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Equilibrium and Optimal Fertility with Increasing Returns to Population and Endogenous Mortality

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Abstract

We present a general equilibrium dynamic model that characterizes the gap between optimal and equilibrium fertility and investment in human capital. In the model, the aggregate production function exhibits increasing returns to population arising from specialization but households face the standard quantity-quality trade-off when deciding how many children they have and how much education these children receive. In the benchmark model, we solve for the equilibrium and optimal levels of fertility and investment per child and show that competitive fertility is too low and investment per child too high. We next introduce mortality of young adults in the model and assume that households have a precautionary demand for children. Human capital investment raises the likelihood that a child survives to the next generation. In this setup, the model endogenously generates a demographic transition but, since households do not internalize the positive effects of a larger population on productivity and the negative effects of human capital on mortality, both the industrial revolution and the demographic transition take place much later than it would have been optimal. Our model can be interpreted as a bridge between the literature on endogenous demographic transitions and papers that study welfare issues associated with fertility and human capital decisions.

1 Introduction

In this paper we first present a theoretical model that characterizes the equilibrium and efficient fertility rates and investment per children. Due to the presence of an agglomeration economy arising from specialization the competitive equilibrium results in a too low number of children and a too high investment per child. When we introduce young mortality that decreases with human capital the model predicts that the onset of both the demographic transition and the industrial revolution occur much later in the competitive solution than in the efficient one. The mechanism through which this occurs is that the planner chooses to accumulate human capital as fast as possible to eradicate young mortality and then, in the balanced growth path, he sets the fertility rate at a higher level than in the competitive problem since he also internalizes the positive effects that a larger population has on specialization and hence aggregate productivity.

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The paper is organized as follows. The next section presents a summary of the related literature. The theoretical model is developed in Section 3. Section 4 adds mortality to the model. In Section 5 we present the numerical solution of the model with mortality and, finally, Section 5 concludes the paper.

2 Related Literature

Our paper relates to two different strands of the literature. The first one is the study of welfare properties associated with fertility and human capital decisions. The concept of an optimal population has long been discussed in Dasgupta (1969), Samuelson (1975), Razin and Ben-Zion (1975), Nerlove et al (1982, 1987), Gigliotti (1983), Willis (1987) and Zimmerman (1989) among others. These papers often discuss how to apply the concept of Pareto optimality to questions related to population, acknowledging the fact that an increase in population by one member rises the welfare of this individual, even if it may lower the welfare of the rest of the existing population (Eckstein and Wolpin, 1985). In our model, parents maximize a utility function that depends on their children's income and this generates an inefficient equilibrium because parents do not internalize the positive effect that a larger population and stock of human capital have on next generation's income. Therefore our model has welfare implications but by focusing on both fertility and human capital decisions and how they are affected by endogenous declines in young mortality it significantly differs from this existing literature.

We also contribute to the theoretical literature on the demographic transition. The demographic transition is typically composed of two stages: a mortality transition, characterized by sharp declines in mortality rates and more or less constant fertility rates, and a fertility transition where fertility rates decline much faster than mortality rates. Altogether, the typical demographic transition then displays a hump-shaped evolution of a country's population growth rate. Most of the existing theoretical literature focuses on analysing the triggers of the fertility transition, which include an increase in technological progress and the demand for human capital (Galor and Weil, 1999, 2000; Galor and Moav, 2002), an increase in income per capita (Becker, 1960; Becker and Lewis, 1973), a reduction in gender gaps (Galor and Weil, 1996; Lagerlof, 2003; Tertilt, 2005, 2006; Doepke and Tertilt, 2009), and a fall in infant mortality rates (Sah, 1991; Kalemli-Ozcan, 2002). The causes of the mortality transition are better understood. Weil (2005) highlights the importance of improvements in the standards of living that resulted from an increase in the quantity and quality of food consumed, advances in housing and more often washing of clothes, and investments in public health (clean water and food), as well as new medical treatments. However, few models of the demographic transition endogenize mortality and analyse the effects that declines in mortality rates have on the fertility transition. This paper is a step forward in the sense that we are able to add to the scarce literature that treats the fertility and mortality transitions in a unified framework. The mechanism that generates the demographic transition in our model is the endogenous decline in young mortality as a result of accumulated human capital. While there is strong empirical evidence that the number of children produced by a household declines as infant and/or child mortality declines, the effect that reductions in mortality have on fertility sharply differs across theoretical models. In the framework of the Barro-Becker model (Becker and Barro, 1988; Barro and Becker, 1989), Doepke (2005) and Fernandez-Villaverde (2005) show that a fall in child mortality reduces the cost of rising survivor children, hence increasing net fertility. In contrast, Boldrin and Jones (2002) show that if one modifies the Barro-Becker model so that parents' consumption when old directly enters the children's utility function (the old-age security hypothesis), it is possible to generate a positive correlation between infant and/or child mortality rates and fertility. One important difference between Boldrin and Jones (2002) and our paper is that child mortality rates are endogenous in our case, decreasing as the stock of human capital rises. Moreover, while our model generates a positive correlation between young adult mortality and fertility as in their paper, we emphasize a different channel, namely a fall in parents precautionary demand for children as young adult mortality falls. The implicit assumption behind this mechanism is that a decline in the likelihood that children die before adulthood induces a reduction in fertility if parents seek to have an optimal number of surviving offspring (Tamura, 1996, 2002, 2006). Kalemli-Ozcan (2002) develops a model in which parents also have a precautionary demand for children and so reductions in infant mortality induce a fall in fertility and an increase in investment per child. One important difference with our paper is that she does not use the model to have predictions on the timing of the demographic transition, nor she compares the optimal and equilibrium problems.[1]

Another important paper that endogenously links the fall in mortality rates to the fertility transition is Jones (2001). In his model, the occasional generation of new ideas in a Malthusian economy eventually translates into a larger population size. This larger population in turn produces more ideas and this raises consumption per capita, reducing mortality rates and hence triggering a demographic transition. In his model, as in ours, it is the case that, had agents taken into account the positive effects of a larger population, both the industrial revolution and the demographic transition would have taken place much earlier. However, Jones does not solve for the optimal problem explicitly and hence it is hard to use his model to derive welfare implications.

3 Theoretical Benchmark

3.1 The Baseline Model

3.1.1 Setup

Consider an economy populated by P_t workers. There is a single consumption good which is produced using the human capital of N_t workers, where $1 \leq N_t < P_t$. These N_t workers use the following reduced form for aggregate output:

$$Y_t = \left\{ \sum_{i=1}^{N_t} h_{it}^{\frac{1}{\omega}} \right\}^{\omega} \tag{1}$$

where Y_t is output at period t, and h_{it} is human capital provided by agent i at period t. The parameter $\omega > 1$ represents the degree of increasing returns to population at the aggregate level, arising from specialization returns as in Rosen (1982) and Tamura (1992, 2002, 2006). Note that this production function exhibits diminishing returns to individual human capital since ω is larger than one, however there are constant returns to the entire distribution of human capital. Within the production coalition, workers are paid the

[1] Soares (2005) develops a model where reductions in mortality (but not infant, child, or young mortality specifically) are the main force behind economic development. His model also generates a demographic transition where gains in life expectancy at birth are followed by reductions in fertility and increases in the rate of human capital accumulation.

marginal product of their human capital. Let y_{jt} be a typical worker j's earnings. Then:

$$\begin{aligned} y_{jt} &= \left\{\sum_{i=1}^{N_t} h_{it}^{\frac{1}{\omega}}\right\}^{\omega-1} \overline{h}_{jt}^{\frac{1-\omega}{\omega}} h_{jt} \\ &= \left\{\sum_{i=1}^{N_t} h_{it}^{\frac{1}{\omega}}\right\}^{\omega-1} h_{jt}^{\frac{1}{\omega}} \end{aligned} \tag{2}$$

where the last equality arises since we consider a symmetric equilibrium where all agents have the same human capital $\overline{h}_{jt} = h_{jt}$. It is evident that earnings of all N_t members of the production coalition exactly exhaust output:

$$Y_t = \sum_{j=1}^{N_t} y_{jt} \tag{3}$$

There are two restrictions on the number of workers in a production coalition. First if cannot fall below 1, which represents autarky. Second we impose a restriction that the number of workers cannot exceed a minority proportion of the total population in the economy. Thus we assume:

$$1 \leq N_t < \lambda_1 P_t. \tag{4}$$

where $\lambda_1 > 0$. Thus while market specialization can increase, that is the number of distinct workers cooperating to produce the single consumption good can increase, it can never exceed the population of the economy.[2] What determines the scope of specialization? We assume that it is determined by the average human capital, $\overline{h}_t$, in the economy:

$$N_t = \begin{cases} 1 & \text{if } \mathrm{L}\overline{h}_t^{\lambda_2} \leq 1 \\ \mathrm{L}\overline{h}_t^{\lambda_2} & \text{if } 1 < \mathrm{L}\overline{h}_t^{\lambda_2} \leq \lambda_1 P_t \\ \lambda_1 P_t & \text{if } \lambda_1 P_t < \mathrm{L}\overline{h}_t^{\lambda_2} \end{cases} \tag{5}$$

where L, $\lambda_2 > 0$. Since the number of workers in the production coalition is determined by the average human capital $\overline{h}_t$, or the population of the economy, P_t, there is an external effect of human capital and population. During autarkic production, human capital accumulation does not affect the number of workers within a production coalition. Absent any other external effects of human capital, there would be no difference in the human capital accumulation of workers in equilibrium versus workers in an efficient allocation. However during the intermediate phase, $1 < \mathrm{L}\overline{h}_t^{\lambda_2} < \lambda_1 P_t$, an efficient allocation would have greater rates of human capital accumulation in order to internalize the gains from rising specialization. Finally in the third branch, when we explicitly assume that specialization gains are capped at a finite proportion of the population, an efficient allocation would have greater population growth than an equilibrium solution path.

