The value of reliability

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Abstract

We derive the value of reliability in the scheduling of an activity of random duration, such as travel under congested conditions. We show that the minimal expected cost is linear in the mean and standard deviation of duration, regardless of the form of the standardized distribution of durations. This insight provides a unification of the scheduling model and models that include the standard deviation of duration directly as an argument in the cost or utility function. The results generalize approximately to the case where the mean and standard deviation of duration depend on the starting time. Empirical illustration is provided.

KEYWORDS: Welfare; Random duration; Time; Scheduling; Reliability; Variability

JEL codes: D01; D81

1 Introduction

In this paper we consider the value of reliability for an agent who wishes to undertake an activity of random duration and must decide when to initiate the activity, knowing only the distribution of its duration. We are concerned with the value of changes to the distribution of the duration. The value of a change in the mean duration is just the value of time, which is a concept with a long history in economics (Becker, 1965; Beesley, 1965; Johnson, 1966; DeSerpa, 1971) and there is a large literature on its measurement.[1] The concept of the value of reliability of duration, i.e. the value of a change in the standard deviation of duration, is less well established but not much less important.

We incorporate reliability by building on the model of Small (1982), who considered the scheduling of commuter work trips when the commuters have scheduling costs as well as time and monetary costs.[2] We formulate the scheduling costs as an opportunity cost of starting early and a greater cost of finishing late relative to some fixed deadline. In contrast to earlier contributions (e.g., Noland and Small, 1995), we are able to derive the optimal expected cost for a general distribution of durations. We obtain the simple result that the optimal head start as well as the optimal expected cost depend linearly on the mean and standard deviation of the distribution of durations, provided the standardized distribution of durations is fixed. Both the optimal head start and the value of reliability depend in a simple

[1] Some recent references on the measurement of the value of time are Small et al. (2005) and Fosgerau (2007).

[2] Some early contributions discussing the cost of stochastic delay are Douglas and Miller (1974) and Anderson and Kraus (1981).

way on the standardized distribution of durations and the optimal probability of being late, which in turn is given by the scheduling costs.

In order to apply the scheduling model without the results in the present paper, it would be necessary to observe the individual times at which lateness costs are incurred. An important point about the present results is that now it suffices just to observe the distribution of durations, which is much easier and can be done at the aggregate level.

Although originally motivated in transport, the structure of the scheduling problem occurs in many situations; for example, deciding when to enter a queue or when to begin a search. For firms, some inventory stock problems are of similar structure, for instance when holding a deteriorating good in stock is costly, with a random cut-off quality with high cost and costs for delayed delivery (Liaoa et al., 2000; Hochman et al., 1990). In health economics, waiting times for patients are random and associated with a cost (Mataria et al., 2007). The structure of the scheduling problem also occurs in the decision of how durable to design a product, given that it should last for at least some fixed period and lacking knowledge of the intensity of use.

In transportation, similar problems occur in several different contexts. Airlines have to decide how much slack to allow in the scheduling of flights. In Brueckner (2004), the passenger must buy the air ticket before knowing his preferred arrival time, and expected scheduled delay differs between airlines. The scheduling problem with delays is also relevant in the tourism literature (Batabyal, 2007).

Increasingly congested road networks have caused travel times to become highly irregular in many places, which makes road transport an important case for the scheduling model. In this paper we find that travel time uncertainty accounts for about 15 per cent of time costs on a typical urban road. Considering the large share of individuals' time budgets that is spent on transport, it is clear that uncertainty of durations represents a significant cost to society in general. The concepts of the value of time and the value of reliability are both of crucial importance for decisions regarding capacity provision, operations, pricing and other regulation of transport networks. Both concepts are similarly important in urban economics, travel costs being a main determinant of urban spatial structure (e.g., Brueckner, 1987).

Transport is also a clear case of time dependent demand with sharp peaks in the morning and the afternoon. In our empirical illustration in Section 3 we test and accept that the dependency of the distribution of durations on the time of day is described completely through the mean and standard deviation of the distribution of durations. In the theoretical analysis we also deal with the case where the mean and standard deviation of the distribution of durations depend on the head start. In transport, this pertains to of the case where a traveler decides to leave earlier or later along the slopes of the peak in order to avoid the worst. This kind of analysis is important for understanding the effects of pricing policies aimed at regulating traffic during the peak (peak spreading).

As a final application of the model we consider the case of a scheduled service, where the head start cannot be chosen freely but must adhere to a fixed schedule. We are able to make some progress with this case, but find that the simple properties of the unconstrained case no longer hold. In particular, the expected cost is no longer linear in the standard deviation of duration.

The timing of activities is also the focus in the literature on the real option model (Dixit and Pindyck, 1994). However, while the option value model focuses on the timing of decision under increasing information, we address a situation where waiting does not allow more information to be gathered. Hence, our problem is not an optimal stopping problem in the sense of the real option model.

