Non-renewable resources and growth, the case of the oil: a simple endogenous model

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Abstract

We present a growth model in which a non-renewable resource enters in the production function. The non-renewable resource is supposed to be sold by an external monopolistic that maximizes his intertemporal discounted cash flow. This approach allows to endogenize the price of the resource. We use the historical data of the oil price and of the oil production to calibrate the model. The forecasts of the model about the evolution of the GDP growth rate, the price and amount of the production of the oil are described.

Keywords: Non-renewable resources, Oil, Endogenous Growth.
JEL Classification: O4, Q3.

1 Introduction

We present an endogenous growth model characterized by a Cobb-Douglas production function of the form

$$y(t) = Ak^{(1-\theta)}(t)q^{\theta}(t)$$

where $k(t)$ is the stock of capital at time t and $q(t)$ is the amount of a non-renewable resource used in the production. The non-renewability of the resource is formalized assuming that, along the evolution of the economy, the following constraint is satisfied (normalizing the global amount of the resource to 1):

$$\int_0^{+\infty} q(t)\,\mathrm{d}t \leq 1. \tag{1}$$

Such a kind of approach was already introduced in the classical models like [13, 14] (see also [5], [10]), [12]). In those works the planner can use freely the non-renewable resource or it is sold in a competitive market. In the recent debate (see, only as example [1], [11], [4] and [2]) the optimistic positions of the seventies give way to more caution and problematic opinions about the autonomous capacity of the market of exploiting the exhaustible resources in a farsighted way.

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In the simple model we present we assume that the planner of the economy has to buy the exhaustible resource from an external monopolistic (like the OPEC in the oil context). So the planner have to deal with the budget constraint

$$y(t) = i(t) + c(t) + q(t)p(t)$$

where $p(t)$ is the unit price of the non-renewable resource (chosen by the monopolistic), $i(t)$ is the investment in new capital (so that $\dot{k}(t) = i(t)$) and $c(t)$ is the amount of consumption. We will not introduce a dynamic optimization problem solved by planner but we will assume to have a constant consumption rate (Subsection 2.3) so that $c(t) = sy(t)$ for some $s \in (0, 1)$ (or $c(t) = 0$ in Subsection 2.1) so the agent has only to choose $i(t)$ and $q(t)$. There are not strong economic arguments in favor of such an old-fashion choice but it can be accepted as the main focus of the model is on the impact of the finiteness of the oil on the growth; a moderate variability of the consumption rate would not change the qualitative behavior of the economy. The choice of a constant consumption rate helps to simplify the mathematic difficulties of the problems, that (it will be clearer in a while) are not trivial.

In the model the monopolistic chooses the evolution of the price $p(t)$ in order to maximize the intertemporal discounted cash flow

$$\max \int_0^{+\infty} e^{-\rho t} p(t)q(t)\, \mathrm{d}t$$

where ρ is a strictly positive constant and $q(t)$ is determined by demand-side. Once the monopolistic has chosen the evolution of the price $p(t) \geq 0$ for $t \geq 0$ the dynamics of the economy is uniquely determined (thanks to equation (8) and (4)). In particular once the monopolistic has fixed $p(t)$ for all $t \geq 0$ we have the evolution of $q(t)$ for all $t \geq 0$ and we can verify if such a $q(t)$ satisfy (1). We will say that an evolution of the price $p(t)$ is admissible if the related $q(t)$ satisfies (1).

So the optimization problem of the monopolistic is to find a price evolution that, together with the relate $q(t)$, maximize the discounted cash flow among the exponential admissible evolutions.

Once the optimization problem is solved we have: the evolution of the price $p(t)$ for $t \geq 0$, the related function $q(t)$ that describe the amount of the sold non-renewable resources for $t \geq 0$ and the evolution of the production $y(t)$.

It is easy to see that we can reduce the maximization problem of finding the maximum among the trajectories that satisfy (1) to that of finding the maximum among the trajectories on "the boundary" an then using the constraint (10).

The exponential case In the study of the model we will first give a general result (Proposition 3.1) that roughly speaking states that, as we aspect, all the admissible evolutions of the (real) price grow to infinity for t that goes to infinity: there do not exist admissible evolutions of the (real) price that remain bounded. We then focus to a particular class of admissible price: the exponential case. So we assume that $p(t) = p_0 e^{\omega t}$ for some $p_0 > 0$ and $\omega > 0$.

Proposition 3.2 shows that for every ω there exists a unique p_0 such that the evolution of the price $p(t) = p_0 e^{\omega t}$ is admissible (that is the related evolution of $q(t)$ satisfies (10)). Proposition 3.4 states that (given the positive constants A,

θ, k_0, s and ρ) there exists an admissible exponential strategy that maximizes the functional among the set of all the admissible exponential strategy.

The choice of the exponential case is mainly due to a technical reason and it can be seen as analogous to the study of the balance growth paths in a neoclassical growth model. Note that, even though the evolution of the price is exponential along the exponential strategy (by definition) the evolution of $y(t)$ and $q(t)$ is more complex.

Calibration and simulation In Section 4 and Section 5 we will focus our attention on the oil case: we will calibrate and "use" of the model. The model allows to endogenize the evolution of the price and so we can calibrate it using the price and the amount of production of the oil. They are surely more reliable data than the evaluations of the different countries about the remaining availability of the oil.

