Revealed time-preference

Dziewulski, Pawel

Department of Economics, University of Oxford

11 June 2014

Revealed time-preference*

Paweł Dziewulski[†]

Preliminary draft: June 2014

Abstract

Consider an experiment in which subjects are asked to choose between pairs consisting of a monetary payment and a time-delay at which the payment is delivered. Given a finite set of observations, under what conditions the choices of an individual agent can be rationalised by a discounted utility function? We develop an axiomatic characterisation of time-preference with various forms of discounting, including weakly present-biased, quasi-hyperbolic, and exponential, and determine the testable restrictions for each specification. Moreover, we discuss identification issues which may arise in this class of experiments.

Keywords: revealed preference, testable restrictions, rationalisation, time-preference, discounted utility

JEL Classification: C14, C60, C61, D11, D12

1 Introduction

Consider an experiment in which, in every trial, a consumer is presented with a finite set of pairs (m, t) consisting of a monetary payment $m \in \mathbb{R}_+$ and a time-delay $t \in \mathbb{N}$ at which the payment is delivered. The agent is allowed to choose exactly one option from the set. Suppose that we can observe both the set of feasible options, denoted by A, and the corresponding choice (m, t). Given a finite number of repetitions of the experiment, under what conditions can the choices of the consumer can be rationalised? In other

*I am especially grateful to John Quah for his invaluable help during writing this paper.

[†]Department of Economics, University of Oxford, Manor Road Building, Manor Road, Oxford OX1 3UQ, United Kingdom. Email: pawel.dziewulski@economics.ox.ac.uk.

words, when is it possible to determine a function $v : \mathbb{R}_+ \times \mathbb{N} \to \mathbb{R}$ such that, for any observable set of options A, we have

$$v(m, t) \geq v(n, s), \text{ for all } (n, s) \in A?$$

Clearly, without any additional conditions imposed on v, the above problem is trivial, as any constant function would rationalise an arbitrary set of observations. For this reason, given our setting, we focus on a class of functions which are strictly increasing with respect to monetary payments and strictly decreasing with time-delays. In particular, we are interested in preferences that are separable with respect to the two variables. That is, we discuss conditions under which the observable choices of agents can be supported by a utility function $v(m, t) := u(m)\gamma(t)$, where $u : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing, while $\gamma : \mathbb{N} \to (0, 1]$ is strictly decreasing. For obvious reasons, we shall refer to γ as to a *discounting function*.

The separable specification of preferences seems to be especially important from the economic point of view. The discounted utility model plays a crucial role throughout the economic analysis and is widely accepted as a valid normative standard for public policies, as well as a descriptively accurate representation of the actual behaviour of economic agents. However, in the recent years an important question was raised concerning the form of the discounting function that reflects the actual time-preferences of consumers. In particular, alternative specifications of hyperbolic and quasi-hyperbolic discounting were proposed, which could explain various observations anomalous in the model of exponential discounting utility, formerly dominant in economics. See Frederick et al. (2002) for a detailed discussion concerning this topic.

We propose an axiomatic characterisation of time-preference in a framework where the domain of choices is restricted to pairs of monetary payments and time-delays. Our prize-time set-up is similar to the one discussed in Fishburn and Rubinstein (1982), Ok and Masatlioglu (2007), or Noor (2011). However, unlike in those papers, we do not take the preference relation of an agent as a primitive, rather, we assume that the observer can monitor only a finite number of choices that the consumer makes. This restriction significantly affects the conditions characterising time-preference. Since the observable choices induce only an incomplete preference ordering over the space of prize-time pairs (m, t), the question is how to extend the relation in a way that is consistent with a certain type of utility function. Whether this is possible or not determines if a given data set can be rationalised by a specific form of time-preference. The main motivation of this paper is to establish the testable restrictions of various models of inter-temporal choice. In particular, we are interested in conditions that would allow us to distinguish between different specifications of the discounted utility, including the hyperbolic, quasi-hyperbolic, and exponential models.

We consider our framework to be particularly relevant from the perspective of empirical applications. There are numerous examples of experiments in which subjects are asked to choose between different monetary payments delivered with various time-delays. This includes an extensive list of studies presented by Frederick et al. (2002, Table 1), as well as the works by Chabris et al. (2008, 2009), Andersen et al. (2008), Benhabib et al. (2010), or Dohmen et al. (2012). The design of the experiments allows us to apply our results directly to the sets of observations they generate.

We begin our discussion in Section 2, where we introduce the notation as well as some preliminary results. Throughout this paper our axiomatic characterisation is imposed on the directly revealed preference relation induced by the set of observations. We say that a pair (m, t) is *directly revealed preferred* to (n, s), whenever there exists at least one observation of the experiment such that both options are available, i.e., they both belong the corresponding feasible set A, and (m, t) is chosen. We shall denote $(m, t)\,\mathcal{R}^*(n, s)$. The main difficulty of our paper is to determine a pre-order that extends the directly revealed relation to the whole domain of the prize-time pairs $\mathbb{R}_+ \times \mathbb{N}$. Moreover, we need to guarantee that the ordering can be represented by a utility function that possesses desirable properties, in particular, monotonicity and separability.

We begin by stating the necessary and sufficient conditions under which the set of observations can be rationalised by a utility function $(m, t) \to v(m, t)$ that is strictly increasing with respect to monetary payments m and strictly decreasing in time-delays t. We define a partial order $\geq_X$ such that $(m, t) \geq_X (n, s)$ whenever $m \geq n$ and $t \leq s$, which is strict if at least one of the above inequalities is strict. We show that the set of observations can be rationalised in the above sense, whenever there is no sequence $\{(m^i, t^i)\}_{i=1}^n$ of pairs observed in the experiment such that every subsequent element dominates the preceding one with respect to $\mathcal{R}^*$ or $\geq_X$, and $(m^1, t^1) >_X (m^n, t^n)$. Therefore, we evoke a special case of *generalised cyclical consistency* discussed in Nishimura et al. (2013), as well as the Rationalisability Theorem II presented in the same paper.

Given the initial result, in Section 3 we concentrate on the axiomatic characterisation of preferences representable by a separable utility function $v(m, t) := u(m)\gamma(t)$. Clearly, as it is a special case of the previous representation, cyclical consistency is still a necessary condition, however, it is no longer sufficient. For this reason we introduce an alternative restriction called *dominance axiom*. Roughly speaking, the condition states that exists no collection of directly revealed preference relations $(m, t)\, \mathcal{R}^*(n, s)$ in which the distribution of payments n, appearing in the inferior options, *first order stochastically dominates* the distribution of prizes m in the preferred pairs, while the distribution of time-delays t *first order stochastically dominates* the distribution of delays s.

Our approach to the characterisation of time-preference is novel. However, the tools we use to show the necessity and sufficiency of our axioms are similar to those applied in the classical literature on intuitive probability and additive plausibility (see, e.g., Kraft et al., 1959 or Scott, 1964). In particular, dominance axiom has a similar flavour to the *cancellation law* used extensively in this area of research. Moreover, our restriction describing the separable formulation of time-preference resembles the condition characterising the expected utility hypothesis, introduced by Border (1992). In his paper, Border concentrates on observable choices over sets of lotteries, and discusses conditions under which they can be rationalised by an expected utility model. The restriction of *ex ante dominance*, that he proposes, hinges on a specific form of the first order stochastic dominance between the observable and an alternative, hypothetical choice function. Even thought the question we consider, the framework we specify, as well as the tools we apply are substantially different from those used by Border, the intuition behind our results is very similar.

Having established the testable restrictions for the separable formulation of preferences, in Section 4 we concentrate on conditions which would allow us to characterise the discounting function γ more precisely. In particular, we provide the axiomatic characterisation of the *weakly present-biased* specification of γ, for which ratio $\gamma(t)/\gamma(t+1)$ is a decreasing function of t. Therefore, under our formulation, the relative discounting between any two dates diminishes as they become more distant in the future. Equivalently, this is to say that the function has a *log-convex* extension to the domain of real numbers. We consider this class to be especially important as it contains all the well-known specifications of discounting, including hyperbolic, quasi-hyperbolic, and exponential.

The condition characterising this class of time-preference is summarised by *cumulative dominance axiom*. Our restriction requires that there exists no collection of directly revealed relations $(m, t)\, \mathcal{R}^*(n, s)$ such that the distribution of payments n in the inferior options *first order order stochastically dominates* the distribution of payments m appearing in the preferred pairs, while the distribution of time-delays t *second order* *stochastically dominates* the distribution of time-delays s. The condition is similar to the dominance axiom. However, as we require for the discounting function γ to be "log-convex", in order to make sure that the cumulative dominance axiom holds, we also need to consider samples in which the distributions of time-delays are ordered with respect to the second order stochastic dominance. Therefore, the cumulative dominance axiom is more restrictive, as we need to verify a larger class of collections of elements of the directly revealed preference relation while performing the test.

Finally, in the second part of Section 4, we draw our attention to an axiomatic characterisations of two specific examples of weakly present-biased discounting functions, namely, quasi-hyperbolic and exponential. The testable implications of the two specifications are similar, however, distinguishable. The essence of the two restrictions is summarised in *strong cumulative dominance axiom*. Loosely speaking, the two specification of time-preference require that there is no collection of directly revealed relations $(m, t)\,\mathcal{R}^*(n, s)$ such that the distribution of monetary payments n in the inferior options (n, s) first order stochastically dominates the analogous distribution of payments m, while the sum of time-delays t appearing in the superior prize-time pairs (m, t) is greater than the sum of delays s on the left hand side.

The results we present in this paper are not the first attempt to axiomatise time-preference in a setting with a finite number of observations. Echenique et al. (2014) characterise various forms of the time-separable model of inter-temporal choice in a framework in which agents choose streams of a single consumption good rather than prize-time pairs. In their setting, an observation consists of a consumption path selected by the subject and the corresponding prices of the commodity in the periods for which the choice is made. The authors specify both the necessary and sufficient conditions under which the set of observations can be rationalised by different forms of time-separable preference. What is crucial to their result, is the assumption concerning concavity of the instantaneous utility function. This allows the authors to constrain their attention to the implications of the first order conditions characterising the solution to the consumer optimisation problem. Therefore, the restrictions they discuss refer to the model of time-separable preferences *with* a concave instantaneous utility function. Our framework allows us to concentrate solely on the core implications of the discounted utility theory. We dispense the assumptions that are not crucial to the hypothesis and characterise these observable restrictions which are pivotal to this class of models. Nevertheless, as our set-up differs substantially from the one adopted by Echenique et al. (2014), our results are not comparable.

2 Preliminaries

We begin the analysis with a formal specification of our framework. Let the *domain* over which the agents determine their choices be defined by $X := \mathbb{R}_+ \times \mathbb{N}$. Each element $x = (m, t)$ of the set consists of a monetary payoff $m \in \mathbb{R}_+$ and a time-delay $t \in \mathbb{N}$ at which the payment is delivered.

Let K be a finite set enumerating the subsequent trials (repetitions) of the experiment. In each trial $k \in K$, an agents is asked to choose one element from a finite *set of feasible options* $A_k \subset X$. An *experiment*, denoted by $\mathcal{E}$, is a collection of sets of feasible options,

$$\mathcal{E} := \{A_k\}_{k \in K}.$$

In every trial $k \in K$ of the experiment the subjects are obliged to choose exactly one element from the corresponding set of feasible options A_k. Therefore, an *observation* is an ordered pair (A_k, x_k), consisting of the set A_k and the option $x_k \in A_k$ chosen by the agent. Given this, the *set of observations* from the experiment is defined by a collection of the ordered pairs

$$\mathcal{O} := \{(A_k, x_k)\}_{k \in K},$$

where $x_k \in A_k$, for all $k \in K$. Note that our framework allows for the agents to make multiple choices from a single set A_k. However, each such choice has to be treated as a separate trial ("with replacement").[1] Finally, define the set of *observable options* by

$$\mathcal{A} := \bigcup_{k \in K} A_k.$$

Hence, set $\mathcal{A}$ contains all the possible pairs of monetary payments and time-delays that the agent was offered at least once during the experiment. Clearly, we have $\mathcal{A} \subset X$. Moreover, since both K and A_k are finite, for every $k \in K$, so is $\mathcal{A}$.

Let the set of *observable payments* be given by

$$\mathcal{M} := \{m \in \mathbb{R}_+ : (m, t) \in \mathcal{A}\}.$$

Therefore, $\mathcal{M}$ is the set of all monetary prizes that appeared in at least one option during the experiment. Throughout the paper we shall denote the cardinality of the set by $|\mathcal{M}|$,

[1] Note that our framework does not allow for the consumers to choose several options simultaneously from a single set of feasible options, as the sequence in which the elements are chosen matters. Suppose that an agents chooses two elements x and y from some set A ("without replacement"), then the corresponding observations are either (A, x) and $(A \backslash \{x\}, y)$, or (A, y) and $(A \backslash \{y\}, x)$, depending on which option was chosen first.

while $\underline{m} := \min \mathcal{M}$ and $\overline{m} := \max \mathcal{M}$. We define the set of *observable time-delays* by

$$\mathcal{T} := \{t \in \mathbb{N} : (m,t) \in \mathcal{A}\},$$

with its cardinality denoted by $|\mathcal{T}|$. Analogously, let the least and the greatest element of the set be denoted by $\underline{t} := \min \mathcal{T}$ and $\bar{t} := \max \mathcal{T}$. Finally, let $\bar{\mathcal{A}} := \mathcal{M} \times \mathcal{T}$.

2.1 Revealed preference relations and mixed-monotonicity

In the following section we discuss properties of preference relations induced by the set of observations $\mathcal{O}$. For any two elements x and y in $\mathcal{A}$, we say that x is *directly revealed preferred* to y, if in at least one trial of the experiment both options x and y were feasible but the agent decided to choose x rather than y. Formally, we will say that the pair (x, y) belongs to set $\mathcal{R}^*$ defined by

$$\mathcal{R}^* := \{(x,y) \in \mathcal{A} \times \mathcal{A} : \text{there exists } A \in \mathcal{E} \text{ such that } x, y \in A \text{ and } (A, x) \in \mathcal{O}\}.$$

For convenience, we will denote $x\,\mathcal{R}^* y$ instead of $(x,y) \in \mathcal{R}^*$. Whenever both (x,y) and (y,x) belong to $\mathcal{R}^*$ we will say that x and y are *directly revealed indifferent.* We denote the symmetric part of the relation by $\mathcal{I}^*$, i.e.,

$$\mathcal{I}^* := \{(x,y) \in \mathcal{A} \times \mathcal{A} : x\,\mathcal{R}^* y \text{ and } y\,\mathcal{R}^* x\}.$$

Similarly, we shall write $x\,\mathcal{I}^* y$ in place of $(x,y) \in \mathcal{I}^*$. Note that we do *not* define the strict counterpart of $\mathcal{R}^*$.

