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Zhu, Ke and Ling, Shiqing

Chinese Academy of Sciences, Hong Kong University of Science and Technology

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MODEL-BASED PRICING FOR FINANCIAL DERIVATIVES

By Ke Zhu and Shiqing Ling

Chinese Academy of Sciences & Hong Kong University of Science and Technology

Assume that S_t is a stock price process and B_t is a bond price process with a constant continuously compounded risk-free interest rate, where both are defined on an appropriate probability space P. Let $y_t = \log(S_t/S_{t-1})$. y_t can be generally decomposed into a conditional mean plus a noise with volatility components, but the discounted S_t is not a martingale under P. Under a general framework, we obtain a risk-neutralized measure Q under which the discounted S_t is a martingale in this paper. Using this measure, we show how to derive the risk neutralized price for the derivatives. Special examples, such as NGARCH, EGARCH and GJR pricing models, are given. Simulation study reveals that these pricing models can capture the "volatility skew" of implied volatilities in the European option. A small application highlights the importance of our model-based pricing procedure.

1. Introduction. After the seminal work of Black and Scholes (1973) and Merton (1973), there has been explosive growth in the trading activities on derivatives in the worldwide financial markets. A fundamental question in finance is how we give a fair price for the derivative, whose payoff is on the evolution of an asset price upon which the derivative is written. Black and Scholes (1973) first fairly valued the option according to the principle of "the absence of arbitrage". Their valuation method relies on "efficient market hypothesis", under which there exists a risk-neutralized probability measure such that the discounted asset price is a martingale, and then a fair price of the derivative is the expected discounted value of its future payoff under this measure. Particularly, the risk-neutralized measure is not unique when the market is incomplete. For more discussions on the principle of "the absence of arbitrage", we refer to Harrison and Kreps (1979) and Harrison and Pliska (1982).

Although Black and Scholes's (1973) pricing model (hereafter, BS model) has achieved a great success in finance, it exhibits some systematic bias. The well-documented evidence is the so-called "volatility smile" in Rubinstein (1985) and Sheikh (1991), from which one may concern that the homoscedastic assumption on an asset return (that is, the asset return follows a geometric Brownian motion) is not reliable any more. This

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motivates people to use other heteroskedastic stochastic processes to model the asset return. The related works in this field are Cox (1975), Merton (1976), Hull and White (1987), Stein and Stein (1991), Heston (1993), and Xiu (2014) to name a few; see also Bates (2003) or Broadie and Detemple (2004) for an overview in this framework.

In this paper, we model the asset return by a discrete stochastic process under the physical probability measure P, which can be decomposed into the conditioanl mean part plus a noise with volatility components. In this case, the discounted asset price is not a martingale under P. By using the method in Gerber and Shiu (1994), we first construct a risk-neutralized Esscher measure Q, under which the discounted asset price is a martingale. Next, we give the structure of our stochastic process of an asset return under Q, when the conditional distribution of the innovation is normal, shift negative gamma, and shift negative inverse gaussian, respectively. Particularly, the option pricing models in Duan (1995), Siu et al. (2004), and Christoffersen et al. (2006) can be easily retrieved from our method. Furthermore, we propose a model-based Monte Carlo pricing procedure and apply it to some special examples, such as NGARCH, EGARCH and GJR pricing models. Simulation study reveals that these pricing models can capture the "volatility skew" of implied volatilities in the European option. A small application highlights the importance of our model-based pricing procedure.

The remainder of the paper is organized as follows. In Section 2, we introduce a risk-neutralized Esscher measure Q. In Section 3, we consider the processes for asset return under Q. A model-based Monte Carlo procedure with application to some pricing models is given in Section 4. Simulation study is reported in Section 5. A small application is given in Section 6. Concluding remarks are offered in Section 7.

2. Risk-neutralized Esscher measure. Let $\{S_t : t = 0, 1, \cdots\}$ be an asset price process and B_t be a bond price process with a constant continuously compounded risk-free interest rate r, where both are defined on the probability space $(\Omega, P, \mathcal{F})$. Assume that the log-return of S_t follows a discrete stochastic process under the physical probability measure P, i.e.,

$$y_t = \log \frac{S_t}{S_{t-1}} \quad \text{and} \quad y_t = \mu_t + \eta_t \sqrt{h_t}, \quad \text{under P}, \tag{2.1}$$

where $\eta_t | \mathcal{F}_{t-1} \sim D(0,1)$, $D(0,1)$ denotes some distribution $F(\cdot)$ with zero mean and unit variance, and $\mathcal{F}_t$ is the information set up to time t; $\mu_t \in \mathcal{F}_{t-1}$ and $h_t \in \mathcal{F}_{t-1}$ are the conditional mean and the conditional variance of y_t, respectively. Process (2.1) gives us enough freedom to model the asset return, and most importantly it includes the ARCH-type models originally introduced by Engle (1982). Nowadays, the ARCH-type models are widely used to analyze economic time series with time-varying volatility; see, e.g., Bollerslev et al. (1992), Berkes et al. (2003), and Francq and Zakoïan (2010).

Since the price of the derivative is sensitive to the volatility of its underlying asset, the ARCH-type model which provides a good prediction on volatility is applicable to value the derivative; see, e.g., Engle and Mustafa (1992), Duan (1995), Ritchken and Trevor (1999), Heston and Nandi (2000), Christoffersen and Jacobs (2004), Garcia et al. (2010), and many others.

Next, we use the method in Gerber and Shiu (1994) to get a risk-neutralized measure, under which the discounted process $\{e^{-rt}S_t : t = 0, 1, \cdots\}$ is a martingale. First, let $M_t(z)$ be the conditional moment-generating function of y_t, given $\mathcal{F}_{t-1}$, i.e.,

$$M_t(z) = E_P[e^{zy_t}|\mathcal{F}_{t-1}] = \int_{-\infty}^{\infty} e^{zx} dF\left(\frac{x-\mu_t}{\sqrt{h_t}}\right). \tag{2.2}$$

Secondly, we define a sequence of the conditional distribution functions as follows:

$$\begin{aligned}\Xi_t(u|\mathcal{F}_{t-1}) &\equiv E_P\left[I\{y_t \le u\}\frac{e^{\theta_t y_t}}{M_t(\theta_t)}|\mathcal{F}_{t-1}\right]\\ &= \frac{1}{M_t(\theta_t)}\int_{-\infty}^{u} e^{\theta_t x} dF\left(\frac{x-\mu_t}{\sqrt{h_t}}\right),\end{aligned}$$

where $I\{\cdot\}$ is the indicator function and $\theta_t \in \mathcal{F}_{t-1}$ be determined subsequently. Furthermore, we define a sequence of conditional distribution functions $\{Q_t : t = 1, 2, \cdots\}$ on $(\Omega, \mathcal{F}_t)$:

$$Q_t(y_i \le u_i : i = 1, 2, \cdots, t) \equiv \int_{-\infty}^{u_1}\int_{-\infty}^{u_2}\cdots\int_{-\infty}^{u_t}\prod_{i=1}^{t}\Xi_i(d\tilde{u}_i|\mathcal{F}_{i-1}).$$

Obviously, $\{Q_t : t = 1, 2, \cdots\}$ is consistent, i.e.,

$$Q_t(y_i \le u_i : i = 1, 2, \cdots, t) = Q_{t+1}(y_{t+1} \in R, y_i \le u_i : i = 1, 2, \cdots, t).$$

By Kolmogorov extension theorem, there exists a probability Q on $(\Omega, \mathcal{F})$ such that

$$Q(y_i \le u_i : i = 1, 2, \cdots, t) = Q_t(y_i \le u_i : i = 1, 2, \cdots, t),$$

for all t and u_i, where $\mathcal{F} = \sigma(\cup_{i=1}^{\infty}\mathcal{F}_i)$. Thus, we have

$$\begin{aligned}Q(y_t \le u) &= Q_t(y_t \le u)\\ &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_{-\infty}^{u}\prod_{i=1}^{t}\Xi_i(d\tilde{u}_i|\mathcal{F}_{i-1})\\ &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left[\prod_{i=1}^{t-1}\Xi_i(d\tilde{u}_i|\mathcal{F}_{i-1})\right]\Xi_t(u|\mathcal{F}_{t-1})\\ &= \Xi_t(u|\mathcal{F}_{t-1}).\end{aligned}$$

Third, let $M_t^{(q)}(z)$ be the conditional moment-generating function of y_t under Q. By (2.2), we can show that

$$\begin{aligned} M_t^{(q)}(z) &= E_Q\left[e^{zy_t}|\mathcal{F}_{t-1}\right] \\ &= \int_{-\infty}^{\infty} e^{zx}\frac{e^{\theta_t x}}{M_t(\theta_t)}dF\left(\frac{x-\mu_t}{\sqrt{h_t}}\right) \\ &= \frac{M_t(z+\theta_t)}{M_t(\theta_t)}. \end{aligned} \tag{2.3}$$

Then, we have

$$\begin{aligned} E_Q\left[e^{-rt}S_t|\mathcal{F}_{t-1}\right] &= e^{-r(t-1)}S_{t-1}E_Q\left[e^{-r+y_t}|\mathcal{F}_{t-1}\right] \\ &= e^{-r(t-1)}S_{t-1}e^{-r}M_t^{(q)}(1). \end{aligned}$$

Thus, under Q, $\{e^{-rt}S_t : t = 0, 1, \cdots\}$ is a martingale iff $e^{-r}M_t^{(q)}(1) = 1$, i.e.,

$$\frac{M_t(1+\theta_t)}{M_t(\theta_t)} = e^r \text{ for all } t = 0, 1, \cdots. \tag{2.4}$$

Now, if equation (2.4) has a solution θ_t which may not be unique, the martingale measure Q associated with this θ_t is called the risk-neutralized Esscher measure. By Proposition 2.6 in Harrison and Pliska (1982), a fair price of any derivative at current time t, denoted by V_t, can be calculated as

$$V_t = E_Q\left[e^{-r(T-t)}W(S_j; j \le T)\Big|\mathcal{F}_{t-1}\right], \tag{2.5}$$

where $W(S_j; j \le T)$ is the payoff of this derivative at future time T. Note that

$$S_j = S_t \exp\left(\sum_{i=t+1}^{j} y_i\right). \tag{2.6}$$

So, to calculate (2.5), it is necessary for us to consider $\{y_t\}$ under Q.