Under symmetry one has

$$Y_t = N_t^{\omega} h_t$$

[2] We have in mind a population P_t that can exceed the population of a small country like Denmark. However it seems reasonable that much specialization can be restricted to a large metropolitan area. Hence any increasing returns to specialization are inherently smaller than a country like the United States.

or, in per capita terms

$$y_t = N_t^{\omega-1} h_t = Z_t h_t$$

In an equilibrium solution, individuals do not take into account the positive external benefit their human capital accumulation or their fertility provides for future specialization gains. Furthermore they do not take into account that their human capital is complementary to other workers, and by extension the human capital investments they make on their children's human capital has effects on wages of all other workers in that generation. Hence they treat the time path of total factor productivity, Z_t as exogenous technological progress.[3] The economy is populated by individuals who live for two periods and a family consists of a parent and his children. In the first period individuals receive education provided by their parent. In the second they form their own household.

We first focus on the stationary balanced growth world without mortality. Later on we will introduce mortality to show that the efficient solution involves a different emphasis altogether. Parents care about their own consumption, c, the number of children, x, they have and the income, y, of their typical child. A parent receives no utility from leisure, so they only work or spend time rearing and raising their children. Thus the typical generation t parent i wishes to maximize:

$$\alpha lnc_{it} + (1-\alpha) lnx_{it} + \alpha\beta lny_{it+1} \tag{6}$$

The typical t period parent's budget constraint is

$$c_{it} = w_{it} h_{it} \left[1 - x_{it}(\theta + \tau_{it})\right] \tag{7}$$

where θ is the unavoidable time cost of child rearing, and τ_t is time spent teaching each child. Human capital next period depends on the investment time per child τ_t i.e.

$$h_{t+1} = A \bar{\bar{h}}_t^{\rho} h_t^{1-\rho} \tau_t^{\mu} \tag{8}$$

where $\bar{\bar{h}}_t = \max\{h\}$ is the maximum human capital in the world and $\rho, \mu \in (0,1)$. $A > 1$ is an efficiency parameter.

3.1.2 Competitive Equilibrium

In the competitive equilibrium we have price taking behavior. Individuals are paid the marginal product of their human capital. The typical parent takes the wage per unit of human capital as given, and does not assume that his investment decisions have any effect on this wage, nor on the wage of any other worker. Given that the production technology has constant returns to scale in the distribution of human capital in the economy, paying each worker their marginal product exactly exhausts output. To see this, we note

[3] Of course in equilibrium we have $\frac{Z_{t+1}}{Z_t} = \left(\frac{N_{t+1}}{N_t}\right)^{\omega-1} = g_n^{\omega-1}$

that for the typical worker j the wage per unit of human capital for worker j is given by:[4]

$$w_{it} = \left\{ \sum_{j=1}^{N_t} h_{jt}^{\frac{1}{\omega}} \right\}^{\omega - 1} h_{it}^{\frac{1}{\omega}-1} \tag{9}$$

$$y_{it} = \left\{ \sum_{j=1}^{N_t} h_{jt}^{\frac{1}{\omega}} \right\}^{\omega - 1} h_{it}^{\frac{1}{\omega}} \tag{10}$$

Observe that the wage per unit of human capital for worker i is *decreasing* in human capital of worker i. However (10) shows that *earnings*, y_{it} for worker i is increasing in worker i's human capital. Utility for a parent i in generation t can be written as:

$$\begin{aligned} V(h_{it}) &= \max_{\{h_{it+1}, x_{it}\}} \{\alpha \ln c_{it} + (1-\alpha)\ln x_{it} + \alpha\beta ln y_{it+1}\} \\ &= \alpha \max_{\{h_{it+1}, x_{it}\}} \left\{ lnw_{it} + \beta lnw_{it+1} + lnh_{it} + ln(1 - x_{it}[\theta + \tau_{it}]) + \frac{1-\alpha}{\alpha} lnx_{it} + \beta lnh_{it+1} \right\} \end{aligned} \tag{11}$$

The first order condition with respect to fertility shows that the fraction of resources spent on the next generation is constant and equal to $1 - \alpha$:

$$\begin{aligned} \frac{\partial}{\partial x_{it}} = 0 \Leftrightarrow \frac{\alpha(\theta + \tau_{it})}{1 - x_{it}(\theta + \tau_{it})} &= \frac{1-\alpha}{x_{it}} \\ x_{it}(\theta + \tau_{it}) &= 1 - \alpha \end{aligned} \tag{12}$$

Now consider the first order condition with respect to a child's human capital:

$$\begin{aligned} \frac{\partial}{\partial h_{it+1}} = 0 \Leftrightarrow \frac{\alpha x_{it}\tau_{it}}{(1 - x_{it}[\theta + \tau_{it}])\mu h_{it+1}} &= \frac{\alpha\beta}{h_{it+1}} \\ x_{it}\tau_{it} &= \alpha\beta\mu \end{aligned} \tag{13}$$

Combining the results in (12) and (13) we find the equilibrium stationary fertility and investment time:

$$x^{eq} = \frac{1 - \alpha - \alpha\beta\mu}{\theta} \tag{14}$$

$$\tau^{eq} = \frac{\alpha\beta\mu\theta}{1 - \alpha - \alpha\beta\mu} \tag{15}$$

One immediate parameter restriction is evident:

$$1 - \alpha - \alpha\beta\mu > 0$$

Intuitively, an increase in β - i.e. if parents care more about their children's income - reduces fertility and increases the investment in children's human capital. Similarly, a higher θ obviously reduces fertility and increases investment per child since the fixed cost of rearing a child is now higher. Finally, an increase in

[4] One can assume that there are N_t different types of workers, each type with an initial measure of 1, and each worker is a set of measure zero of the number of workers of their type. In the equilibrium solution, a parent completely ignores the effect of human capital investment on their children's wage or the wage of any one of that generation.

μ also increases investment in children's human capital, since the return to this investment is now higher.[5]

3.1.3 Efficient Solution

In the efficient solution, we consider the case in which a social planner internalizes the positive externality of population growth on TFP growth, as well as the positive externality of human capital investment on the next generation wages. For convenience and without loss of generality we focus on the efficient solution in which all parents are treated equally.[6] The social planner's problem can be written as:

$$\max_{\{c_{jt},x_{jt},h_{jt+1}\}_{j=1}^{N_t}} \left\{ \frac{1}{N_t} \sum_{j=1}^{N_t} [\alpha ln c_{jt} + (1-\alpha) ln x_{jt} + \alpha\beta ln y_{jt+1}] \right\} \tag{16}$$

subject to the resource constraint:

$$\sum_{j=1}^{N_t} c_{jt} \leq \left\{ \sum_{j=1}^{N_t} h_{jt}^{\frac{1}{\omega}} (1 - x_{jt}[\theta + \tau_{jt}])^{\frac{1}{\omega}} \right\}^{\omega} \tag{17}$$

Let Λ be the multiplier on the resource constraint. The first order condition with respect to consumption for the typical parent i is given by:

$$\frac{\partial}{\partial c_{it}} = 0 \Leftrightarrow \frac{\alpha}{N_t c_{it}} = \Lambda \tag{18}$$