The main purpose of our paper is to establish conditions under which the set of observations $\mathcal{O}$ can be rationalised by a specific form of utility function. Clearly, one of the necessary conditions for rationalisation is existence of a transitive closure of $\mathcal{R}^*$ over $\mathcal{A}$. A complete, transitive, and reflexive pre-order $\mathcal{R} \subseteq \mathcal{A} \times \mathcal{A}$ is *consistent* with the directly revealed preference relation $\mathcal{R}^*$ if for any two x, $y \in \mathcal{A}$, we have $x\,\mathcal{R}^* y \Rightarrow x\,\mathcal{R}\,y$, or equivalently $\mathcal{R}^* \subseteq \mathcal{R}$. We shall denote the strict component of $\mathcal{R}$ by $\mathcal{P}$, i.e.,

$$\mathcal{P} := \{(x,y) \in \mathcal{A} \times \mathcal{A} : x\,\mathcal{R}\,y \text{ and } \neg(y\,\mathcal{R}\,x)\}.$$

As previously, we shall write $x\,\mathcal{P}\,y$ instead of $(x,y) \in \mathcal{P}$. Finally, the symmetric part of $\mathcal{R}$ will be denoted by $\mathcal{I}$. Hence, $x\,\mathcal{I}\,y$ if and only if $x\,\mathcal{R}\,y$ and $y\,\mathcal{R}\,x$.

For the purposes of this paper, we shall concentrate on a specific class of consistent pre-orders. Let $\geq_X$ denote a partial order on X such that, for any $x = (m,t)$ and

$y = (n, s)$ in X, we have $x \geq_X y$ whenever $m \geq n$ and $t \leq s$. Moreover, the relation is strict, and denoted by $x >_X y$, if at least one of the above inequalities is strict. Loosely speaking, we will say that option x is greater than y with respect to $\geq_X$, if it offers a higher payment at a shorter delay. A pre-order $\mathcal{R}$ is *mixed-monotone*, whenever for any two elements x and y in $\mathcal{A}$, $x \geq_X y$ implies $x\,\mathcal{R}\,y$. In addition, if $x >_X y$ then $x\,\mathcal{P}\,y$. The definition suggests that whenever an agent is presented with two options such that one of them has a (weakly) higher payoff and a (weakly) shorter delay than the other one, then the former option should be preferred. Clearly, not every set of observations admits a consistent mixed-monotone pre-order. In the following section we discuss conditions under which there exists such an extension of the directly revealed preference relation.

2.2 Mixed-monotone rationalisation

Set $\mathcal{O}$ is *rationalisable* if there exists a function $v : X \to \mathbb{R}$, strictly increasing with respect to the partial order $\geq_X$,[2] such that for all $(A, x) \in \mathcal{O}$,

$$v(x) \geq v(y), \text{ for all } y \in A.$$

In the remainder of the paper we focus on conditions under which the set of observations can be rationalised by a utility function v that strictly increases in the value of the monetary payment $m \in \mathbb{R}_+$ and strictly decreases with respect to the time-delay $t \in \mathbb{N}$. We begin by introducing the following axiom.

Cyclical consistency axiom. *Set $\mathcal{O}$ is* cyclically consistent*, if for any $\{x^i\}_{i=1}^n$ in $\mathcal{A}$ such that $x^{i+1}\mathcal{R}^*x^i$ or $x^{i+1} \geq_X x^i$, for $i = 1, \ldots, n-1$, and $x^1 \geq_X x^n$, we have $x^1 = x^n$.*

The above axiom is a special case of *generalised cyclical consistency* condition formulated by Nishimura et al. (2013). It requires that whenever there exists a sequence of observable options such that every subsequent element is directly revealed preferred or greater with respect to $\geq_X$ to the previous one, then it cannot be that the first element of the sequence is strictly greater than the ultimate one. Clearly, the violation of this condition excludes the existence of a consistent, mixed-monotone pre-order on $\mathcal{A}$. In fact, by Nishimura et al. (2013, Rationalisability Theorem I), cyclical consistency is also a sufficient condition for the existence of such a pre-order. In order to make our presentation more transparent, we consider the following example.

[2] That is, for any $x, y \in X$, whenever $x >_X y$ then $v(x) > v(y)$.

Example 1. Suppose that we observe the following directly revealed preference relation:

$$(5,3)\,\mathcal{R}^*(15,4),\;\;(15,2)\,\mathcal{R}^*(10,1),\;\;(15,1)\,\mathcal{R}^*(25,3),\;\;\text{and}\;(25,4)\,\mathcal{R}^*(20,2).$$

It is easy to check that the set of observations inducing the above relation is cyclically consistent. In fact, given Nishimura et al. (2013, Rationalisability Theorem I), it is both necessary and sufficient to propose a consistent, mixed-monotone relation $\mathcal{R}$ defined over the observable options. For example

$$(15,1)\,\mathcal{I}\,(25,3)\,\mathcal{P}\,(25,4)\,\mathcal{I}\,(20,2)\,\mathcal{P}\,(15,2)\,\mathcal{I}\,(10,1)\,\mathcal{P}\,(5,3)\,\mathcal{P}\,(15,4).$$

Clearly, the relation is both consistent and mixed-monotone.

Proposition 1. *Set $\mathcal{O}$ is rationalisable if and only if it is cyclically consistent.*

Proposition 1 is a special case of the result by Nishimura et al. (2013, Rationalisability Theorem II), who establish the necessity and sufficiency of the generalised cyclical consistency for the existence of a utility function rationalising the choice data in a general class of partially ordered spaces. Unfortunately, their argument supporting the claim is not constructive, which implies that the only way of verifying whether $\mathcal{O}$ is rationalisable is by referring directly to the definition of cyclical consistency. Even though finite, this method may be highly inconvenient for applications, especially when the set of observations is large. For this reason, in the Appendix, we present an alternative, constructive proof of Proposition 1, which introduces a more convenient method of verifying rationalisability of the set of observations $\mathcal{O}$ in our framework.

The necessity of cyclical consistency for rationalisation is straightforward. Clearly, for any function v rationalising $\mathcal{O}$, and any sequence $\{x^i\}_{i=1}^n$ specified as in the definition of the axiom, we have

$$v(x^n) \geq v(x^{n-1}) \geq \ldots \geq v(x^2) \geq v(x^1)\;\;\text{and}\;\;v(x^1) \geq v(x^n),$$

which can be satisfied only if $x_1 = x_n$. On the other hand, the "sufficiency" part of the proof is more demanding. We show the result in three steps (see the Appendix). First, in Lemma A.1 we argue that cyclical consistency implies existence of a consistent, mixed-monotone pre-order $\mathcal{R}$ over $\mathcal{A}$. In Lemma A.2, we show that whenever such a pre-order exists, we can always find a sequence of real numbers $\{v^t_m\}_{(m,t)\in\bar{\mathcal{A}}}$ such that for any two options (m,t), $(n,s) \in \bar{\mathcal{A}}$, whenever $(m,t) >_X (n,s)$ or $(m,t)\,\mathcal{P}\,(n,s)$ then $v^t_m > v^s_n$, while $(m,t)\,\mathcal{I}\,(n,s)$ implies $v^t_m = v^s_n$. Finally, in Lemma A.3, we use any such sequence of numbers to construct a function rationalising the set of observations. Every step of our argument is constructive. Therefore, it presents a direct method of verifying whether an arbitrary set of observations is rationalisable.

3 Discounted utility rationalisation

Set $\mathcal{O}$ is rationalisable by a *discounted utility* function whenever there is a strictly increasing instantaneous utility function $u : \mathbb{R}_+ \to \mathbb{R}_+$ and a strictly decreasing discounting function $\gamma : \mathbb{N} \to (0, 1]$, with $\gamma(0) = 1$, such that $v(m, t) := u(m)\gamma(t)$ rationalises $\mathcal{O}$. Clearly, cyclical consistency is a necessary condition for this form of representation. However, it is no longer sufficient. The following section is devoted to determining both the necessary and sufficient conditions which would allow for such a representation.

3.1 Dominance axiom

A *sample* of the directly revealed preference relation $\mathcal{R}^*$ is a finite, indexed collection $\{(x^i, y^i)\}_{i\in I}$ of elements in $\mathcal{R}^*$, where we denote $x^i = (m^i, t^i)$ and $y^i = (n^i, s^i)$, for some m^i, $n^i \in \mathcal{M}$ and t^i, $s^i \in \mathcal{T}$. We allow for the samples to be generated "with replacement". That is, a single element of $\mathcal{R}^*$ may appear more than once in a sample.

Dominance axiom. *For any sample $\{(x^i, y^i)\}_{i\in I}$ of $\mathcal{R}^*$, where we denote $x^i = (m^i, t^i)$ and $y^i = (n^i, s^i)$, such that for any $m \in \mathcal{M}$ and $t \in \mathcal{T}$, we have*

$$|\{i \in I : m^i \leq m\}| \;\geq\; |\{i \in I : n^i \leq m\}| \;\text{ and }\; |\{i \in I : t^i \leq t\}| \;\leq\; |\{i \in I : s^i \leq t\}|,$$

all of the above conditions hold with equality.

The above axiom requires that whenever there exists a sample such that the distribution of monetary payments n^i, in the inferior options y^i, first order stochastically dominates the corresponding distribution of payments m^i, in the preferred options x^i, while the distribution of time-delays t^i, appearing on the left hand side of $\mathcal{R}^*$, first order stochastically dominates the distribution of s^i, then both distributions of monetary payments and time-delays have to be equal. Therefore, the axiom is violated whenever there exists a sample $\{(x^i, y^i)\}_{i\in I}$ of the directly revealed preference relation, with $x^i = (m^i, t^i)$ and $y^i = (n^i, s^i)$, such that the distribution of n^i stochastically dominates the distribution of m^i, the distribution of t^i stochastically dominates the distribution of s^i, and at least one of the two relations is strict. In order to make our presentation more transparent, we discuss the following example.

Example 2. Consider the directly revealed preference relation analysed in Example 1. We claim that the set of observations inducing the relation fails to satisfy the dominance axiom. In order to show this, we need to find at least one sample which violates the condition specified in the definition of the axiom. Take $\mathcal{R}^*$. Clearly, the set is a sample of itself. Note that, given the support $\{5, 10, 15, 20, 25\}$, the distribution of payments in the preferred $x^i = (m^i, t^i)$ and the inferior options $y^i = (n^i, s^i)$ are respectively $(1, 0, 2, 0, 1)$ and $(0, 1, 1, 1, 1)$. Similarly, given the support $\{1, 2, 3, 4\}$, the distributions of the corresponding time-delays are both equal to $(1, 1, 1, 1)$. Therefore, there exists a sample of $\mathcal{R}^*$ for which the distribution of payments n^i *strictly* first order stochastically dominates the distribution of m^i, while the distributions of the time-delays are equal. Hence, we conclude that the set of observations $\mathcal{O}$ violates the dominance axiom.

The above example indicates that dominance is a stronger condition than cyclical consistency, as there exist sets of observations satisfying the latter but failing the former. On the other hand the dominance axiom implies cyclical consistency, as we will show in the remainder of this section.

Interestingly, the axiom is both a necessary and sufficient condition for the set of observations $\mathcal{O}$ to be rationalisable by a discounted utility function. We summarise this result in the following theorem. The proof is presented in the Appendix.

Theorem 1. *Set of observations $\mathcal{O}$ is rationalisable by a discounted utility function if and only if it obeys the dominance axiom.*

In order to understand why the dominance axiom is a necessary condition, suppose that $\mathcal{O}$ is rationalisable by a discounted utility function $v(m, t) := u(m)\gamma(t)$. This implies that the set is at the same time rationalisable by function $w(m, t) := \phi(m) + \varphi(t)$, where $\phi := \log(u)$ and $\varphi := \log(\gamma)$. Moreover, under this transformation functions ϕ and φ preserve the strict monotonicity of u and γ respectively.

Take any sample $\{(x^i, y^i)\}_{i\in I}$ of the directly revealed preference relation $\mathcal{R}^*$, where $x^i = (m^i, t^i)$ and $y^i = (n^i, s^i)$. By the definition of rationalisation, for any element of the sample, we have $\phi(m^i) + \varphi(t^i) \geq \phi(n^i) + \varphi(s^i)$. In particular, once we sum up all the inequalities with respect to $i \in I$, we obtain

$$\sum_{i\in I} \phi(m^i) + \sum_{i\in I} \varphi(t^i) \geq \sum_{i\in I} \phi(n^i) + \sum_{i\in I} \varphi(s^i).$$

Suppose that the sample is specified as in the definition of the axiom. Then, the corresponding distribution of monetary payments n^i first order stochastically dominates the distribution of m^i. Since function ϕ is strictly increasing, it must be that $\sum_{i\in I}\phi(m^i)\leq\sum_{i\in I}\phi(n^i)$. On the other hand, we know that the distribution of time-delays t^i first order stochastically dominates the distribution of s^i. By monotonicity of φ, this implies that $\sum_{i\in I}\varphi(t^i)\leq\sum_{i\in I}\varphi(s^i)$. However, given the previous condition, the two inequalities hold only if they are satisfied with equality. This, on the other hand, requires that the distribution of monetary payments m^i is equal to the distribution of n^i, while the distribution of time-delays t^i coincides with the distribution of s^i.

The above argument highlights the form of consistency which is expected from a discounted utility maximiser. As it was shown above, whenever the set of observations is rationalisable by a discounted utility function, it can also be rationalised by an additive utility function $w(m,t):=\phi(m)+\varphi(t)$, where ϕ is strictly increasing and φ strictly decreasing. Take any sample $\{(x^i,y^i)\}_{i\in I}$ of $\mathcal{R}^*$, where $x^i=(m^i,t^i)$ and $y^i=(n^i,s^i)$, and construct two lotteries: one with the support corresponding to the preferred pairs (m^i,t^i) and probabilities equal to the frequencies with which they appear in the sample; the other one supported by options (n^i,s^i) and probabilities defined by the frequencies with which they appear in $\{(x^i,y^i)\}_{i\in I}$. Whenever the agent is an expected utility maximiser, his preference over the two lotteries should be consistent with his choices over individual pairs. Hence, the gamble supported by the superior options (m^i,t^i) should be preferred. On the other hand, due to separability and monotonicity of the utility function, any violation of dominance axiom would imply that the consumer would rather choose the lottery supported by (n^i,s^i). However, such behaviour cannot be reconciled with the discounted utility maximisation.