The model allows a choice of A, θ, ρ and k_0 such that the optimal exponential strategy fits quite precisely with historical series of the price of the oil, with the historical series of the oil supply and with the growth rate of the global GDP (see Figure 2, Figure 3 and Figure 4). In Section 5 we look at the predictions of the calibrated model. We show the forecasted evolution of the oil production (Figure 5) and of the GDP growth rate (Figure 6). The evolution of the oil production has a maximum in the 2008 and then begin to decrease. On the other hand the forecasted evolution of the growth rate of the production is only slightly decreasing. So, in the model, the economy continues to maintain an high growth rate of the production also if the price of the oil strongly grows and the production of the oil reduces. This actually is in line with the macroeconomic data of the last years (see Section 5 on this point). Of course this is only a simple model and the predictions are only qualitative, in particular the forecasts can be significant only in a not too long interval of time in which we can assume constant the technology and the oil-dependency of the economy (that are modeled by the constants A and θ).

For other remarks on the results see Section 5.

In Figure 7 we represent the prediction of the evolution of the oil sold on the market in the period 1990-2100. It is a very long period of time but we chose to show all the qualitative behavior of the oil production predicted by the model (it is a way to describe the model more than the reality). As discussed in Section 5, the represented curve cannot be considered as an endogenous version of the Hubbert curve (see for example [3]). The phenomenon we represent has some similarities and some relevant differences with a Hubbert curve: both are due to the finiteness of the oil and they have a similar shape but while a Hubbert peak of the oil has a physical and geological reasons here the reason of the decrease of the oil supply is an effect of merely economic considerations: the monopolistic chooses to gradually decrease the oil supply because this is the most profitable strategy.

Other comments and observations on the results and on the simulations are in Section 5.

2 The model

2.1 The demand-side

We assume to have a Cobb Douglas production function, with constant returns to scale, in the two factors $k(t)$ (stock of capital at time t) and $q(t)$ (the amount of the exhaustible resource used in the production at time t)

$$y(t) = Ak^{(1-\theta)}(t)q^{\theta}(t) \tag{2}$$

where A is a positive constant and $\theta \in (0,1)$. The agent has to choose at each time how to split the production among investment in new capital, consumption and exhaustible resource:

$$y(t) = i(t) + c(t) + q(t)p(t) \tag{3}$$

where $i(t)$ and $c(t)$ are the amount of investment in new capital and the consumption at time t, $q(t)p(t)$ is the cost for the non-renewable resource (being $p(t)$ the unit price of the resource). The demand side is price taker, $q(t)$ is adjusted in order to maximize $y(t) - p(t)q(t)$. So the first order condition gives

$$q(t) = \left(\frac{p(t)}{\theta A}\right)^{-\frac{1}{1-\theta}} k(t). \tag{4}$$

We impose, as in the common neoclassical models, the dynamic of the capital accumulation to be

$$\dot{k}(t) = i(t), \tag{5}$$

some depreciation factor can be included in A. As already announced we choose to ignore the dynamics of the consumption and to consider $c(t) = 0$, in Subsection 2.3 we will see how this approach covers the more general case of a constant consumption rate $c(t) = sy(t)$.

From (2), (3) and (5) we have

$$\dot{k}(t) = y(t) - q(t)p(t) = Ak^{(1-\theta)}(t)q^{\theta}(t) - q(t)p(t).$$

Using (4) we have

$$\begin{aligned}\dot{k}(t) = Ak(t)\left(\frac{p(t)}{\theta A}\right)^{\frac{\theta}{\theta-1}} - k(t)\left(\frac{1}{\theta A}\right)^{\frac{1}{\theta-1}} p(t)^{\frac{\theta}{\theta-1}} = \\ = k(t)\left(\left(A\left(\frac{1}{\theta A}\right)^{\frac{\theta}{\theta-1}} - \left(\frac{1}{\theta A}\right)^{\frac{1}{\theta-1}}\right)p(t)^{\frac{\theta}{\theta-1}}\right).\end{aligned} \tag{6}$$

So, calling $B = \left(A\left(\frac{1}{\theta A}\right)^{\frac{\theta}{\theta-1}} - \left(\frac{1}{\theta A}\right)^{\frac{1}{\theta-1}}\right)$, that we will assume to be positive, we can write

$$\dot{k}(t) = k(t)\left(Bp(t)^{\frac{-\theta}{1-\theta}}\right). \tag{7}$$

Eventually, if $k_0 > 0$ is the stock of capital at time 0 we have

$$k(t) = k_0 \exp\left(\int_0^t \left(Bp(s)^{\frac{-\theta}{1-\theta}}\right) \mathrm{d}s\right). \tag{8}$$

2.2 The supply-side

The monopolistic acts to maximize the functional[1]

$$J(p(t)) := \int_0^{+\infty} e^{-\rho t} p(t) q(t)\, \mathrm{d}t \tag{9}$$

subject to (4) and (5), among the trajectories that are admissible in sense that[2]

$$\int_0^{+\infty} q(t)\, \mathrm{d}t = 1. \tag{10}$$

Equation (10) models the finiteness of the resource: a monopolistic strategy for the price $p(t)$ is admissible if the related trajectory for the amount of the sold resource $q(t)$, obtained replacing $p(t)$ in (5) and (4), satisfies (10).