Thus all parents receive the same consumption.[7] Assume for the time being the possibility that production coalition size is bigger than 1, and is already determined by the third line of (5). That is the production coalition is a constant proportion of the population. We will focus on the homogeneous population case, but for now we allow the social planner to pick individual parental fertility by agent type, $1, ..., N_t$. The first order condition for optimal fertility can be written as:

$$\frac{\partial}{\partial x_{it}} = 0 \Leftrightarrow$$

$$\frac{1}{N_t} \left\{ \frac{1-\alpha}{x_{it}} + \sum_{j=1}^{N_t} \frac{(\omega-1) \left\{ \sum_{s=1}^{N_t} x_{st} h_{st+1}^{\frac{1}{\omega}} \right\}^{\omega-2} h_{jt+1}^{\frac{1}{\omega}} h_{it+1}^{\frac{1}{\omega}}}{y_{jt+1}} \right\} =$$

$$\Lambda \left\{ \sum_{j=1}^{N_t} h_{jt}^{\frac{1}{\omega}} (1 - x_{jt}[\theta + \tau_{jt}])^{\frac{1}{\omega}} \right\}^{\omega-1} h_{it}^{\frac{1}{\omega}} (1 - x_{it}[\theta + \tau_{it}])^{\frac{1}{\omega}-1} [\theta + \tau_{it}] \tag{19}$$

The middle line represents the marginal benefits of fertility. The second term in the curly brackets is the full effect of additional fertility on the earnings of the generation t+1 adults. Notice that we can keep the number of types of workers constant, and adjust the population of each type by x_{jt}. The third line represents the marginal cost of fertility in terms of foregone current output. The first order condition with

[5] These results are similar to those in Tamura (2002).

[6] It is simple to show that this has no effect on the human capital investment decision in the cases with unequal Pareto weights.

[7] Again, if the Pareto weights were different, then two parents could receive different consumption values, but it would not affect the accumulation path of human capital.

respect to human capital investment can be written as:

$$\frac{\partial}{\partial h_{it+1}} = 0 \Leftrightarrow$$

$$\frac{\alpha\beta}{\omega N_t}\left\{\sum_{j=1}^{N_t}\frac{(\omega-1)\left\{\sum_{s=1}^{N_t} x_{st}h_{st+1}^{\frac{1}{\omega}}\right\}^{\omega-2} h_{jt+1}^{\frac{1}{\omega}} x_{it} h_{it+1}^{\frac{1}{\omega}-1}}{y_{jt+1}} + \frac{\left\{\sum_{s=1}^{N_t} x_{st}h_{st+1}^{\frac{1}{\omega}}\right\}^{\omega-1} h_{it+1}^{\frac{1}{\omega}-1}}{y_{it+1}}\right\} =$$

$$\Lambda\left\{\sum_{j=1}^{N_t} h_{jt}^{\frac{1}{\omega}}(1-x_{jt}[\theta+\tau_{jt}])^{\frac{1}{\omega}}\right\}^{\omega-1}\frac{h_{it}^{\frac{1}{\omega}}(1-x_{it}[\theta+\tau_{it}])^{\frac{1}{\omega}-1}x_{it}\tau_{it}}{\mu h_{it+1}} \tag{20}$$

Looking at the marginal benefits term, the middle line of equation (20), the first term in the curly brackets is the effect of higher human capital for the ith type generation t+1 adult on all wages of t+1 workers. The second term in the curly brackets is the direct effect of higher human capital on earnings of the ith type generation t+1 worker. In both cases the social planner internalizes the effect on both the wage per unit of human capital as well as the direct effect, the second term.

We now impose symmetry, that is we assume that all individuals are of the same type, $h_{it} = h_{jt}, \forall i, j$. Utilizing the definition of y_{jt+1}, our Euler equation with respect to fertility can now be written as:

$$\frac{1-\alpha+\alpha\beta(\omega-1)}{x_t} = \Lambda N_t^{\omega} h_t(\theta+\tau_t) \tag{21}$$

Similarly we can write our Euler equation with respect to human capital investment as:

$$\alpha\beta\mu = \Lambda y_t N_t x_t \tau_t \tag{22}$$

Using the Euler equation for consumption and the resource constraint we can solve for Λ:

$$\Lambda = \frac{\alpha}{N_t^{\omega} h_t(1-x_t[\theta+\tau_t])} \tag{23}$$

The fraction of resources spent on the next generation is given by:

$$x_t(\theta+\tau_t) = \frac{1-\alpha+\alpha\beta(\omega-1)}{1+\alpha\beta(\omega-1)} \tag{24}$$

Using this we can solve for the efficient fertility and efficient investment rate:

$$x^{eff} = \frac{1-\alpha-\alpha\beta\mu+\alpha\beta(\omega-1)}{\theta[1+\alpha\beta(\omega-1)]} \tag{25}$$

$$\tau^{eff} = \frac{\alpha\beta\mu\theta}{1-\alpha-\alpha\beta\mu+\alpha\beta(\omega-1)} \tag{26}$$

It is clear that an increase in ω induces a higher fertility rate since the positive external effect of population on output per capita and wages is now higher. Of course, the corresponding level of investment per child decreases as ω increases. Note that when $\omega = 1$ this solution coincides with the competitive one.

3.1.4 Comparing Both Setups

In this section we compare the different solutions of the equilibrium problem and the efficient problem. It is easy to show that fertility is higher in the efficient solution than in the equilibrium solution. We can also show that human capital investment is slower an utility higher in the efficient solution compared with the equilibrium solution. Nontrivially however, economic growth can be higher in the equilibrium solution than in the efficient solution. However for large enough gains from specialization, higher ω, the growth rate in the efficient solution exceeds that of the equilibrium solution.

Comparing fertility between the two cases:

$$x^{eff} = \frac{1-\alpha-\alpha\beta\mu+\alpha\beta(\omega-1)}{\theta[1+\alpha\beta(\omega-1)]} \quad > \quad \frac{1-\alpha-\alpha\beta\mu}{\theta} = x^{eq} \tag{27}$$

$$\iff 0 \quad > \quad -\alpha(1+\beta\mu)\alpha\beta(\omega-1), \tag{28}$$

which holds $\forall\omega > 1$. Next we compare human capital investment rates from the two cases:

$$\tau^{eff} = \frac{\alpha\beta\mu\theta}{1-\alpha-\alpha\beta\mu+\alpha\beta(\omega-1)} \quad < \quad \frac{\alpha\beta\mu\theta}{1-\alpha-\alpha\beta\mu} = \tau^{eq} \tag{29}$$

$$\iff \alpha\beta(\omega-1) \quad > \quad 0, \tag{30}$$

which holds $\forall\omega > 1$. Next we show that utility is always higher in the efficient solution compared with the equilibrium solution: Assume that the population is identical between the efficient and equilibrium problems to start, all individuals have the same human capital, h_t, and that the production coalition is the same size as well, $N_t = \lambda_1 P_t$. Writing out utility of the typical parent with h_t human capital for both equilibrium and efficient problems, and canceling out identical terms we end up with:

$$V^{eff}(h_t) - V^{eq}(h_t) = (1-\alpha-\alpha\beta\mu+m)ln(\frac{1-\alpha-\alpha\beta\mu+m}{1-\alpha-\alpha\beta\mu}) - (1+m)ln(1+m) \tag{31}$$

$$m = \alpha\beta(\omega-1)$$

Observe that when there is no agglomeration return to specialization, $\omega = 1$, then $m = 0$, and there is no difference between the efficient utility and equilibrium utility. Taking the derivative of (31) with respect to m we get:

$$\frac{\partial(V^{eff}(h_t) - V^{eq}(h_t)}{\partial m} = ln(\frac{1-\alpha-\alpha\beta\mu+m}{1-\alpha-\alpha\beta\mu}) + 1 - ln(1+m) - 1$$

$$\frac{\partial(V^{eff}(h_t) - V^{eq}(h_t)}{\partial m} = ln(\frac{1-\alpha-\alpha\beta\mu+m}{(1-\alpha-\alpha\beta\mu)(1+m)}) > 0$$

$$\iff m > 0$$

So for all values with positive agglomeration economy gains from specialization, $\omega > 1, m > 0$, we have that the gap between the efficient utility and the equilibrium utility is positive.

Finally we show that growth in output per worker cannot be so easily ordered. For small enough gains in specialization, equilibrium growth can exceed efficient growth. For larger values of specialization gains, efficient growth can exceed equilibrium growth.