Showing that the dominance axiom is a sufficient condition for this form of rationalisation is more demanding, hence, we place the proof in the Appendix. Nevertheless, we present the main observation used in our argument in the following proposition. The following result plays a crucial role in the applicability of our method to empirics.

Proposition 2. *Set $\mathcal{O}$ obeys dominance axiom if and only if there exists a strictly increasing sequence $\{\phi_m\}_{m\in\mathcal{M}}$ and a strictly decreasing sequence $\{\varphi_t\}_{t\in\mathcal{T}}$ of real numbers such that $(m,t)\,\mathcal{R}^*(n,s)$ implies $\phi_m+\varphi_t\geq\phi_n+\varphi_s$.*

This result is implied by Lemmas A.4 and A.5 in the Appendix, as well as the necessity of the dominance axiom for rationalsiation by a discounted utility function. To support the above proposition we apply a variation of Farka's lemma, commonly known as *Motzkin's rational trasposition*. Using the result, we show that the system of inequalities implied by the directly revealed preference relation fails to have a solution only if the set of observations violates the dominance axiom.

It is worth pointing out the importance of Proposition 2 for the applicability of Theorem 1. The result presents an alternative way of verifying whether the set of observations obeys the dominance axiom. In fact, the proposition states that the axiom is equivalent to the existence of a solution to a linear system of inequalities. Since such systems are in general solvable, i.e., there exist algorithms which allow to determine in a finite number of steps whether a given system has a solution or not, we find the alternative method of verifying the axiom in a much more convenient manner.

In the remaining sections of the paper we determine conditions under which the class of preferences rationalising the data can be characterised more precisely. In particular, we focus on finer restrictions imposed on the form of the discounting function γ. Before we proceed with our analysis, we would like to discuss a family of experiments for which a narrower specification of time-preference is *never* possible.

3.2 Anchored experiments and identification

An experiment $\mathcal{E}$ is *anchored*, whenever it consists solely of *binary* sets of feasible choices A, and there exists some $x^* \in X$ such that for all $A \in \mathcal{E}$, we have $x^* \in A$. In other words, in each trial of an anchored experiment the subjects are asked to choose between one fixed option x^* and some other element in X. We shall denote $x^* = (m^*, t^*)$.

There are several notable examples of anchored experiments that were performed in the literature, including Kirby and Marakovic (1964), Coller and Williams (1999), or Kirby et al. (1999), Harrison et al. (2002). Therefore, this class of experiments is relevant from the empirical point of view. An important advantage of these tests is that the choices the subjects face are relatively simple, which minimises the chance of errors made by the agents. Nevertheless, as we show in this section, the simplicity of the experiment substantially reduces the informativeness of the observations it generates.

Proposition 3 (Indeterminacy)**.** *For any anchored experiment the following statements are equivalent.*

(i) Set $\mathcal{O}$ is rationalisable.

(ii) Set $\mathcal{O}$ is rationalisable by a discounted utility function.

(iii) For any discounting function $\gamma : \mathbb{N} \to (0, 1]$, with $\gamma(0) = 1$, there is a strictly increasing function $u : \mathbb{R}_+ \to \mathbb{R}_+$ such that $v(m, t) := u(m)\gamma(t)$ rationalises $\mathcal{O}$.

The above proposition states that, given observations from an anchored experiment, one can only determine if the choices of the subject are rationalisable according to the definition in Section 2.2. Therefore, the design of the experiment does not allow to verify whether the observable choices can be rationalised by a narrower class of preferences. In particular, Proposition 3(iii) implies that once the set of observations is rationalisable, it can also be rationalised by virtually *any* form of discounting. Hence, we consider this class of experiments to be rather weak, as the data they produce do not allow for a conclusive specification of time-preference explaining the observable choices.

The argument supporting the above claim is three-fold. In order to prove implication (i) $\Rightarrow$ (ii), we show that in any anchored experiment, whenever the set of observations is cyclically consistent, it may violate the dominance axiom only if it contains an infinite number of elements. Clearly, by definition this can never hold. Implication (ii) $\Rightarrow$ (iii) follows from the fact that any directly revealed preference relation can only be expressed with respect to the option (m^*, t^*). Therefore, given any discounting function γ, while constructing the utility function u we simply need to assign a value $u(m)$ to every element m of $\mathcal{M}$ that satisfies $u(m) \geq \gamma(t^*)/\gamma(t)u(m^*)$, whenever $(m, t)\,\mathcal{R}^*(m^*, t^*)$, and $u(m) \leq \gamma(t^*)/\gamma(t)u(m^*)$ otherwise, which is always possible. Implication (iii) $\Rightarrow$ (i) is obvious. The full proof of the result is presented in the Appendix

4 Weakly present-biased rationalisation

In the previous section we have determined the necessary and sufficient conditions under which the set of observations can be rationalised by a discounted utility function. We devote the remainder of the paper to determine restrictions which allow for a narrower characterisation of the discounting function γ. In particular, we focus on a class of preferences that exhibit some degree of present-biased. We will say that a strictly decreasing discounting function $\gamma : \mathbb{N} \to (0, 1]$, with $\gamma(0) = 1$, is *weakly present-biased*, whenever function $\vartheta : \mathbb{N} \to \mathbb{R}_{++}$,

$$\vartheta(t) := \frac{\gamma(t)}{\gamma(t+1)},$$

is decreasing. In other words, we require that the relative discounting between any two subsequent periods decreases as the two dates are further away in the future. Equivalently, this is to say that there exists a *log-convex* extension of function γ to the domain of the real numbers. Our interest in the above class is justified by the fact that it contains the most commonly used forms of discounting. In particular, exponential, quasi-hyperbolic, and hyperbolic discounting models are included in this family.

4.1 Cumulative dominance axiom

A set $\mathcal{O}$ is rationalisable by a *weakly present-biased discounted* utility function, whenever there exists a strictly increasing instantaneous utility function $u : \mathbb{R}_+ \to \mathbb{R}_+$ and a strictly decreasing, weakly present-biased discounting function $\gamma : \mathbb{N} \to (0, 1]$, with $\gamma(0) = 1$, such that $v(m, t) := u(m)\gamma(t)$ rationalises $\mathcal{O}$.

Cumulative dominance axiom. *For any sample* $\{(x^i, y^i)\}_{i\in I}$ *of* $\mathcal{R}^*$*, where* $x^i = (m^i, t^i)$ *and* $y^i = (n^i, s^i)$*, such that, for any* $m \in \mathcal{M}$ *and* $t \in \mathcal{T}$*,*

$$|\{i \in I : m^i \leq m\}| \geq |\{i \in I : n^i \leq m\}| \text{ and } \int_{\underline{t}}^{t} |\{i \in I : t^i \leq z\}|dz \leq \int_{\underline{t}}^{t} |\{i \in I : s^i \leq z\}|dz,$$

we have $|\{i \in I : m^i \leq m\}| = |\{i \in I : n^i \leq m\}|$ *and* $|\{i \in I : t^i \leq t\}| = |\{i \in I : s^i \leq t\}|$*, for all* $m \in \mathcal{M}$ *and* $t \in \mathcal{T}$*.*

The cumulative dominance axiom requires that whenever there exists a sample such that the distribution of monetary payments n^i in the inferior options *first order stochastically dominates* the distribution of payments m^i in the preferred prize-time pairs, while the distribution of time-delays t^i appearing on the left hand side of $\mathcal{R}^*$ *second order stochastically dominates* the distribution of s^i, then the distributions of monetary payments and delays have to be equal. Roughly speaking, the axiom is violated whenever there exists at least one sample $\{(x^i, y^i)\}_{i\in I}$ of the directly revealed preference relation, with $x^i = (m^i, t^i)$ and $y^i = (n^i, s^i)$, such that the distribution of n^i *first* order stochastically dominates the distribution of m^i, and the distribution of t^i *second* order stochastically dominates the distribution of s^i, while at least one of the two relations is strict.

Note that the cumulative dominance axiom is stronger than the dominance axiom. Suppose that the set of observations satisfies the former condition. Then, there exists no sample of the directly revealed preference relation such that the monetary payments in the inferior options first order stochastically dominate the prizes in the preferred options, while the superior time-delays second order stochastically dominate the delays appearing on the right hand side of the relation. Since first order stochastic dominance implies the second, this means that, at the same time, there exists no sample such the latter relation is preserved under the first order stochastic dominance. Therefore, the set of observations obeys the dominance axiom as well. However, the opposite implication does not hold, as there might exist a collection of elements in $\mathcal{R}^*$ such that the corresponding distributions of time-delays are not ordered with respect to the first order stochastic dominance, but are ordered with respect to the second. In other words, the cumulative dominance axiom requires verifying a larger set of samples than the dominance axiom.

Theorem 2. *Set $\mathcal{O}$ is rationalisable by a weakly present-biased discounted utility function if and only if it obeys the cumulative dominance axiom.*

In order to show the necessity of cumulative dominance axiom for the narrower form of rationalisation, suppose that the choices of an agent can be explained by some function $v(m,t) := u(m)\gamma(t)$, where u is strictly increasing, while γ is strictly decreasing and weakly present-biased. Clearly, one can always rationalise the same set of observations by function $w(m,t) := \phi(m)+\varphi(t)$, with $\phi := \log(u)$ and $\varphi := \log(\gamma)$. The transformation preserves the monotonicity of the two functions, while φ has a convex extension to $\mathbb{R}_+$.

By definition, for any sample $\{(x^i, y^i)\}_{i\in I}$ of $\mathcal{R}^*$, where $x^i = (m^i, t^i)$ and $y^i = (n^i, s^i)$, we have $\phi(m^i) + \varphi(t^i) \geq \phi(n^i) + \varphi(s^i)$, for all $i \in I$. In particular,

$$\sum_{i\in I} \phi(m^i) + \sum_{i\in I} \varphi(t^i) \geq \sum_{i\in I} \phi(n^i) + \sum_{i\in I} \varphi(s^i).$$

Suppose that the sample is specified as in the definition of cumulative dominance axiom. As ϕ is strictly increasing, we have $\sum_{i\in I} \phi(m^i) \leq \sum_{i\in I} \phi(n^i)$. Moreover, since the distribution of time-delays t^i second order stochastically dominates the distribution of s^i, the existence of a convex extension of φ implies that $\sum_{i\in I} \varphi(t^i) \leq \sum_{i\in I} \varphi(s^i)$. However, the two inequalities can be consistent with the initial condition only if they are satisfied with equality which, by the strict monotonicity of ϕ and φ, requires that the distributions of monetary payments and time-delays on the two sides of the directly revealed preference relation are equivalent.

The form of consistency that has to be satisfied by a weakly present-biased discounted utility maximiser is similar to the one discussed in Section 3. Take any sample $\{(x^i, y^i)\}_{i\in I}$ of $\mathcal{R}^*$, where $x^i = (m^i, t^i)$ and $y^i = (n^i, s^i)$, and construct two lotteries: one with the support corresponding to the preferred pairs (m^i, t^i) and probabilities equal to the frequencies with which they appear in the sample; the other one supported by options (n^i, s^i) and probabilities defined by the frequencies with which they show up in $\{(x^i, y^i)\}_{i\in I}$. Clearly, by construction, any expected utility maximiser prefers the former gamble to the latter. However, whenever the Bernoulli utility function is specified by $w(m,t) := \phi(m) + \varphi(t)$, where ϕ is strictly increasing, while φ is strictly decreasing an "convex", whenever the cumulative axiom is violated, it would be possible to construct a sample such that the lottery over the inferior options would be preferred to the gamble over the superior prize-time pairs.

The "sufficiency" part of the proof of Theorem 2 is more demanding. Similarly as in the case discounted utility, our argument consists of two steps. First, in Lemma A.7 we show that once the set of observations satisfies the cumulative dominance axiom, there always exists a solution to a specific system of linear inequalities. In the second step, see Lemma A.8 in the Appendix, we use the solution to construct an instantaneous utility function u and a log-convex discounting function γ which rationalise the data. We present our key observation made in the proof of the theorem in form of a proposition.

Proposition 4. *Set of observations $\mathcal{O}$ obeys cumulative dominance axiom if and only if there is a strictly increasing sequence $\{\phi_m\}_{m\in\mathcal{M}}$, a strictly decreasing sequence $\{\varphi_t\}_{t\in\mathcal{T}}$, and a strictly negative sequence $\{v_t\}_{t\in\mathcal{T}}$ of real numbers such that $(m,t)\,\mathcal{R}^*(n,s)$ implies $\phi_m + \varphi_t \geq \phi_n + \varphi_s$, while for all $t \in \mathcal{T}$, we have $\varphi_t + v_t(s-t) \leq \varphi_s$, for all $s \in \mathcal{T}$.*

It is worth mentioning that the proposition is important for the applicability of our main result, as it presents an alternative method of verifying the cumulative dominance axiom. Moreover, we consider it to be much more convenient than applying the definition of the axiom directly.

4.2 Quasi-hyperbolic discounting

In the following section we concentrate on quasi-hyperbolic discounting functions, which constitute a narrower class of weakly present-biased preferences. We say that the set of observations is rationalisable by a *quasi-hyperbolic discounted utility* function, whenever there exist a strictly increasing function $u : \mathbb{R}_+ \to \mathbb{R}_+$, numbers β, $\delta \in (0,1)$, and some time-delay $t^\circ \in \mathbb{N}$ such that $v(m,t) := u(m)\gamma(t)$ rationalises $\mathcal{O}$, where

$$\gamma(t) := \begin{cases} \beta^t \delta^t & \text{for } t < t^\circ, \\ \beta^{t^\circ} \delta^t & \text{otherwise.} \end{cases}$$

Note that our definition of a quasi-hyperbolic discounting function generalises the standard notion for which $t^\circ = 1$. By allowing for the "threshold" time-delay t° to vary, we are able to analyse a wider class of preferences. In particular, as t° denotes a time-delay which separates the dates perceived by the agent as "present" from those regarded as "future", we allow in our test for this parameter to be determined endogenously.

Since in the case of a quasi-hyperbolic discounting function $\vartheta(t) := \gamma(t)/\gamma(t+1)$ takes the value of $(\beta\delta)^{-1}$, for $t < t^\circ - 1$, and δ^{-1} otherwise, the quasi-hyperbolic specification is weakly present-biased. Hence, any set of observations rationalisable by this more specific form of time-preference obeys the cumulative dominance axiom. However, it is no longer sufficient. In this section we discuss the condition that fully characterises this form of the discounted utility model.

