A condition for determinacy of optimal strategies in zero-sum convex polynomial games

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Abstract

The main purpose of this paper is to prove that if there is a non-expansive map relating the sets of optimal strategies for a convex polynomial game, then there exists only one optimal strategy for solving that game. We introduce the remark that those sets are semi-algebraic. This is a natural and important property deduced from the polynomial payments. This property allows us to construct the space of strategies with an infinite number of semi-algebraic curves. We semi-algebraically decompose the set of strategies and relate them with non-expansive maps. By proving the existence of an unique fixed point in these maps, we state that the solution of zero-sum convex polynomial games is determined in the space of strategies.

Key words: determinacy, polynomial game, semi-algebraic set and function

JEL classification: C63, C73, L13

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1 Introduction

Two-person zero-sum games are widely studied. Those games represent strategical situations where the income of an agent equals the looses of the other. It is commonly used the linear programming in solving those games. The most remarkable result, Von-Neumann's minimax theorem, guarantees the existence of a particular solution, but it does not solve the determinacy or computation of that solution. One novel way to treat those problems is by using real algebraic algorithms.

There have been important advances in the linkages of real algebraic geometry and game theory. Blume and Zame (1994) use them for analysing the relationship between sequential and perfect equilibria, and Bolte, Gaubert and Vigeral (2013) in the study of stochastic games. They exploit algebraic properties of the fundamental sets to get information of the equilibrium sets. Empirically, it allows to compute solutions; theoretically, it allows to determine a particular solution.

In this note we are interested in studying the determinacy of zero-sum polynomial games by using real algebraic geometry. These games are those whose payments are linear or non-linear polynomials. We restrict our analysis to real polynomials to be able to use tame topology and real algebraic algorithms. This is a natural way to approach the solution of those games, because semi-algebraic sets are defined by polynomial inequalities and algebraic sets and functions are also semi-algebraic.

This question appears directly once the problem of existence is solved. It is theoretically relevant since we would be searching for an strategy that constitute the unique solution for the game. It is also a relevant empirical question. With multiplicity it is hard and expensive to find a way for choosing the best solution. Blume and Zame (1992) and Kubler and Schmedders (2010) show the uses of real algebraic geometry in solving that problem, by studing local determinacy of general equilibrium.

We start up by assuming that the sets of optimal strategies are convex and compact. They are also semi-algebraic, because they are defined by polynomial inequalities. We use the Karlin's (1992,V.II, P.52) algorithm to find out the fixed points. The semi-algebraicness of those sets allows us to decompose them in semi-algebraic subsets. We relate them with non-expansive maps whose unique fixed point is the unique solution to the game. It determines the solution for a particular strategy.

This note is as follows. After this, we state some definitions on two-person zero-sum convex polynomial games. Then, we construct the space of optimal strategies with semi-algebraic geodesic curves in an example. The result of global determinacy follows, and finally the references.

2 Strategies

I and II are individuals. Suppose $\Lambda \subseteq \mathbb{R}^n$ is the space of strategies being compact and convex. I selects $\boldsymbol{x} = (x_1, ..., x_n) \in \Lambda$ and II selects $\boldsymbol{y} = (y_1, ..., y_n) \in \Lambda$. Assume $\boldsymbol{y} = \boldsymbol{x}$ by $y_1 = x_1, y_2 = x_2, ..., y_n = x_n$.

Let $\|p_{ij}\|_{n \cdot n}$ be a $(n \cdot n)$-matrix with positive coordinates. Let $P(\boldsymbol{x}, \boldsymbol{y})$ be the pay-off to individual I. It is a linear polynomial function of $(\boldsymbol{x}, \boldsymbol{y})$

$$P(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} x_i y_j$$

Let $x_i(\boldsymbol{\alpha}), y_j(\boldsymbol{\beta})$ be semi-algebraic continuous functions in Λ. Let us suppose $0 \leq x_i, y_j \leq 1$ and $x_i = \int_0^1 x_i(\boldsymbol{\alpha}) dz_i(\boldsymbol{\alpha})$ and $y_i = \int_0^1 y_j(\boldsymbol{\beta}) dz_j(\boldsymbol{\beta})$ with $\sum_{i,j} z_{i,j} = 1$. Then,

$$P(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} x_i(\boldsymbol{\alpha}) y_j(\boldsymbol{\beta})$$

Definition 1 *A zero-sum convex polynomial game is a tuple* $\{\Lambda, P(\boldsymbol{x}, \boldsymbol{y})\}$ *with* $\boldsymbol{x}, \boldsymbol{y} \in \Lambda$.