Case 1: Assume the following parameter configuration:

$$\alpha = \frac{5}{8}, \beta = \frac{3}{5}, \mu = \frac{1}{2}, \theta = \frac{1}{8}, A = 5, \omega = 1.35$$
$$x^{eq} = 0.98, \tau^{eq} = .2066, x^{eff} = 1.85, \tau^{eff} = .096$$
$$\Gamma^{eq} = (x^{eq})^{\omega-1} A(\tau^{eq})^{\mu} = 2.2568$$
$$\Gamma^{eff} = (x^{eff})^{\omega-1} A(\tau^{eff})^{\mu} = 1.9198$$

Case 2: Assume the following parameter configuration:

$$\alpha = \frac{5}{8}, \beta = \frac{3}{5}, \mu = \frac{1}{2}, \theta = \frac{1}{8}, A = 5, \omega = 1.75$$
$$x^{eq} = 0.98, \tau^{eq} = .2066, x^{eff} = 2.62, \tau^{eff} = .059$$
$$\Gamma^{eq} = (x^{eq})^{\omega-1} A(\tau^{eq})^{\mu} = 2.2387$$
$$\Gamma^{eff} = (x^{eff})^{\omega-1} A(\tau^{eff})^{\mu} = 2.5060$$

Thus we have the interesting possibility that while contemporaneous utility is higher under an efficient solution compared with the contemporaneous equilibrium solution, if economic growth is more rapid under the equilibrium solution eventually those born in the future would be happier due to their higher human capital compared with those equivalent generation arriving from the efficient solution.[8]

4 Mortality

In this section we add mortality of young adults and we assume that human capital investment raises the likelihood that a child survives to the next generation. Second we modify the preferences to introduce a precautionary demand for fertility, as in Tamura (2006), Tamura and Simon (2013) and Tamura, Simon and Murphy (2014). These are inspired by the seminal work of Kalemli-Ozcan (2002, 2003).

4.1 Equilibrium Solution

Assume that parents care about their own consumption, expected surviving children, and the earnings of their surviving adult children. Assume preferences can be written as:

$$\alpha ln c_{it} + (1-\alpha) ln[x_{it}(1-\delta_{it})] - \frac{\delta_{it}}{2x_{it}(1-\delta_{it})} + \alpha\beta ln y_{it+1} \tag{32}$$

We assume as in Tamura (2006) that parents must educate all their children, and after the education investment has made, only a fraction $1-\delta_{it}$, of the children survive to adulthood. The term $\frac{\delta_{it}}{2x_{it}(1-\delta_{it})}$ represents a precautionary demand for children (see Kalemli-Ozcan, 2002). Intuitively, for a given mortality of young adults, increases in fertility reduce the disutility generated by the death of a family's child. The

[8]Recall that the efficient solution maximizes the utility of the representative parent alive today. The typical parent only cares about the future through fertility and the income of the typical child. Had a parent cared about the infinitely lived dynasty, then this result could be overturned.

budget constraint for the typical parent is given, as before, by (7):

$$c_{it} = w_{it} h_{it} \left[1 - x_{it}(\theta + \tau_{it})\right],$$

where w_{it} is given above by (9). We assume that human capital accumulation remains as in (8):

$$h_{t+1} = A\bar{\bar{h}}_t^{\rho} h_t^{1-\rho} \tau_t^{\mu}$$

Importantly, we assume that cumulative mortality of young adults is a declining function of the average human capital of their generation:[9]

$$\delta_{it} = \Delta exp(-\lambda_3 \bar{h}_{it+1}^{\lambda_4}), \tag{33}$$

where $\bar{h}_{it+1}$ is the average human capital of adult generation $t+1$ and λ_3 and λ_4 are positive parameters. Hence whether a child born of a t generation parent, and hence a $t+1$ generation adult, survives is a function of the average human capital of their generation.[10] Labelling terms that are not affected by parental choices as $\hat{U}$, the optimization problem for the parent can be written as:

$$\begin{aligned} V(h_{it}) &= \max_{\{h_{it+1}, x_{it}\}} \left\{ \alpha \ln c_{it} + (1-\alpha) \ln x_{it} - \frac{\delta_{it}}{2x_{it}(1-\delta_{it})} + \alpha\beta ln(y_{it+1}) \right\} \\ &= \max_{\{h_{it+1}, x_{it}\}} \left\{ \hat{U} + \alpha ln(1 - x_{it}[\theta + \tau_{it}]) + (1-\alpha) ln x_{it} - \frac{\delta_{it}}{2x_{it}(1-\delta_{it})} + \alpha\beta ln h_{it+1} \right\} \end{aligned} \tag{34}$$

The first order condition with respect to fertility produces the following:

$$\frac{\partial}{\partial x_{it}} = 0 \Leftrightarrow \frac{\alpha[\theta + \tau_{it}]}{1 - x_{it}[\theta + \tau_{it}]} = \frac{1-\alpha}{x_{it}} + \frac{\delta_{it}}{2x_{it}^2(1-\delta_{it})} \tag{35}$$

and the first order condition for human capital investment becomes

$$\frac{\partial}{\partial h_{it+1}} = 0 \Leftrightarrow \frac{\alpha x_{it} \tau_{it}}{1 - x_{it}[\theta + \tau_{it}]} = \alpha\beta\mu \tag{36}$$

Observe that the first order condition on optimal human capital investment in the equilibrium model is identical whether mortality is non-zero or not. This is precisely the case since we assumed that the entire effect of human capital investment on young adult mortality is external to the parent.

In order to analyze this model we can solve the model backwards in time. That is for a given value of h_{it+1} we can solve the model for (x_{it}, τ_{it}).[11] Taking the ratio of the two Euler equations and solving for τ_{it} as a function of x_{it} produces:

$$\tau_{it} = \frac{\alpha\beta\mu\theta}{1 - \alpha - \alpha\beta\mu + \frac{\delta(h_{it+1})}{2[1-\delta(h_{it+1})]x_{it}}} \tag{37}$$

[9] This is similar to what is assumed in Tamura (2006).

[10] Tamura (2006) argues that this is the case if modern sanitation and modern personal hygiene are best at reducing early mortality. He allows for international spillovers like development of antibiotics and vaccines, which are abstracted from here.

[11] This method was used in Tamura (2006).

Substituting this back into (36) produces the following quadratic equation in x_{it}:

$$
\begin{aligned}
ax_{it}^2 + bx_{it} + c &= 0 && (38)\\
a &= \theta && (39)\\
b &= \frac{\theta\delta(h_{it+1})}{2[1-\delta(h_{it+1})]} - 1 + \alpha + \alpha\beta\mu && (40)\\
c &= \frac{-\delta(h_{it+1})}{2[1-\delta(h_{it+1})]} && (41)
\end{aligned}
$$

So for a given value of h_{it+1}, we can solve for the fertility, x_{it}, and human capital investment rate, τ_{it}. From these we can then produce the value of parental human capital, h_{it} via:

$$h_{it} = \left\{ \frac{h_{it+1}}{A\bar{h}_t^{\rho}\tau_{it}^{\mu}} \right\}^{\frac{1}{1-\rho}} \tag{42}$$

We assume that $\bar{\bar{h}}_t = h_{it}$, so the law of motion becomes:

$$h_{it} = \frac{h_{it+1}}{A\tau_{it}^{\mu}} \tag{43}$$

Returning to the fertility equation (35), we can examine fertility in the region where mortality is essentially stationary.[12] Observe that the coefficient on the quadratic term is strictly positive, $\theta > 0$, and the constant term is strictly negative, $c < 0$. Thus there is only one positive solution for fertility, x_i. This is given by:

$$
\begin{aligned}
x_{eq} &= \frac{-b_{eq} + \sqrt{b_{eq}^2 - 4a_{eq}c_{eq}}}{2a_{eq}} && (44)\\
a_{eq} &= \theta && (45)\\
b_{eq} &= \frac{\theta\Delta}{2[1-\Delta]} - 1 + \alpha + \alpha\beta\mu && (46)\\
c_{eq} &= -\frac{\Delta}{2[1-\Delta]} && (47)
\end{aligned}
$$

4.2 Efficient Solution

Here we solve the planner's problem. In particular we also focus on the equal weight solution, and hence the representative parent. There are several regions that must be solved in order to characterize the efficient path. In the earliest days, human capital will be sufficiently low that no specialization occurs, and hence autarky is the method of production. Human capital accumulation may affect the mortality of the young adult, but not determine the scale of the production team. In the intermediate stage, human capital is sufficient to allow specialization, but not so large as to be bound by $\lambda_1 P_t$. The final stage in the efficient allocation is the region where the optimal production coalition is a constant proportion of the population, $N_t = \lambda_1 P_t$.

[12] For very low values of h_{it+1} mortality is given essentially by Δ.