Strong cumulative dominance axiom. *There exists some $t' \in \mathcal{T}$ such that, for any sample $\{(x^i, y^i)\}_{i\in I}$ of $\mathcal{R}^*$, where $x^i = (m^i, t^i)$ and $y^i = (n^i, s^i)$, which satisfies*

(i) $|\{i \in I : m^i \leq m\}| \geq |\{i \in I : n^i \leq m\}|$, *for all* $m \in \mathcal{M}$;

(ii) $\sum_{i\in I} t^i \geq \sum_{i\in I} s^i$;

(iii) $\sum_{i\in I} \min\{t^i, t'\} \geq \sum_{i\in I} \min\{s^i, t'\}$,

all the above conditions hold with equality.

The above condition is stronger than cumulative dominance. Clearly, in order to verify whether the above axiom is not violated, we need to consider a wider class of samples of the directly revealed preference relation. As previously, the samples of interest need to satisfy condition (i). However, additionally, they have to obey restrictions (ii) and (iii), which impose a weaker requirement on the relation between the distribution of time-delays appearing on both sides of the directly revealed preference relation in the sample. Therefore, a greater number of samples satisfies the two conditions then in case of the cumulative dominance axiom.

Proposition 5. *Set $\mathcal{O}$ is rationalisable by a quasi-hyperbolic discounted utility function if and only if it obeys the strong cumulative dominance axiom.*

The necessity of the axiom can be proven similarly as in the previous sections. Suppose that the set of observations is rationalisable by a quasi-hyperbolic discounting function $v(m, t) := u(m)\gamma(t)$, where γ is specified as at the beginning of this section for some β, δ in $(0, 1)$, and a time-delay t°. This implies that the set of observations is also rationalisable by function $w(m,t) := \phi(m) + \min\{t, t^\circ\}\hat{\beta} + t\hat{\delta}$, where $\phi := \log(u)$, $\hat{\beta} := \log(\beta)$, and $\hat{\delta} := \log(\delta)$. By definition, for any sample $\{(x^i, y^i)\}_{i\in I}$ of the directly revealed preference relation, with $x^i = (m^i, t^i)$ and $y^i = (n^i, s^i)$, we have

$$\sum_{i\in I}\phi(m^i) + \hat{\beta}\sum_{i\in I}\min\{t^i, t^\circ\} + \hat{\delta}\sum_{i\in I}t^i \geq \sum_{i\in I}\phi(n^i) + \hat{\beta}\sum_{i\in I}\min\{s^i, t^\circ\} + \hat{\delta}\sum_{i\in I}s^i.$$

Whenever the sample is specified as in the strong cumulative dominance axiom, it must be that $\sum_{i\in I}\phi(m^i) \leq \sum_{i\in I}\phi(n^i)$, as well as $\hat{\beta}\sum_{i\in I}\min\{t^i, t^\circ\} \leq \hat{\beta}\sum_{i\in I}\min\{s^i, t^\circ\}$ and $\hat{\delta}\sum_{i\in I}t^i \leq \hat{\delta}\sum_{i\in I}s^i$, as $\hat{\beta}$ and $\hat{\delta}$ are strictly negative. However, these conditions can be consistent with the initial inequality only if they all hold with equality. Moreover, the argument remains unchanged once we substitute t° with $t' := \min\{t \in \mathcal{T} : t \geq t^\circ\}$. This requires for the strong cumulative dominance axiom to be satisfied.

The "sufficiency" part of the proof is presented in the Appendix. Our argument is constructed around one important observation, which we summarise in the following lemma. The result plays also a crucial role in the implementation of Proposition 5.

Lemma 1. *Set $\mathcal{O}$ obeys the strong cumulative dominance axiom for some $t' \in \mathcal{T}$ if and only if there exists a strictly increasing sequence $\{\phi_m\}_{m\in\mathcal{M}}$ and strictly negative numbers $\hat{\beta}$ and $\hat{\delta}$ such that $(m,t)\,\mathcal{R}^*\,(n,s)$ implies $\phi_m + \min\{t,t'\}\hat{\beta} + t\hat{\delta} \geq \phi_n + \min\{s,t'\}\hat{\beta} + s\hat{\delta}$.*

Proposition 5 requires some comment. First of all, observe that the value of the time-delay t' for which set $\mathcal{O}$ obeys the axiom, determines the empirical "kink" of the quasi-hyperbolic discounting function. In particular, this means that we do not assume prior to the test the "threshold" date which separates the perceived "present" from the "future", but determine it endogenously. In fact, it is possible that one set of observations admits various forms of quasi-hyperbolic discounting, not only with respect to the values of the discount factors β and δ, but also with respect to the pivotal time-delay t'.

Second of all, Lemma 1 proposes an alternative method of verifying whether the set of observations obeys the axiom. As in Propositions 2 and 4, it hinges on the existence of a solution to a system of linear inequalities conditional on t'. Since the set of the observable time-delays is finite, the test can be performed in a finite number of steps.

4.3 Exponential discounting

Finally, we draw our attention to the exponential discounting models. We say that set $\mathcal{O}$ is rationalisable by *exponential discounted utility* function whenever there is a strictly increasing instantaneous utility function $u : \mathbb{R}_+ \to \mathbb{R}_+$ and some $\delta \in (0,1)$ such that function $v(m,t) := \delta^t u(m)$ rationalises the set of observations.

Proposition 6. *Set $\mathcal{O}$ is rationalisable by an exponential discounted utility function if and only if for any* <u>*subset*</u> *$\{(x^i, y^i)\}_{i\in I}$ of $\mathcal{R}^*$, where $x^i = (m^i, t^i)$ and $y^i = (n^i, s^i)$, such that $|\{i \in I : m^i \leq m\}| \geq |\{i \in I : n^i \leq m\}|$, for all $m \in \mathcal{M}$, and $\sum_{i\in I} t^i \geq \sum_{i\in I} s^i$, all the above conditions hold with equality.*

The above propositions states the necessary and sufficient condition under which a set of observations can be rationalised by an exponential discounting function. Note that the requirement significantly resembles the one stated in the definition of the strong cumulative dominance axiom. In fact, the only distinguishable implications of the quasi-hyperbolic and exponential model are implied by condition (iii) stated in the definition of the axiom. Clearly, Proposition 6 imposes a stronger condition on the set of observations, as it admits a larger class of samples that might violate it. However, whenever the strong cumulative dominance axiom is satisfied for t' equal to the least or the greatest observable time-delay, then the two requirements are equivalent. Observe that, given an arbitrary sample $\{(x^i, y^i)\}_{i\in I}$ of the directly revealed preference relation, condition (iii) stated in the axiom is trivially satisfied whenever $t' = \underline{t}$. On the other hand, if $t' = \bar{t}$, then conditions (ii) and (iii) coincide. This implies, that in these extreme cases, it is impossible to distinguish between the quasi-hyperbolic and exponential rationalisation. We summarise the result in the following corollary.

Corollary 1. *Set $\mathcal{O}$ is rationalisable by an expected utility function if and only if it obeys the strong cumulative dominance axiom for $t' = \underline{t}$ or $t' = \bar{t}$.*

We omit the proof. The condition stated in Proposition 6 differs additionally in one substantial aspect from the strong cumulative dominance axiom. Observe that in order to rationalise the set of observations by an exponentially discounted utility function, we need to verify the requirement stated in the proposition only for *subsets* of the directly revealed preference relation, and not samples. Clearly, this substantially simplifies the test and reduces the number of steps required to verify the condition.

4.4 Indeterminacy of discount factors

In the following section we discuss some indeterminacy issues that arise while rationalising the set of observations by quasi-hyperbolic or exponential discounted utility functions.

We begin with the following proposition.

Proposition 7. *Set of observations $\mathcal{O}$ is rationalisable by a discounted utility function $v(m,t) := u(m)\gamma(t)$ if and only if, for any $a > 0$, it is rationalisable by a discounted utility function $\hat{v}(m,t) := \hat{u}(m)\hat{\gamma}(t)$, where $\hat{u} := u^a$ and $\hat{\gamma} := \gamma^a$.*

We omit the proof. The above result simply states that whenever a set of observations is rationalisable by some discounted utility function, then it is also rationalisable by its positive exponential transformation. This observation is not new, and was already noted by Fishburn and Rubinstein (1982) in their representation theorem. However, the above proposition implies a much stronger conclusion. Namely, any property of function γ that is preserved under positive exponential transformations, is also satisfied by function $\hat{\gamma}$. In particular, this means that whenever γ is weakly present-biased (respectively quasi-hyperbolic or exponential) then so is $\hat{\gamma}$. This plays an important role in the next result.

Corollary 2. *Set $\mathcal{O}$ is rationalisable by a quasi-hyperbolic discounted utility function if and only if for any β (or δ) in $(0,1)$ there is some δ (respectively β) in $(0,1)$, a strictly increasing utility function $u : \mathbb{R}_+ \to \mathbb{R}_+$, and a time-delay t° such that $v(m,t) := \gamma(t)u(m)$ rationalises $\mathcal{O}$, where $\gamma(t) = \beta^t\delta^t$, for $t < t^\circ$, and $\gamma(t) = \beta^{t^\circ}\delta^t$ otherwise.*

Proof. We show ($\Rightarrow$). There exists some strictly increasing function $u : \mathbb{R}_+ \to \mathbb{R}_+$, discount factors β, δ in $(0,1)$, as well as time-delay t° such that $v(m,t) := u(m)\gamma(t)$ rationalises $\mathcal{O}$, where function γ is defined as at the beginning of this section. Take any $\beta' \in (0,1)$, and define $a := \log(\beta')/\log(\beta)$. Clearly, function $\hat{v}(m,t) := \hat{u}(m)\hat{\gamma}(t)$, where $\hat{u} = u^a$ and $\hat{\gamma} = \gamma^a$, also rationalises $\mathcal{O}$. Moreover, $\hat{\gamma}(t) = (\beta')^t(\delta^a)^t$, for $t < t^\circ$, and $\hat{\gamma}(t) = (\beta')^{t^\circ}(\delta^a)^t$ otherwise. We present an analogous argument for the claim inside the brackets. Implication ($\Leftarrow$) is trivial. □

Given the nature of our framework and the above result, there is no testable restriction for the values of the discount factors β or δ, as long as we consider the two parameters separately. However, there exists a restriction for pairs (β, δ) of the two discount factors. In fact, a straightforward application of the above result allows to show that the restriction can be imposed on the ratio $\log(\delta)/\log(\beta)$.

On the other hand, there are no observable implications for the value of the discount factor δ rationalising the set of observations under exponential discounting. In fact, once the choice data can be rationalised for one value of the discount factor, it can be rationalised for virtually any other value $\delta \in (0,1)$. This observation is directly implied by Corollaries 1 and 2.

Corollary 3. *Set $\mathcal{O}$ is rationalisable by exponential discounted utility function if and only if for any $\delta \in (0,1)$ there exists a strictly increasing utility function $u : \mathbb{R}_+ \to \mathbb{R}_+$ such that $v(m,t) := \delta^t u(m)$ rationalises $\mathcal{O}$.*

Even though it is impossible to determine the value of the discount factor while rationalising the set of observations by an exponential discounted utility function, we are still able to impose the testable restriction on this class of models. Therefore, the conditions stated in Proposition 6 allow us to evaluate the shape of the discounting function, but not the parameter characterising it.

Appendix

In the following section we present proofs of the results stated in the main body of the paper. Moreover, we include some auxiliary results that are used in our arguments. First we show that cyclical consistency axiom implies the existence of a consistent, mixed-monotone pre-order on $\mathcal{O}$.

Lemma A.1. *Whenever the set of observations $\mathcal{O}$ is cyclically consistent, it admits a consistent, mixed-monotone pre-order $\mathcal{R}$ over $\mathcal{A}$.*

Proof. First, we define an equivalence relation on $\mathcal{A}$. We will say that elements x and y are related to each other whenever $x\,\mathcal{I}^*y$. Denote a typical equivalence class by $[x]$. We prove our claim by induction on the number of equivalence classes. If there is a single equivalence class of $\mathcal{A}$, then let $x\,\mathcal{R}\,y$ for any two elements in $\mathcal{A}$, which is consistent with the revealed preference relation and mixed-monotone. Otherwise, there would be some x, y in $\mathcal{A}$ such that $x\,\mathcal{R}^*y$, $y\,\mathcal{R}^*x$, and $x >_X y$, which would violate cyclical consistency.

Suppose that the claim holds for L equivalence classes of $\mathcal{A}$. We will show that it also holds for $L+1$ classes. Define a relation $\succeq$ over the equivalence classes in the following way. We say that $[x]' \succeq [x]$ whenever there exists some $x' \in [x]'$, $x \in [x]$, and a sequence $\{x^i\}_{i=1}^n$ in $\mathcal{X}$ such that: (i) $x^{i+1}\mathcal{R}^*x^i$ or $x^{i+1} \geq_X x^i$, (ii) $x'\,\mathcal{R}^*x^1$ or $x' \geq_X x^1$, and (iii) $x^n\,\mathcal{R}^*\,x$ or $x^n \geq_X x$. Clearly, the relation is transitive. In order to show that it is anti-symmetric, assume the opposite, i.e., that there exist two distinct equivalence classes $[x]'$ and $[x]$ such that $[x]' \succeq [x]$ and $[x] \succeq [x]'$. The first relation implies that there exist some

$x' \in [x]'$, $x \in [x]$, and a sequence $\{x^i\}_{i=1}^n$ satisfying conditions (i), (ii), and (iii). The second relation implies that there exist some $y \in [x]$, $y' \in [x]'$, and a sequence $\{y^i\}_{i=1}^m$ satisfying conditions (i), (ii), and (iii). Hence, there exists a sequence $\{z^i\}_{i=1}^k$ in $\mathcal{A}$ such that every subsequent element of the sequence dominated the preceding one with respect to $\mathcal{R}^*$ or $\geq_X$, and $z^1 \geq_X z^k$. If the elements are ordered only with respect to $\mathcal{R}^*$, then it must be $[x]' = [x]$. This yields a contradiction, since by assumption the two equivalence classes are distinct. Otherwise, we have $z^1 >_X z^k$, which violates cyclical consistency.