It should be noted that we take the individual perspective, where the distribution of durations is seen as exogenous. This is in contrast with the literature that investigates, in the context of road transport, the properties of equilibrium where the travel time distribution is dependent on the individual departure time choices, and where individuals apply scheduling considerations. Notably, there is the bottleneck model of Vickrey (1969), which has been developed, e.g., in Arnott et al. (1993). A few contributions include stochastic capacity such that travel times become random, e.g., Daniel (1995) and Arnott et al. (1999). Furthermore, there is also a literature on learning the equilibrium in congestion games (Sandholm, 2002, 2005) that goes some way in handling stochastic delays. However, none of the papers mentioned in this paragraph address the value of reliability.

The layout of the paper is as follows. Section 2 does the first bit of theoretical work presenting the simplest model where the duration distribution is independent of the head start. Then Section 3 measures some characteristics of observed travel times by car on a typical congested urban road. As this example illustrates, the standardized travel time distribution may be independent of the time of day but the mean and standard deviation are not. Therefore the model is extended in Section 4 to the cases where the mean and the standard deviation but not the standardized duration distribution depend on head start as could be the case through a peak period. Section 5 considers briefly the case of a scheduled service, where the head start cannot be chosen freely. Section 6 concludes the paper. The more complicated derivations are placed in the appendix.

2 A simple model

Consider an agent about to undertake an activity of uncertain duration. Express the duration in the following convenient form.

$$T = \mu + \sigma X, \tag{1}$$

where X is a standardized random variable with mean 0, variance 1, density ϕ, and cumulative distribution Φ.[3] The problem of the agent is to choose when to initiate the activity given a preferred time of completing the activity. Say that his preferred completion time is 0 and that he begins at time $-D$, such that D is the head start.

[3]We have assumed that the mean of X exists. We also assume that X has convex support such that the inverse distribution exists. The assumption that X has a variance is not strictly necessary. At the cost of some complication, it is possible to replace σ by another measure of scale such as the interquartile range.

We assume a cost function consisting of three terms.

$$C(D,T) = \alpha D + \omega T + \beta (T - D)^{+}, \tag{2}$$

where the + notation denotes the function $x^{+} = x$ if $x > 0$ and zero otherwise.

The first term is the cost of starting early. We may think of this as the opportunity cost of interrupting a prior activity. The second term captures the cost of time spent in the activity. The third term is the cost of being late.[4] We assume that the agent chooses D so as to minimise expected cost, i.e.

$$EC^{*} = \min_{D} EC(D,T) = \min_{D} \left[\alpha D + \omega \mu + \beta \int_{\frac{D-\mu}{\sigma}}^{\infty} (\mu + \sigma x - D)\phi(x)dx \right]. \tag{3}$$

Appendix A.1 shows that the first order condition for the agent's cost minimisation problem is

$$\Phi\left(\frac{D-\mu}{\sigma}\right) = 1 - \frac{\alpha}{\beta}. \tag{4}$$

We note from (4) that $\frac{\alpha}{\beta}$ is the optimal probability of being late (Bates et al., 2001). Rewriting the first order condition we find that

$$D = \mu + \sigma \Phi^{-1}\left(1 - \frac{\alpha}{\beta}\right). \tag{5}$$

This shows that the distribution of X only enters the head start though its $1 - \frac{\alpha}{\beta}$ quantile. So even though the scheduling cost function has a kink, the optimally

[4] This model is equivalent to the Small (1982) formulation except that we omit his discrete lateness penalty. See also footnote 12.

chosen head start is linear in μ and σ. By inserting the optimal head start (5) into the expected cost it is possible to obtain the optimal minimum expected cost (see appendix A.1).

$$EC^{*} = (\alpha + \omega)\mu + \beta\sigma \int_{1-\frac{\alpha}{\beta}}^{1} \Phi^{-1}(s)ds$$

Define the functional $H(\phi, \frac{\alpha}{\beta}) = \int_{1-\frac{\alpha}{\beta}}^{1} \Phi^{-1}(s)ds$. This is the average time late in the standardized distribution of durations. Then the expected cost of an agent who faces a given distribution of durations and who optimally chooses his head start is given by

$$EC^{*} = (\alpha + \omega)\mu + \beta H(\Phi, \frac{\alpha}{\beta})\sigma. \tag{6}$$

We observe that the optimal expected cost is linear in the mean and standard deviation of duration provided the standardized distribution of durations Φ is constant. It should be emphasized that this holds for *any* fixed standardized duration distribution.[5]

The first term, $(\alpha + \omega)$ is the cost of the duration of the activity, the value of time. It includes the effect of the activity itself as well as the loss from interrupting a previous activity. The value of reliability, $\beta H(\phi, \frac{\alpha}{\beta})$, is not a constant preference parameter but depends on the standardized duration distribution ϕ as well as on the scheduling parameters α and β. This is an important realization as it shows that the value of reliability depends on the setting in which it is applied. This fact has been overlooked in the past, where the value of reliability has been estimated

[5]This result has been hinted at in the literature, but until now it has only been established in some special cases (Noland and Small, 1995).

under one ϕ and then applied in another setting with a different ϕ. Given estimates of the scheduling parameters, we may calculate the value of reliability for any standardized duration distribution using (6).