To be more formal, given a[3]

$$p(t) \in \mathcal{S} := \left\{ p(t) \in L^{1,+}_{\mathrm{loc}}([0,+\infty)) \ : \ (p(t))^{\frac{-\theta}{1-\theta}} \in L^1_{\mathrm{loc}}([0,+\infty)) \right\}$$

we call $k_p(t)$ the related (continuous) solution of (8) and $q_p(t)$ the (measurable) expression given by (4) and we define the set of the admissible evolutions of the price as:

$$\mathcal{A} := \left\{ p(t) \in \mathcal{S} \ : \ \int_0^{\infty} q_p(t)\, \mathrm{d}t = 1 \right\}.$$

We normalize the global amount of the resource to 1.

2.3 The constant consumption rate case

We could introduce a consumption $c(t)$ in the economy and impose the resource constraint

$$y(t) = i(t) + c(t) + q(t)p(t).$$

instead of (3). If we consider a generic consumption $c(t)$ the model becomes hardly treatable (we would need to introduce another functional to model the decisions of the planner). Anyway we can observe that the setting we used allow to treat the case of an exogenous constant consumption rate so that $c(t) = sy(t)$ the problem would be

$$\dot{k}(t) = (1-s)y(t) - q(t)p(t) = (1-s)Ak^{(1-\theta)}(t)q^{\theta}(t) - q(t)p(t)$$

and then, calling $\tilde{A} = A(1-s)$

$$\dot{k}(t) = \tilde{A}\left(k^{(1-\theta)}(t)q^{\theta}(t)\right) - q(t)p(t).$$

[1]The "right" notation should be $J(p(\cdot))$ since J is a functional that associates to the function $p(\cdot)$ a real number, but we will write, using a imprecise but diffuse notation, $p(t)$ to mean both the function $p(\cdot)$ and its value at point t.

[2]As we observed in the introduction we can substitute the constraint $\int_0^{+\infty} q(t)\,\mathrm{d}t \le 1$ with the constraint $\int_0^{+\infty} q(t)\,\mathrm{d}t = 1$.

[3]We call $L^1_{\mathrm{loc}}([0,+\infty))$ the set of the locally integrable function (that is the set of the function $f\colon [0,+\infty) \to \mathbb{R}$ s.t. $\int_a^b |f(t)|\mathrm{d}t < +\infty$ for all $0 \le a < b < \infty$) and $L^{1,+}_{\mathrm{loc}}([0,+\infty))$ the set of the set of the locally integrable function that are positive.

Eventually we would obtain $B = \left(\tilde{A}\left(\frac{1}{\theta A}\right)^{\frac{\theta}{\theta-1}} - \left(\frac{1}{\theta A}\right)^{\frac{1}{\theta-1}}\right)$ and

$$\dot{k}(t) = k(t)\left(Bp(t)^{\frac{-\theta}{1-\theta}}\right).$$

that is the same of (5) with a different value for the B. So the problem with a constant consumption rate can be studied exactly in the same way.

3 The study of the model

Using (4) and (8) we can write (10) as

$$1 = \int_0^{+\infty} q(t)\,\mathrm{d}t = \int_0^{+\infty} \left(\frac{p(t)}{\theta A}\right)^{\frac{-1}{1-\theta}} k(t)\,\mathrm{d}t =$$
$$= \int_0^{+\infty} \left(\frac{p(t)}{\theta A}\right)^{\frac{-1}{1-\theta}} k_0 \exp\left(\int_0^t \left(Bp(s)^{\frac{-\theta}{1-\theta}}\right) \mathrm{d}s\right) \mathrm{d}t. \quad (11)$$

Such an expression can be difficultly treat in the general case anyway note that

Proposition 3.1. *If $p(t) \in \mathcal{A}$ then $\sup_{t\in[0,\infty)} p(t) = +\infty$. Moreover, if we call μ the Lebesgue measure on $\mathbb{R}$, for every $M > 0$ we have that $\mu\{t \in [0,+\infty) \,:\, p(t) \leq M\} < \infty$.*

Proof. We can argue by contradiction: suppose that there exists a $p(t) \in \mathcal{A}$ and $M > 0$ s.t., if we define

$$S_M := \{t \in [0,+\infty) \,:\, p(t) \leq M\},$$

we have $\mu(S_M) = +\infty$. Then, observing that (from (8)) $k_p(t) \geq k_0$ for all $t \geq 0$ we have (from (4))

$$q_p(t) \geq c := \left(\frac{1}{\theta A}\right)^{\frac{-1}{1-\theta}} \left(\frac{1}{M}\right)^{\frac{1}{1-\theta}} k_0 > 0$$

for all the $t \in S_M$. And then

$$\int_0^{\infty} q_p(t)\,\mathrm{d}t \geq \int_{S_M} q_p(t)\,\mathrm{d}t \geq c\mu(S_M) = +\infty$$

and so we have the contradiction (if $p \in \mathcal{A}$ by definition $\int q_p = 1 \neq +\infty$) and the claim. $\square$

3.1 The exponential case

So Proposition 3.1 states, roughly speaking, that the price has to grow to infinity, moreover it could be also proven that "light" growths of the prices are not enough: for example it can be proven that linear evolutions of the price cannot be admissible strategy.