Given the above argument, it follows that there exists an equivalence class $\underline{[x]}$ such that there is no other equivalence class $[x]$ with $\underline{[x]} \succeq [x]$. Moreover, the exclusion of the equivalence class from $\mathcal{A}$ does not affect the relations between the remaining equivalence classes. Construct the new set of observations $\bar{\mathcal{O}}$ in the following way. Whenever $(A, x) \in \mathcal{O}$ and $x \in \underline{[x]}$, we have $(A, x) \notin \bar{\mathcal{O}}$. For any $(A, x) \in \mathcal{O}$ such that $x \notin \underline{[x]}$, define set $\bar{A} := \{x \in A \mid x \notin \underline{[x]}\}$, and let $(\bar{A}, x) \in \bar{\mathcal{O}}$. Note that the set of observable choices corresponding to $\bar{\mathcal{O}}$ is $\bar{\mathcal{A}} = \mathcal{A} \backslash \underline{[x]}$. Since $\bar{\mathcal{A}}$ has only K equivalence classes, by the induction hypothesis we conclude that there exists a mixed-monotone pre-order $\mathcal{R}$ on $\bar{\mathcal{A}}$ consistent with the revealed relations generated by $\bar{\mathcal{O}}$. We can extend the pre-order to $\mathcal{A}$ by letting $x' \, \mathcal{P} \, x$ for all $x' \in \bar{\mathcal{A}}$ and $x \in \underline{[x]}$, and $x' \, \mathcal{I} \, x$, for x', $x \in \underline{[x]}$. □

Lemma A.2. *Let $\mathcal{O}$ admit a consistent, mixed-monotone pre-order $\mathcal{R}$ on $\mathcal{A}$. Then, there are some real numbers $\{v_m^t\}_{(m,t)\in\bar{\mathcal{A}}}$ such that for any (m, t), $(n, s) \in \bar{\mathcal{A}}$, (i) whenever $(m, t) >_X (n, s)$ or $(m, t) \, \mathcal{P} \, (n, s)$ then $v_m^t > v_n^s$, and (ii) $(m, t) \, \mathcal{I} \, (n, s)$ implies $v_m^t = v_n^s$.*

Proof. Since $\mathcal{O}$ admits a mixed-monotone pre-order $\mathcal{R}$ consistent with the revealed relations, for any two elements x, y in $\mathcal{A}$, we have either $x \mathcal{P} y$ or $x \mathcal{I} y$. First, we determine numbers $\{v_m^t\}_{(m,t)\in\mathcal{A}}$ in a recursive manner. Take any $(n, s) \in \mathcal{A}$ such that for all $(m, t) \in \mathcal{A}$ we have $(m, t) \, \mathcal{R} \, (n, s)$. Clearly, such element exists. Assign any strictly positive value to v_n^s and define $\mathcal{X}_1 := \{(n, s)\}$ as well as $V_1 := \{v_n^s, 0\}$.

For any $j \geq 1$, assume that $\mathcal{X}_j \subseteq \mathcal{A}$ is non-empty, and for all $(m, t) \in \mathcal{A} \backslash \mathcal{X}_j$, we have $(m, t) \, \mathcal{R} \, (n, s)$, for any $(n, s) \in \mathcal{X}_j$. Moreover, assume that set V_j is a finite set of strictly positive real numbers and 0. Take any $(n, s) \in \mathcal{A} \backslash \mathcal{X}_j$ such that for all $(m, t) \in \mathcal{A} \backslash \mathcal{X}_j$, we have $(m, t) \, \mathcal{R} \, (n, s)$. If there exists a $(m, t) \in \mathcal{X}_j$ such that $(n, s) \mathcal{I} (m, t)$, let $v_n^s = \max V_j$. Otherwise, set v_n^s to be any number strictly greater than $\max V_j$. Denote $\mathcal{X}_{j+1} := \mathcal{X}_j \cup \{(n, s)\}$ and $V_{j+1} := V_j \cup \{v_n^s\}$. The algorithm terminates whenever $\mathcal{A} = \mathcal{X}_j$. In this case, denote $V := V_j$. By construction, for any (m, t), $(n, s) \in \mathcal{A}$, if $(m, t) \, \mathcal{P} \, (n, s)$ then $v_m^t > v_n^s$, and $(m, t) \, \mathcal{I} \, (n, s)$ implies $v_m^t = v_n^s$.

In order to complete the proof, we need to determine the values of the remaining elements of $\{v_m^t\}_{(m,t)\in\bar{\mathcal{A}}}$. If $(\overline{m},\underline{t})$ it belongs to $\mathcal{A}$, let $v_{\overline{m}}^{\underline{t}}$ take the value assigned previously. Otherwise, let $v_{\overline{m}}^{\underline{t}}$ be any number strictly greater than $\max V$. We determine the remaining numbers in the following way. If $(m,t)\in\mathcal{A}$, then v_m^t takes the value as determined above. Otherwise, define $m^+ := \min\{m' \in \mathcal{M} : m' > m\}$ and $t^- := \max\{t' \in \mathcal{T} : t' < t\}$.[3] Set $v_m^t = v$, where v is any number satisfying

$$\min\left\{v_{m^+}^t, v_m^{t^-}\right\} > v > \max\left\{v' \in V : \min\left\{v_{m^+}^t, v_m^{t^-}\right\} > v'\right\}.$$

Finally, we show that, for any (m,t), both $v_m^t < v_{m^+}^t$ and $v_m^t < v_m^{t^-}$. If (m,t) does not belong to $\mathcal{A}$, the claim holds by construction. Therefore, it suffices to consider the case when $(m,t)\in\mathcal{A}$. We prove the claim by contradiction. Suppose that $v_m^t \geq v_{m^+}^t$ or $v_m^t \geq v_m^{t^-}$. This would imply, that there exists some $n \geq m$ and $s \leq t$ such that $(n,s)\in\mathcal{A}$ and $v_n^s \leq v_m^t$. By construction of $\{v_m^t\}_{(m,t)\in\mathcal{A}}$, this would imply that $(m,t)\,\mathcal{R}\,(n,s)$. However, unless $(m,t)=(n,s)$, we have $(n,s) >_X (m,t)$. Since $\mathcal{R}$ is mixed-monotone, this yields a contradiction. □

Lemma A.3. *Set $\mathcal{O}$ is rational if there are numbers $\{v_m^t\}_{(m,t)\in\bar{\mathcal{A}}}$ such that, for any (m,t), $(n,s)\in\bar{\mathcal{A}}$, if $(m,t) >_X (n,s)$ then $v_m^t > v_n^s$, and $(m,t)\,\mathcal{R}^*(n,s)$ implies $v_m^t \geq v_n^s$.*

Proof. Whenever $\underline{m} \neq 0$, construct a decreasing sequence $\{v_0^t\}_{t\in\mathcal{T}}$ such that, for all $t\in\mathcal{T}$, we have $v_0^t < v_{\underline{m}}^t$. Similarly, whenever $\underline{t} \neq 0$, let $\{v_m^0\}_{m\in\mathcal{M}}$ be a decreasing sequence with $v_m^0 > v_m^{\underline{t}}$, for all $m\in\mathcal{M}$. If both $\underline{m}\neq 0$ and $\underline{t}\neq 0$ let v_0^0 be such that $v_0^{\underline{t}} < v_0^0 < v_{\underline{m}}^0$. Finally, denote $\mathcal{M}_0 := \mathcal{M}\cup\{0\}$ and $\mathcal{T}_0 := \mathcal{T}\cup\{0\}$.

For all $m\in\mathcal{M}_0\backslash\{\overline{m}\}$, and $t\in\mathcal{T}$, let $\lambda_m^t := (v_{m^+}^t - v_m^t)/(m^+ - m)$, where we define $m^+ := \min\{m'\in\mathcal{M} : m' > m\}$. Similarly, for any $t\in\mathcal{T}_0\backslash\{\bar{t}\}$ and $m\in\mathcal{M}$, we shall denote $\mu_m^t := (v_m^{t^+} - v_m^t)/(t^+ - t)$, where $t^+ := \min\{t'\in\mathcal{T} : t' > t\}$. Finally, let $\{\lambda_{\overline{m}}^t\}_{t\in\mathcal{T}_0}$ be any increasing sequence of strictly positive real numbers, while $\{\mu_m^{\bar{t}}\}_{m\in\mathcal{M}_0}$ denotes any decreasing, strictly negative sequence of real numbers. Define function $v : X\to\mathbb{R}$ by

$$v(n,s) := \sum_{(m,t)\in\mathcal{M}_0\times\mathcal{T}_0}\left[v_m^t + \lambda_m^t(n-m) + \mu_m^t(s-t)\right]\chi_{X_m^t}(n,s),$$

with χ being the indicator function,[4] where for all $m\in\mathcal{M}_0\backslash\{\overline{m}\}$ and $t\in\mathcal{T}_0\backslash\{\bar{t}\}$, we have

$$X_m^t := \left\{(m',t')\in X : m'\in[m,m^+) \text{ and } t\leq t' < t^+\right\},$$

[3]That is, m^+ is the immediate successor of m, while t^- is the immediate predecessor of t (with respect to the increasing order on $\mathcal{M}$ and $\mathcal{T}$ respectively).

[4]That is, for any set B, we have $\chi_B(x) = 1$ whenever $x\in B$, and $\chi_B(x) = 0$ otherwise.

and for all $t \in \mathcal{T}_0$, we define $X^t_{\overline{m}} := \{(m', t') \in X : m' \geq \overline{m} \text{ and } t \leq t' < t^+\}$, while for all $m \in \mathcal{M}_0$, we have $X^{\bar{t}}_m := \left\{(m', t') \in X : m' \in [m, m^+) \text{ and } t' \geq \bar{t}\right\}$, with m^+ and t^+ defined as previously.

Clearly, the above function is strictly increasing with respect to the partial order $\geq_X$. Moreover, for any $(m, t) \in \mathcal{A}$, we have $v(m, t) = v^t_m$. Hence, whenever $x\,\mathcal{R}^* y$ then $v(x) \geq v(y)$. This completes the proof. □

In the remainder of our paper we shall often refer to the so called *Motzkin's rational transposition*. In order to make our paper self-contained, we state the theorem below.

Theorem A.1 (Motzkin's rational transposition)**.** *Let A be an $m \times n$ rational matrix, B be an $l \times n$ rational matrix, and C be an $r \times n$ rational matrix, where B or C can be omitted (but not A). Exactly one of the following alternatives is true.*

(i) There exists $x \in \mathbb{R}^n$ such that $A \cdot x \gg 0$, $B \cdot x \geq 0$, and $C \cdot x = 0$.

(ii) There exist $\theta \in \mathbb{Z}^r$, $\lambda \in \mathbb{Z}^l$, and $\pi \in \mathbb{Z}^m$ such that $\theta \cdot A + \lambda \cdot B + \pi \cdot C = 0$, where $\theta > 0$ and $\lambda \geq 0$.

The proof of the above theorem can be found in Stoer and Witzgall (1970). We proceed with the following lemma.

Lemma A.4. *Whenever $\mathcal{O}$ obeys the dominance axiom there is a strictly increasing sequence $\{\phi_m\}_{m \in \mathcal{M}}$ and a strictly decreasing sequence $\{\varphi_t\}_{t \in \mathcal{T}}$ of real numbers such that $(m, t)\,\mathcal{R}^*(n, s)$ implies $\phi_m + \varphi_t \geq \phi_n + \varphi_s$.*

Proof. Enumerate the elements of $\mathcal{R}^*$ so that it is equal to $\{(x^j, y^j)\}_{j \in J}$, where we denote $x^j = (m^j, t^j)$ and $y^j = (n^j, s^j)$. Let $\mu_m \in \{0, 1\}^{|\mathcal{M}|}$, $m \in \mathcal{M}$, be a vector equal to 1 at the coordinate corresponding to m and 0 elsewhere. Analogously define $\tau_t \in \{0, 1\}^{|\mathcal{T}|}$, $t \in \mathcal{T}$.

Let $\mathbb{I}$ denote a $|\mathcal{M}| + |\mathcal{T}|$ by $|\mathcal{M}| + |\mathcal{T}|$ identity matrix. Moreover, let B be a $|J|$ times $|\mathcal{M}| + |\mathcal{T}|$ matrix such that, for any $j \in J$, the j'th row of the matrix is equal to $\left(\sum_{k \leq m^j} \mu_k - \sum_{k \leq n^j} \mu_k, \sum_{k \geq t^j} \tau_k - \sum_{k \geq s^j} \tau_k\right)$. We claim that if $\mathcal{O}$ obeys dominance axiom, there are vectors $\xi \in \mathbb{R}^{|\mathcal{M}|}$ and $\vartheta \in \mathbb{R}^{|\mathcal{T}|}$ such that $\mathbb{I} \cdot (\xi, \vartheta) \gg 0$ and $B \cdot (\xi, \vartheta) \geq 0$.

We prove the claim by contradiction. Suppose that $\mathcal{O}$ obeys the axiom, but there are no such vectors. By Theorem A.1, there are some $\theta \in \mathbb{Z}^{|\mathcal{M}|+|\mathcal{T}|}$ and $\lambda \in \mathbb{Z}^{|J|}$ such that $\theta \cdot \mathbb{I} + \lambda \cdot B = 0$ $(\star)$, with $\theta > 0$ and $\lambda \geq 0$. Take any such θ and λ, and let $\lambda = (\lambda_j)_{j \in J}$.

For all $j \in J$, take λ_j copies of pair $(x^j, y^j) \in \mathcal{R}^*$ and construct a sample $\{(x^i, y^i)\}_{i \in I}$, where $x^i = (m^i, t^i)$ and $y^i = (n^i, s^i)$. Since $\theta > 0$, for equation $(\star)$ to hold it must be that

$\sum_{i\in I}\sum_{k\leq m^i}\mu_k \leq \sum_{i\in I}\sum_{k\leq n^i}\mu_k$ and $\sum_{i\in I}\sum_{k\geq t^i}\tau_k \leq \sum_{i\in I}\sum_{k\geq s^i}\tau_k$ is satisfied, with at least one inequality being strict ($<$). Clearly, this is possible only if

$$|\{i \in I : m^i \geq m\}| \leq |\{i \in I : n^i \geq m\}| \quad \text{and} \quad |\{i \in I : t^i \leq t\}| \leq |\{i \in I : s^i \leq t\}|,$$

for all $m \in \mathcal{M}$ and $t \in \mathcal{T}$, where the inequality is strict for some m of t. However, this violates dominance axiom, which yields a contradiction.