Finally, it is worth noting that by (2.2) and (2.4), we have

$$E_P\left[e^r \cdot Z_{t,t-1}|\mathcal{F}_{t-1}\right] = 1 \text{ and } E_P\left[e^{y_t} \cdot Z_{t,t-1}|\mathcal{F}_{t-1}\right] = 1,$$

where $Z_{t,t-1} = e^{\theta_t y_t}\left\{E_P\left[e^{(1+\theta_t)y_t}|\mathcal{F}_{t-1}\right]\right\}^{-1}$. Thus, our method to construct Q can be viewed as a special case of stochastic discount factor (SDF) methods with SDF equals to $Z_{t,t-1}$. Particularly, Chorro et al. (2012) used the SDF method to get the same martingale measure Q as ours, and they further applied it to get the ARCH-type option pricing model when η_t is conditional generalized hyperbolic distributed. For more discussions on SDF methods, we refer to Jagannathan and Wang (2001), Smith and Wickens (2002), Monfort and Pegoraro (2011), and references therein.

3. Processes for an asset return under Q. After the empirical studies in Mandelbrot (1963), Fama (1965), Bollerslev (1987), and Bollerslev et al. (1992), the valuation of the derivative with non-normal innovation has drawn more and more attentions. For example, Duan (1999) and Christoffersen et al. (2010) studied the case when the conditional innovation is a generalized error distribution; Siu et al. (2004) explored an option pricing model when the conditional innovation has a gamma distribution; and Christoffersen et al. (2006) gave an analytic pricing form when the conditional innovation has an inverse gaussian distribution; see also Chorro et al. (2012), Xi (2013), and references therein.

In this section, we consider the processes of an asset return under Q when η_t is conditionally normal, shift negative gamma (SNG) or shift negative inverse gaussian (SNIG) distributed. When η_t is conditionally normal, we retrieve the result in Duan (1995). When η_t is the conditional SNG or SNIG, the Gamma-GARCH process in Siu et al. (2004) or IG-GARCH process in Christoffersen et al. (2006) can also be easily derived, respectively. Meanwhile, it is worth noting that our method is different from Duan's (1999) method for dealing with a non-normal innovation η_t. The method in Duan (1999) needs to transform the non-normal innovation into another innovation which is standard normal with a shift in mean under the local risk-neutralized measure. However, our method skips that transformation and keeps the distribution of the non-normal innovation unchanged under the risk-neutralized Esscher measure Q. It not only seems to be more reasonable, but avoids the cumbersome numerical problem arisen from the transformation as shown in Duan (1999) and Christoffersen et al. (2010).

3.1. *Conditional normal innovation.* Suppose that $\eta_t|\mathcal{F}_{t-1} \sim N(0,1)$. Then, we have

$$M_t(z) = \exp\left(z\mu_t + \frac{z^2 h_t}{2}\right).$$

By (2.4), it follows that

$$\theta_t = \frac{1-\mu_t}{h_t} - \frac{1}{2}.$$

With this specified θ_t and relation (2.3), we have

$$M_t^{(q)}(z) = \exp\left[z\left(r - \frac{h_t}{2}\right) + \frac{z^2 h_t}{2}\right].$$

Thus, under Q,

$$y_t = r - \frac{h_t}{2} + \varepsilon_t, \tag{3.1}$$

where $\varepsilon_t|\mathcal{F}_{t-1} \sim N(0, h_t)$. This is the same result as the one in Duan (1995), who first obtained it under the local risk-neutralized measure.

3.2. *Conditional SNG innovation.* Suppose that $\eta_t = (\xi_t + a_t)/\sqrt{a_t}$, where $a_t \in \mathcal{F}_{t-1}$ is positive, $\xi_t|\mathcal{F}_{t-1} \sim -G(a_t, 1)$, and $G(a, b)$ is a random variable having the density function

$$g(x) = b^a x^{a-1}[e^{bx}\Gamma(a)]^{-1}, \quad \text{for } x \geq 0.$$

In this case, we call that η_t is conditionally SNG distributed, and denote it by $\eta_t|\mathcal{F}_{t-1} \sim SNG(a_t)$. Note that the conditional skewness and kurtosis of η_t are

$$skew(\eta_t|\mathcal{F}_{t-1}) = -2a_t^{-1/2} \quad \text{and} \quad kurt(\eta_t|\mathcal{F}_{t-1}) = 6a_t^{-1},$$

respectively. Thus, by using a_t, we can describe the time-varying conditional skewness or kurtosis of η_t.

When $\eta_t|\mathcal{F}_{t-1} \sim SNG(a_t)$, model (2.1) reduces to

$$y_t = \mu_t + \varepsilon_t \quad \text{under } P, \tag{3.2}$$

where $\varepsilon_t = \sqrt{a_t h_t} + \sqrt{h_t/a_t}\xi_t$. By (2.2), a direct calculation gives us

$$M_t(z) = \frac{a_t^{a_t/2}\exp\left[z(\mu_t + \sqrt{a_t h_t})\right]}{\left(\sqrt{a_t} + z\sqrt{h_t}\right)^{a_t}}, \quad \text{for } z > -\sqrt{\frac{a_t}{h_t}}.$$

By (2.4), it follows that $\theta_t = b_t - \sqrt{a_t/h_t}$, where

$$b_t = \left[\exp\left(\frac{\mu_t - r + \sqrt{a_t h_t}}{a_t}\right) - 1\right]^{-1}.$$

With this specified θ_t and relation (2.3), if $b_t > 0$, it follows that

$$M_t^{(q)}(z) = \frac{\exp\left[z\left(\mu_t + \sqrt{a_t h_t}\right)\right]}{(1 + z/b_t)^{a_t}}, \quad \text{for } z > -b_t.$$

Thus, under Q,

$$y_t = \mu_t + \varepsilon_t^*, \tag{3.3}$$

where $\varepsilon_t^* = \sqrt{a_t h_t} + \xi_t^*$ with $\xi_t^*|\mathcal{F}_{t-1} \sim -G(a_t, b_t)$.

Particularly, when $\mu_t = r + \nu\sqrt{h_t} - h_t/2$ and $h_t = \omega + \alpha\varepsilon_{t-1}^2 + \beta h_{t-1}$ with $\eta_t = (\xi_t - a)/\sqrt{a}$ and $\xi_t|\mathcal{F}_{t-1} \sim G(a, 1)$ for some constant $a > 0$, by (3.2), we have under P,

$$\begin{cases} y_t = r + \nu\sqrt{h_t} - h_t/2 + \varepsilon_t, \\ \varepsilon_t = \xi_t\sqrt{h_t/a} - \sqrt{a h_t} \text{ and } h_t = \omega + \alpha\varepsilon_{t-1}^2 + \beta h_{t-1}; \end{cases} \tag{3.4}$$

and by using the same method as for (3.3), we have under Q,

$$\left\{\begin{array}{l} y_t = r + \nu\sqrt{h_t} - h_t/2 + \varepsilon_t^*, \\ \varepsilon_t^* = \xi_t^* - \sqrt{ah_t} \text{ with } \xi_t^*|\mathcal{F}_{t-1} \sim G(a, b_t^*), \\ h_t = \omega + \alpha(\varepsilon_{t-1}^*)^2 + \beta h_{t-1}, \end{array}\right. \tag{3.5}$$

where

$$b_t^* = \left[1 - \exp\left(\frac{\nu\sqrt{h_t} - h_t/2 - \sqrt{ah_t}}{a}\right)\right]^{-1}.$$

Models (3.4)-(3.5) are the Gamma-GARCH models in Siu et al. (2004).

3.3. *Conditional SNIG innovation.* Suppose that $\eta_t = (\xi_t + \delta_t)/\sqrt{\delta_t}$, where $\delta_t \in \mathcal{F}_{t-1}$ is positive, $\xi_t|\mathcal{F}_{t-1} \sim -IG(\delta_t)$, and $IG(\delta)$ is a random variable having the density function

$$g(x) = \frac{\delta}{\sqrt{2\pi x^3}} \exp\left\{-\frac{(x-\delta)^2}{2x}\right\}, \quad \text{for } x > 0;$$

see Barndorff-Nielsen (1998). In this case, we call that η_t is conditional SNIG distributed, and denote it by $\eta_t|\mathcal{F}_{t-1} \sim SNIG(\delta_t)$. As for the SNG case, the conditional skewness and kurtosis of η_t are both time-varying in this case, and they satisfy

$$skew(\eta_t|\mathcal{F}_{t-1}) = -3\delta_t^{-1/2} \quad \text{and} \quad kurt(\eta_t|\mathcal{F}_{t-1}) = 15\delta_t^{-1},$$

respectively.