Definition 2 *A vector* $(\boldsymbol{x}^0, \boldsymbol{y}^0) \in \Lambda$ *is an optimal strategy for* $\{\Lambda, P(\boldsymbol{x}, \boldsymbol{y})\}$ *if it satisfies the following condition:*

$$max_{\boldsymbol{x} \in \Lambda} P(\boldsymbol{x}, \boldsymbol{y}^0) = min_{\boldsymbol{y} \in \Lambda} P(\boldsymbol{x}^0, \boldsymbol{y}) = P(\boldsymbol{x}^0, \boldsymbol{y}^0)$$

Let $F : \Lambda^2 \rightrightarrows \Lambda^2$ be an upper hemi-continuous correspondence with $(\boldsymbol{x}, \boldsymbol{y}) \in \Lambda^2$ in domain, and $F(\boldsymbol{x}, \boldsymbol{y}) = (\Lambda(\boldsymbol{x}), \Lambda(\boldsymbol{y})) \in \Lambda^2$ in co-domain. An optimal strategy is a fixed point of $F(\boldsymbol{x}, \boldsymbol{y})$.

Definition 3 *The sets of optimal strategies* $\Lambda(\boldsymbol{x}^0), \Lambda(\boldsymbol{y}^0) \subseteq \Lambda$ *are defined by the following conditions*

$$\Lambda(\boldsymbol{x}^0) = \{\boldsymbol{y} \in \Lambda : P(\boldsymbol{x}^0, \boldsymbol{y}) \leq P(\boldsymbol{x}^0, \boldsymbol{y}') \forall \boldsymbol{y} \neq \boldsymbol{y}' \in \Lambda\}$$

$$\Lambda(\boldsymbol{y}^0) = \{\boldsymbol{x} \in \Lambda : P(\boldsymbol{x}, \boldsymbol{y}^0) \geq P(\boldsymbol{x}', \boldsymbol{y}^0) \forall \boldsymbol{x} \neq \boldsymbol{x}' \in \Lambda\}$$

The sets of optimal strategies are clearly compact and convex. Additionally, they are semi-algebraic[1]. To see that, just notice they are written by polynomial inequalities with $P(\boldsymbol{x}, \boldsymbol{y})$.

[1] A set is semi-algebraic of finite dimension if it is possible to write it as a combination of polynomial equations and real inequalities. Any algebraic set is also semi-algebraic. A function is semi-algebraic if its graph is semi-algebraic.

3 Example

Let us consider the classical and easy example of ***matching pennies***. It is a two-person zero-sum polynomial game. Table 1 shows the pay-off's of the individual I in the two strategies: H and T.

Table 1:
I's payoff

		II: H	II: T
I	H	1	−1
	T	−1	1

Suppose player I chooses H with probability x and T with probability $1-x$. Also, player II chooses H with probability y and T with probability $1-y$. Now, the space of the strategies is convex.

By using Von-Newmann's expected utility we write the I's expected pay-off as $P(\boldsymbol{x},\boldsymbol{y}) = 4(x-.5)(y-.5)$. It is a non-linear polynomial function. We can approximate its surface by semi-algebraic curves.

3.1 Asymmetric case

Define the parametric regular curve $r(u,v)$ for the surface $P(\boldsymbol{x},\boldsymbol{y})$. This curve is $r(u,v) = [u, v, 4uv - 2u - 2v + 1]$. We assume $x \neq y$ which is equivalent to $u \neq v$.