We reduce here the study of the optimal strategy for the exhaustible resource price only among the set of the exponential strategy. This is of course a restriction of the problem presented in (9) and (10) and this kind of study is mainly due to technical reasons. This kind of approach is an analogous of studying of the BGPs for a neoclassical growth model.

We consider a price of the form $p(t) = p_0 e^{\omega t}$, we define

$$\mathcal{U} = \left\{ t \mapsto p_0 e^{wt} \; : \; p_0 > 0, \; \omega > 0 \right\}.$$

A strategy of $\mathcal{U}$ to be admissible has to satisfy the (10) that in our case can be expressed as (11). If $p(t) = p_0 e^{\omega t}$ the expression of the constraint becomes

$$\begin{aligned} 1 = \left(\frac{p_0}{\theta A}\right)^{\frac{-1}{1-\theta}} k_0 \int_0^{+\infty} e^{\frac{-\omega}{1-\theta} t} \exp\left(B p_0^{\frac{-\theta}{1-\theta}} \int_0^t \left(e^{\frac{-\omega\theta}{1-\theta} s} \right) \mathrm{d}s \right) \mathrm{d}t = \\ = \left(\frac{p_0}{\theta A}\right)^{\frac{-1}{1-\theta}} k_0 \int_0^{+\infty} e^{\frac{-\omega}{1-\theta} t} \exp\left(B p_0^{\frac{-\theta}{1-\theta}} \frac{1-\theta}{\omega\theta} \left(1 - e^{\frac{-\omega\theta}{1-\theta} t} \right) \right) \mathrm{d}t = \end{aligned} \tag{12}$$

calling $\beta = \left(B p_0^{\frac{-\theta}{1-\theta}} \frac{1-\theta}{\omega\theta} \right) > 0$ and with the change of variable $y = \beta e^{\frac{-\omega\theta}{1-\theta} t}$

$$\begin{aligned} = \frac{\left(\frac{p_0}{\theta A}\right)^{-\frac{1}{1-\theta}} k_0 \, e^{\beta}}{\beta} \frac{1-\theta}{\omega\theta} \int_0^{\beta} e^{-y} \left(\frac{y}{\beta}\right)^{\frac{1-\theta}{\theta}} \mathrm{d}y = \\ = \frac{\left(\frac{p_0}{\theta A}\right)^{-\frac{1}{1-\theta}} k_0 \, e^{\beta}}{\beta^{1/\theta}} \left(\frac{1-\theta}{\omega\theta}\right) \int_0^{\beta} e^{-y} y^{\frac{1}{\theta}-1} \, \mathrm{d}y \end{aligned} \tag{13}$$

The last integral is always finite for every positive θ and β (it is a part of the integral defining the Euler Gamma[4] of $1/\theta$). We call

$$W(\beta) := \int_0^{\beta} e^{-y} y^{\left(\frac{1}{\theta}-1\right)} \, \mathrm{d}y < \infty. \tag{14}$$

Proposition 3.2. *For every positive ω there exists a unique positive $p_0 = I(\omega)$ such that (13) is satisfied. Moreover the function I that associate ω to p_0 is strictly decreasing and if we call*

$$\bar{\beta}(\omega, p_0) = \left(B p_0^{\frac{-\theta}{1-\theta}} \frac{1-\theta}{\omega\theta} \right) \tag{15}$$

we have that

$$\bar{\beta}(\omega, I(\omega)) \xrightarrow{\omega \to 0^+} +\infty \tag{16}$$

$$\bar{\beta}(\omega, I(\omega)) \xrightarrow{\omega \to +\infty} 0. \tag{17}$$

We also have

$$I(\omega) \xrightarrow{\omega \to 0^+} +\infty. \tag{18}$$

Proof. In the proof we will first introduce β and then I, we reversed the order in the statement to make it clearer.

In (13) we can express $\left(\frac{p_0}{\theta A}\right)^{-1/(1-\theta)}$ in terms of β as

$$\left(\frac{p_0}{\theta A}\right)^{\frac{-1}{1-\theta}} = \frac{\beta^{1/\theta}}{\left(B \frac{1-\theta}{\theta\omega}\right)^{1/\theta}} \left(\frac{1}{\theta A}\right)^{-1/(1-\theta)}.$$

[4]Indeed sometimes the function $(x, \beta) \mapsto \int_0^{\beta} e^{-y} y^{(x-1)} \, \mathrm{d}y$ is called incomplete Gamma function.