When $\eta_t|\mathcal{F}_{t-1} \sim SNIG(\delta_t)$, model (2.1) reduces to

$$y_t = \mu_t + \varepsilon_t \quad \text{under } P, \tag{3.6}$$

where $\varepsilon_t = \sqrt{\delta_t h_t} + \sqrt{h_t/\delta_t}\xi_t$. Furthermore, by (2.2)-(2.4), a direct calculation gives us

$$M_t^{(q)}(z) = \exp\left\{\left(\mu_t + \sqrt{\delta_t h_t}\right) z + \delta_t \left[\sqrt{1 + 2\sqrt{\frac{h_t}{\delta_t}}\theta_t} - \sqrt{1 + 2\sqrt{\frac{h_t}{\delta_t}}(z + \theta_t)}\right]\right\},$$

where $\theta_t \in \mathcal{F}_{t-1}$ satisfies

$$1 + 2\sqrt{\frac{h_t}{\delta_t}}\theta_t = \frac{1}{4}\left(\frac{r - \mu_t - \sqrt{\delta_t h_t}}{2\delta_t} - \frac{2\sqrt{\delta_t h_t}}{r - \mu_t - \sqrt{\delta_t h_t}}\right)^2 \triangleq c_t.$$

Thus, it follows that

$$y_t = \mu_t + \varepsilon_t^* \quad \text{under Q}, \tag{3.7}$$

where $\varepsilon_t^* = \sqrt{\delta_t h_t} + c_t^{-1}\sqrt{h_t/\delta_t}\xi_t^*$ with $\xi_t^*|\mathcal{F}_{t-1} \sim -IG(\sqrt{c_t}\delta_t)$.

Particularly, when $\mu_t = r + \lambda h_t$ and $h_t = \omega + \alpha h_{t-1} + \beta\zeta_{t-1} + \gamma h_{t-1}^2/\zeta_{t-1}$ with $\zeta_t|\mathcal{F}_{t-1} \sim IG(\delta_t)$ and $\delta_t = h_t/\eta^2$ for some $\eta > 0$, by (3.6), we have under P,

$$\begin{cases} y_t = r + \nu h_t - \eta\zeta_t, \\ h_t = \omega + \alpha h_{t-1} + \beta\zeta_{t-1} + \gamma h_{t-1}^2/\zeta_{t-1}, \end{cases} \tag{3.8}$$

where $\nu = \lambda + 1/\eta$; and by (3.7), we have under Q,

$$\begin{cases} y_t = r + \nu^* h_t^* - \eta^*\zeta_t^*, \\ h_t^* = \omega^* + \alpha h_{t-1}^* + \beta^*\zeta_{t-1}^* + \gamma^* {h_{t-1}^*}^2/\zeta_{t-1}^*, \end{cases} \tag{3.9}$$

where $\zeta_t^*|\mathcal{F}_{t-1} \sim IG(h_t^*/\eta^{*2})$, $\zeta_t^* = c\zeta_t$, $h_t^* = h_t/c^{3/2}$, $\nu^* = \nu c^{3/2}$, $\eta^* = \eta/c$, $\omega^* = \omega/c^{3/2}$, $\beta^* = \beta/c^{5/2}$, $\gamma^* = \gamma c^{5/2}$ and $c = [1/(\nu\eta) - (\nu\eta^2)/4]^2$. Models (3.8)-(3.9) are the IG-GARCH models in Christoffersen et al. (2006).

4. Model-based pricing procedure. In this section, we give a model-based pricing procedure to calculate V_t in (2.5). Since V_t has no closed form in general, our pricing procedure is fulfilled by Monte Carlo method as follows:

Step 1. fit the historical data set $\{y_i\}_{i\le t}$ by a specified model in (2.1) under P;

Step 2. obtain the corresponding fitted model under Q;

Step 3. generate a sequence of data set $\{y_i\}_{i=t+1}^T$ from the fitted model in Step 2, and then obtain a value of v_t through

$$v_t = e^{-r(T-t)}W(S_j; j \le T),$$

where S_j is calculated from (2.6);

Step 4. repeat Step 3 by N times to get a sequence $\{v_t^{(i)}\}_{i=1}^N$, and eventually approximate V_t by

$$\tilde{V}_t = \frac{1}{N}\sum_{i=1}^N v_t^{(i)}.$$

Clearly, the value of $\tilde{V}_t$ is model-based, because we need to choose a specified pricing model in Step 1. For μ_t in (2.1), the usual choices are the GARCH-in-mean (GIM) model in Duan (1995) and ARMA model. For the conditional variance h_t in (2.1), we can choose it from the ARCH family or other nonlinear models as long as $h_t > 0$ and $h_t \in \mathcal{F}_{t-1}$. Some special choices are the nonlinear NGARCH model in Engle and Ng (1993), EGARCH model in Nelson (1991), and GJR model in Glosten et al. (1993). These three models are to capture the "leverage effect" in volatility (see, e.g, Rubinstein (1994) and Xiu (2014)), and their practical usefulness in asset pricing has been verified in Schmitt (1996), Heston and Nandi (2000), Duan and Zhang (2001), Barone-Adesi et al. (2008), and many others.

4.1. *GIM-type pricing models.* In this subsection, we give three price models when μ_t in (2.1) is the GIM model.

EXAMPLE 4.1. *(GIM-NGARCH pricing models) Suppose μ_t in (2.1) follows a GIM model and h_t in (2.1) follows a NGARCH$(1,1)$ model, i.e., under P,*

$$\left\{\begin{array}{l} y_t = r + \nu\sqrt{h_t} - h_t/2 + \varepsilon_t, \\ \varepsilon_t = \eta_t\sqrt{h_t} \ \ and \ \ h_t = \omega + \alpha\left(\varepsilon_{t-1} - \theta\sqrt{h_{t-1}}\right)^2 + \beta h_{t-1}, \end{array}\right. \tag{4.1}$$

where ν is the unit risk premium, $\omega, \alpha > 0$, $\beta \in (0,1)$, and $\eta_t|\mathcal{F}_{t-1} \sim N(0,1), SNG(a)$, or $SNIG(\delta)$. Based on the historical data set $\{y_i\}_{i\leq t}$, the vector of parameters $(\nu, \omega, \alpha, \beta, \theta)$ can be estimated by its QMLE $(\hat{\nu}, \hat{\omega}, \hat{\alpha}, \hat{\beta}, \hat{\theta})$ as in Francq and Zakoian (2004). Denote the realized error and conditional variance by $\hat{\varepsilon}_t$ and $\hat{h}_t$, respectively. Note that $E(\eta_t^3|\mathcal{F}_{t-1}) = -2/\sqrt{a}$ when $\eta_t|\mathcal{F}_{t-1} \sim SNG(a)$, and $E(\eta_t^3|\mathcal{F}_{t-1}) = -3/\sqrt{\delta}$ when $\eta_t|\mathcal{F}_{t-1} \sim SNIG(\delta)$. Then, by the method of moments approach, the parameters a and δ can be estimated by $\hat{a}$ and $\hat{\delta}$, respectively, where

$$\hat{a} = \left(\frac{2\sum \hat{h}_t^{3/2}}{\sum \hat{\varepsilon}_t^3}\right)^2 \quad and \quad \hat{\delta} = \left(\frac{3\sum \hat{h}_t^{3/2}}{\sum \hat{\varepsilon}_t^3}\right)^2.$$

Hereafter, we will use the proceeding method to obtain the estimators for all models.

By (3.1), when $\eta_t|\mathcal{F}_{t-1} \sim N(0,1)$ and under Q, model (4.1) reduces to

$$\left\{\begin{array}{l} y_t = r - h_t/2 + \varepsilon_t^*, \\ \varepsilon_t^* \sim N(0, h_t) \ \ and \ \ h_t = \omega + \alpha\left[\varepsilon_{t-1}^* - (\nu+\theta)\sqrt{h_{t-1}}\right]^2 + \beta h_{t-1}. \end{array}\right. \tag{4.2}$$

By (3.3), when $\eta_t|\mathcal{F}_{t-1} \sim SNG(a)$ and under Q, model (4.1) reduces to

$$\left\{\begin{array}{l} y_t = r + \nu\sqrt{h_t} - h_t/2 + \varepsilon_t^*, \\ \varepsilon_t^* = \sqrt{ah_t} + \xi_t^* \ \ with \ \ \xi_t^* \sim -G(a, b_t), \\ h_t = \omega + \alpha\left(\varepsilon_{t-1}^* - \theta\sqrt{h_{t-1}}\right)^2 + \beta h_{t-1}, \end{array}\right. \tag{4.3}$$

where

$$b_t = \left[\exp\left(\frac{\nu\sqrt{h_t} - h_t/2 + \sqrt{ah_t}}{a}\right) - 1\right]^{-1}.$$

By (3.7), when $\eta_t|\mathcal{F}_{t-1} \sim SNIG(\delta)$ and under Q, model (4.1) reduces to

$$\left\{\begin{array}{l} y_t = r + \nu\sqrt{h_t} - h_t/2 + \varepsilon_t^*, \\ \varepsilon_t^* = \sqrt{\delta h_t} + c_t^{-1}\sqrt{h_t/\delta}\xi_t^* \ \ with \ \ \xi_t^*|\mathcal{F}_{t-1} \sim -IG(\delta\sqrt{c_t}), \\ h_t = \omega + \alpha\left(\varepsilon_{t-1}^* - \theta\sqrt{h_{t-1}}\right)^2 + \beta h_{t-1}, \end{array}\right. \tag{4.4}$$

where

$$c_t = \frac{1}{4}\left(\frac{h_t/2 - \nu\sqrt{h_t} - \sqrt{\delta h_t}}{2\delta} - \frac{2\sqrt{\delta h_t}}{h_t/2 - \nu\sqrt{h_t} - \sqrt{\delta h_t}}\right)^2.$$