3 Empirical illustration

In this section we use a travel time dataset to first estimate the mean and standard deviation of travel time as a function of the time of day. Results show that these functions are not constant as was the maintained assumption in the previous section. The standardized travel time distribution ϕ does however appear to be constant over the day. We will use this information to take a look at H.

We use data recorded over the period January 16 to May 8, 2007 at a congested radial road in Greater Copenhagen. Based on cameras and numberplate recognition, the data provide minute by minute observations of the average travel time in minutes for an 11.260 km section. We consider weekdays between 6 AM and 10 PM and discard observations where no traffic was recorded. We use data for the direction towards the city center, where there is a distinct peak in the morning. This dataset has 24 527 observations. Label these by (T_i, t_i), where T_i is travel time in minutes for the i'th observation and t_i is the time of day in minutes since midnight.

Figure 1 shows first a nonparametric kernel regression of travel time against time of day.[6] The resulting curve is an estimate of $\mu(t)$. The figure also shows the

[6]This regression has been performed using a normal second-order kernel and a bandwidth of 2.6146 minutes chosen by least squares cross-validation (Li and Racine, 2007). It seems that this

95 per cent pointwise confidence band. It is fairly tight indicating that μ is quite precisely estimated. There is a sharp morning peak at 8 AM and a smaller peak in the afternoon between 4 and 5 PM.

The lower curve estimates the standard deviation of travel time as a function of the time of day, $\sigma(t) = \sqrt{E[(T - \mu(t))^2]}$. This is achieved by performing a nonparametric regression of squared residuals $(T_i - \mu(t_i))^2$ against time of day and then taking the square root of the result.[7]

The relationship between μ and σ has some strong features as is evident from the scatter plot of $\sigma(t)$ against $\mu(t)$ in Figure 2. Most of the day there is a strong correlation between μ and σ. The characteristic bubble on the scatter plot corresponds to the end of the morning peak where the standard deviation remains at a high level while the mean travel time decreases.[8] There are clear variations in μ and σ over the day.

Using these estimated functions we have computed standardized travel times by $X_i = (T_i - \mu(t_i))/\sigma(t_i)$ such that these have zero mean and unit variance conditional on the time of day. Figure 3 shows a nonparametric estimate of the density of standardized travel time conditional on the time of day. Note that we have deliberately undersmoothed in the time of day dimension so as to exaggerate the impression of any dependency on the time of day.[9] It is possible to use the

leads to some undersmoothing outside the peak. All programming is in Ox (Doornik, 2001). The code is available on request.

[7] The cross-validated bandwidth is here 2.6873 minutes. The confidence band around the mean is computed using the bandwidth from the mean regression.

[8] This pattern has been noticed before. See, e.g., Taylor (2007).

[9] The bandwidth here is 28.8 minutes or 3 per cent of the period 6 AM to 10 PM.

cross-validation procedure to test for independence of the standardized travel time of the time of day (Li and Racine, 2007). The idea is that if the standardized travel time distribution depends on the time of day, then the cross-validation procedure will select a bandwidth in the time of day dimension that becomes small in large samples. If, on the other hand, the standardized travel time distribution does not depend on the time of day then the cross-validation procedure will select a large bandwidth in the time of day dimension.[10] Using maximum likelihood cross-validation on the density of the standardized travel times conditional on the time of day, we have accepted independence of the standardized travel time and time of day, since the bandwidth corresponding to the time of day increases without bound. This is a potentially important result. If it generalizes to other roads, then it means that the standardized travel time distribution Φ may be treated as fixed even as μ and σ vary.

We have then estimated the density of the standardized travel times.[11] The resulting estimate is shown in Figure 4, where the dashed curve indicates the lower bound of the 95 per cent pointwise confidence interval. The estimated density has a heavy right tail and resembles a lognormal or a gamma density but it is clear from the figure that it is neither.

We have computed $H(\phi, \frac{\alpha}{\beta})$ for various values of $\frac{\alpha}{\beta}$. We have done this both for the empirical distribution of standardized travel times and for the standard

[10] The bandwidth exceeds any bound with a probability that approaches 1 as sample size increases.

[11] Using again a normal second-order kernel and a bandwidth of 0.21025 chosen by least squares cross-validation.

normal distribution. The results are shown in Table 1. In this example there are large differences in H between the normal and the empirical travel time distribution, showing the importance of accounting for the actual distribution of durations. The differences are largest for small values of $\frac{\alpha}{\beta}$ when the optimal probability of being late is low. This is due to the fat right tail of the empirical travel time distribution relative to the normal distribution. However, the difference is also quite large when the optimal probability of lateness is about one half.

4 Time varying mean and standard deviation of duration

So far we have considered the distribution of durations to be independent of the head start. This is not true in general, as the previous section showed. Indeed, in that example, the mean and standard deviation of durations did depend on the head start, while the distribution of standardized durations could still be assumed to be independent of the head start. Moreover μ and σ in the example seemed to vary (more or less) linearly with the time of day for long periods of time.