Considering first the region with autarkic production, $N_t = 1$, the objective of the social planner is:

$$\max_{x_{it},h_{it+1}} \left\{ \alpha ln(1 - x_{it}[\theta + \tau_{it}]) + (1-\alpha)[ln(1-\delta(h_{it+1})) + lnx_{it}] - \frac{\delta(h_{it+1})}{2x_{it}[1-\delta(h_{it+1})]} + \alpha\beta ln h_{it+1} \right\} \quad (48)$$

The first order conditions for fertility and human capital investment can be written as:

$$\frac{\alpha(\theta+\tau_{it})}{1-x_{it}[\theta+\tau_{it}]} = \frac{1-\alpha}{x_{it}} + \frac{\delta(h_{it+1})}{2x_{it}^2[1-\delta(h_{it+1})]} \quad (49)$$

$$\frac{\alpha x_{it}}{1-x_{it}[\theta+\tau_{it}]} = \frac{\alpha\beta\mu}{\tau_{it}} - \frac{1}{1-\delta(h_{it+1})}\frac{\partial\delta(h_{it+1})}{\partial\tau_{it}} \left\{1-\alpha+\frac{1}{2x_{it}[1-\delta(h_{it+1})]}\right\} \quad (50)$$

$$\frac{\partial\delta(h_{it+1})}{\partial\tau_{it}} = -\frac{\delta(h_{it+1})\lambda_3\lambda_4\mu h_{it+1}^{\lambda_4}}{\tau_{it}} \quad (51)$$

Taking the ratio of the Euler equations and solving for τ_{it} as a function of fertility, x_{it}, and human capital of the child, h_{it+1} produces:

$$\tau_{it} = \frac{\theta \mathrm{Z}_t}{1-\alpha+\alpha\beta(\omega-1)+\frac{\delta(h_{it+1})}{2x_{it}[1-\delta(h_{it+1})]} - \mathrm{Z}_t} \quad (52)$$

$$\mathrm{Z}_t = \alpha\beta\mu + \frac{\delta(h_{it+1})\lambda_3\lambda_4\mu h_{it+1}^{\lambda_4}}{1-\delta(h_{it+1})}\left\{1-\alpha+\frac{1}{2x_{it}[1-\delta(h_{it+1})]}\right\}. \quad (53)$$

Again following the solution strategy from the equilibrium problem we end up with a quadratic equation in x_{it}:

$$ax_{it}^2 + bx_{it} + c = 0 \quad (54)$$

$$a_{eff}^{N=1} = \theta \quad (55)$$

$$b_{eff}^{N=1} = \frac{\delta(h_{it+1})}{1-\delta(h_{it+1})}\left\{\frac{\theta}{2} + \lambda_3\lambda_4\mu h_{it+1}^{\lambda_4}[1-\alpha]\right\} - 1 + \alpha + \alpha\beta\mu \quad (56)$$

$$c_{eff}^{N=1} = \frac{-\delta(h_{it+1})}{2[1-\delta(h_{it+1})]}\left\{1 - \frac{\lambda_3\lambda_4\mu h_{it+1}^{\lambda_4}}{1-\delta(h_{it+1})}\right\} \quad (57)$$

When human capital is sufficiently small, $\delta_t \approx \Delta$, and these coefficients become:

$$ax_{it}^2 + bx_{it} + c = 0 \quad (58)$$

$$a_{eff}^{N=1} = \theta \quad (59)$$

$$b_{eff}^{N=1} = \frac{\Delta}{1-\Delta}\left\{\frac{\theta}{2} + \lambda_3\lambda_4\mu h_{it+1}^{\lambda_4}[1-\alpha]\right\} - 1 + \alpha + \alpha\beta\mu \quad (60)$$

$$c_{eff}^{N=1} = \frac{-\Delta}{2[1-\Delta]}\left\{1 - \frac{\lambda_3\lambda_4\mu h_{it+1}^{\lambda_4}}{1-\Delta}\right\} \quad (61)$$

Comparing these with the equilibrium solution we find:

$$a_{eq} = a_{eff}^{N=1} \tag{62}$$
$$b_{eq} < b_{eff}^{N=1} \tag{63}$$
$$c_{eq} < c_{eff}^{N=1} \tag{64}$$

Thus we have that $x_{eff}^{N=1} < x_{eq}$, and we can also show numerically that $\tau_{eft}^{N=1} > \tau_{eq}$. Now consider the region where the scope of specialization is given by $N_t = \text{L}h_t^{\lambda_2}$. Ignoring constants, the objective of the social planner is:

$$\max_{x_{it},h_{it+1}} \left\{ \alpha ln(1 - x_{it}[\theta + \tau_{it}]) + (1-\alpha)[ln(1-\delta(h_{it+1})) + lnx_{it}] - \frac{\delta(h_{it+1})}{2x_{it}[1-\delta(h_{it+1})]} + \alpha\beta[1+\lambda_2(\omega-1)]lnh_{it+1} \right\} \tag{65}$$

The first order conditions for fertility and human capital investment can be written as:

$$\begin{aligned}
\frac{\alpha(\theta+\tau_{it})}{1-x_{it}[\theta+\tau_{it}]} &= \frac{1-\alpha}{x_{it}} + \frac{\delta(h_{it+1})}{2x_{it}^2[1-\delta(h_{it+1})]} && (66)\\
\frac{\alpha x_{it}}{1-x_{it}[\theta+\tau_{it}]} &= \frac{\alpha\beta\mu[1+\lambda_2(\omega-1)]}{\tau_{it}} - \frac{1}{1-\delta(h_{it+1})}\frac{\partial\delta(h_{it+1})}{\partial\tau_{it}} \left\{1-\alpha+\frac{1}{2x_{it}[1-\delta(h_{it+1})]}\right\} && (67)\\
\frac{\partial\delta(h_{it+1})}{\partial\tau_{it}} &= -\frac{\delta(h_{it+1})\lambda_3\lambda_4\mu h_{it+1}^{\lambda_4}}{\tau_{it}} && (68)
\end{aligned}$$

Taking the ratio of the Euler equations and solving for τ_{it} as a function of fertility, x_{it}, and human capital of the child, h_{it+1} produces:

$$\begin{aligned}
\tau_{it} &= \frac{\theta \text{Y}_t}{1-\alpha+\alpha\beta(\omega-1)+\frac{\delta(h_{it+1})}{2x_{it}[1-\delta(h_{it+1})]} - \text{Y}_t} && (69)\\
\text{Y}_t &= \alpha\beta\mu[1+\lambda_2(\omega-1)] + \frac{\delta(h_{it+1})\lambda_3\lambda_4\mu h_{it+1}^{\lambda_4}}{1-\delta(h_{it+1})}\left\{1-\alpha+\frac{1}{2x_{it}[1-\delta(h_{it+1})]}\right\}. && (70)
\end{aligned}$$

Again following the solution strategy from the equilibrium problem we end up with a quadratic equation in x_{it}:

$$\begin{aligned}
ax_{it}^2 &+ bx_{it} + c = 0 && (71)\\
a &= \theta && (72)\\
b &= \frac{\delta(h_{it+1})}{1-\delta(h_{it+1})}\left\{\frac{\theta}{2} + \lambda_3\lambda_4\mu h_{it+1}^{\lambda_4}[1-\alpha]\right\} - 1 + \alpha + \alpha\beta\mu[1+\lambda_4(\omega-1)] && (73)\\
c &= \frac{-\delta(h_{it+1})}{2[1-\delta(h_{it+1})]}\left\{1 - \frac{\lambda_3\lambda_4\mu h_{it+1}^{\lambda_4}}{1-\delta(h_{it+1})}\right\} && (74)
\end{aligned}$$

Finally in the region where the scope of specialization is given by $N_t = \lambda_1 P_t$, the objective of the social planner is:

$$\max_{x_{it},h_{it+1}} \left\{ \alpha ln(1 - x_{it}[\theta + \tau_{it}]) + (1-\alpha+\alpha\beta(\omega-1))[ln(1-\delta(h_{it+1})) + lnx_{it}] - \frac{\delta(h_{it+1})}{2x_{it}(1-\delta(h_{it+1}))} + \alpha\beta \ln h_{it+1} \right\} \tag{75}$$

The coefficient on log expected surviving children contains two parts, the direct utility from surviving children, $1-\alpha$, and the gain from agglomeration returns to specialization, $\alpha\beta(\omega-1)$. The first order conditions for fertility and human capital investment can be written as:

$$\frac{\alpha(\theta+\tau_{it})}{1-x_{it}[\theta+\tau_{it}]} = \frac{1-\alpha+\alpha\beta(\omega-1)}{x_{it}} + \frac{\delta(h_{it+1})}{2x_{it}^2[1-\delta(h_{it+1})]} \tag{76}$$

$$\frac{\alpha x_{it}}{1-x_{it}[\theta+\tau_{it}]} = \frac{\alpha\beta\mu}{\tau_{it}} - \frac{1}{1-\delta(h_{it+1})}\frac{\partial\delta(h_{it+1})}{\partial\tau_{it}}\left\{1-\alpha+\alpha\beta(\omega-1)+\frac{1}{2x_{it}[1-\delta(h_{it+1})]}\right\} \tag{77}$$

$$\frac{\partial\delta(h_{it+1})}{\partial\tau_{it}} = -\frac{\delta(h_{it+1})\lambda_3\lambda_4\mu h_{it+1}^{\lambda_4}}{\tau_{it}} \tag{78}$$