Using a user-chosen initial variance h_t*, then we can generate the data set* $\{y_i\}_{i=t+1}^T$ *in Step 3 from model (4.2), (4.3) or (4.4). Particularly, when* $\theta = 0$*, our GIM-NGARCH pricing model (4.2) reduces to the GIM-GARCH pricing model in Duan (1995).*

EXAMPLE 4.2. *(GIM-EGARCH pricing models) Suppose* μ_t *in (2.1) follows a GIM model and* h_t *in (2.1) follows an EGARCH*$(1,1)$ *model, i.e., under* P*,*

$$\text{(4.5)} \qquad \begin{cases} y_t = r + \nu\sqrt{h_t} - h_t/2 + \varepsilon_t \text{ and } \varepsilon_t = \eta_t\sqrt{h_t}, \\ \log h_t = \omega + \alpha(\varepsilon_{t-1}/\sqrt{h_{t-1}}) + \theta(|\varepsilon_{t-1}/\sqrt{h_{t-1}}| - \sqrt{2/\pi}) + \beta\log h_{t-1}, \end{cases}$$

where $\beta \in (-1,1)$ *and* $\eta_t|\mathcal{F}_{t-1} \sim N(0,1), SNG(a)$*, or* $SNIG(\delta)$*. By (3.1), (3.3), and (3.7), model (4.5) under* Q *reduces to*

$$\text{(4.6)} \qquad \begin{cases} y_t = r - h_t/2 + \varepsilon_t^* \text{ and } \varepsilon_t^* \sim N(0,h_t), \\ \log h_t = \omega + \alpha(\frac{\varepsilon_{t-1}^*}{\sqrt{h_{t-1}}} - \nu) + \theta(|\frac{\varepsilon_{t-1}^*}{\sqrt{h_{t-1}}} - \nu| - \sqrt{2/\pi}) + \beta\log h_{t-1}, \end{cases}$$

$$\text{(4.7)} \qquad \begin{cases} y_t = r + \nu\sqrt{h_t} - h_t/2 + \varepsilon_t^* \text{ and } \varepsilon_t^* = \sqrt{ah_t} + \xi_t^* \text{ with } \xi_t^* \sim -G(a, b_t), \\ \log h_t = \omega + \alpha(\varepsilon_{t-1}^*/\sqrt{h_{t-1}}) + \theta(|\varepsilon_{t-1}^*/\sqrt{h_{t-1}}| - \sqrt{2/\pi}) + \beta\log h_{t-1}, \end{cases}$$

$$\text{(4.8)} \qquad \begin{cases} y_t = r + \nu\sqrt{h_t} - h_t/2 + \varepsilon_t^*, \\ \varepsilon_t^* = \sqrt{\delta h_t} + c_t^{-1}\sqrt{h_t/\delta}\xi_t^* \text{ with } \xi_t^*|\mathcal{F}_{t-1} \sim -IG(\delta\sqrt{c_t}), \\ \log h_t = \omega + \alpha(\varepsilon_{t-1}^*/\sqrt{h_{t-1}}) + \theta(|\varepsilon_{t-1}^*/\sqrt{h_{t-1}}| - \sqrt{2/\pi}) + \beta\log h_{t-1}, \end{cases}$$

respectively, when $\eta_t|\mathcal{F}_{t-1} \sim N(0,1), SNG(a)$ *and* $SNIG(\delta)$*. Using a user-chosen initial variance* h_t*, then we can generate the data set* $\{y_i\}_{i=t+1}^T$ *in Step 3 from model (4.6), (4.7) or (4.8).*

EXAMPLE 4.3. *(GIM-GJR pricing models) Suppose* μ_t *in (2.1) follows a GIM model and* h_t *in (2.1) follows a GJR*$(1,1)$ *model, i.e., under* P*,*

$$\text{(4.9)} \qquad \begin{cases} y_t = r + \nu\sqrt{h_t} - h_t/2 + \varepsilon_t \text{ and } \varepsilon_t = \eta_t\sqrt{h_t}, \\ h_t = \omega + \alpha\varepsilon_{t-1}^2 I(\varepsilon_{t-1} > 0) + \theta\varepsilon_{t-1}^2 I(\varepsilon_{t-1} \le 0) + \beta h_{t-1}, \end{cases}$$

where $\omega, \alpha, \theta > 0$*,* $\beta \in (0,1)$*, and* $\eta_t|\mathcal{F}_{t-1} \sim N(0,1), SNG(a)$*, or* $SNIG(\delta)$*. By (3.1), (3.3), and (3.7), model (4.9) under* Q *reduces to*

$$\text{(4.10)} \qquad \begin{cases} y_t = r - h_t/2 + \varepsilon_t^* \text{ and } \varepsilon_t^* \sim N(0,h_t), \\ h_t = \omega + \alpha(\varepsilon_{t-1}^* - \nu\sqrt{h_{t-1}})^2 I(\varepsilon_{t-1}^* > \nu\sqrt{h_{t-1}}) \\ \qquad +\theta(\varepsilon_{t-1}^* - \nu\sqrt{h_{t-1}})^2 I(\varepsilon_{t-1}^* \le \nu\sqrt{h_{t-1}}) + \beta h_{t-1}, \end{cases}$$

$$\begin{cases} y_t = r + \nu\sqrt{h_t} - h_t/2 + \varepsilon_t^* \text{ and } \varepsilon_t^* = \sqrt{a h_t} + \xi_t^* \text{ with } \xi_t^* \sim -G(a, b_t), \\ h_t = \omega + \alpha\varepsilon_{t-1}^{*2} I(\varepsilon_{t-1}^* > 0) + \theta\varepsilon_{t-1}^{*2} I(\varepsilon_{t-1}^* \leq 0) + \beta h_{t-1}, \end{cases} \tag{4.11}$$

$$\begin{cases} y_t = r + \nu\sqrt{h_t} - h_t/2 + \varepsilon_t^*, \\ \varepsilon_t^* = \sqrt{\delta h_t} + c_t^{-1}\sqrt{h_t/\delta}\xi_t^* \text{ with } \xi_t^* | \mathcal{F}_{t-1} \sim -IG(\delta\sqrt{c_t}), \\ h_t = \omega + \alpha\varepsilon_{t-1}^{*2} I(\varepsilon_{t-1}^* > 0) + \theta\varepsilon_{t-1}^{*2} I(\varepsilon_{t-1}^* \leq 0) + \beta h_{t-1}, \end{cases} \tag{4.12}$$

respectively, when $\eta_t | \mathcal{F}_{t-1} \sim N(0,1), SNG(a)$ *and* $SNIG(\delta)$. *Using a user-chosen initial variance* h_t, *then we can generate the data set* $\{y_i\}_{i=t+1}^T$ *in Step 3 from model (4.10), (4.11) or (4.12).*

4.2. *ARMA-type pricing models.* In this subsection, we give three price models when μ_t in (2.1) is the ARMA model.

Example 4.4. *(ARMA-NGARCH pricing models) Suppose* μ_t *in (2.1) follows an ARMA*(p, q) *model and* h_t *in (2.1) follows a NGARCH*$(1,1)$ *model, i.e., under* P,

$$\begin{cases} y_t = \phi_0 + \sum_{i=1}^p \phi_i y_{t-i} + \sum_{i=1}^q \psi_i \varepsilon_{t-i} + \varepsilon_t \text{ and } \varepsilon_t = \eta_t\sqrt{h_t}, \\ h_t = \omega + \alpha\left(\varepsilon_{t-1} - \theta\sqrt{h_{t-1}}\right)^2 + \beta h_{t-1}, \end{cases} \tag{4.13}$$

where $\omega, \alpha > 0$, $\beta \in (0,1)$, *and* $\eta_t | \mathcal{F}_{t-1} \sim N(0,1), SNG(a)$, *or* $SNIG(\delta)$. *Hereafter, we assume that* $\Phi(z) \neq 0$ *and* $\Psi(z) \neq 0$ *when* $|z| \leq 1$, *and* $\Psi(z)$ *and* $\Psi(z)$ *have no common root with* $\psi_p \neq 0$ *or* $\psi_q \neq 0$, *where* $\Phi(z) = 1 - \sum_{i=1}^p \phi_i z^i$ *and* $\Psi(z) = 1 + \sum_{i=1}^q \psi_i z^i$. *Next, by (3.1), (3.3), and (3.7), model (4.13) under* Q *reduces to*

$$\begin{cases} y_t = r - h_t/2 + \varepsilon_t^* \text{ and } \varepsilon_t^* \sim N(0, h_t), \\ h_t = \omega + \alpha\left(z_{t-1}^* - \theta\sqrt{h_{t-1}}\right)^2 + \beta h_{t-1}, \end{cases} \tag{4.14}$$

$$\begin{cases} y_t = \phi_0 + \sum_{i=1}^p \phi_i y_{t-i} + \sum_{i=1}^q \psi_i \varepsilon_{t-i}^* + \varepsilon_t^*, \\ \varepsilon_t^* = \sqrt{a h_t} + \xi_t^* \text{ with } \xi_t^* \sim -G(a, \bar{b}_t), \\ h_t = \omega + \alpha\left(z_{t-1}^* - \theta\sqrt{h_{t-1}}\right)^2 + \beta h_{t-1}, \end{cases} \tag{4.15}$$