To complete the argument, take any vectors ξ and ϑ such that $\mathbb{I}\cdot(\xi,\vartheta) \gg 0$ and $B\cdot(\xi,\vartheta) \geq 0$. Define sequences $\{\xi_m\}_{m\in\mathcal{M}}$ and $\{\vartheta_t\}_{t\in\mathcal{T}}$ by $\xi_m := \mu_m\cdot\xi$ and $\vartheta_t := \tau_t\cdot\vartheta$. By construction, $(m,t)\,\mathcal{R}^*(n,s)$ implies $\sum_{k\leq m}\xi_k + \sum_{k\geq t}\vartheta_k \geq \sum_{k\leq n}\xi_k + \sum_{k\geq s}\vartheta_k$. Let $\phi_m := \sum_{k\leq m}\xi_k$ and $\varphi_t := \sum_{k\geq t}\vartheta_k$, for all $m\in\mathcal{M}$ and $t\in\mathcal{T}$. Since sequences $\{\xi_m\}_{m\in\mathcal{M}}$ and $\{\vartheta_t\}_{t\in\mathcal{T}}$ are strictly positive, both $\{\phi_m\}_{m\in\mathcal{M}}$ and $\{\varphi_t\}_{t\in\mathcal{T}}$ are strictly monotone. Moreover, $(m,t)\,\mathcal{R}^*(n,s)$ implies $\phi_m+\varphi_t \geq \phi_n+\varphi_s$. □

Lemma A.5. *Set $\mathcal{O}$ is rationalisable by a discounted utility function whenever there is a strictly increasing sequence $\{\phi_m\}_{m\in\mathcal{M}}$ and a strictly decreasing sequence $\{\varphi_t\}_{t\in\mathcal{T}}$ of real numbers such that $(m,t)\,\mathcal{R}^*(n,s)$ implies $\phi_m+\varphi_t \geq \phi_n+\varphi_s$.*

Proof. Take any number $\phi_0 < \phi_{\underline{m}}$ whenever $\underline{m} \neq 0$, and let $\phi_0 = \phi_{\underline{m}}$ otherwise. Denote $\mathcal{M}_0 = \mathcal{M}\cup\{0\}$. In addition, for any $m \in \mathcal{M}\cup\{0\}$ different from $\overline{m}$, define the immediate successor of m by $m^+ := \min\{n\in\mathcal{M} : n > m\}$. Let $\phi : \mathbb{R}\to\mathbb{R}$ be specified by

$$\phi(n) := \sum_{m\in\mathcal{M}_0} [\phi_m + \lambda_m(n-m)]\chi_{N_m}(n), \tag{A1}$$

where χ is the indicator function, $\lambda_m := (\phi_{m^+}-\phi)/(m^+-m)$ for all m different from $\overline{m}$, $\lambda_{\overline{m}}$ is any strictly positive number, while $N_m := [m, m^+)$ for $m \neq \overline{m}$, and $N_{\overline{m}} := [\overline{m}, \infty)$. Clearly, the function is continuous and strictly increasing. Moreover, for any $m\in\mathcal{M}$, we have $\phi(m) = \phi_m$.

If $\underline{t} = 0$, define sequence $\{\tilde{\varphi}_t\}_{t\in\mathcal{T}}$ by $\tilde{\varphi}_t := \varphi_t - \varphi_{\underline{t}}$. Otherwise, let $\{\tilde{\varphi}_t\}_{t\in\mathcal{T}} := \{\varphi_t\}_{t\in\mathcal{T}}$. Denote $\tilde{\varphi}_0 = 0$, $\mathcal{T}_0 := \mathcal{T}\cup\{0\}$, and let the immediate successor of t in $\mathcal{T}$ be defined by $t^+ := \min\{s\in\mathcal{T} : s > t\}$, for any t different from $\overline{t}$. Define function $\varphi : \mathbb{N}\to\mathbb{R}_-$ by

$$\varphi(s) := \sum_{t\in\mathcal{T}\cup\{0\}} [\tilde{\varphi}_t + v_t(s-t)]\chi_{S_t}(s)$$

where χ is the indicator function, $v_t := (\tilde{\varphi}_{t^+}-\tilde{\varphi}_t)/(t^+-t)$ for all $t\in\mathcal{T}\cup\{0\}$ different from $\overline{t}$, $v_{\overline{t}}$ is any strictly negative number, while $S_t := \{s\in\mathbb{N} : t\leq s < t^+\}$ for $t\neq\overline{t}$, and $S_{\overline{t}} := \{s\in\mathbb{N} : \overline{t}\leq s\}$. Note that the function is strictly decreasing and takes only negative values with $\varphi(0) = 0$. Moreover, we have $\varphi(t) = \tilde{\varphi}_t$, for any $t\in\mathcal{T}$.

Define functions $u : \mathbb{R}_+ \to \mathbb{R}_+$ and $\gamma : \mathbb{N} \to (0, 1]$ by $u := \exp(\phi)$ and $\gamma := \exp(\varphi)$. Clearly the two functions are strictly monotone, while $\gamma(0) = 1$. Finally, for any (m, t) in $\mathcal{A}$, we have $v(m, t) = \exp(\phi_m + \tilde{\varphi}_t)$. Hence, $(m, t)\, \mathcal{R}^*(n, s)$ implies $v(m, t) \geq v(n, s)$. □

The next result states an important property of anchored experiments. We apply the following lemma in the proof of Proposition 3, presented in the remainder of this section.

Lemma A.6. *Let $\mathcal{E}$ be an anchored experiment. Whenever $\mathcal{O}$ is rationalisable by a discounted utility function, there is a strictly increasing sequence $\{n_t\}_{t\in\mathcal{T}}$ in $\mathbb{R}_+$ such that $(m, t)\, \mathcal{R}^*(m^*, t^*)$ implies $m \geq n_t$, and $(m^*, t^*)\, \mathcal{R}^*(m, t)$ implies $n_t \geq m$.*

Proof. Since set $\mathcal{O}$ is rationalisable by a discounted utility function, there is a strictly increasing utility function u and a discounting function γ such that $(m, t)\, \mathcal{R}^*(m^*, t^*)$ implies $u(m)\gamma(t) \geq u(m^*)\gamma(t^*)$, and $(m^*, t^*)\, \mathcal{R}^*(m, t)$ implies $u(m^*)\gamma(t^*) \geq u(m)\gamma(t)$. Let $u^* := u(m^*)$ and $\gamma^* := \gamma(t^*)$, while for all $t \in \mathcal{T}$, $n_t \in \{m \in \mathbb{R}_+ : u(m) = u^*\gamma^*/\gamma(t)\}$. Since u is continuous and strictly increasing, n_t is uniquely defined, for all $t \in \mathcal{T}$. Moreover, since γ is a strictly decreasing function, $\{n_t\}_{t\in\mathcal{T}}$ is a strictly increasing sequence.

Suppose that $(m, t)\, \mathcal{R}^*(m^*, t^*)$. This implies that $u(m)\gamma(t) \geq u(m^*)\gamma(t^*)$. Therefore, $u(m) \geq u^*\gamma^*/\gamma(t) = u(n_t)$. By strict monotonicity of u, it must be that $m \geq n_t$. Analogously, we show that $(m^*, t^*)\, \mathcal{R}^*(m, t)$ implies $u(m^*)\gamma(t^*) \geq u(m)\gamma(t)$. □

Before we proceed with the proof Proposition 3, we need to introduce one additional notion. We will say that collection $\{(x^i, y^i)\}_{i\in I}$, where we denote $x^i = (m^i, t^i)$ and $y^i = (n^i, s^i)$, is a *dominant* sample of $\mathcal{R}^*$ whenever for all $m \in \mathcal{M}$ and $t \in \mathcal{T}$, we have

$$|\{i \in J : m^i \leq m\}| \geq |\{i \in J : n^i \leq m\}| \;\text{ and }\; |\{i \in J : t^i \leq t\}| \leq |\{i \in J : s^i \leq t\}|.$$

We say that a sample is *strictly dominant*, whenever it is dominant, and there exists no subset $J \subseteq I$, such that for all $m \in \mathcal{M}$ and $t \in \mathcal{T}$, we have

$$|\{i \in J : m^i \leq m\}| \leq |\{i \in J : n^i \leq m\}| \;\text{ and }\; |\{i \in J : t^i \leq t\}| \geq |\{i \in J : s^i \leq t\}|.$$

Clearly, whenever the sample is dominant and such subset exists, then $\{(x^i, y^i)\}_{i\in I\setminus J}$ is also dominant. It is straightforward to show that for any dominant sample $\{(x^i, y^i)\}_{i\in I}$ such that either $|\{i \in J : m^i \geq m\}| < |\{i \in J : n^i \geq m\}|$, for some $m \in \mathcal{M}$, or $|\{i \in J : t^i \leq t\}| < |\{i \in J : s^i \leq t\}|$, for some $t \in \mathcal{T}$, has a strictly dominant sub-sample.

Proof of Proposition 3. We prove (i) ⇒ (ii) by contradiction. Suppose that $\mathcal{O}$ is cyclically consistent, but there is a sample $\{(x^i, y^i)\}_{i\in I}$, where $x^i = (m^i, t^i)$ and $y^i = (n^i, s^i)$, such that for all $m \in \mathcal{M}$ and $t \in \mathcal{T}$, we have $|\{i \in J : m^i \geq m\}| \geq |\{i \in J : n^i \leq m\}|$ and $|\{i \in J : t^i \leq t\}| < |\{i \in J : s^i \leq t\}|$, where at least one inequality is strict. In particular, this implies that there exists a strictly dominant sub-sample $\{(x^j, y^j)\}_{j \in J}$, $J \subseteq I$.

Take any element (x^j, y^j) of the sub-sample. Since it is strictly dominant, it cannot be that $x^j \geq_X y^j$. Otherwise, we would have $|\{i \in \{j\} : m^i \leq m\}| \leq |\{i \in \{j\} : n^i \leq m\}|$ and $|\{i \in \{j\} : t^i \leq t\}| \geq |\{i \in \{j\} : s^i \leq t\}|$, for all $m \in \mathcal{M}$ and $t \in \mathcal{T}$, which would violate that $\{(x^j, y^j)\}_{j \in J}$ is strictly dominant. On the other hand, since $\mathcal{O}$ is cyclically consistent, it cannot be that $y^j >_X x^j$. Otherwise, this would imply that $x^j \, \mathcal{R}^* y^j$ and $y^j >_X x^j$, which would violate the axiom. Hence, for any element (x^j, y^j) of the sample, x^j and y^j must be unordered with respect to the partial order $\geq_X$. Therefore, either (i) $m^j > n^j$ and $t^j > s^j$, or (ii) $m^j < n^j$ and $t^j < s^j$. Finally, since $\mathcal{E}$ is an anchored experiment, for any element of the sample, we have either $x^j = x^*$ and $y^j \neq x^*$, or $x^j \neq x^*$ and $y^j = x^*$. Take any pair (x^j, y^j) from the sample and consider the following claims.

Claim 1: If $x^j = x^*$, $n^j > (<)\ m^*$, and $s^j > (<)\ t^*$, then there is some (x^k, y^k) in the sample such that $m^k \geq (\leq)\ n^j$ and $t^k > (<)\ s^j$, or $m^k > (<)\ n^j$ and $t^k \geq (\leq)\ s^j$. We prove the claim outside the brackets. Since the sample is dominant, it contains some (x^k, y^k) such that $t^k \geq s^j$. Since $\mathcal{E}$ is an anchored experiment, this implies that $y^k = x^*$. First, suppose that $t^k \geq s^j$ and $m^k < n^j$. Then $y^j >_X x^k$. However, since $x^k \, \mathcal{R}^* \, x^* \, \mathcal{R}^* y^j$, this would violate cyclical consistency. On the other hand, whenever $m^k = n^j$ and $t^k = s^j$, we have $x^k = y^j$ and $y^k = x^j$. In particular, this implies that for all $m \in \mathcal{M}$ and $t \in \mathcal{T}$, we have $|\{i \in \{j,k\} : m^i \leq m\}| = |\{i \in \{j,k\} : n^i \leq m\}|$ and $|\{i \in \{j,k\} : t^i \leq t\}| = |\{i \in \{j,k\} : s^i \leq t\}|$, which contradicts that $\{(x^j, y^j)\}_{j \in J}$ is strictly dominant. Therefore, it must be either $m^k \geq n^j$ and $t^k > s^j$, or $m^k > n^j$ and $t^k \geq s^j$. In order to prove the version in the brackets, note that the sample must contain some (x^k, y^k) such that $m^k \leq n^j$. The rest of the argument is analogous.

Claim 2: If $y^j = x^*$, $m^j > (<)\ m^*$, and $t^j > (<)\ t^*$, then there is some (x^k, y^k) in the sample such that $n^k \geq (\leq)\ m^j$ and $s^k > (<)\ t^j$, or $n^k > (<)\ m^j$ and $s^k \geq (\leq)\ t^j$. We prove the claim analogously to Claim 1.

Given Claims 1 and 2, there exists a sequence $\{x^k\}_{k=1}^K$ in $\mathcal{A}$ such that for any two subsequent elements $x^k = (m^k, t^k)$ and $x^{k+1} = (m^{k+1}, t^{k+1})$, we have $m^k \geq m^{k+1}$ and $t^k \geq t^{k+1}$, where at least one of the above inequalities is strict. Clearly, since $\mathcal{A}$ is finite, there exists the final element x^K of the sequence, for which there is no $y = (n, s)$ in $\mathcal{A}$ such that $m^K \geq n$ and $t^K \geq s$ (with at least one inequality being strict). However, by Claims 1 and 2, this contradicts the existence of a properly embedded sample of $\mathcal{R}^*$. Therefore, given that $\mathcal{O}$ obeys cyclical consistency, there is no dominant sample $\{(x^i, y^i)\}_{i\in I}$ of $\mathcal{R}^*$ such that, for some $m \in \mathcal{M}$ or $t \in \mathcal{T}$, we have $|\{i \in J : m^i \geq m\}| < |\{i \in J : n^i \geq m\}|$ or $|\{i \in J : t^i \leq t\}| < |\{i \in J : s^i \leq t\}|$. Hence, dominance axiom is satisfied. By Theorem 1, this implies that set of observations $\mathcal{O}$ is rationalisable by a discounted utility function.