As before we can take the ratio of the Euler equations and solve for τ_{it} as a function of fertility, x_{it}, and human capital of the child, h_{it+1}.[13] Thus we have:

$$\tau_{it} = \frac{\theta\Omega_t}{1-\alpha+\alpha\beta(\omega-1)+\frac{\delta(h_{it+1})}{2x_{it}[1-\delta(h_{it+1})]}-\Omega_t} \tag{79}$$

$$\Omega_t = \alpha\beta\mu + \frac{\delta(h_{it+1})\lambda_3\lambda_4\mu h_{it+1}^{\lambda_4}}{1-\delta(h_{it+1})}\left\{1-\alpha+\alpha\beta(\omega-1)+\frac{1}{2x_{it}[1-\delta(h_{it+1})]}\right\} \tag{80}$$

Again following the solution strategy from the equilibrium problem we end up with a quadratic equation in x_{it}:

$$ax_{it}^2 + bx_{it}+c=0 \tag{81}$$

$$a = \theta\left\{1+\alpha\beta(\omega-1)\right\} \tag{82}$$

$$b = \frac{\delta(h_{it+1})}{1-\delta(h_{it+1})}\left\{\frac{\theta}{2}+\lambda_3\lambda_4\mu h_{it+1}^{\lambda_4}[1-\alpha+\alpha\beta(\omega-1)]\right\}-1+\alpha+\alpha\beta\mu-\alpha\beta(\omega-1) \tag{83}$$

$$c = \frac{-\delta(h_{it+1})}{2[1-\delta(h_{it+1})]}\left\{1-\frac{\lambda_3\lambda_4\mu h_{it+1}^{\lambda_4}}{1-\delta(h_{it+1})}\right\} \tag{84}$$

While the three branches of production specialization seem daunting, our numerical method of solution is straightforward. We solve the model backward in time. We start with a human capital value that places the economy at the highest level of specialization, that is in period U, $N_U=\lambda_1 P_U$. The first order conditions going forward, are assumed to be on the balanced growth path of the efficient solution, $\delta_t=0$. We solve for period $U-1$ values of τ_{U-1} and x_{U-1}. The law of motion going backward for P_{U-1} and h_{U-1} is applied. As long as $N_{U-1}=\lambda_1 P_{U-1} < \mathrm{L}h_{U-1}^{\lambda_2}$, we remain on the same branch of Euler equations. Once we reach a solution for τ_s and x_s, such that $N_s=\mathrm{L}h_s^{\lambda_2}<\lambda_1 P_s$, period $s-1$ will use the Euler equations in the branch given by (59)-(61). Finally when we reach period w, $N_w=1>Lh_w^{\lambda_2}$, the Euler equations are given by (49)-(51).

5 Numerical Solution

In this section we present a numerical solution to compare and contrast the equilibrium time series with the efficient one. To analyze this issue we present the results of two numerical solutions with time varying

[13] Recall that we are solving the model backwards in time, so h_{it+1} is known.

Table 1: Parameter Values for Numerical Solution

parameter		parameter
Efficient Grows Faster		
A_t		
$=2.479720000$	if $h_t < 15$	$\alpha = 0.45$
$=3.318780657$	if $h_t \geq 15$	
ω_t		$\beta = 1.75$
$=1.00000$	if $h_t < 1$	
$=1.00100$	if $1 \leq h_t < 2$	
$=1.00500$	if $2 \leq h_t < 32$	
$=1.25000$	if $32 \leq h_t < 62.5$	$\mu = 0.471$
$=1.76500$	if $h_t \geq 62.5$	
		$\theta = 0.16886792$
$\delta(h_t) = min\left\{.48125, .65exp(-1.0 * 10^{-6} h_t^{2.53})\right\}$		$\lambda_1 = .001$
$L = .33$	$\lambda_2 = 8.75$	
Equilibrium Grows Faster		
A_t		
$=2.479720000$	if $h_t < 15$	$\alpha = 0.45$
$=3.318780657$	if $h_t \geq 15$	
ω_t		$\beta = 1.75$
$=1.00000$	if $h_t < 1$	
$=1.00100$	if $1 \leq h_t < 2$	
$=1.00500$	if $2 \leq h_t < 32$	
$=1.25000$	if $32 \leq h_t < 62.5$	$\mu = 0.471$
$=1.60000$	if $h_t \geq 62.5$	
		$\theta = 0.16886792$
$\delta(h_t) = min\left\{.48125, .65exp(-1.0 * 10^{-6} h_t^{2.53})\right\}$		$\lambda_1 = .001$
$L = .33$	$\lambda_2 = 8.75$	

ω and time varying A. In the first case, in the upper panel of Table 1, we assume that the balanced growth rate in the efficient solution is greater than the balanced growth rate in the equilibrium solution. In the second case, in the lower panel of Table **??**, we change only the final value of the gains from specialization, ω, to produce a balanced growth rate in the efficient solution that is smaller than the balanced growth rate in the equilibrium solution. We focus on these cases in order to produce plausible initial income values, as well as plausible rising gains from specialization during the industrial transformation.

Tables 2 and 3 contain the comparisons of balanced growth path fertilities, human capital investment rates, and income and population growth rates for these ω values. The tables display the classical period, when production remains in the family, $N = 1$, the transition dynamics during which production specialization involves market trade, but is determined by the level of human capital in the economy, $N = Lh^{\lambda_2}$, and the period inclusive of the demographic transition, where production specialization becomes a fixed proportion of the total population, $N = \lambda_1 P$. We also present the time series of these variables, in the following figures, in order to provide a feel for the transition dynamics that occur in these models. Equilibrium fertility falls from 1.98, a total fertility rate of 3.96, to 1.06, a total fertility rate of 2.12 for both long run cases, $\omega = 1.765$ and $\omega = 1.60$. In contrast the efficient fertility rate increases from 2.01, a total fertility

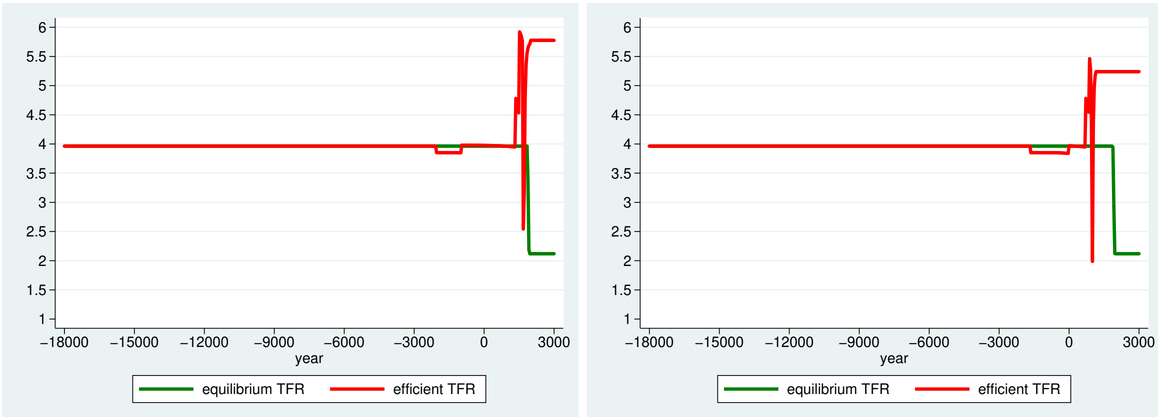
Table 2: Equilibrium and Efficient Solutions: Efficient Grows Faster

	equilibrium	efficient
Classical autarky, $N = 1$, years	18000 BC to 2000 BC	18000 BC to 2040 BC
Fertility, x	1.9820206	1.981883
Annualized population growth rate	0.06947%	0.06928%
Classical Investment, τ	0.1516836	0.1517062
Annualized income growth rate	0.04436%	0.04454%
Transition dynamics, $N_t = Lh_t^{\lambda_2}$ or $\delta_t > 0.43$	1960 BC to 1880 AD	2000 BC to 1720 AD
Fertility, x	1.979059	2.012232
Annualized population growth rate	0.07174%	0.11027%
Investment, τ	0.1520828	0.1544187
Annualized income growth rate	0.041365%	0.41658%
Balanced growth, $N_t = \lambda_1 P_t$	post 1880 AD	post 1720 AD
Fertility, x	1.060228	2.881947
Annualized population growth rate	0.14595%	2.64517%
Balanced Growth Investment, τ	0.349911	0.0807417
Annualized Income Growth Rate	1.87388%	2.0575%

rate of 4.02, to 2.9, a total fertility rate of 5.8 in the terminal value $\omega = 1.765$ case and from 3.96 to 5.23 in the terminal value $\omega = 1.60$ case. The mirror image of this is the change in the human capital investment rate. For the equilibrium solution, the investment rate increases from 0.15 in the Classical period to 0.35 along the balanced growth path. In contrast for the efficient solution there is a a decline in the investment rate from 0.15 in the Classical period to 0.08 along the balanced growth path.