$$\begin{cases} y_t = \phi_0 + \sum_{i=1}^p \phi_i y_{t-i} + \sum_{i=1}^q \psi_i \varepsilon_{t-i}^* + \varepsilon_t^*, \\ \varepsilon_t^* = \sqrt{\delta h_t} + \bar{c}_t^{-1}\sqrt{h_t/\delta}\xi_t^* \text{ with } \xi_t^* | \mathcal{F}_{t-1} \sim -IG(\delta\sqrt{\bar{c}_t}), \\ h_t = \omega + \alpha\left(z_{t-1}^* - \theta\sqrt{h_{t-1}}\right)^2 + \beta h_{t-1}, \end{cases} \tag{4.16}$$

respectively, when $\eta_t | \mathcal{F}_{t-1} \sim N(0,1), SNG(a)$ *and* $SNIG(\delta)$. *Here,* $z_t^* = \Psi(B)^{-1}[\varepsilon_t^* + r - h_t/2 - \psi_0 - \sum_{i=1}^p \phi_i y_{t-i}]$,

$$\bar{b}_t = \left[\exp\left(\frac{\bar{\mu}_t^* - r + \sqrt{a h_t}}{a}\right) - 1\right]^{-1} \text{ and } \bar{c}_t = \frac{1}{4}\left(\frac{r - \bar{\mu}_t^* - \sqrt{\delta h_t}}{2\delta} + \frac{2\sqrt{\delta h_t}}{r - \bar{\mu}_t^* - \sqrt{\delta h_t}}\right)^2$$

with $\bar{\mu}_t^* = \phi_0 + \sum_{i=1}^p \phi_i y_{t-i} + \sum_{i=1}^q \psi_i \varepsilon_{t-i}^*$. *Using a user-chosen initial variance* h_t, *then we can generate the data set* $\{y_i\}_{i=t+1}^T$ *in Step 3 from model (4.14), (4.15) or (4.16).*

EXAMPLE 4.5. *(ARMA-EGARCH pricing models) Suppose μ_t in (2.1) follows an ARMA(p,q) model and h_t in (2.1) follows an EGARCH$(1,1)$ model, i.e., under P,*

$$\left\{\begin{array}{l} y_t = \phi_0 + \sum_{i=1}^p \phi_i y_{t-i} + \sum_{i=1}^q \psi_i \varepsilon_{t-i} + \varepsilon_t \ \textit{ and } \ \varepsilon_t = \eta_t \sqrt{h_t}, \\ \log h_t = \omega + \alpha(\varepsilon_{t-1}/\sqrt{h_{t-1}}) + \theta(|\varepsilon_{t-1}/\sqrt{h_{t-1}}| - \sqrt{2/\pi}) + \beta \log h_{t-1}, \end{array}\right. \tag{4.17}$$

where $\beta \in (-1,1)$ and $\eta_t|\mathcal{F}_{t-1} \sim N(0,1), SNG(a)$, or $SNIG(\delta)$. By (3.1), (3.3), and (3.7), model (4.17) under Q reduces to

$$\left\{\begin{array}{l} y_t = r - h_t/2 + \varepsilon_t^* \ \textit{ and } \varepsilon_t^* \sim N(0,h_t), \\ \log h_t = \omega + \alpha(\frac{z_{t-1}^*}{\sqrt{h_{t-1}}} - \nu) + \theta(|\frac{z_{t-1}^*}{\sqrt{h_{t-1}}} - \nu| - \sqrt{2/\pi}) + \beta \log h_{t-1}, \end{array}\right. \tag{4.18}$$

$$\left\{\begin{array}{l} y_t = \phi_0 + \sum_{i=1}^p \phi_i y_{t-i} + \sum_{i=1}^q \psi_i \varepsilon_{t-i}^* + \varepsilon_t^*, \\ \varepsilon_t^* = \sqrt{a h_t} + \xi_t^* \ \textit{ with } \xi_t^* \sim -G(a, \bar{b}_t), \\ \log h_t = \omega + \alpha(z_{t-1}^*/\sqrt{h_{t-1}}) + \theta(|z_{t-1}^*/\sqrt{h_{t-1}}| - \sqrt{2/\pi}) + \beta \log h_{t-1}, \end{array}\right. \tag{4.19}$$

$$\left\{\begin{array}{l} y_t = \phi_0 + \sum_{i=1}^p \phi_i y_{t-i} + \sum_{i=1}^q \psi_i \varepsilon_{t-i}^* + \varepsilon_t^*, \\ \varepsilon_t^* = \sqrt{\delta h_t} + \bar{c}_t^{-1}\sqrt{h_t/\delta}\xi_t^* \ \textit{ with } \xi_t^*|\mathcal{F}_{t-1} \sim -IG(\delta\sqrt{\bar{c}_t}), \\ \log h_t = \omega + \alpha(z_{t-1}^*/\sqrt{h_{t-1}}) + \theta(|z_{t-1}^*/\sqrt{h_{t-1}}| - \sqrt{2/\pi}) + \beta \log h_{t-1}, \end{array}\right. \tag{4.20}$$

respectively, when $\eta_t|\mathcal{F}_{t-1} \sim N(0,1), SNG(a)$ and $SNIG(\delta)$. Using a user-chosen initial variance h_t, then we can generate the data set $\{y_i\}_{i=t+1}^T$ in Step 3 from model (4.18), (4.19) or (4.20).

EXAMPLE 4.6. *(ARMA-GJR pricing models) Suppose μ_t in (2.1) follows an ARMA(p,q) model and h_t in (2.1) follows a GJR$(1,1)$ model, i.e., under P,*

$$\left\{\begin{array}{l} y_t = \phi_0 + \sum_{i=1}^p \phi_i y_{t-i} + \sum_{i=1}^q \psi_i \varepsilon_{t-i} + \varepsilon_t \ \textit{ and } \ \varepsilon_t = \eta_t \sqrt{h_t}, \\ h_t = \omega + \alpha \varepsilon_{t-1}^2 I(\varepsilon_{t-1} > 0) + \theta \varepsilon_{t-1}^2 I(\varepsilon_{t-1} \leq 0) + \beta h_{t-1}, \end{array}\right. \tag{4.21}$$

where $\omega, \alpha, \theta > 0$, $\beta \in (0,1)$, and $\eta_t|\mathcal{F}_{t-1} \sim N(0,1), SNG(a)$, or $SNIG(\delta)$. By (3.1), (3.3), and (3.7), model (4.21) under Q reduces to

$$\left\{\begin{array}{l} y_t = r - h_t/2 + \varepsilon_t^* \ \textit{ and } \varepsilon_t^* \sim N(0,h_t), \\ h_t = \omega + \alpha[z_{t-1}^*]^2 I(z_{t-1}^* > 0) + \theta[z_{t-1}^*]^2 I(z_{t-1}^* \leq 0) + \beta h_{t-1}, \end{array}\right. \tag{4.22}$$

$$\left\{\begin{array}{l} y_t = \phi_0 + \sum_{i=1}^p \phi_i y_{t-i} + \sum_{i=1}^q \psi_i \varepsilon_{t-i}^* + \varepsilon_t^*, \\ \varepsilon_t^* = \sqrt{a h_t} + \xi_t^* \ \textit{ with } \xi_t^* \sim -G(a, \bar{b}_t), \\ h_t = \omega + \alpha[z_{t-1}^*]^2 I(z_{t-1}^* > 0) + \theta[z_{t-1}^*]^2 I(z_{t-1}^* \leq 0) + \beta h_{t-1}, \end{array}\right. \tag{4.23}$$

$$\left\{\begin{array}{l} y_t = \phi_0 + \sum_{i=1}^p \phi_i y_{t-i} + \sum_{i=1}^q \psi_i \varepsilon_{t-i}^* + \varepsilon_t^*, \\ \varepsilon_t^* = \sqrt{\delta h_t} + \bar{c}_t^{-1}\sqrt{h_t/\delta}\xi_t^* \ \textit{ with } \xi_t^*|\mathcal{F}_{t-1} \sim -IG(\delta\sqrt{\bar{c}_t}), \\ h_t = \omega + \alpha[z_{t-1}^*]^2 I(z_{t-1}^* > 0) + \theta[z_{t-1}^*]^2 I(z_{t-1}^* \leq 0) + \beta h_{t-1}, \end{array}\right. \tag{4.24}$$

respectively, when $\eta_t|\mathcal{F}_{t-1} \sim N(0,1), SNG(a)$ and $SNIG(\delta)$. Using a user-chosen initial variance h_t, then we can generate the data set $\{y_i\}_{i=t+1}^{T}$ in Step 3 from model (4.22), (4.23) or (4.24).

5. Simulation study. In this section, we examine the finite sample performance of our GIM-NGARCH, GIM-EGARCH, and GIM-GJR pricing models in Section 4 and the GIM-GARCH pricing model in Duan (1995). For brevity, we only consider European call option written on daily Hang Seng Index (HSI). To choose the values of parameters in GARCH, NGARCH, EGARCH, and GJR models, we fit these four models to the log-return of a historical HSI data set, which has a total of 1001 observations taken from January 13, 2009 to December 31, 2012. The estimated results are reported in Table 1. Since our major interest in this section is to see how the implied volatility of the European call option varies according to different models and distributions of η_t, we use these estimators for simulations without considering model-checking. In application, model checking should be important, and we will consider it in Section 6.

TABLE 1
Estimated results for all models.