We are therefore motivated to consider the situation where the mean and standard deviation of durations depend linearly on the head start. We use the following parametrization where linear functions are pivoted around the optimal head start corresponding to constant values of μ and σ.

$$\begin{aligned} D_0 &= \mu_0 + \sigma_0 \Phi^{-1}\left(1 - \frac{\alpha}{\beta}\right) \\ \mu &= \mu_0 + \mu'(D - D_0) \\ \sigma &= \sigma_0 + \sigma'(D - D_0) \end{aligned}$$

The mathematical derivations for this case are somewhat involved and are given in Appendix A.2. It turns out that the optimal expected cost is still linear in mean duration. That is,

$$\frac{dEC^*}{d\mu_0} = (\alpha + \omega),$$

which is the same result as in the case where the distribution of durations is independent of the head start. Thus in computing the marginal expected cost of mean duration we may ignore that the mean and standard deviation functions depend on the endogenous head start D.

The corresponding result for the standard deviation is more complicated. Write the value of reliability, i.e. the derivative of the optimal expected cost with respect to σ_0, as a function of the slopes μ' and σ':

$$VoR(\mu', \sigma') = \frac{dE^*}{d\sigma_0}.$$

Like the value of time, the value of reliability does not depend on the levels of the mean and standard deviation of duration in μ_0 and σ_0. Define for convenience the standardized head starts $Y = \frac{D-\mu}{\sigma}$ and $Y_0 = \Phi^{-1}(1 - \frac{\alpha}{\beta})$, where the latter is the optimal head start in the case of constant $\mu = \mu_0$ and $\sigma = \sigma_0$. Then we find that the value of reliability is given by

$$VoR(\mu', \sigma') = \alpha Y_0 - \beta Y_0(1 - \Phi(Y)) + \beta \int_Y^\infty x\phi(x)dx.$$

As would be expected, this expression reduces to $VoR(0,0) = \beta \int_{Y_0}^\infty x\phi(x)dx$, which is the same result as in Section 2. The appendix shows that $VoR(\mu', \sigma') \leq VoR(0,0)$, such that the value of reliability is overestimated if dependency of the distribution of durations on the head start is ignored. This is true regardless of the signs of μ' and σ', so it does not matter whether the upward or the downward slope of a peak is considered.

Using the independence assumption as an approximation may, however, not lead to a large error as can be seen from the following approximation, derived in Appendix A.2.

$$\frac{VoR(\mu', \sigma') - VoR(0,0)}{VoR(0,0)} \approx -\frac{1}{2\phi(Y_0)H} \left(\frac{\alpha + \omega}{\beta} \mu' + H\sigma' \right)^2$$

This formula may be used to correct an estimate of VoR based on constant μ and σ. If the discrepancy is small we may alternatively just use $VoR(0,0)$ and ignore the error. For the example in Section 3 we find the following figures. Observe from Figure 1 that $\mu' \approx 10/120 \approx 0.08$ and that $\sigma' \approx 4/120 \approx 0.03$. Use the values $(\alpha, \omega, \beta) = (1, 1, 5)$ based roughly on Small (1982).[12] From Table

[12]Note that our model is parameterized differently than that of Small. Small defines the cost function by $-\eta T - \gamma(D-T)^+ - \delta(T-D)^+$, where $(D-T)^+$ is schedule delay early and $(T-D)^+$ is schedule delay late. Note that $(D-T)^+ = (T-D)^+ - T + D$. Insert this to write

1 we find $H \approx 0.31$. Furthermore, $\phi(Y_0) \approx 0.22$. With these numbers it turns out that the relative approximation error is about -0.012, which must be considered small given the precision with which the preference parameters can be estimated.

Given that the approximation error from applying the independence assumption is small, we may use (6) to compute the share of the time costs due to reliability for a traveler in the empirical example in the previous section. Figure 5 shows this share over the day. It varies around 15 per cent which must be considered significant. Even so, it is quite conceivable that this share is higher in places with more serious congestion.

5 The case of a scheduled service

Consider now the situation where the agent is not able to choose his head start freely but has to choose from a fixed set of head starts with a fixed interval (headway) of $2h$. This case arises for example when the issue is reliability of rail or bus services (Bates et al., 2001). We assume that the timing of head starts is unrelated to his preferred completion time and retain the assumption that duration is random given by $\mu + \sigma X$, where μ and σ are now again fixed. In this situation it seems not possible to solve the cost minimization problem explicitly for general duration distributions. Still, it is possible to say something.

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the cost function as $C = -(\eta-\gamma)T - \gamma D - (\gamma+\delta)(T-D)^+$. From Small we take the parameters as roughly $(\gamma, \eta, \delta) = (1, 2, 4)$. Then in our formulation we have $\omega = \eta - \gamma = 1$, $\alpha = \gamma = 1$ and $\beta = \gamma + \delta = 5$.