The Christoffel symbols of the first fundamental form are

$$\Gamma^1_{11} = \Gamma^2_{11} = \Gamma^1_{22} = \Gamma^2_{22} = 0$$
$$\Gamma^1_{12} = \frac{8(4v-2)}{2\left[1+(4u-2)^2+(4v-2)^2\right]}$$
$$\Gamma^2_{12} = \frac{8(4u-2)}{2\left[1+(4u-2)^2+(4v-2)^2\right]}$$

Let s be the line element. The geodesic curves are:

$$\frac{d^2u}{ds^2} = -\frac{8(4v-2)}{1+(4u-2)^2+(4v-2)^2}\frac{du}{ds}\frac{dv}{ds}$$
$$\frac{d^2v}{ds^2} = -\frac{8(4u-2)}{1+(4u-2)^2+(4v-2)^2}\frac{dv}{ds}\frac{du}{ds}$$

These functions are clearly continuous semi-algebraic. There is an approximation to this curves in the figure 1.

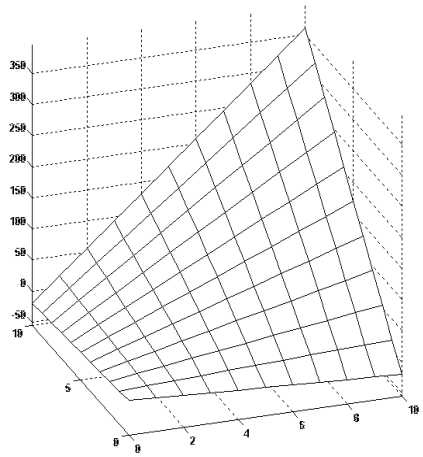

Figure 1: Utility surface

3.2 Symmetric case

In this case $x = y = t$ or equivalently $u = v = t$. The parametric curve is $r(t) = [t, t, 4t^2 - 4t + 1]$. The Christoffel symbols of the first fundamental form are:

$$\Gamma^1_{11} = \Gamma^2_{11} = \Gamma^1_{22} = \Gamma^2_{22} = 0$$
$$\Gamma^1_{12} = \Gamma^2_{12} = \frac{8(4t-2)}{2\left[1 + 2(4t-2)^2\right]}$$

The geodesic curves are:

$$\frac{d^2t}{ds^2} = -\frac{8(4t-2)}{1 + 2(4t-2)^2}\left(\frac{dt}{ds}\right)^2$$

The equation of the line element is $ds = \sqrt{2 + 4(4t-2)^2}dt$. With this, we are able to compute the distance between any two points. In $[0, 1]$ the distance is the solution of a semi-algebraic equation:

$$s = \int_0^1 \frac{ds}{dt}$$
$$= \frac{log(2\sqrt{2} + 3)}{4} + \frac{3\sqrt{2}}{2} \approx 2.5$$

The unique optimal strategy is $x = 0.5 = 1 - x = y = 0.5 = 1 - y$. In this symmetrical solution we have $P(\boldsymbol{x}, \boldsymbol{y}) = 0$ $\square$

4 Determinacy

We are going to analyse the determinacy of optimal strategies in zero-sum polynomial games. Let $\varphi, \psi : \Lambda \to \Lambda$ be continuous maps. Let us suppose they are non-expansive, that is, $\|\varphi(\boldsymbol{x}^0) - \varphi(\boldsymbol{x}^1)\| \leq \|\boldsymbol{x}^0 - \boldsymbol{x}^1\|$ for every $\boldsymbol{x}^0, \boldsymbol{x}^1 \in \Lambda$ and $\|\psi(\boldsymbol{y}^0) - \psi(\boldsymbol{y}^1)\| \leq \|\boldsymbol{y}^0 - \boldsymbol{y}^1\|$ for every $\boldsymbol{y}^0, \boldsymbol{y}^1 \in \Lambda$. We will restrict the domain and range of these maps to the optimal strategy sets, and show there is only one fixed point for them. Notice we can not use the contraction mapping theorem directly for this particular case.