	$\hat{\nu}$	$\hat{w}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{a}$	$\hat{\delta}$
GARCH	0.0595	0.4×10^{-6}	0.1024	0.8855	———	2.484×10^3	5.590×10^3
NGARCH	0.0264	0.4×10^{-6}	0.0932	0.8654	0.5584	1.264×10^3	2.843×10^3
EGARCH	0.0387	-0.5290	-0.0874	0.9385	0.2281	7.702×10^3	1.733×10^4
GJR	0.0289	0.3×10^{-6}	0.0260	0.9061	0.1395	1.720×10^3	3.870×10^3

Next, by using the parameters in Table 1, we calculate the price of the European call option $\tilde{V}$ by the Monte Carlo procedure in Section 4 with a control-variate technique in Boyle et al. (1998). Here, as in Duan (1995), we set the risk-free rate $r = 0$, the strike price $K = 1$, and the repetition time $M = 50,000$. The moneyness (S/K) is from 0.8 to 1.2, the time to maturity (TM) is 30, 90 or 120 days, and the initial variance (IV) h_1 is $(0.8\sigma)^2$, σ^2 or $(1.2\sigma)^2$, where σ $(= 0.0149)$ is the standard deviation of the log-return series. As usual, the call option is out-of-the-money (OTM), at-the-money (ATM), and in-the-money (ITM) if $S/K << 1$, $S/K \approx 1$, and $S/K >> 1$, respectively.

As a comparison, we also consider the BS price V_{BS}, which is calculated by

$$V_{BS} = BS(S, K, \sigma_{BS}, TM/365, r),$$

where $\sigma_{BS} = 28.5\%$ is the annualized volatility of the log-return series, and

$$BS(S, K, \sigma, T, r) = S \cdot N(d_1) - Ke^{-rT} N(d_2)$$

is the BS price formula with

$$d_1 = \frac{\log(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}.$$

Furthermore, we calculate the annualized implied volatility σ_{im} according to

$$\tilde{V} = BS(S, K, \sigma_{im}, TM/365, r),$$

and then compare V_{BS} and $\tilde{V}$ in terms of σ_{BS} and σ_{im} through a conventional way. Moreover, Figures 1-3 plot σ_{im} along with different pricing models, distributions of η_t, TMs, and IVs. Since the results based on SNG and SNIG innovations are similar, we only report the results when η_t is conditional SNIG. From Figures 1-3, our findings are as follows:

(i) In all cases, the choice of IV will determine the position of the σ_{im} curves. Specifically, when IV gets large, the σ_{im} curves will shift up. Thus, how to choose a suitable IV should be very important in practice.

(ii) For the GIM-GARCH pricing model, the price of ITM option based on $\eta_t|\mathcal{F}_{t-1} \sim N(0, 1)$ is significantly higher than that based on $\eta_t|\mathcal{F}_{t-1} \sim SNIG(\delta)$. However, the price based on GIM-NGARCH, GIM-EGARCH and GIM-GJR models is less impacted by the distribution of η_t.

(iii) For the GIM-GARCH pricing model with $\eta_t|\mathcal{F}_{t-1} \sim N(0, 1)$, the relationship between V_{BS} and $\tilde{V}$ is consistent to that in Duan (1995). For other cases, when IV is smaller, V_{BS} is higher than $\tilde{V}$ for the OTM and ATM options, while it is smaller than $\tilde{V}$ for the ITM option; and when IV is larger, V_{BS} is smaller than $\tilde{V}$.

(iv) For the GIM-GARCH pricing model, the U-shape of the σ_{im} curves (i.e., "volatility smile") exists when $\eta_t|\mathcal{F}_{t-1} \sim N(0, 1)$, while the U-shape of the σ_{im} curves is skew-to-left (i.e., "volatility skew") when $\eta_t|\mathcal{F}_{t-1} \sim SNIG(\delta)$. For other three pricing models, the σ_{im} curves are always skew-to-left. The reason is because except the GIM-GARCH pricing model with $\eta_t|\mathcal{F}_{t-1} \sim N(0, 1)$, all of our pricing models can capture the "leverage effect", meaning that positive returns are associated with decreases of volatility and vice versa. So, the OTM option which needs larger positive returns to end up in the money at maturity, tends to have a smaller σ_{im}.

(v) Except the GIM-GARCH pricing model with $\eta_t|\mathcal{F}_{t-1} \sim N(0, 1)$, the U-shape of each σ_{im} curve fades away as TM becomes longer. The reason is that when TM is shorter, a big movement in stock price is highly possible, and hence the OTM option is more likely to become the ITM option. If this really happens, the OTM option will produce higher return but with lower capital than ATM or ITM option. Therefore, the speculators in the market will buy OTM options to take advantage of the potential big movement in stock price, and consequently, this will cause the higher σ_{im} in OTM options. When TM becomes longer, the possibility of extreme stock movement tends to be smaller, and eventually it will cause the vanish of the upwards movement of σ_{im} in OTM options.

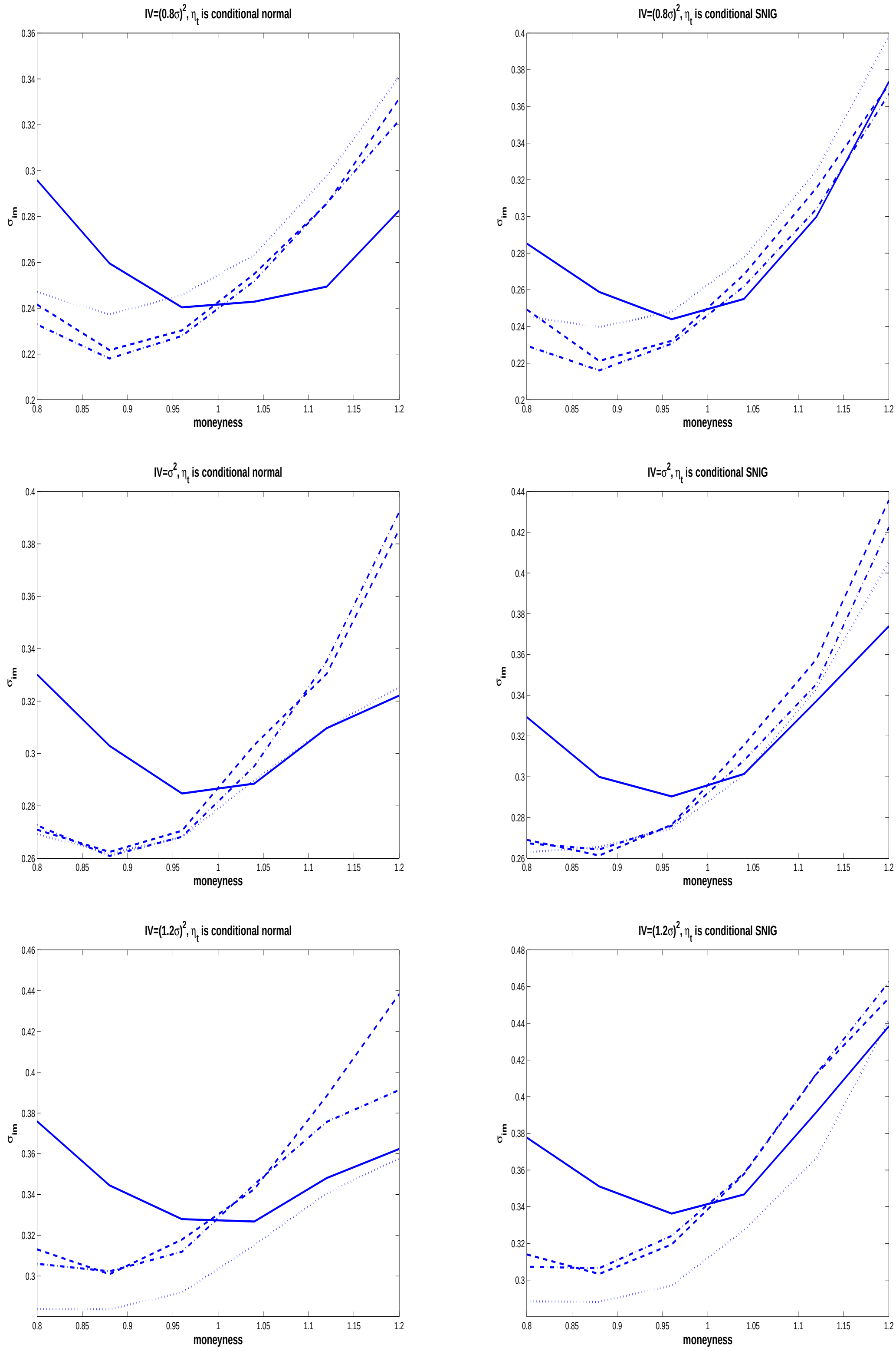

FIG 1. *Annualized implied volatility for $TM = 30$ under GIM-GARCH model (solid line), GIM-NGARCH model (dashed line), GIM-EGARCH model (dotted line), and GIM-GJR model (dot-dashed line).*