Consider the expected cost function as a function of head start:

$$EC(D) = \alpha D + \omega\mu + \beta \int_{\frac{D-\mu}{\sigma}}^{\infty} (\mu + \sigma x - D)\phi(x)dx.$$

This is globally convex since $\frac{d^2EC(D)}{dD^2} = \frac{\beta}{\sigma}\phi(\frac{D-\mu}{\sigma}) > 0$. The expected cost minimizing head start will therefore always be the head start in the interval defined by the equation

$$EC(D_h - h) = EC(D_h + h),$$

since this equation identifies the interval of length $2h$ of minimal expected cost. We have fixed the preferred completion time at time 0 and taken the scheduling of head starts to be independent of everything else. We may therefore view the scheduling of head start as a uniformly distributed random variable over the interval $[D_h - h, D_h + h]$. The expected cost under such a schedule is therefore given by the following expression.[13]

$$E(EC(D)) = \frac{1}{2h} \int_{D_h-h}^{D_h+h} EC(D)dD$$

It seems not possible to find a general explicit solution for this for a general duration distribution. It is however possible in some cases to derive $E(EC(D))$ under specific assumptions about the duration distribution. It turns out that the resulting expression for $E(EC(D))$ is rather complex, and in particular it is not in general linear in the mean and standard deviation of duration. Appendix A.3 presents an

[13]Expectation is formed both with respect to the location of the schedule of head start and with respect to the duration distribution.

example of this for the case of an exponentially distributed duration.

6 Conclusion

We have established a simple relationship between the fundamental quantities from which cost or disutility is derived in Small's scheduling model and the mean and standard deviation of a distribution of durations under the optimally chosen head start. Given the marginal costs in the scheduling model, it is then possible to compute the value of reliability for any given duration distribution. Moreover, it is possible to translate the value of reliability from one duration distribution to another. The result remains a good approximation when the mean and standard deviation of duration depend on the head start while the standardized distribution must be constant.

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A Mathematical appendix

A.1 A simple model

This appendix refers to section 2. The first point is to find the first order condition for the minimization of the expected cost in (3). Recall the following general formula:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)dt = b'(x)f(x,b) - a'(x)f(x,a) + \int_{a(x)}^{b(x)} \frac{df(x,t)}{dx}dt.$$

Use this to differentiate (3) with respect to the head start D and set to zero to find the first order condition. Note here that the derivative with respect to the lower integral limit is zero, since the integrand is zero at the lower bound. The derivative with respect to the upper integral limit is also zero, since the upper integral limit is constant. The first order condition then becomes

$$\alpha = \beta\left(1 - \Phi\left(\frac{D-\mu}{\sigma}\right)\right).$$

Insert the optimal head start (5) into the expected cost to obtain the optimal expected cost.

$$
\begin{aligned}
EC^* = \quad & \alpha\left[\mu+\sigma\Phi^{-1}\left(1-\frac{\alpha}{\beta}\right)\right]+\omega\mu+\beta\int_{\Phi^{-1}\left(1-\frac{\alpha}{\beta}\right)}^{\infty}\left(\sigma x-\sigma\Phi^{-1}\left(1-\frac{\alpha}{\beta}\right)\right)\phi(x)dx \\
= \quad & (\alpha+\omega)\mu+\alpha\sigma\Phi^{-1}\left(1-\frac{\alpha}{\beta}\right) \\
& -\beta\sigma\Phi^{-1}\left(1-\frac{\alpha}{\beta}\right)\int_{\Phi^{-1}\left(1-\frac{\alpha}{\beta}\right)}^{\infty}\phi(x)dx+\beta\sigma\int_{\Phi^{-1}\left(1-\frac{\alpha}{\beta}\right)}^{\infty}x\phi(x)dx \\
= \quad & (\alpha+\omega)\mu+\alpha\sigma\Phi^{-1}\left(1-\frac{\alpha}{\beta}\right) \\
& -\beta\sigma\Phi^{-1}\left(1-\frac{\alpha}{\beta}\right)\left(1-\Phi\left(\Phi^{-1}\left(1-\frac{\alpha}{\beta}\right)\right)\right)+\beta\sigma\int_{\Phi^{-1}\left(1-\frac{\alpha}{\beta}\right)}^{\infty}x\phi(x)dx \\
= \quad & (\alpha+\omega)\mu+\beta\sigma\int_{1-\frac{\alpha}{\beta}}^{1}\Phi^{-1}(s)ds
\end{aligned}
$$

A.2 Approximation to the value of reliability

This appendix refers to Section 4 where the mean and standard deviation of duration are linear in the head start. We defined standardized head starts $Y = \frac{D-\mu}{\sigma}$ and $Y_0 = \Phi^{-1}(1-\frac{\alpha}{\beta})$ The first-order condition for the choice of head start can be expressed in compact form.

$$
0 = \alpha + \omega\mu' + \beta\int_{Y}^{\infty}(\mu' + \sigma' x - 1)\phi(x)dx
$$