So we can rewrite (13) only in terms of ω and β. It becomes:

$$1 = \left(\frac{\omega\theta}{1-\theta}\right)^{\frac{1-\theta}{\theta}} \left(\frac{\left(\frac{1}{\theta A}\right)^{-1/(1-\theta)} k_0}{B^{1/\theta}}\right) e^{\beta} W(\beta). \tag{19}$$

This relation, since both $\beta \mapsto e^{\beta}$ and $\beta \mapsto W(\beta)$ are positive and strictly growing and $\omega \mapsto \left(\frac{\omega\theta}{1-\theta}\right)^{\frac{1-\theta}{\theta}}$ is a strictly growing function that holds 0 in 0 has limit $+\infty$ for $\omega \to \infty$, ensure that for every ω there exists a unique $\beta = \tilde{\beta}(\omega)$ satisfying (19), that $\omega \mapsto \tilde{\beta}(\omega)$ is strictly decreasing and satisfy

$$\tilde{\beta}(\omega) \xrightarrow{\omega \to 0^+} +\infty \qquad \textit{and} \qquad \tilde{\beta}(\omega) \xrightarrow{\omega \to +\infty} 0. \tag{20}$$

Thanks to the expression of $\bar{\beta}(\cdot,\cdot)$ given in (15) we can easily see that for given ω and β there exists a unique $p_0 = I(\omega)$ satisfying $\beta = \bar{\beta}(\omega, p_0)$. We have of course $\tilde{\beta}(\omega) = \bar{\beta}(\omega, I(\omega))$ and so (16) and (17) follow from (20).

To see that $\omega \mapsto I(\omega)$ is a strictly decreasing function we have only to observe that $(\omega, p_0, t) \mapsto p(t) = p_0 e^{\omega t}$ is, for every $t > 0$ a strictly increasing function in ω and p_0, so, thanks to the expression for the capital (8) the amount of capital at time t is strictly decreasing in ω and p_0, and, thanks to (4) we finally see that $q(t)$ (for every $t > 0$) is strictly decreasing in ω and p_0, so (referring to the initial formulation of the constraint (10) we have the claim.

To prove the (18) we can consider (13):

$$1 = k_0 \left(\frac{p_0}{\theta A}\right)^{-\frac{1}{1-\theta}} \left(\frac{1-\theta}{\omega\theta}\right) \left(\frac{e^{\beta}}{\beta^{1/\theta}} \int_0^{\beta} e^{-y} y^{\frac{1}{\theta}-1}\, \mathrm{d}y\right) \tag{21}$$

we have that

$$\left(\frac{1-\theta}{\omega\theta}\right) \xrightarrow{\omega \to 0} +\infty,$$

moreover we have already see that $\beta(\omega, I(\omega)) \xrightarrow{\omega \to 0} +\infty$ so

$$\left(\frac{e^{\beta}}{\beta^{1/\theta}} \int_0^{\beta} e^{-y} y^{\frac{1}{\theta}-1}\, \mathrm{d}y\right) \xrightarrow{\omega \to 0} +\infty$$

and then to satisfy (21) we need

$$\left(\frac{p_0}{\theta A}\right)^{-\frac{1}{1-\theta}} \xrightarrow{\omega \to 0^+} 0$$

and then the (18). So also the last claim is proved. $\square$

Remark 3.3. *We have proven that for every ω we can find a positive $p_0 = I(\omega)$ such that (13) is satisfied. The opposite is not true. Namely there can exist some $p_0 > 0$ (in particular some "too small" p_0) such that there does not exist any $\omega > 0$ such that $p_0 = I(\omega)$.*

So we define now a subset of $\mathcal{U}$ given by the admissible strategies, that is the strategies satisfying (13):

$$\mathcal{U}_{ad} = \left\{ t \mapsto I(\omega) e^{\omega t} \ : \ \omega > 0, \right\}$$

Proposition 3.4. *The functional $J(p(t))$ defined in (9) admits a maximum in the set $\mathcal{U}_{ad}$, namely there exists an admissible exponential strategy $p^o(t) = I(\omega^o)e^{\omega^o t}$ such that $J(p^o(t)) \geq J(p(t))$ for all $p(t) \in \mathcal{U}_{ad}$.*

Proof. To prove the statement we consider the value of the functional (9) on a trajectories of $\mathcal{U}_{ad}$ $p(t) = I(\omega)e^{\omega t}$ for $\omega > 0$. We write $J(\omega)$ for $J(I(\omega)e^{\omega t})$ and we call p_ω and q_ω the trajectories of the price of the oil and of the quantity of the oil related to the price $p(t) = I(\omega)e^{\omega t}$. So

$$J(\omega) = \int_0^{+\infty} e^{-\rho t} q_\omega(t) p_\omega(t)\,\mathrm{d}t.$$

We show now that $J(\omega) \xrightarrow{\omega \to 0^+} 0$ and $J(\omega) \xrightarrow{\omega \to +\infty} 0$ and this proves the claim. We first check the case $\omega \to 0^+$, we have (from (4) and (8))

$$J(\omega) = \int_0^{+\infty} e^{-\rho t} q_\omega(t) p_\omega(t)\,\mathrm{d}t = \int_0^{+\infty} e^{-\rho t} p_\omega(t)^{\frac{-\theta}{1-\theta}} k_0 e^{\int_0^t B p_\omega(s)^{\frac{-\theta}{1-\theta}}\,\mathrm{d}s}\,\mathrm{d}t \leq$$
$$\int_0^{+\infty} e^{-\rho t} I(\omega)^{\frac{-\theta}{1-\theta}} k_0 e^{tBI(\omega)^{\frac{-\theta}{1-\theta}}\,\mathrm{d}s}\,\mathrm{d}t \leq \quad (22)$$

(since we are considering $\omega \to \infty$ we can assume, for (18), that $Bp_0^{\frac{-\theta}{1-\theta}} \leq \rho/2$)

$$\leq k_0 \int_0^{+\infty} e^{-\rho t/2} I(\omega)^{\frac{-\theta}{1-\theta}}\,\mathrm{d}t$$

that goes to zero for the dominate convergence theorem when $\omega \to 0^+$ thanks to (from 18).