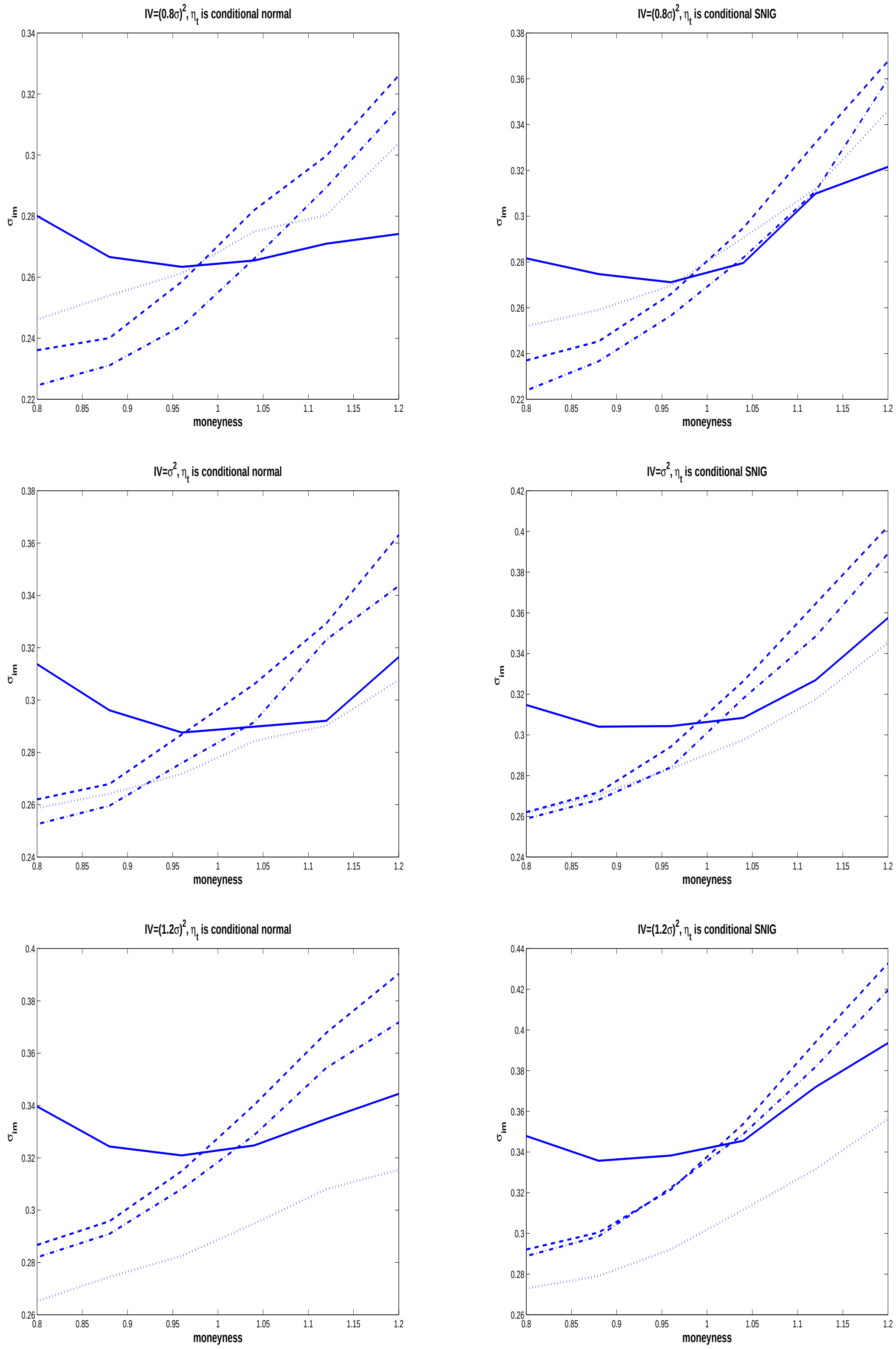

FIG 2. *Annualized implied volatility for $TM = 90$ under GIM-GARCH model (solid line), GIM-NGARCH model (dashed line), GIM-EGARCH model (dotted line), and GIM-GJR model (dot-dashed line).*

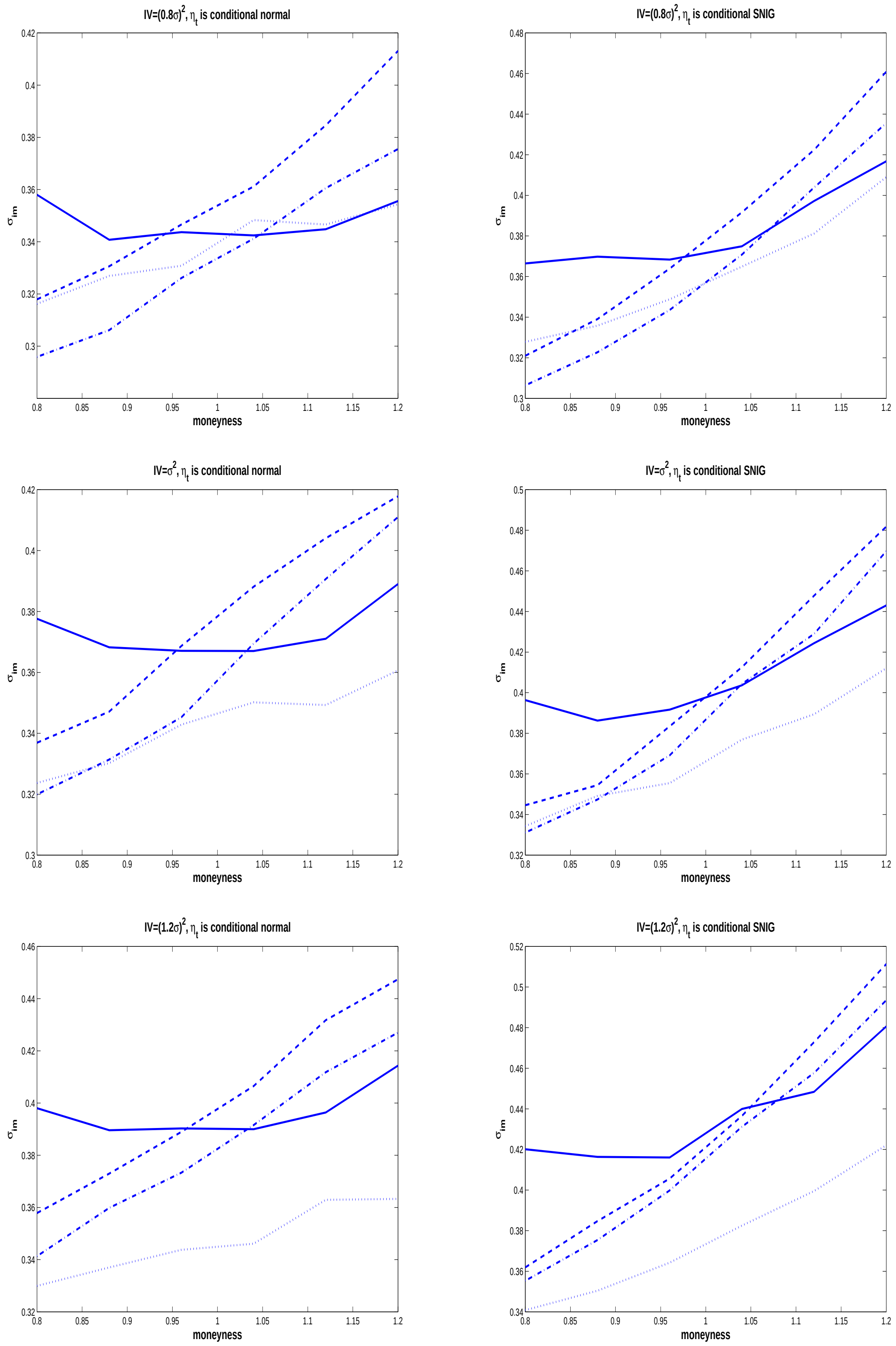

FIG 3. *Annualized implied volatility for* $TM = 120$ *under GIM-GARCH model (solid line), GIM-NGARCH model (dashed line), GIM-EGARCH model (dotted line), and GIM-GJR model (dot-dashed line).*

Overall, our pricing models can capture the "volatility skew" phenomenon in the market and should be useful in practice.

6. Application. In this section, we assess the performance of six different pricing models (see Table 3) by comparing our model-based prices with the real market prices. For brevity, we only consider the traded European S&P 500 call option data on April 18, 2002. This data set from Schoutens (2003) includes a total number of 53 call options with $TM = 22, 46, 109, 173$ or 234 days and K ranging from 975 to 1325; see Table 2 for more details. The closing price S_0 is 1124.47. The annual risk-free interest rate r is 1.9%, and the dividend yield d is 1.2%. So, the annual effective interest rate r_0 is 0.7% in all calculations. The parameters of all pricing models are estimated using the log-return of daily closing price of S&P 500 from January 04, 1988 to April 17, 2002 (a total of 3606 observation), and their results are reported in Table 3. To check the model adequacy, the p-values of the Ljung and Box tests $Q(M)$ and Li and Mak tests $Q^2(M)$ are also reported in the same table. From Table 3, we find that all of ARMA-type models are adequate to fit this log-return series, while all of GIM-type models are inadequate to fit the conditional mean of this log-return series.