Figure 1 contains the time series for fertility for both $\omega = 1.765$ and $\omega = 1.60$. There are demographic transitions in both equilibrium examples. In contrast the efficient solutions have a brief collapse in fertility, a total fertility rate of 2.5 in the case of the long run specialization gain, $\omega = 1.765$, and an even more severe decline in fertility, a total fertility rate of 2 in the case of long run specialization gain, $\omega = 1.60$. In both efficient cases, however, long total fertility rates exceed the classical period values. In the $\omega = 1.765$ case, a total fertility rate of 5.8, and in the $\omega = 1.60$ case, a total fertility rate of 5.2.

Figure 2 contains the time series for human capital investment, τ. We see something like the inverse of

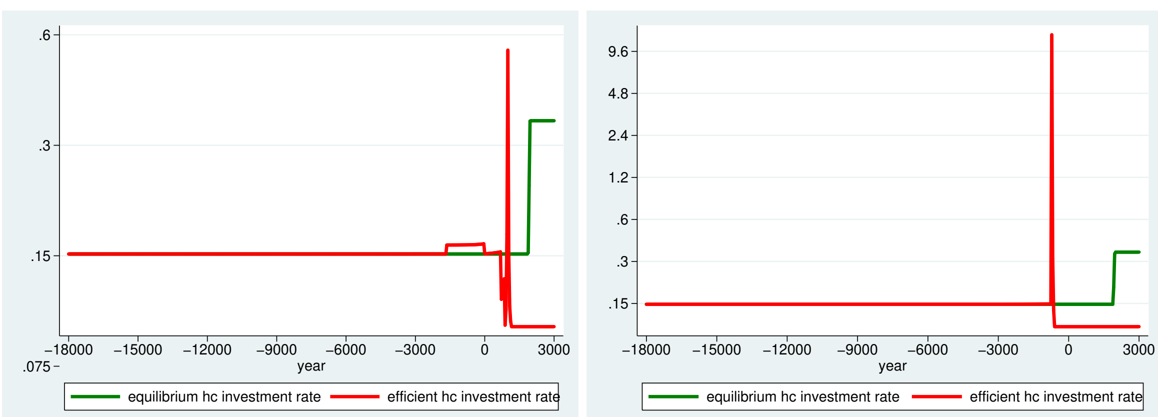


(a) Efficient Faster than Equilibrium

(b) Equilibrium Faster than Efficient

Figure 1: Equilibrium and Efficient Fertility

the fertility graph. The demographic transition has accelerated human capital investment in the efficient solution before falling to a lower balanced growth path value. Notice that the higher the returns to specialization in the long run, the lower the rate of human capital investment. However the social planner does accelerate human capital investment during the period in which mortality is elastically responding to human capital. The growth rate of human capital mirrors the time series of human capital investment, as



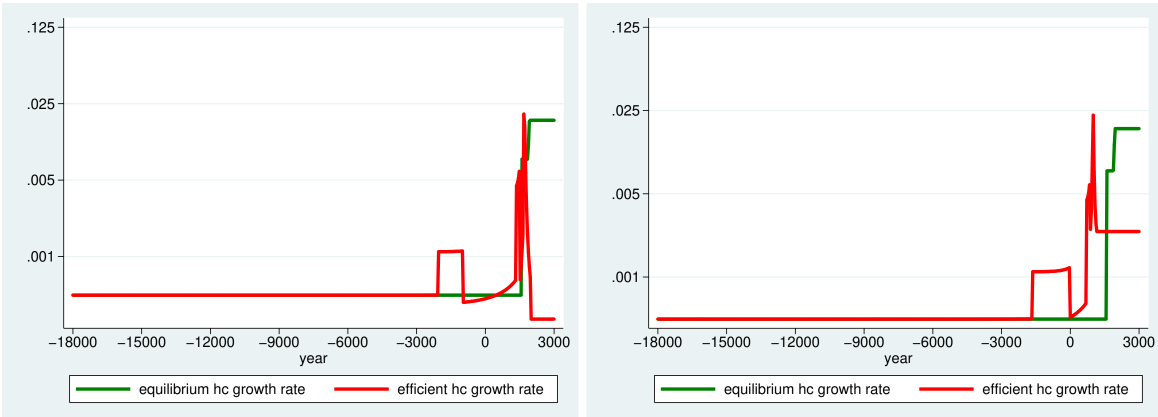
(a) Efficient Faster than Equilibrium

(b) Equilibrium Faster than Efficient

Figure 2: Equilibrium and Efficient Human Capital Investment Rate

shown in Figure 3. During the demographic transition, the efficient growth rate of human capital exceeds the long run efficient growth rate of human capital. This occurs in both cases. In fact the transitions have higher growth rates of human capital even in comparison to the long run equilibrium growth rate of human capital.

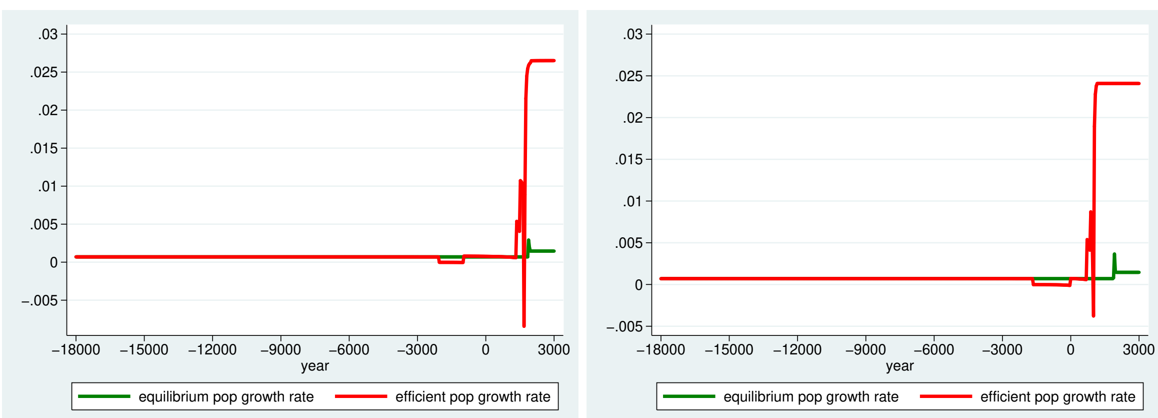
Figure 4 contains the time series growth rate of population, that is the fertility rate multiplied by the survival rate. The Demographic Transition is evident in all cases. However only the equilibrium solutions



(a) Efficient Faster than Equilibrium (b) Equilibrium Faster than Efficient

Figure 3: Equilibrium and Efficient Human Capital Growth Rate

have the traditional shape of falling population growth rate after the mortality revolution. In the efficient solutions, the population growth rate accelerates to its long run balanced path value, which exceeds its value during high mortality.[14] Notice that during the transition, the efficient solution has rapid population decline for a brief period, as the investment rate in human capital is increased greatly.



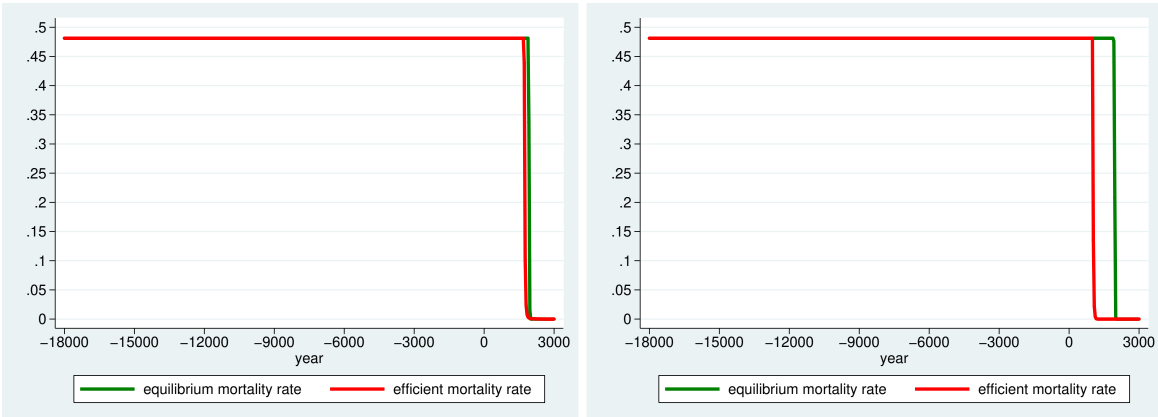
(a) Efficient Faster than Equilibrium (b) Equilibrium Faster than Efficient

Figure 4: Equilibrium and Efficient Population Growth Rate

Figure 5 contains the mortality rate for these solutions. Here the standard demographic transition via the mortality revolution is evident for both the efficient solutions and the equilibrium solutions.