Otherwise when $\omega \to +\infty$ we have, thanks to (4), (8)

$$J(\omega) = \int_0^{+\infty} e^{-\rho t} q_\omega(t) p_\omega(t)\,\mathrm{d}t \leq \int_0^{+\infty} q_\omega(t) p_\omega(t)\,\mathrm{d}t =$$
$$\int_0^{+\infty} k_0 p_\omega(t)^{\frac{-\theta}{1-\theta}} e^{\int_0^t B p_\omega(s)^{\frac{-\theta}{1-\theta}}\,\mathrm{d}s}\,\mathrm{d}t = \frac{k_0}{B} \int_0^\beta e^r\,\mathrm{d}r, \quad (23)$$

but the last integral, thank to (17) goes to 0 for $\omega \to \infty$. □

Remark 3.5. *It is natural to wonder whether we have the uniqueness of the minimum. In the simulations the minimum always happened to be unique but we cannot prove formally this fact at this step.*

A numerical example We consider the set of parameters described in Table (1).

θ	0.248
ρ	0.05
A	0.025
k_0	4000

Table 1: Set of the parameters for the oil case

Now we want to show how the functions $\omega \mapsto I(\omega) = p_0(\omega)$ and $\omega \mapsto J(p_0(\omega)e^{\omega t})$ appear with our choice of parameters. We obtained Figure 1 with the Matlab code `GRAPHS_beta_and_J.m`[5]. To summarize: the first graph of

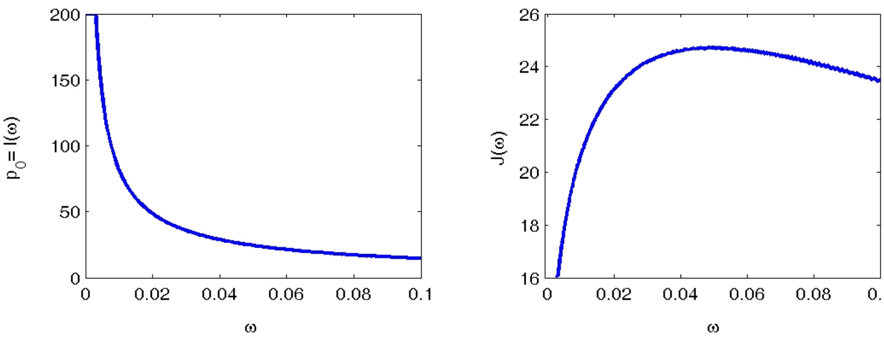

Figure 1: Graph of the function $\omega \mapsto I(\omega)$ and $\omega \mapsto J(\omega)$

Figure 1 represents the function that associate to every ω the only $p_0 = I(\omega)$ such that $p_0 e^{\omega t}$ is a admissible function. The second one[6] represents the value of the functional J on the admissible strategies $I(\omega)e^{\omega t}$ varying ω. Since the monopolistic acts to maximize the functional J among the admissible exponential strategies, the chosen strategy will be determine by the maximum of the second graph.

The maximum of $J(\omega)$ is obtained in $\bar{\omega} = 4,91\%$ and the related $p_0(\bar{\omega}) = I(\bar{\omega})$ is 24.6\$.

4 Calibration of the model

In the model the total amount of the oil is normalized to 1. This is generic normalization factor that is of course inconsistent if we want for example to measure the amount of the oil in barrels (the oil is scarce but not so much!). As a consequence, when we calibrate the model we cannot aspect that the scaling parameters A and k_0 have realistic value and we will be indeed interested mainly in proportions and rates.

Nevertheless we can choose a variable to have a value consistent with the real data, we will choose to calibrate the model to obtain an acceptable value for the price of the oil (dollars a barrel). We will also calibrate the model w.r.t the growth rate of the oil supply and the growth rate of the GDP.

To calibrate the model we have chosen the period 1990-2007. The choice of the period is quite problematic, indeed the short run variations in the oil price have mainly political and financial reasons. So the interval we use to calibrate the model has to be large enough to show an underlying long run behavior due to economic factors. On the other hand the period we consider cannot be too

[5] All the codes used in the paper can be found in the web page of the author: *http://docenti.luiss.it/fabbri*.

[6] Note that the second graph of Figure 1 is incomplete because, as we have proven, $J(\omega) \xrightarrow{\omega \to 0} 0$.

long because in the model the technology and the oil-dependency of the economy (that era modeled by A and θ) are constant.

We have chosen to exclude in the calibration the pick in the oil price of the late-seventies/early eighties. Nevertheless in the early nineties the price of the oil is strongly influenced by the first Gulf War with a pick in the price in the 1990-1991 and a period of unnatural low prices that continues until the end of the nineties. We considered these facts fitting the data. Of course focusing of the data following the 1990-1991 has also the meaning of avoid the data of the "two blocs" age, in which the economical and political situation was completely different from those of today.

We use the set of parameters of Table 1.