It is only Y in this expression that depends on μ_0 and σ_0. So we can conclude that the derivatives of Y with respect to μ_0 and σ_0 are zero. This insight allows us to derive the marginal expected costs of μ_0 and σ_0. Multiply the first-order condition by $D - D_0$ and subtract from the expected cost (3) to obtain

$$\begin{aligned} EC^* &= \alpha D_0 + \omega\mu_0 + \beta \int_Y^\infty (\mu_0 + \sigma_0 x - D_0)\phi(x)dx \\ &= (\alpha+\omega)\mu_0 + \alpha\sigma_0 Y_0 - \beta\sigma_0 Y_0(1-\Phi(Y)) + \beta\sigma_0 \int_Y^\infty x\phi(x)dx \end{aligned}$$

We find that $\frac{dEC^*}{d\mu_0} = (\alpha+\omega)$ for any value of μ', σ'. This is the same result as in the case when μ and σ are constant.

The next point is to find the value of reliability. Differentiate the expected cost above with respect to σ_0 to obtain

$$\frac{dEC^*}{d\sigma_0} = \alpha Y_0 - \beta Y_0(1-\Phi(Y)) + \beta \int_Y^\infty x\phi(x)dx$$

Recall that the value of reliability is $\beta \int_{Y_0}^\infty x\phi(x)dx$ in the case when $\mu' = \sigma' = 0$ and note that the expression above reduces to this when $\mu' = \sigma' = 0$. As an approximation to $\frac{dEC^*}{d\sigma_0}$ it is natural to consider using $\beta \int_{Y_0}^\infty x\phi(x)dx$ since this does not require computation of Y. It is therefore of interest to consider the size of the error in using such an approximation

Denote the value of reliability by $VoR(\mu', \sigma') = \frac{dEC^*}{d\sigma_0}$. We are then concerned with the relative difference $\frac{VoR(\mu',\sigma') - VoR(0,0)}{VoR(0,0)}$ and we would like to show that this is small when μ', σ' are small.

We may obtain from the FOC that

$$\frac{dY}{d\mu'}(\mu' = \sigma' = 0) = -\frac{\alpha+\omega}{\beta\phi(Y_0)} < 0$$

and

$$\frac{dY}{d\sigma'}(\mu' = \sigma' = 0) = -\frac{\int_{1-\frac{\alpha}{\beta}}^{1} \Phi^{-1}(s)ds}{\phi(Y_0)} < 0.$$

Let z_1 be one of μ', σ'. Then

$$\frac{dVoR}{dz_1} = -\beta(Y - Y_0)\phi(Y)\frac{dY}{dz_1}.$$

This is zero at $\mu' = \sigma' = 0$ since then $Y = Y_0$. Hence the change in the marginal cost of standard deviation is small when μ', σ' are small. Differentiate again again to find

$$\frac{d^2VoR}{dz_1dz_2} = -\beta\phi(Y)\frac{dY}{dz_1}\frac{dY}{dz_2} - \beta(Y - Y_0)\phi'(Y)\frac{dY}{dz_1}\frac{dY}{dz_2} - \beta(Y - Y_0)\phi(Y)\frac{d^2Y}{dz_1dz_2}.$$

At $Y = Y_0$ this equals $-\beta\phi(Y_0)\frac{dY}{dz_1}\frac{dY}{dz_2} < 0$, so the value of reliability is locally concave in z with a local maximum at $z = 0$. This means that the value of reliability at $z \neq 0$ is overestimated by using the value at $z = 0$, regardless of the signs of z.

Given that we will be making a systematic error by using the formula derived under constant μ and σ, it is useful the assess the size of the error if we use the value of reliability at Y_0 at small values of z. Using a quadratic approximation we find that

$$\begin{aligned}
\frac{VoR(\mu',\sigma')-VoR(0,0)}{VoR(0,0)} &\approx \frac{1}{2VoR(0,0)}\left(\frac{d^2VoR}{d\mu'^2}\mu'^2+2\frac{d^2VoR}{d\mu'd\sigma'}\mu'\sigma'+\frac{d^2VoR}{d\sigma'^2}\sigma'^2\right)\\
&= -\frac{\beta\phi(Y_0)}{2\beta H}\left(\frac{dY}{d\mu'}\mu'+\frac{dY}{d\sigma'}\sigma'\right)^2\\
&= -\frac{1}{2\phi(Y_0)H}\left(\frac{\alpha+\omega}{\beta}\mu'+H\sigma'\right)^2.
\end{aligned}$$

A.3 Example with a scheduled service

This appendix presents an example of a scheduled service with an exponentially distributed duration, where $T=\mu+X$ and $X\sim\phi(x)=\lambda e^{-\lambda x}$. We note that $\Phi(x)=1-e^{-\lambda x}$ and that $\Psi(x):=\int_0^x x\phi(x)dx=\frac{1-e^{-\lambda x}}{\lambda}-xe^{-\lambda x}$. Then it may be verified that ($\omega=0$ and $\alpha=1$ are omitted)

$$EC(D)=D+\frac{\beta}{\lambda}e^{-\lambda(D-\mu)}.$$

Moreover the midpoint of the interval from which head start is chosen is

$$\mu+\frac{1}{\lambda}log(\frac{\beta}{2\lambda h}(e^{\lambda h}-e^{-\lambda h})),$$

such that the expected expected cost becomes

$$E(EC(D))=(\mu+\frac{1}{\lambda})+\frac{1}{\lambda}log(\frac{\beta}{2\lambda h}(e^{\lambda h}-e^{-\lambda h})).$$