To show (ii) $\Rightarrow$ (iii), assume that (ii) holds. By Lemma A.6, there exists a strictly increasing sequence of numbers $\{n_t\}_{t\in\mathcal{T}}$ such that for any $x \in \mathcal{A}$, where $x = (m, t)$, whenever $x\,\mathcal{R}^*x^*$ then $m \geq n_t$, and $x^*\mathcal{R}^*x$ implies $m \leq n_t$. Take any number $u^* > 0$ and a discounting function $\gamma : \mathbb{N} \to (0, 1]$, with $\gamma(0) = 1$. Let $\gamma^* := \gamma(t^*)$ and $u_t := u^*\gamma^*/\gamma(t)$. As γ is strictly decreasing, sequence $\{u_t\}_{t\in\mathcal{T}}$ is strictly increasing and positive.

Whenever $n_{\underline{t}} \neq 0$, set $n_0 = 0$, and let u_0 be any strictly positive number such that $u_0 < u_{\underline{t}}$. Otherwise, let $n_0 = n_{\underline{t}}$ and $u_0 = u_{\underline{t}}$. Denote $\mathcal{T}_0 := \mathcal{T} \cup \{0\}$. Finally, for all $t \in \mathcal{T}_0$ different from $\bar{t}$, let the immediate successor of t be defined by $t^+ := \min\{s \in \mathcal{T} : s > t\}$. We define function $u : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$u(m) := \sum_{t\in\mathcal{T}_0} [u_t + \lambda_t(m - n_t)]\chi_{N_t}(m),$$

where $\lambda_j = (u_{t^+} - u_t)/(n_{t^+} - n_t)$ for all $t \neq \bar{t}$, $\lambda_{\bar{t}}$ is any strictly positive number, and $N_t := [n_t, n_{t^+})$, for all $t \neq \bar{t}$, while $N_{\bar{t}} := [n_{\bar{t}}, \infty)$. Clearly, the function is continuous and strictly increasing. Moreover, by construction, for all $t \in \mathcal{T}$, we have $u(n_t) = u_t$.

To complete this part of the proof, suppose that for some $x \in \mathcal{A}$, we have $x\,\mathcal{R}^*x^*$, where $x = (m, t)$. By assumption, it must be that $m \geq n_t$. By monotonicity of u, we have $u(m) \geq u(n_t) = u^*\gamma^*/\gamma(t)$, which implies that $u(m)\gamma(t) \geq \phi^*\gamma^* = u(m^*)\gamma(t^*)$. Analogously, we show that if $x^*\,\mathcal{R}^*x$ then $u(m^*)\gamma(t^*) \geq u(m)\gamma(t)$.

In order to complete the proof, note that implication (iii) $\Rightarrow$ (i) holds trivially, since any set rationalisable by a discounted utility function is rational. □

In the following two lemmas we prove the sufficiency of the cumulative dominance axiom for rationalisation by a weakly present-biased discounted utility function.

Lemma A.7. *Whenever $\mathcal{O}$ obeys the cumulative dominance axiom, there is a strictly increasing sequence $\{\phi_m\}_{m\in\mathcal{M}}$, a strictly decreasing sequence $\{\varphi_t\}_{t\in\mathcal{T}}$, and a strictly negative sequence $\{v_t\}_{t\in\mathcal{T}}$ of real numbers such that $(m, t)\,\mathcal{R}^*(n, s)$ implies $\phi_m + \varphi_t \geq \phi_n + \varphi_s$, while for all $t \in \mathcal{T}$, we have $\varphi_t + v_t(s - t) \leq \varphi_s$, for all $s \in \mathcal{T}$.*

Proof. Enumerate the elements of $\mathcal{R}^*$ such that it is equal to $\{(x^j, y^j)\}_{j\in J}$, where $x^j = (m^j, t^j)$ and $y^j = (n^j, s^j)$. In addition, define the immediate successor of t in $\mathcal{T}$ by $t^+ := \min\{t' \in \mathcal{T} : t' > t\}$, for all $t \in \mathcal{T}$ different from $\bar{t}$, while $\bar{t}^+ := \bar{t} + 1$.

For any $m \in \mathcal{M}$, let $\mu_m \in \{0,1\}^{|\mathcal{M}|}$ be a vector that takes the value of 1 at the coordinate corresponding to m, and 0 elsewhere. Moreover, for any $t \in \mathcal{T}$, let $\tau_t \in \{0,1\}^{|\mathcal{T}|}$ be a vector taking the value of $(t^+ - t)$ at the coordinate corresponding to t, and 0 in all the remaining entries.

By $\mathbb{I}$ we denote a $|\mathcal{M}| + |\mathcal{T}|$ by $|\mathcal{M}| + |\mathcal{T}|$ identity matrix. Moreover, let B_1 be a $|J|$ times $|\mathcal{M}|$ matrix such that, for any $j \in J$, the j'th row of the matrix is equal to $\left(\sum_{k \le m^j} \mu_k - \sum_{k \le n^j} \mu_k\right)$. Similarly, let B_2 be a $|J|$ by $|\mathcal{T}|$ matrix where the j'th row is equal to $\left(\sum_{k \ge t^j} \tau_k - \sum_{k \ge s^j} \tau_k\right)$. Define B_3 as a $|\mathcal{T}| - 1$ times $|\mathcal{M}|$ matrix with every entry equal to 0. Finally, B_4 is a matrix of dimensions $|\mathcal{T}| - 1$ by $|\mathcal{T}|$, where each column corresponds to one element in $\mathcal{T}$ and each row corresponds to an element in $\mathcal{T} \backslash \{\underline{t}\}$. Moreover, for each row corresponding to time-delay t, the entry in the column corresponding to t is equal to 1, while the entry in the column corresponding to t^+ is equal to -1. We set all the remaining entries to be equal to 0. We use the above matrices to construct a $|J| + |\mathcal{T}| - 1$ times $|\mathcal{M}| + |\mathcal{T}|$ matrix B, defined by

$$B := \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}.$$

We claim that whenever $\mathcal{O}$ obeys the cumulative dominance axiom, there exist vectors $\xi \in \mathbb{R}^{|\mathcal{M}|}$ and $\vartheta \in \mathbb{R}^{|\mathcal{T}|}$ such that $\mathbb{I} \cdot (\xi, \vartheta) \gg 0$ and $B \cdot (\xi, \vartheta) \ge 0$.

We prove the claim by contradiction. Suppose that $\mathcal{O}$ obeys the cumulative dominance axiom, but there are no such vectors. Theorem A.1 implies that there are some $\theta \in \mathbb{Z}^{|\mathcal{M}|+|\mathcal{T}|}$ and $\lambda \in \mathbb{Z}^{|J|+|\mathcal{T}|-1}$ such that $\theta \cdot \mathbb{I} + \lambda \cdot B = 0$, where $\theta > 0$ and $\lambda \ge 0$. Take any such vectors and denote $\lambda = (\overline{\lambda}, \underline{\lambda})$, where $\overline{\lambda} = (\lambda_j)_{j \in J}$ and $\underline{\lambda} = (\lambda_t)_{t \in \mathcal{T} \backslash \{\underline{t}\}}$, so that every coordinate of $\overline{\lambda}$ corresponds to a single element in $\mathcal{R}^*$, while each entry of $\underline{\lambda}$ corresponds to a time-delay $t \neq \underline{t}$. Moreover, let $\overline{B} := [B_1 \ B_2]$, while $\underline{B} := [B_3 \ B_4]$. Clearly, we have $\theta \cdot \mathbb{I} + \overline{\lambda} \cdot \overline{B} + \underline{\lambda} \cdot \underline{B} = 0$ $(\star)$.

Construct a sample $\{(x^i, y^i)\}_{i \in I}$ of $\mathcal{R}^*$ by taking λ_j copies of pair (x^j, y^j) from the directly revealed preference relation, for each $j \in J$. Since $\theta > 0$, for condition $(\star)$ to hold, it must be that $\underline{\lambda} \cdot \underline{B} < -\overline{\lambda} \cdot \overline{B}$. This implies that, $\sum_{i \in I} \sum_{k \le m^i} \mu_k \le \sum_{i \in I} \sum_{k \le n^i} \mu_k$. Hence, for all $m \in \mathcal{M}$, we have $|\{i \in I : m^i \le m\}| \ge |\{i \in I : m^i \le m\}|$.

Moreover, we require that $\underline{\lambda} \cdot \underline{B} \le \sum_{i \in I} \sum_{k \ge s^i} \tau_k - \sum_{i \in I} \sum_{k \ge t^i} \tau_t$. Note that by definition, $\sum_{i \in I} \sum_{k \ge t^i} \tau_t = \int_t^{t^+} |\{i \in I : t^i \le z\}| dz$, for all $t \in \mathcal{T}$. Hence, in particular, the initial condition implies that

$$\lambda_{\underline{t}} \le \int_{\underline{t}}^{\underline{t}^+} |\{i \in I : s^i \le z\}| dz - \int_{\underline{t}}^{\underline{t}^+} |\{i \in I : t^i \le z\}| dz.$$

Since $\lambda_{\underline{t}} \geq 0$, we have $\int_{\underline{t}}^{t^+} |\{i \in I : t^i \leq z\}|dz \leq \int_{\underline{t}}^{t^+} |\{i \in I : s^i \leq z\}|dz$. Denote the immediate predecessor of t in $\mathcal{T}$ by $t^- := \max\{s \in \mathcal{T} : s < t\}$. Take any $t \in \mathcal{T}$, different from $\underline{t}$, and suppose that

$$\lambda_{t^-} \leq \int_{\underline{t}}^{t} |\{i \in I : s^i \leq z\}|dz - \int_{\underline{t}}^{t} |\{i \in I : t^i \leq z\}|dz. \tag{A2}$$

For inequality $(\star)$ to hold, we require that

$$\lambda_t - \lambda_{t^-} \leq \int_{t}^{t^+} |\{i \in I : s^i \leq z\}|dz - \int_{t}^{t^+} |\{i \in I : t^i \leq z\}|dz.$$

Given the initial condition (A2), this implies that

$$\lambda_t \leq \int_{\underline{t}}^{t^+} |\{i \in I : s^i \leq z\}|dz - \int_{\underline{t}}^{t^+} |\{i \in I : t^i \leq z\}|dz.$$

Since $\lambda_{t^+} \geq 0$, we have $\int_{\underline{t}}^{t^+} |\{i \in I : t^i \leq z\}|dz \leq \int_{\underline{t}}^{t^+} |\{i \in I : s^i \leq z\}|dz$. By induction, we conclude that the condition holds for all $t \in \mathcal{T}$.

Recall that $\theta > 0$. Therefore, for $(\star)$ to hold, it must be that at least one of the above inequalities is strict. However, this violates the cumulative dominance axiom. Contradiction. Therefore, we conclude that there exist vectors ξ and ϑ such that $\mathbb{I} \cdot (\xi, \vartheta) \gg 0$ and $B \cdot (\xi, \vartheta) \geq 0$.

Take any pair (ξ, ϑ) satisfying the above system of inequalities. Define $\{\xi_m\}_{m \in \mathcal{M}}$ and $\{\vartheta_t\}_{t \in \mathcal{T}}$ by $\xi_m := \mu_m \cdot \xi$ and $\vartheta_t := \tau_t \cdot \vartheta$ respectively. Clearly, both sequences are strictly positive, while $(m,t)\, \mathcal{R}^*(n,s)$ implies $\sum_{k \leq m} \xi_k + \sum_{k \geq t} \vartheta_k \geq \sum_{k \leq n} \xi_k + \sum_{k \geq s} \vartheta_k$. Next, define $\{\phi_m\}_{m \in \mathcal{M}}$ by $\phi_m := \sum_{k \leq m} \xi_k$. As $\{\xi_m\}_{m \in \mathcal{M}}$ is strictly positive, the above sequence is strictly increasing. Similarly, construct $\{\varphi_t\}_{t \in \mathcal{T}}$, where $\varphi_t := \sum_{k \geq t} \vartheta_k$, which is strictly decreasing. Finally, $(m,t)\, \mathcal{R}^*(n,s)$ implies $\phi_m + \varphi_t \geq \phi_n + \varphi_s$.

In order to complete the proof, define $v_t := -\vartheta_t/(t^+ - t)$, for all $t \in \mathcal{T}$. By definition of $\{\vartheta_t\}_{t \in \mathcal{T}}$, sequence $\{v_t\}_{t \in \mathcal{T}}$ is strictly negative and (weakly) increasing. We need to show that for all $t \in \mathcal{T}$, we have $\varphi_t + v_t(s-t) \leq \varphi_s$, for all $s \in \mathcal{T}$. Clearly, the inequality holds whenever $s = t$. Assume that $s > t$. By definition of φ_t, we have

$$\begin{aligned}
\varphi_t + v_t(s-t) &= -\sum_{r \geq t}(r^+ - r)v_r + v_t(s-t) \\
&= -\sum_{r \geq t}(r^+ - r)v_r + v_t \sum_{s > r \geq t}(r^+ - r) \\
&= \sum_{s > r \geq t}(r^+ - r)(v_t - v_r) - \sum_{r \geq s}(r^+ - r)v_r \\
&\leq -\sum_{r \geq s}(r^+ - r)v_r \\
&= \varphi_s,
\end{aligned}$$

where the inequality follows form the fact that $\{v_t\}_{t\in\mathcal{T}}$ is an increasing sequence. Using a similar argument, we can show that the same condition holds for any $s < t$. □

Lemma A.8. *Set $\mathcal{O}$ is rationsalisable by a weakly present-biased discounted utility function, whenever there is a strictly increasing sequence $\{\phi_m\}_{m\in\mathcal{M}}$, a strictly decreasing sequence $\{\varphi_t\}_{t\in\mathcal{T}}$, and a strictly negative sequence $\{v_t\}_{t\in\mathcal{T}}$ of real numbers such that $(m,t)\,\mathcal{R}^*(n,s)$ implies $\phi_m+\varphi_t \geq \phi_n+\varphi_s$, while for all $t\in\mathcal{T}$, we have $\varphi_t + v_t(s-t) \leq \varphi_s$, for all $s \in \mathcal{T}$.*

Proof. Define function $\phi : \mathbb{R}_+ \to \mathbb{R}$ as in (A1). Whenever $\underline{t} = 0$, construct a sequence $\{\tilde{\varphi}_t\}_{t\in\mathcal{T}}$, where $\tilde{\varphi}_t := \varphi_t - \varphi_{\underline{t}}$. Otherwise, let $\{\tilde{\varphi}_t\}_{t\in\mathcal{T}}$ be equal to $\{\varphi_t\}_{t\in\mathcal{T}}$. Moreover, denote $\tilde{\varphi}_0 = 0$ and $\mathcal{T}_0 := \mathcal{T} \cup \{0\}$. Define function $\bar{\varphi} : \mathbb{R} \to \mathbb{R}_-$, by

$$\bar{\varphi}(s) := \max_{t\in\mathcal{T}_0}\{\tilde{\varphi}_t + v_t(s-t)\}.$$

Note that, by definition of $\{\varphi_t\}_{t\in\mathcal{T}}$ and $\{v_t\}_{t\in\mathcal{T}}$, the above function is strictly decreasing, and convex. Hence, function $\vartheta(t) := \varphi(t) - \varphi(t+1)$ is also decreasing. Moreover, we have $\bar{\varphi}(t) = \tilde{\varphi}_t$, for all $t \in \mathcal{T}$. Finally, the above properties hold once we restrict $\bar{\varphi}$ to $\mathbb{N}$.