Figure 2 show the fitting of the model (the red line) with the historical series of the price of the crude oil. The historical data are from [7] (they are the price of the imported crude oil in US, the prices are expressed in dollars of October 2007), the simulation for the model is obtained using the file `GRAPHS_price_and_quantity.m`. In Figure 3 the historical series (the blue bars)

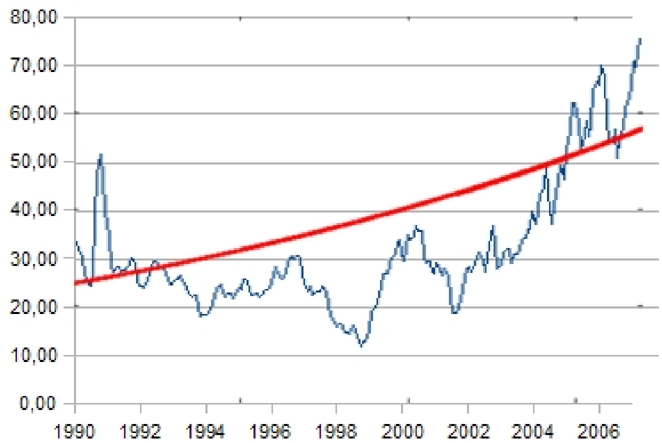

Figure 2: Price of the oil ($ a barrel) : simulation and historical data

of the oil production (from [6], they represent the world total crude oil supply) in the period 1990-2006 are compared with the result obtained in the simulations[7] (the red line). In the model the price clears the market and then there is not difference between the demand and the supply. Otherwise in the historical data there are small differences, we have chosen the supply side that is the main actor in the model. Note that the results of the simulations are multiplied by a constant to be comparable with the historical data, since in the model the total amount of the oil is 1. The production is expressed in BPD (barrels per day). In figure 4 the biennium 2006-2007 is represented, the data are from [8].

5 Outlook and conclusions

In this section we present some forecasts that arise from our approach. We use the calibration presented in Section 4 (that is the set of parameters of Table

[7] Using the file *GRAPHS_price_and_quantity.m*.

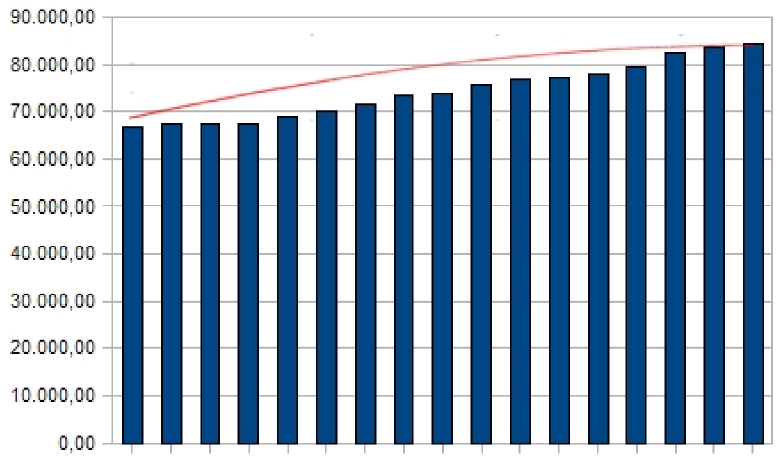

Figure 3: Supply of the oil (1990-2006): simulation and historical data

1). Using such a set of constants the 2007 growth rate predicted by the model is 4.8% (the growth rate forecasted by the IMF is 4.9% see [9]). We show the forecasts of the model for the period 2007-2040. Maybe it is a long period of time, but it is useful to see the qualitative evolution suggested by the model. Figure 5 presents the predicted evolution of the oil price and of the production of oil. The evolution of the price is of course exponential, more precisely it is $p(t) = p_0 e^{\bar{\omega} t}$ where p_0 is 24.6$, $\bar{\omega} = 0.0491 = 4.91\%$ [8]. The evolution of the oil production is obtained using the expression of the predicted oil price in (8) and then in (4). In Figure 6 we represent the forecasted GDP growth rate, obtained using the evolution of the oil production and of the capital in production function (2) and then computing the growth rate. Of course this is only a simple model and the predictions are only qualitative.

The period 2005-2007 The data of the last quarters are of particular interest. These are the EIA data, see [8] and [7], the prices are in dollars of the October 2007:

	2006	2007
Q1	85.4 MB/d	85.4 MB/d
Q2	84.9 MB/d	85.1 MB/d
Q3	85.5 MB/d	85.1 MB/d
Q4	85.3 MB/d	-

	2005	2006	2007
Q1	40.99$	57,35$	54,43$
Q2	45.86$	65,89$	62,91$
Q3	56.78$	65,60$	72,18$
Q4	52.04$	55,21$	-

We also add (same source: [8]) that the average total oil production in the 2005 was 85.3 MB/d.

So the last two years (2006 and 2007) were characterized by:

1- A constant trend in the oil supply

2- A strong global GDP growth: 4.8% in 2005, 5.4% in 2006 and 4.9 % in 2007 (see [9])

[8]The time t is expressed in years (and the initial point $t = 0$ is in 2007).