We may interpret the first term as relating to the mean duration while the second term relates to the standard deviation of duration $\frac{1}{\lambda}$. But we note that the parameter λ that characterizes the exponential distribution also appears inside a complicated expression that multiplies the standard deviation. So in contrast to the case when head start can be chosen freely, we do not obtain that expected cost is linear in the standard deviation of duration in the case of a scheduled service.

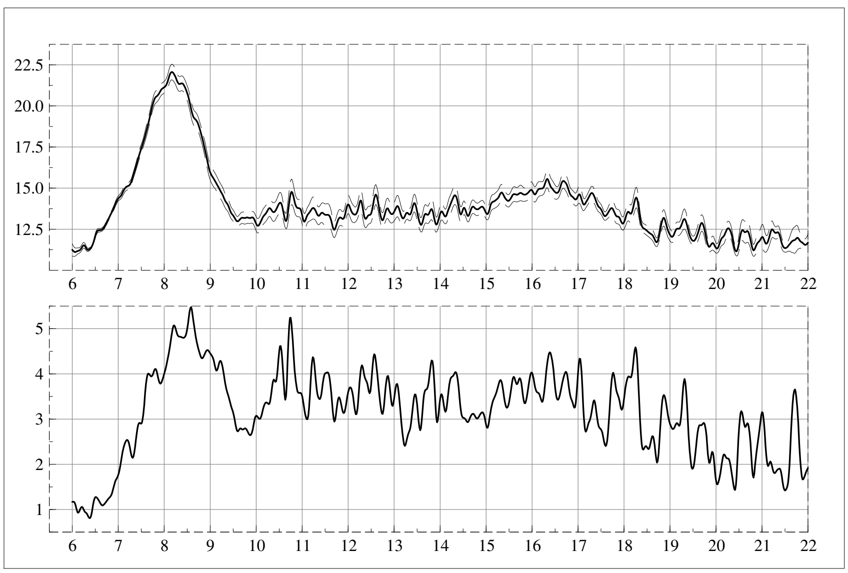

Figure 1: Mean and standard deviation of travel time (in minutes) over a weekday