We state our result in the following way:

Theorem 1 *If $\Lambda(\boldsymbol{y}^0) \rightleftarrows \Lambda(\boldsymbol{x}^0)$ are continuous non-expansive maps then, given $\boldsymbol{x}^0 \in \Lambda(\boldsymbol{y}^0)$(resp. $\boldsymbol{y}^0 \in \Lambda(\boldsymbol{x}^0)$), $\boldsymbol{y}^0 \in \Lambda(\boldsymbol{x}^0)$(resp. $\boldsymbol{x}^0 \in \Lambda(\boldsymbol{y}^0)$) is the unique optimal strategy for the game $\{\Lambda, P(\boldsymbol{x}, \boldsymbol{y})\}$ with $\boldsymbol{x}, \boldsymbol{y} \in \Lambda$.*

The proof of our result is organized as follows. First, we decompose the set of strategies in semi-algebraic subsets by using the semi-algebraic decomposition theorem. Second, we apply these subsets onto each other throughout semi-algebraic non-expansive maps. Third, we find out the optimal strategy that constitutes the fixed point of those maps. We do it by selecting an appropriate semi-algebraic curve. Finally, we verify that this is the unique optimal strategy for the polynomial game.

We state the proof in the following way:

***Proof of Theorem* 1**:

1. ***Cylindrical algebraic decomposition***

In this part we follow the algorithm in Karlin (1992,V.II, P.52) adapted to the semi-algebricity of $\Lambda(\boldsymbol{x}^0), \Lambda(\boldsymbol{y}^0)$. Let $f_j(\boldsymbol{x}), g_i(\boldsymbol{y})$ be semi-algebraic linear functions of $\boldsymbol{x}, \boldsymbol{y} \in \Lambda$ respectively.

Decompose the sets $\Lambda(\boldsymbol{x}^0), \Lambda(\boldsymbol{y}^0)$ in disjunct, compact, convex and semi-algebraic subsets $\Lambda_j(\boldsymbol{x}^0)$,$\Lambda_i(\boldsymbol{y}^0)$ by using the hyperplanes $f_j(\boldsymbol{x}) = 0$ and $g_i(\boldsymbol{y}) = 0$ respectively for each $i, j = 1, 2, ..., n$. Then, we have:

$$\Lambda(\boldsymbol{x}^0) = \Lambda_1(\boldsymbol{x}^0) \cup \Lambda_2(\boldsymbol{x}^0) \cup \cdots \cup \Lambda_n(\boldsymbol{x}^0)$$
$$\Lambda(\boldsymbol{y}^0) = \Lambda_1(\boldsymbol{y}^0) \cup \Lambda_2(\boldsymbol{y}^0) \cup \cdots \cup \Lambda_n(\boldsymbol{y}^0)$$

In the next part of the proof we are going to map semi-algebraically and non-expansively each $\Lambda_j(\boldsymbol{x}^0)$ onto each $\Lambda_i(\boldsymbol{y}^0)$. It is possible because $\Lambda(\boldsymbol{x}^0), \Lambda(\boldsymbol{y}^0)$ are semi-algebraic.

2. $\boldsymbol{Non-expansive\ maps}$

Take a collection of onto continuous maps $[\varphi_j : \Lambda \to \Lambda; \psi_i : \Lambda \to \Lambda]$ for $i, j = 1, ..., n$, with $\varphi_j : \Lambda_j(\boldsymbol{x}^0) \to \Lambda_i(\boldsymbol{y}^0)$ and $\psi_i : \Lambda_i(\boldsymbol{y}^0) \to \Lambda_j(\boldsymbol{x}^0)$. Let $\|\varphi_j(\boldsymbol{x}^0) - \varphi_j(\boldsymbol{x}^1)\| \leq \|\boldsymbol{x}^0 - \boldsymbol{x}^1\|$ and $\|\psi_i(\boldsymbol{y}^0) - \psi_i(\boldsymbol{y}^1)\| \leq \|\boldsymbol{y}^0 - \boldsymbol{y}^1\|$ for every $\boldsymbol{x}^0, \boldsymbol{x}^1, \boldsymbol{y}^0, \boldsymbol{y}^1 \in \Lambda$. These maps are:

$$\varphi_1 : \Lambda_1(\boldsymbol{x}^0) \rightleftarrows \Lambda_1(\boldsymbol{y}^0) : \psi_1$$

$$\varphi_2 : \Lambda_2(\boldsymbol{x}^0) \rightleftarrows \Lambda_2(\boldsymbol{y}^0) : \psi_2$$

$$\vdots$$

$$\varphi_n : \Lambda_n(\boldsymbol{x}^0) \rightleftarrows \Lambda_n(\boldsymbol{y}^0) : \psi_n$$