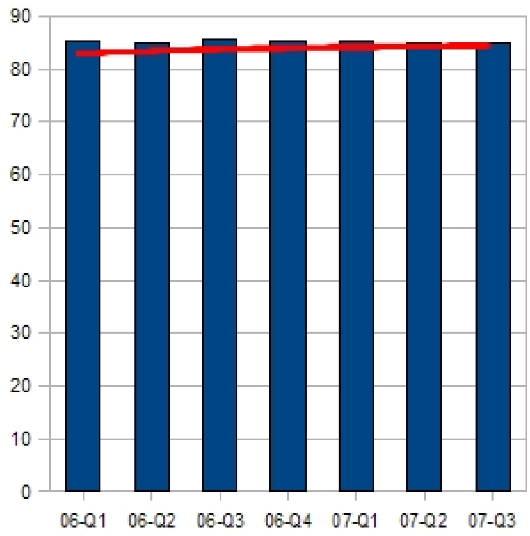

Figure 4: Supply of the oil (2006-2007): simulation and historical data

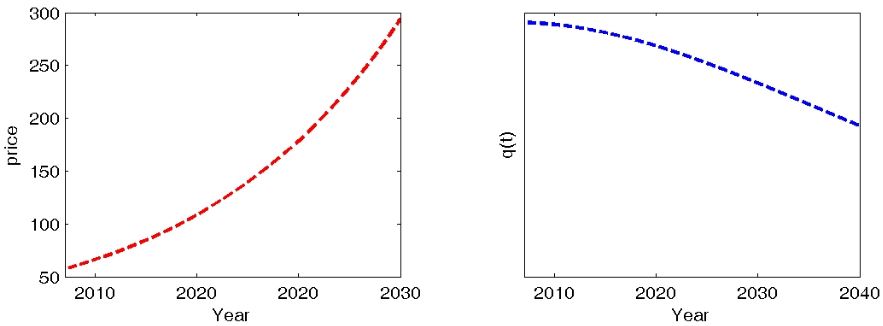

Figure 5: Prediction for future price and global production

3- An high (and growing) price of the oil.

These data seem to outline a new scenario. Indeed in the past (at least from the late eighties) reductions in oil production corresponded to periods of low GDP growth and, usually, to a reduction in the oil price. While a reduction (or constant trend) of the oil in a period of low-growing GDP and a decreasing oil price suggests a reduction of the demand of oil, a reduction (or constant trend) of the oil in a period of high-growing GDP and a increasing oil price is an unambiguous sign that something happening on the supply side. The model we present suggest the economic mechanism that underly to such a behavior. Using the calibration we have suggested, the maximum of the oil production is reached in the 2008 but, since the curve is smooth, in all the period around the maximum the variations in the oil supply are small. The calibrated model fits, on this point, with the historical data.

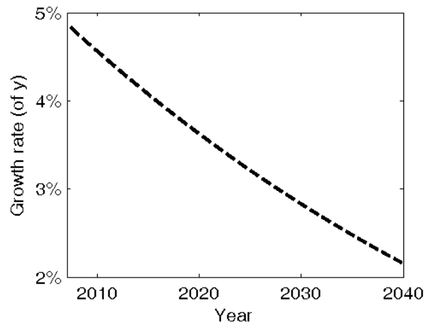

Figure 6: Prediction for future growth

A maximum without tragedies The forecast of the model can be in some sense surprising: it suggests a strong growth in the price of the oil and only a light and slow decrease in the qualitative behavior of the GDP growth rate. Anyway such a result is completely in line with the recent macroeconomic data: we live in a period of high price of the oil and strong growth of the global GDP.

Of course it is a qualitative behavior that does not consider the business cycle or the political events, but it anyway suggests the qualitative contribution of the finiteness of a non-renewable resource, as the oil, in the growth rate. As we have already stressed the model considers a constant dependency of the economy on the oil and so we can aspect that, if there will be enough investments in the research of new technology less dependent on the use of the oil, the qualitative decrease of the growth rate predicted by the model will be stopped.

A non-Hubbert peak In Figure 7 we represent the behavior of the oil production in the period 1990-2100 predicted by the model. This of course is a very long period but we chose to present the whole picture to show the qualitative behaviour suggested by the model. The oil production has a maximum in 2008

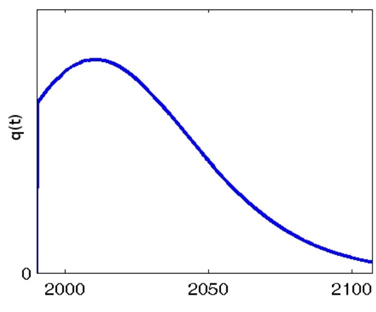

Figure 7: The supply forecasted in the period 1990-2100

and then begins to decrease. As already observed in the introduction it cannot be considered an Hubbert peak. The two phenomena are of course connected, indeed both arise from the observation that the oil is a non-renewable resource, and then the extraction cannot increase forever.

The difference is given by the different reason that generates the peak. While in the Hubbert model the decrease in the production (extraction) of the oil is mainly due to physical and geological reasons (the extraction of the oil decreases because the oil wells have less pressure and give a fewer amount of oil) here the decrease if a free choice of the monopolistic: he could extract more oil but he do not want to. He increase and then decrease the oil supply because this is the most profitable strategy.

Note also that while we can speck of Hubbert peak also "locally" (many oil wells have reached their Hubbert peak and many have been exhausted) we can have the phenomenon we model only globally.

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