Define functions $u : \mathbb{R}_+ \to \mathbb{R}_+$ and $\gamma : \mathbb{N} \to (0,1]$ by $u := \exp(\phi)$ and $\gamma := \exp(\bar{\varphi})$ respectively. Clearly, the two functions are strictly monotone, while γ is log-convex with $\gamma(0) = 1$. Moreover, we have $v(m,t) := u(m)\gamma(t) = \exp(\phi_m + \tilde{\varphi}_t)$, for any $(m,t) \in \mathcal{A}$. Whenever $(m,t)\,\mathcal{R}^*(n,s)$ then $\phi_m+\tilde{\varphi}_t \geq \phi_n+\tilde{\varphi}_s$, which implies that $v(m,t) \geq v(n,s)$. □

Next, we present the two lemmas that support the sufficiency of the strong cumulative dominance axiom for rationalisability. At the same time, the two results complete the proof of Proposition 5.

Lemma A.9. *Whenever $\mathcal{O}$ obeys strong cumulative dominance axiom for some $t' \in \mathcal{T}$, there exists a strictly increasing sequence $\{\phi_m\}_{m\in\mathcal{M}}$ and strictly negative numbers $\hat{\beta}$ and $\hat{\delta}$ such that $(m,t)\,\mathcal{R}^*\,(n,s)$ implies $\phi_m + \min\{t,t'\}\hat{\beta} + t\hat{\delta} \geq \phi_n + \min\{s,t'\}\hat{\beta} + s\hat{\delta}$.*

Proof. Take any $t \in \mathcal{T}$ for which $\mathcal{O}$ obeys strong cumulative dominance axiom. Take any sample $\{(x^j, y^j)\}_{j\in J}$ of $\mathcal{R}^*$, where $x^j = (m^j, t^j)$ and $y^j = (n^j, s^j)$. For any $m \in \mathcal{M}$, let $\mu_m \in \{0,1\}^{|\mathcal{M}|}$ be a vector that takes the value of 1 at the coordinate corresponding to m, and 0 everywhere else. Let $\mathbb{I}$ denote a $|\mathcal{M}|+2$ by $|\mathcal{M}|+2$ a diagonal matrix, where for the first $|\mathcal{M}|$ rows the corresponding entries are equal to 1, while for the last two are equal to -1. Let B be a $|J|$ times $|\mathcal{M}|+2$ matrix such that, for any $j \in J$, its j'th row is equal to $(\sum_{k\leq m^j}\mu_k-\sum_{k\leq n^j}\mu_k, t^j-s^j, \min\{t^j,t'\}-\min\{s^j,t'\})$. We claim that there exists a vector $\xi\in\mathbb{R}$ and real numbers $\hat{\beta}$ and $\hat{\delta}$ such that $\mathbb{I}\cdot(\xi,\hat{\delta},\hat{\beta})\gg 0$ and $B\cdot(\xi,\hat{\delta},\hat{\beta})\geq 0$.

We prove the claim by contradiction. Suppose that there are no such vectors. By Theorem A.1, there is some $\theta\in\mathbb{Z}^{|\mathcal{M}|+2}$ and $\lambda\in\mathbb{Z}^{|J|}$ such that $\theta\cdot\mathbb{I}+\lambda\cdot B=0$, where $\theta>0$ and $\lambda\geq 0$. Take any such vectors and denote $\lambda=(\lambda_j)_{j=1}^J$. Construct sample $\{(x^i,y^i)\}_{i\in I}$ by taking λ_j copies of pair (x^j,y^j) from of $\mathcal{R}^*$, for all $j\in J$. Since $\theta>0$, for condition $(\star)$ to hold, it must be that $\sum_{i\in I}\sum_{k\leq m^i}\mu_k\leq\sum_{i\in I}\sum_{k\leq n^i}\mu_m$, as well as $\sum_{i\in I}t^i\geq\sum_{i\in I}s^j$, $\sum_{i\in I}\min\{t^i,t'\}\geq\sum_{i\in I}\min\{s^i,t'\}$, with at least one inequality being strict. However, this contradicts the strong cumulative dominance axiom.

Take any ξ and $\hat{\delta}$, $\hat{\beta}$ satisfying $\mathbb{I}\cdot(\xi,\hat{\delta},\hat{\beta})\gg 0$ and $B\cdot(\xi,\hat{\delta},\hat{\beta})\geq 0$. Define sequence $\{\xi_m\}_{m\in\mathcal{M}}$ by $\xi_m:=\mu_m\cdot\xi$. By construction of matrix B, whenever $(m,t)\,\mathcal{R}^*(n,s)$ then $\sum_{k\leq m}\xi_k+t\hat{\delta}+\min\{t,t'\}\hat{\beta}\geq\sum_{k\leq n}\xi_k+s\hat{\delta}+\min\{s,t'\}\hat{\beta}$. Define sequence $\{\phi_m\}_{m\in\mathcal{M}}$ by $\phi_m:=\sum_{k\leq m}\xi_k$. Since $\{\xi_m\}_{m\in\mathcal{M}}$ is strictly positive, $\{\phi_m\}_{m\in\mathcal{M}}$ is strictly increasing and satisfies the property specified in the lemma. □

Lemma A.10. *Set $\mathcal{O}$ is rationalisable by a quasi-hyperbolic discounting function if there is some $t'\in\mathcal{T}$, a strictly increasing sequence $\{\phi_m\}_{m\in\mathcal{M}}$ and strictly negative numbers $\hat{\beta}$ and $\hat{\delta}$ such that $(m,t)\,\mathcal{R}^*\,(n,s)$ implies $\phi_m+t\hat{\delta}+\min\{t,t'\}\hat{\beta}\geq\phi_n+s\hat{\delta}+\min\{s,t'\}\hat{\beta}$.*

Proof. Construct function $\phi:\mathbb{R}_+\to\mathbb{R}$ as in (A1). Recall that the function is continuous and strictly increasing. Moreover, for any $m\in\mathcal{M}$, we have $\phi(m)=\phi_m$. Define function $u:\mathbb{R}_+\to\mathbb{R}_+$ by $u:=\exp(\phi)$. Moreover, let $\delta:=\exp(\hat{\delta})$, $\beta:=\exp(\hat{\beta})$, and $t^\circ:=t'$. Define function $\gamma:\mathbb{N}\to(0,1]$ by $\gamma(t):=\beta^t\delta^t$, whenever $t<t^\circ$, and $\gamma(t):=\beta^{t^\circ}\delta^t$ otherwise. Clearly, we have $\gamma(0)=1$. Moreover, for any prize-time pair $(m,t)\in\mathcal{A}$, we have $v(m,t):=u(m)\gamma(t)=\exp(\phi_m+t\hat{\delta}+\min\{t,t^\circ\}\hat{\beta})$. Therefore, whenever $(m,t)\,\mathcal{R}^*(n,s)$ holds, then $v(m,t)\geq v(n,s)$. The proof is complete. □

Proof of Proposition 6. We can show the necessity of the conditions stated in the proposition analogously to quasi-hyperbolic case. In order to show sufficiency, enumerate the elements of $\mathcal{R}^*$ such that is is equal to $\{(x^j,y^j)\}_{j\in J}$, where $x^j=(m^j,t^j)$ and $y^j=(n^j,s^j)$. For each $m\in\mathcal{M}$, let $\mu_m\in\{0,1\}^{|\mathcal{M}|}$ be a vector with all entries equal to zero, apart from the one corresponding to m equal to 1. Moreover, denote the immediate successor of m in $\mathcal{M}$ by $m^+:=\min\{n\in\mathcal{M}:n>m\}$, for all $m\neq\overline{m}$. Let A be a $|\mathcal{M}|$ by $|\mathcal{M}|+1$ matrix constructed as follows. Each of the first $|\mathcal{M}|-1$ rows is equal to $(\mu_{m^+}-\mu_m,0)$, while the $|\mathcal{M}|+1$'th entry of the $|\mathcal{M}|$'th row is equal to -1. Let B be a $|J|$ by $|\mathcal{M}|+1$ matrix, where for each $j \in J$, its j'th row is equal to $(\mu_{m^j} - \mu_{n^j}, t^j - s^j)$. We claim that whenever $\mathcal{O}$ obeys the condition stated in the proposition, there is a vector $\phi \in \mathbb{R}^{|\mathcal{M}|}$ and a number $\hat{\delta}$ such that $A \cdot (\phi, \hat{\delta}) \gg 0$ and $B \cdot (\phi, \hat{\delta}) \geq 0$.

We prove the claim by contradiction. Suppose that $\mathcal{O}$ the condition, but the above system of inequalities has no solution. By Theorem A.1, there exists $\theta \in \mathbb{Z}^{|\mathcal{M}|}$ and $\lambda \in \mathbb{Z}^{|J|}$, with $\theta > 0$ and $\lambda \geq 0$, such that $\theta \cdot A + \lambda \cdot B = 0$ $(\star)$. Take any such vectors and denote $\lambda = (\lambda_j)_{j=1}^J$. Construct a sample $\{(x^i, y^i)\}_{i \in I}$ of $\mathcal{R}^*$, by taking λ_j times each pair (x^j, y^j). By definition of matrices A and B, whenever $(\star)$ is satisfied, the above sample can be partitioned into subsets $\{(x^k, y^k)\}_{k \in K}$ of $\mathcal{R}^*$ such that, for all $m \in \mathcal{M}$, we have $|\{k \in K : m^k \leq m\}| \geq |\{k \in K : n^k \leq m\}|$. Moreover, it must be that $\sum_{i \in I} t^i \geq \sum_{i \in I} s^i$. However, this contradicts the condition stated in the axiom.

Take any ϕ and $\hat{\delta}$ such that $A \cdot (\phi, \hat{\delta}) \gg 0$ and $B \cdot (\phi, \hat{\delta}) \geq 0$, and define sequence $\{\phi_m\}_{m \in \mathcal{M}}$ by $\phi_m := \mu_m \cdot \phi$. Clearly, it is strictly increasing, while $(m,t)\, \mathcal{R}^*(n,s)$ implies $\phi_m + \hat{\delta} t \geq \phi_n + \hat{\delta} s$. Construct function $\phi : \mathbb{R}_+ \to \mathbb{R}$ as in (A1), which is continuous and strictly increasing and, for any $m \in \mathcal{M}$, we have $\phi(m) = \phi_m$. Define function $u : \mathbb{R}_+ \to \mathbb{R}_+$ by $u := \exp(\phi)$, and let $\delta = \exp(\hat{\delta})$. Clearly, for any $(m,t) \in \mathcal{A}$, we have $v(m,t) := \delta^t u(m) = \exp(\phi_m - t\hat{\delta})$. Hence, $(m,t)\, \mathcal{R}^*(n,s)$ implies $v(m,t) \geq v(n,s)$. $\square$

References

Andersen, S., G. W. Harrison, M. I. Lau, and E. E. Rutström (2008). Eliciting risk and time preferences. *Econometrica 76*(3), 583–618.

Benhabib, J., A. Bisin, and A. Schotter (2010). Present-bias, quasi-hyperbolic discounting, and fixed costs. *Games and Economic Behavior 69*(2), 205–223.

Border, K. C. (1992). Revealed preference, stochastic dominance, and the expected utility hypothesis. *Journal of Economic Theory 56*, 20–42.

Chabris, C. F., D. Laibson, C. L. Morris, J. P. Schuldt, and D. Taubinsky (2008). Individual laboratory-measured discount rates predict field behavior. *Journal of Risk and Uncertainty 37*(2), 237–269.

Chabris, C. F., D. Laibson, C. L. Morris, J. P. Schuldt, and D. Taubinsky (2009). The

Allocation of Time in Decision-Making. *Journal of the European Economic Association 7*(2-3), 628–637.

Coller, M. and M. Williams (1999). Eliciting individual discount rates. *Experimental Economics 2*(2), 107–127.

Dohmen, T., A. Falk, D. Huffman, and U. Sunde (2012). Interpreting time horizon effects in inter-temporal choice. Discussion Paper No. 6385, IZA.

Echenique, F., T. Imai, and K. Saito (2014). Testable implications of quasi-hyperbolic and exponential discounting. Caltech SS Working Paper 1388.

Fishburn, P. C. and A. Rubinstein (1982). Time Preference. *International Economic Review 23*(3), 677–94.

Frederick, S., G. Loewenstein, and T. O'Donoghue (2002). Time discounting and time preference: A critical review. *Journal of Economic Literature 40*(2), 351–401.

Harrison, G. W., M. I. Lau, and M. B. Williams (2002). Estimating individual discount rates in denmark: A field experiment. *American Economic Review 92*(5), 1606–1617.

Kirby, K. N. and N. N. Marakovic (1964). Delay-discounting probabilistic rewards: Rates decrease and amounts increase. *Psychonomic Bulletin & Review 3*, 100–104.

Kirby, K. N., N. M. Petry, and W. K. Bickel (1999). Heroin addicts have higher discount rates for delayed rewards than non-drug-using controls. *Journal of Experimental Psychology: General 128*, 78–87.

Kraft, C. H., J. W. Pratt, and A. Seidenberg (1959). Intuitive probability on finite sets. *Annals of Mathematical Statistics 30*, 408–419.

Nishimura, H., E. A. Ok, and J. K.-H. Quah (2013). A Unified Approach to Revealed Preference Theory: The Case of Rational Choice. Economics Series Working Papers 686, University of Oxford, Department of Economics.

Noor, J. (2011). Intertemporal choice and the magnitude effect. *Games and Economic Behavior 72*(1), 255–270.

Ok, E. A. and Y. Masatlioglu (2007). A theory of (relative) discounting. *Journal of Economic Theory 137*(1), 214–245.

Scott, D. (1964). Measurement structures and linear inequalities. *Journal of Mathematical Psychology 1*, 233–247.

Stoer, J. and C. Witzgall (1970). *Convexity and optimization in finite dimensions I.* Springer Verlag, Berlin.