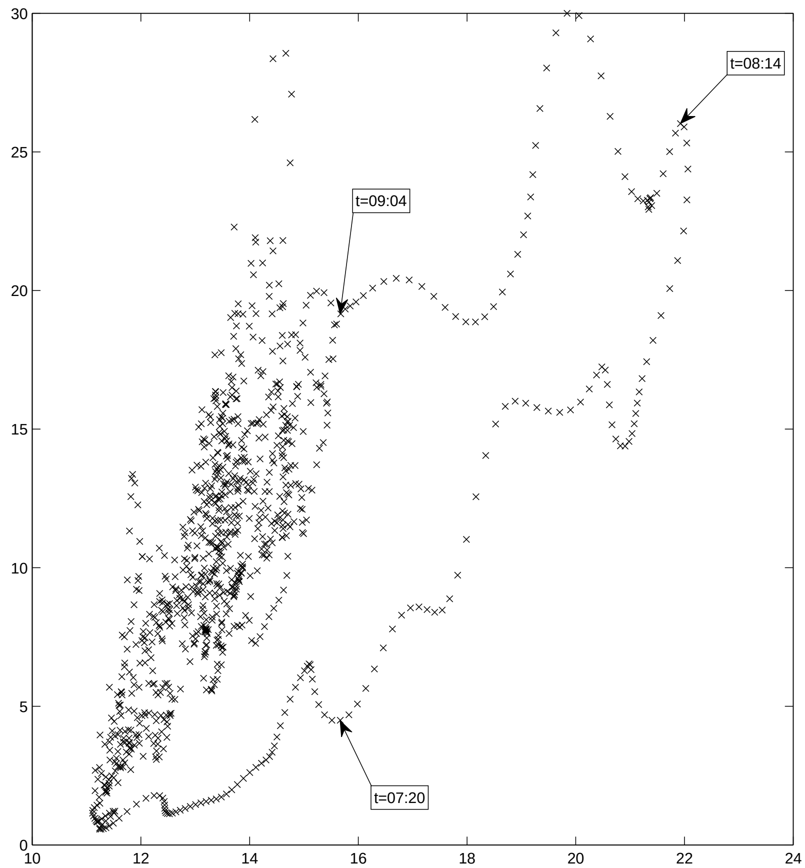

Figure 2: Scatter plot of mean and standard deviation of travel time

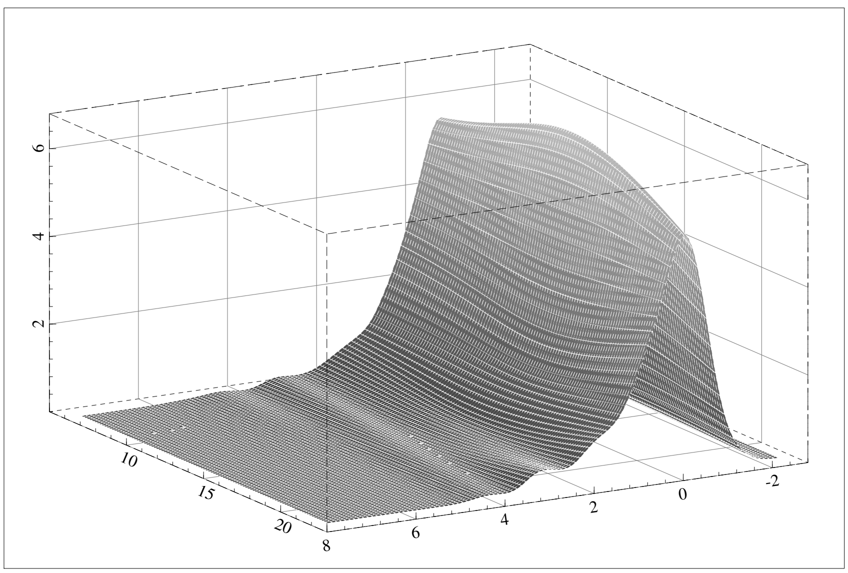

Figure 3: Density of standardized travel time conditional on time of day

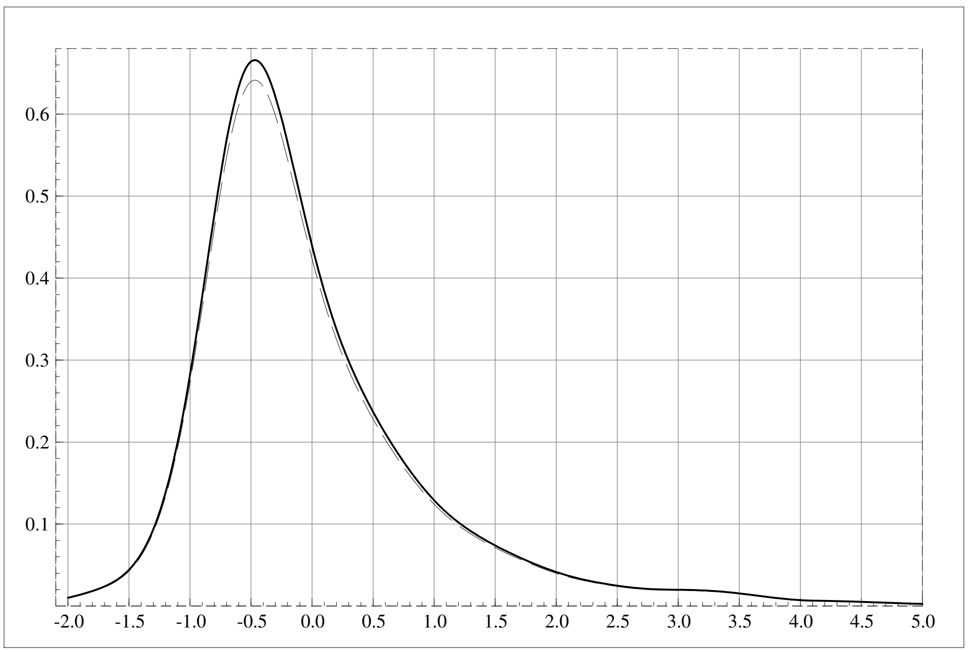

Figure 4: Density of standardized travel time

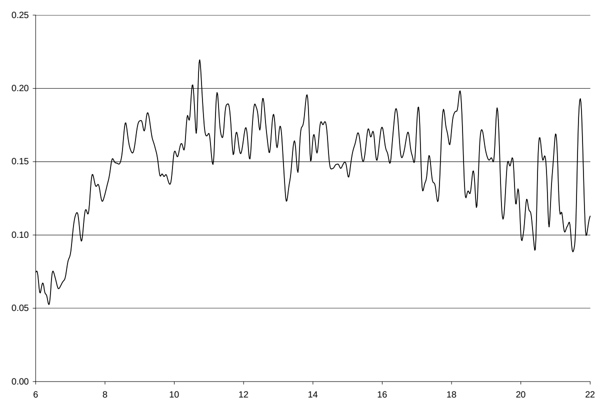

Figure 5: The share of the value of reliability in the total time cost

Table 1: H at various values of $\frac{\alpha}{\beta}$

$\frac{\alpha}{\beta}$	Normal	Empirical
0.50	0.39545	0.32838
0.25	0.31561	0.32999
0.20	0.27809	0.30831
0.15	0.23154	0.27517
0.10	0.17419	0.22593
0.05	0.10264	0.15009