The semi-algebraicness of those maps follows from $\Lambda(\boldsymbol{x^0})$, $\Lambda(\boldsymbol{y^0})$ being semi-algebraic. Their existence follows by $\Lambda(\boldsymbol{x^0})$, $\Lambda(\boldsymbol{y^0})$ being Urysohn. In the next part of the proof we are going to find out the optimal strategies throughout the fixed points of the latter maps.

3. $\boldsymbol{Existence\ of\ a\ fixed\ point}$

We are going to prove that there exists an unique fixed point for each of the latter semi-algebraic maps. Without loss of generality, let us use the map φ_j. The procedure is the same for the others.

First, we are going to give a definition of semi-algebraic curve. A semi-algebraic curve in Λ is a semi-algebraic application $\iota : (a, b) \to \Lambda$ for any $(a, b) \in \mathbb{R}_+$. We require ι to be continuous.

Second, we are going to use a criterion for selecting a semi-algebraic curve. For every point $a \in Cl(\Lambda) \setminus \Lambda$, there is a semi-algebraic continuous curve $\iota : (0, b) \to \Lambda$ such that $lim_{t \to 0} \iota(t) = a$, where Cl is closure.

Third, we are going to apply the criterion. Let $\iota : (0, a) \to \Lambda_j(\boldsymbol{x}^0)$ be an injective semi-algebraic curve with $a > 0$. The semi-algebraic set $\{\|\varphi_j(\iota(b)) - \iota(b)\| = b \geq 0\}$ has arbitrary small elements.

Fix $0 < \epsilon \notin (0, a)$. Let $\iota' : (0 + \epsilon^2, a + \epsilon^2) \to \Lambda_j(\boldsymbol{x}^0)$ be another injective curve. There exists a limit for each one. Take that limit as the $lim_{b \to 0} \iota'(b) = lim_{b \to 0} \iota(b + \epsilon^2) = \iota(\epsilon^2) = \epsilon^2$.

Then, we have $\|\varphi_j(\iota(0)) - \iota(0)\| = \epsilon^2$ from the compactness of $\Lambda_j(\boldsymbol{x}^0)$, but it is a contradiction because $\|\varphi_j(\iota(0)) - \iota(0)\| = 0$ from definition, and also $\epsilon \neq 0$. The curve must be defined over 0, its fixed point.

In the following part we are going to prove the uniqueness of the fixed point of those maps.

4. ***Uniqueness of the fixed point***

In this part we prove the uniqueness of the optimal strategy. Let us suppose φ_j, ψ_i have two different fixed points: $\boldsymbol{x}^0, \boldsymbol{x}^1$ and $\boldsymbol{y}^0, \boldsymbol{y}^1$. It implies $\|\boldsymbol{x}^0 - \boldsymbol{x}^1\| = \|\varphi_j(\boldsymbol{x}^0) - \varphi_j(\boldsymbol{x}^1)\|$ and $\|\boldsymbol{y}^0 - \boldsymbol{y}^1\| = \|\varphi_j(\boldsymbol{y}^0) - \varphi_j(\boldsymbol{y}^1)\|$. However, because φ_j, ψ_i are non-expansive, $\|\boldsymbol{x}^0 - \boldsymbol{x}^1| \geq \|\varphi_j(\boldsymbol{x}^0) - \varphi_j(\boldsymbol{x}^1)\|$ and $\|\boldsymbol{y}^0 - \boldsymbol{y}^1\| \geq \|\psi_i(\boldsymbol{y}^0) - \psi_i(\boldsymbol{y}^1)\|$ contradict the first part. $\square$

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