preferred for a more preferred item would have the same effect. This is the difference in usage referred to in ftn. 1.

As we showed earlier, the application of PR, based on the ranking of minimal bundles, gives a complete EF, PO, and MX allocation of (126, 354). Clearly, preference rankings of minimal bundles are more informative than preference rankings of individual items; in Example 8, they yield a complete allocation when AL fails to find one.

The second 2-person fair-division algorithm we describe, and apply to Example 8, is called "adjusted winner" (AW) (Brams and Taylor, 1996, 1999). Under AW, the two players distribute 100 points across all the items according to how much they value each. Because the value of a set of items is the sum of the points of the items it contains, the points can be interpreted as utilities that are additive.

AW is not strictly applicable to the fair division of indivisible items, because one item must generally be divided between A and B to make the allocation *equitable* (each player gets the same number of its points) as well as EF and PO. We illustrate its application to Example 8 by assuming that the points that A and B place on the six individual items are the same as the utilities we earlier assumed the players have for them:

Item	**1**	**2**	**3**	**4**	**5**	**6**
A	<u>25</u>	24	23	21	4	3
B	4	<u>25</u>	<u>24</u>	21	<u>23</u>	3

Initially, AW allocates to each player the item(s) on which each puts more points (underscored), giving A item 1 (25 points) and B items 2, 3, and 5 (72 points). Because A is behind in points, AW next gives items 4 and 6 (24 points), on which the players tie, to A, so A now has a total of 49 points and B still has 72 points. To equalize the points of the two players and thereby satisfy equitability, B must give back an additional item, or a part thereof, to A.

We start with the item on which the ratio of B's points to A's points is the smallest, which is item 2 (ratio: 25/24). We equalize the points of each player by subtracting from B's total of 72 points the fraction x of item 2, which B values at 25, and add this fraction of item 2, which A values at 24, to A's total of 49 points:

$$72 - 25x = 49 + 24x$$

This equation simplifies to $49x = 23$, so $x = 23/49 \approx 0.469$, which is the fraction of item 2 that B loses and A gains.

This division of item 2 gives each player a total of

$$72 - 25(0.469) = 49 + 24(0.469) \approx 60.3 \text{ points}$$

or more than 3/5 of the total value of all items. By comparison, the minimal-bundle allocation of (126, 354) to (A, B) gives 52 utiles to A and 68 utiles to B.[17]

Thereby AW wipes out the 16-utile difference between A and B if item 2 can be divided in the manner AW prescribes. But if this is not possible, then PR gives the allocation closest to being equitable because it is MX. For comparison to the other procedures, AW gives the allocation (146, 34), with item 2 to be split between A and B in the ratio 23:26.

AW is the most demanding of the three algorithms in requiring that the players provide cardinal information, whereas AL is least demanding in requiring only ordinal

[17] Note that we have assumed not only that item 2 is divisible, but also that a player's utility for any portion of item 2 is equal to that same fraction of the player's utility for the entirety of item 2.

information about the individual items. PR stands in between in requiring the specification and ranking of minimal bundles. (We assumed that this information could be determined from summing the utilities of individual items, but utilities are not necessary. In fact, the assumption of additive utilities would be violated if there are synergies among the items.)

Synergies, if present and different for different players or items, would facilitate complete allocations. Although AL has the advantage of giving partial EF allocations for as many items as possible, it may miss a complete EF allocation that could be found by PR, which does not use pairwise matchings. But the most important difference between PR, and AL and AW, is that PR is applicable to *n*-person fair division, whereas AL and AW are strictly limited to 2-person fair division. In the case of AW, it generally designates one item to be divided, which cannot be determined in advance.

6. Summary and Conclusions

We have shown, using players' rankings of their minimal bundles, that proportional allocations (partial or complete) may or may not satisfy EF, PO, or MX. The last property ensures that the most preferred minimal bundle of the worst-off player is ranked as highly as possible.

As we noted, there may, paradoxically, be a complete proportional allocation that is EF but not PO, while a PO allocation that Pareto-dominates the EF allocation is not itself EF. This conflict between EF and PO is probably rare, but it is worth noting for those who prize EF over PO. In dividing an inheritance, for example, the heirs may prefer to avoid envy rather than achieve efficiency if the envy would lead to recrimination and conflict.

Likewise, there may be a conflict between EF and MX. In this situation, a choice seems hard, because both properties ameliorate, albeit in different ways, the problem of inequality. Finally, while there may be a conflict between PO and MX, this conflict is rendered less severe by Theorem 4, which guarantees that if there is a complete allocation, then there is always one that satisfies both properties. Furthermore, in the 2-player case, this complete allocation must satisfy all three properties.

When each player can obtain, at most, one minimal bundle, PR gives a PO-MX allocation to as many players as possible. However, if players can be assigned more than one minimal bundle, there may be a non-PR allocation that Pareto-dominates a PR allocation, as Example 4 showed. It may be possible to extend PR to the allocation of excess items or minimal bundles so as to satisfy PO, but we have not done so here.

Because a player can do worse by indicating a larger-than-minimal bundle is minimal, exaggeration of one's claims may backfire under PR. But manipulative strategies, such as truncating one's list of minimal bundles to exclude those that are less preferred can also help, so being truthful about one's minimal bundles is not always an optimal strategy.

If there are many items, players may have difficulty in ranking all their minimal bundles. This is true even if there are only two players, because minimal bundles may include (i) relatively few top-ranked items, (ii) relatively many bottom-ranked items, and (iii) many mixes in between. Also, if preferences are positively correlated because the players tend to desire the same items, identifying and allocating non-overlapping minimal bundles becomes more difficult.[18]

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[18] The computational complexity of PR, under different conditions, deserves study that we have not undertaken here.

Just as a plethora of items can tax players in using divide-and-choose, the specifying and ranking of all one's minimal bundles can be burdensome under PR. A practical solution to this problem would be to carry out an allocation in stages, so that relatively few items are allocated to the players at each stage. To be sure, the players may have difficulty in agreeing on which items should be allocated at each stage.

In the case of two players, AL and AW in general give different solutions from PR. Which solution is best suited to resolve a dispute will depend in part on what information the parties in the dispute are able and willing to provide about their preferences.

By way of conclusion, EF—certainly the most studied property in fair division—is not only a difficult property to satisfy, especially when $n > 2$, but also may be at odds with PO and MX because of incompatibilities among the three properties. Therefore, it seems sensible to make proportional allocations, which are less demanding than EF allocations, the starting point.

In allocating only minimal bundles, which tend to be small, PR ensures, insofar as possible, that as many players as possible receive proportional subsets. Unlike AL and AW, in which players rank or score individual items, PR allows for synergies, which players' rankings of minimal bundles can reflect. Finally, the allocation of single minimal bundles that PR yields is PO and MX—and EF as well if there are only two players.

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