We present the growth rates of output per worker in Figure 6. Both cases illustrate both the classical period of nearly zero growth, an industrial transformation with growth rates that can actually exceed the long run balanced growth rate, and then the balanced growth rate. Finally in Figure 7 we present the growth rate of specialization, that is $g_N = \frac{N_{t+1}}{N_t}$. During the Classical period, there is no growth in

[14]This result is similar to that contained in Tamura (2002).

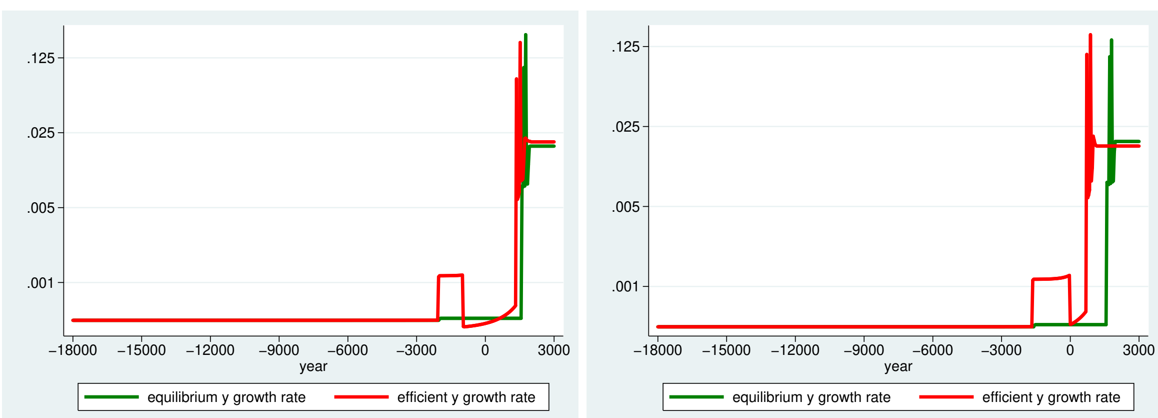


(a) Efficient Faster than Equilibrium

(b) Equilibrium Faster than Efficient

Figure 5: Equilibrium and Efficient Mortality Rate

specialization as $N_{t+1} = N_t = 1$. During the period in which specialization is determined the human capital in the economy, there is a more rapid growth rate in specialization, which helps to produce the industrial revolution. Finally along the long run balanced growth path, the growth in specialization equals the population growth rate.



(a) Efficient Faster than Equilibrium

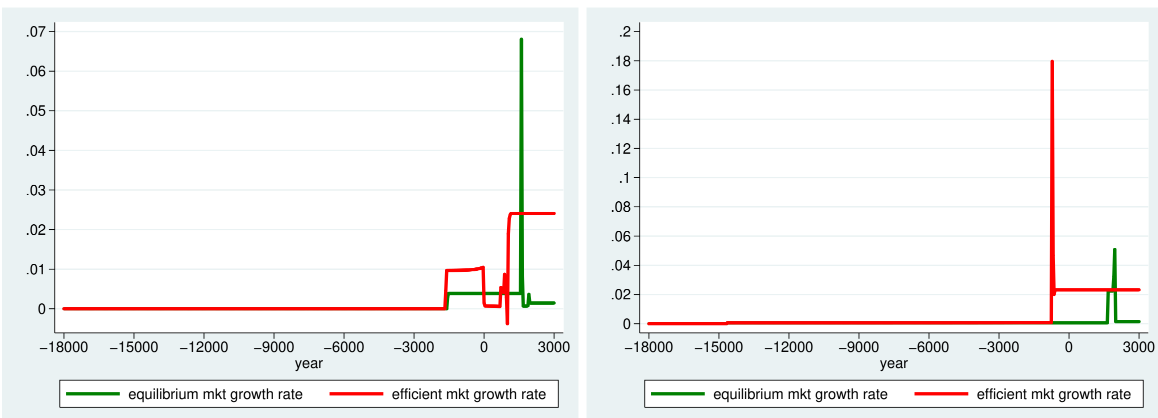
(b) Equilibrium Faster than Efficient

Figure 6: Equilibrium and Efficient Income Growth Rate

5.1 Long Run Growth Differences, and Timing of the Demographic Transition to Balanced Growth

The two different cases produce balanced growth rate differences. In the equilibrium economy, the transition to the balanced growth path is complete after 1880. From that time onward, the economy behaves as the balanced growth path. For the high specialization gains case, $\omega = 1.765$, this produces a long run growth rate of 1.87% per year. In the low specialization gains case, $\omega = 1.60$, this produces a long run growth rate of 1.85% per year. Thus there is little difference between these two equilibrium cases. However there is a much more dramatic change in the efficient solutions. For the high specialization gains case $\omega = 1.765$, the long run efficient growth rate is 2.05% per year. This contrasts with the efficient growth rate of 1.69% per year in the low specialization gains case, $\omega = 1.60$.

Our solutions also produce differential dates of the onset of the demographic transition - industrial transformation of the economy. Where the efficient solutions differ tremendously with the equilibrium solutions is the timing of the transition out of the Classical economy towards the balanced growth economy. Whereas the equilibrium solutions have increasing specialization starting after 2000 BC, and completing the transformation in 1880 AD, the efficient solutions have increasing specialization starting in 2040 BC, and finishing in 1720 AD (1600 BC to 1000 AD for the terminal value $\omega = 1.60$ case). For both examples, the efficient solution produces a balanced growth path between 160 years and 880 years earlier than the equilibrium solutions.



(a) Efficient Faster than Equilibrium

(b) Equilibrium Faster than Efficient

Figure 7: Equilibrium and Efficient Market Growth Rate

6 Conclusions

This paper develops a general equilibrium model that characterizes the gap between optimal and equilibrium fertility and investment in human capital in a context where there exist increasing returns to population driven by specialization. In the benchmark model we show that a benevolent social planner would choose a higher fertility rate than the competitive equilibrium but a lower investment in human capital. This is the case because the planner internalizes the positive effect that a larger population has on specialization and hence on aggregate productivity. We next introduce young mortality in the model and assume that it is a decreasing function of the average human capital in the economy and the presence of a precautionary demand for children in the parent's utility function. Since agents do not internalize the effect of their human capital decisions on mortality rates, they accumulate human capital too slowly. A benevolent social planner would choose to accumulate human capital as fast as possible to completely eliminate mortality

Table 3: Equilibrium and Efficient Solutions: Equilibrium Grows Faster*

	equilibrium	efficient
Classical autarky, $N = 1$, years	18000 BC to 1560 BC	18000 BC to 1640 BC
Fertility, x	1.982026	1.981886
Annualized population growth rate	0.06947%	0.06929%
Classical Investment, τ	0.1516836	0.1517056
Annualized income growth rate	0.04436%	0.04453%
Transition dynamics, $N_t = Lh_t^{\lambda_2}$ or $\delta > .43$	1520 BC to 1920 AD	1600 BC to 1000 AD
Fertility, x	1.975909	1.977512
Annualized population growth rate	0.07298%	0.06895%
Investment, τ	0.1526333	0.160277
Annualized income growth rate	0.39505%	0.57678%
Balanced growth, $N_t = \lambda_1 P_t$	post 1920 AD	post 1000 AD
Fertility, x	1.060011	2.616217
Annualized population growth rate	0.14569%	2.40358%
Balanced Growth Investment, τ	0.3499955	0.096618
Annualized Income Growth Rate	1.84993%	1.68778%

and after that it would choose a fertility rate in the balanced growth path that is above the competitive one. This implies that in the efficient problem both the industrial revolution i.e. the sharp increase in income per capita and the demographic transition would take place much earlier than in the competitive solution.

One could argue that the main results in the paper would change if we introduce some type of congestion cost in the model. For instance, one could introduce land or some other fixed input that would potentially limit the positive effects of population growth. However, as shown in Tamura (2006), this would not affect the main conclusion of the paper. In that model, the long run growth rate of TFP is constant in spite of the fact of the presence of land as a fixed input. In our model, a larger population translates into more specialization and hence faster TFP growth, so the same logic as in Tamura (2006) would apply if we introduced congestion costs.

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