TABLE 2

Market prices of all traded S&P 500 call options on April 18, 2002

K	TM=22	TM=46	TM=109	TM=173	TM=234
975			161.6	173.3	
995			144.8	157	
1025			120.1	133.1	146.5
1075		84.5	100.7	114.8	
1090	43.1				
1100	35.6		65.5	81.2	
1110		39.5			
1120	22.9	33.5			
1125	20.2	30.7	51	66.9	81.7
1130		28			
1135		25.6	45.5		
1140	13.3	23.2		58.9	
1150		19.1	38.1	53.9	68.3
1160		15.3			
1170		12.1			
1175		10.9	27.7	42.5	56.6
1200			19.6	33	46.1
1225			13.2	24.9	36.9
1250				18.3	29.3
1275				13.2	22.5
1300					17.2
1325					12.8

Next, we calculate the model-based prices of those call options in Table 2, and use the average relative error (ARE) criterion to measure the performances of our model-based

Table 3
Estimators for all pricing models

Models	Estimators	$Q(6)$	$Q(12)$	$Q^2(6)$	$Q^2(12)^\dagger$
GIM-NGARCH	$\hat{\nu}=0.0393, \hat{\omega}=6.6\times10^{-6}, \hat{\alpha}=0.1360, \hat{\beta}=0.7668, \hat{\theta}=0.5505,$ $\hat{a}=42.6, \hat{\delta}=95.9$	0.0065	0.0038	0.2830	0.7527
GIM-EGARCH	$\hat{\nu}=0.0445, \hat{\omega}=-0.4846, \hat{\alpha}=-0.1162, \hat{\beta}=0.9480, \hat{\theta}=0.1704,$ $\hat{a}=4482.0, \hat{\delta}=1.008\times10^4$	0.0185	0.0073	0.9122	0.9920
GIM-GJR	$\hat{\nu}=0.0498, \hat{\omega}=8.0\times10^{-6}, \hat{\alpha}=0.0670, \hat{\beta}=0.7772, \hat{\theta}=0.2240,$ $\hat{a}=37.6, \hat{\delta}=84.7$	0.0300	0.0116	0.9174	0.9959
AR(3)-NGARCH	$\hat{\phi}_0=0.0001, \hat{\phi}_1=0.0431, \hat{\phi}_2=-0.0021, \hat{\phi}_3=-0.0412, \hat{\omega}=6.7\times10^{-6},$ $\hat{\alpha}=0.1382, \hat{\beta}=0.7623, \hat{\theta}=0.5809, \hat{a}=77.9, \hat{\delta}=175.3$	0.5562	0.1833	0.2998	0.7764
AR(3)-EGARCH	$\hat{\phi}_0=0.0002, \hat{\phi}_1=0.0430, \hat{\phi}_2=0.0174, \hat{\phi}_3=-0.0245, \hat{\omega}=-0.4789,$ $\hat{\alpha}=-0.1210, \hat{\beta}=0.9484, \hat{\theta}=0.1727, \hat{a}=2.8\times10^6, \hat{\delta}=6.4\times10^6$	0.4608	0.1512	0.9328	0.9935
AR(3)-GJR	$\hat{\phi}_0=0.0002, \hat{\phi}_1=0.0383, \hat{\phi}_2=-0.0013, \hat{\phi}_3=-0.0457, \hat{\omega}=8.3\times10^{-6},$ $\hat{\alpha}=0.0657, \hat{\beta}=0.7739, \hat{\theta}=0.2316, \hat{a}=59.8, \hat{\delta}=134.5$	0.5398	0.1583	0.6446	0.9538

† p-values of Ljung-Box test statistics $Q(M)$ and Li-Mak test statistics $Q^2(M)$.

Table 4
The values of ARE for all pricing models.

Models	IV	$\eta_t\|\mathcal{F}_{t-1}\sim N(0,1)$						$\eta_t\|\mathcal{F}_{t-1}\sim SNG(a)$						$\eta_t\|\mathcal{F}_{t-1}\sim SNIG(\delta)$					
		TM						TM						TM					
	κ	22	46	109	173	234	All	22	46	109	173	234	All	22	46	109	173	234	All
GIM-NGARCH	0.7	1.40	0.83	2.34	4.77	6.39	3.31	1.60	1.07	1.32	2.94	4.87	2.39	1.74	0.78	2.47	3.65	5.50	2.90
GIM-EGARCH	0.8	0.75	1.30	2.74	5.20	7.42	3.76	0.80	1.23	3.00	4.67	6.87	3.56	2.21	1.03	1.19	2.59	4.35	**2.22**
GIM-GJR	0.7	1.76	1.54	3.36	5.46	6.71	3.98	0.72	1.69	2.71	4.51	6.61	3.50	1.62	3.61	3.62	5.48	8.58	4.86
AR(3)-NGARCH	0.7	1.17	0.85	2.48	4.09	6.12	3.10	2.35	1.79	1.41	3.41	4.86	2.77	1.51	0.91	1.76	3.40	4.80	2.55
AR(3)-EGARCH	0.9	4.93	1.84	4.03	6.75	11.24	5.70	2.08	2.04	2.51	4.62	7.00	3.77	7.15	5.73	1.94	2.75	4.51	3.92
AR(3)-GJR	0.7	1.77	1.78	3.82	5.90	7.23	4.36	0.62	0.98	2.86	4.59	6.03	3.28	1.19	2.04	2.60	4.12	5.92	3.37
Benchmarks‡	IV	TM																	
	κ	22	46	109	173	234	All												
BS	1.0	2.53	3.93	5.95	8.28	10.09	6.57												
GIM-GARCH	0.6	6.42	4.34	5.26	7.79	10.42	6.80												

† The ARE is based on all options, and the smallest ARE is in bold.
‡ The ARE of BS pricing model and GIM-GARCH pricing model with $\eta_t|\mathcal{F}_{t-1}\sim N(0,1)$ is calculated with IV=$(\kappa\sigma_{BS})^2$ and $(\kappa\sigma_e)^2$, respectively. Here, σ_{BS} is the annualized volatility of the log-return series.

prices, where

$$\mathrm{ARE} = \frac{1}{N}\sum_{j=1}^{N}\frac{\left|V_j^{\mathrm{model}} - V_j^{\mathrm{market}}\right|}{V_j^{\mathrm{market}}} \times 100,$$

and N is the total number of options considered, V_j^{market} is the market price of j-th option, and V_j^{model} is the model-based price of j-th option. For each pricing model, since V_j^{model} depends on the choice of IV h_1, we choose h_1 to be $(\kappa\sigma_e)^2$, where σ_e is the estimated volatility of the last day of the log-return series, and κ is taken as follows:

$$\kappa = \min_{\kappa_0 \in \{0.1, 0.2, \cdots, 2.0\}} \widetilde{\mathrm{ARE}}(\kappa_0),$$

where $\widetilde{\mathrm{ARE}}(\kappa_0)$ is the ARE of this pricing model with $h_1 = (\kappa_0\sigma_e)^2$ and $\eta_t|\mathcal{F}_{t-1} \sim N(0,1)$. Based on this choice of IV, Table 4 reports the detailed results of ARE along with different models, distributions of η_t, and TMs. From Table 4, we find that (i) except the GIM-GJR model with $\eta_t|\mathcal{F}_{t-1} \sim SNIG(\delta)$, each pricing model with non-normal innovation has a smaller ARE than that with normal innovation, even κ is optimally chosen for the normal innovation; (ii) the performance of each pricing model becomes worse when the value of TM increases; (iii) for each pricing model, the performance of two non-normal cases is comparative; (iv) the performance of the GIM-type pricing models and ARMA-type pricing models is also comparative, and although the GIM-EGARCH pricing model is inadequate, the GIM-EGARCH pricing model with $\eta_t|\mathcal{F}_{t-1} \sim SNIG(\delta)$ has the best performance; see Figure 4 for the plot of difference of model-based prices and market prices in this case. Overall, all of our pricing models have a good performance no matter of model-adequacy and much better than the options based on the BS pricing formula and Duan's standard GARCH(1,1) model with $\eta_t|\mathcal{F}_{t-1} \sim N(0,1)$. Our best ARE is 2.22 less than 2.36, which is the best ARE based on the ESS-TGARCH-M pricing model in Xi (2013).

7. Concluding remarks. In this paper, we construct a risk-neutralized Esscher measure for the asset return which can be decomposed into the conditional mean plus a noise with time-varying volatility components under a physical probability measure P. Using this risk-neutralized measure, six ARCH-type model-based pricing procedures are proposed to value the derivatives. Simulation studies show that our pricing models can capture the "volatility skew" of implied volatilities in the European option. A small application to the European HSI option highlights the importance of our model-based pricing procedure with non-normal innovations. As the empirical studies suggested, the performance of our pricing procedure varies in terms of IV, TM, and the distribution of innovation. Hence, two promising directions for future study are (i) choosing an "optimal" IV by the range of TM in some sense, and (ii) estimating the distribution of innovation non-parametrically.

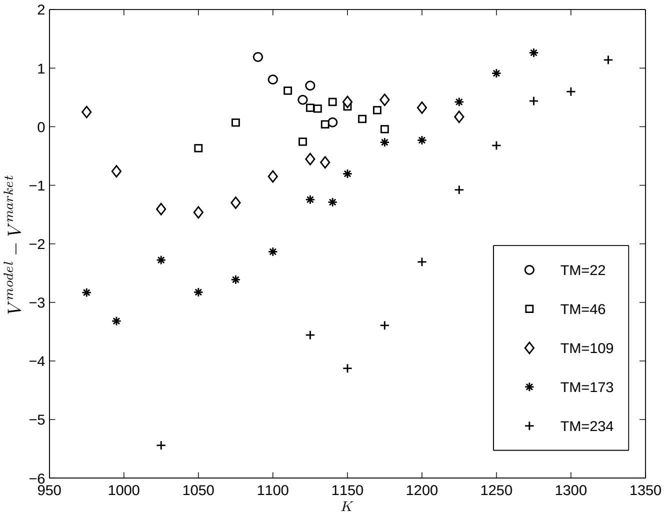

FIG 4. *The values of $V^{model} - V^{market}$ based on the GIM-EGARCH pricing model with $\eta_t|\mathcal{F}_{t-1} \sim SNIG(\delta)$.*

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Chinese Academy of Sciences
Institute of Applied Mathematics
Haidian District, Zhongguancun
Beijing, China
E-mail: kzhu@amss.ac.cn

Department of Mathematics
Hong Kong University of Science and Technology
Clear Water Bay, Kowloon
Hong Kong
E-mail: maling@ust.hk