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The Representative Household Assumption Requires Sustainable Heterogeneity in Dynamic Models

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Abstract

The assumption of the representative household defined as the average of all households is impossible in dynamic models if households are heterogeneous in their time preference rates because, as is well known, the most patient household eventually prevails. Because time preference rates are unquestionably heterogeneous across economies and time periods, macroeconomics studies using the representative household assumption in dynamic models are fallacious. I present an alternative definition of the representative household based on the concept of sustainable heterogeneity. By this definition, use of the representative household assumption becomes possible in dynamic models.

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Keywords: The representative household; Sustainable heterogeneity; Dynamic models; The rate of time preference; Macroeconomics

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1 INTRODUCTION

The concept of the representative household is a necessity in macroeconomic studies. It is used as a matter of course, but its theoretical foundation is fragile. The representative household has been used given the assumption that all households are identical or that there exists one specific individual household, the actions of which are always average among households (I call such a household "the average household" in this paper). The assumption that all households are identical seems to be too strict; therefore, it is usually assumed explicitly or implicitly that the representative household is the average household. However, the average household can exist only under very strict conditions. Antonelli (1886) showed that the existence of an average household requires that all households have homothetic and homogeneous utility functions. This type of utility function is not usually assumed in macroeconomic studies because it is very restrictive and unrealistic. If more general utility functions are assumed, however, the assumption of the representative household as the average household is inconsistent with the assumptions underlying the utility functions.

Nevertheless, the assumption of the representative household has been widely used, probably because it has been believed that the representative household can be interpreted as an approximation of the average household. Particularly in static models, the representative household can be seen to approximate the average household. However, in dynamic models, it is hard to accept the representative household as an approximation of the average household because, if the time preference rates of households are heterogeneous, there is no steady state where all of the optimality conditions of the heterogeneous households are satisfied (Becker, 1980). Therefore, macroeconomic studies using dynamic models are fallacious if the representative household is assumed to approximate the average household.

I offer an alternative definition of the representative household that can be used in dynamic models. This definition is based on the concept of sustainable heterogeneity shown by Harashima (2010, 2012). If sustainable heterogeneity is achieved, all of the optimality conditions of the heterogeneous households are satisfied. In addition, if it is achieved, all heterogeneous households appear to be behaving collectively as a combined supra-household. This supra-household does not fall into the dire state Becker (1980) predicts. Furthermore, the supra-household's behavior is time-consistent. Therefore, the supra-household can be seen as representing all heterogeneous households. If the representative household is defined as this supra-household, the assumption of the representative household can be introduced even in dynamic models.

This paper is organized as follows. In Section 2, the assumption of the representative household as the average household is examined. Section 3 shows the nature of sustainable heterogeneity. In Section 4, an alternative definition of the representative household that can be used in dynamic models is presented. Finally, concluding remarks are offered in Section 5.

2 IMPOSSIBILITY OF THE REPRESENTATIVE HOUSEHOLD IN DYNAMIC MODELS

2.1 The representative household in static models

Static models are usually used to analyze comparative statics. If the average household is represented by one specific unique household for any static state, there will be no problem in assuming the representative household as an approximation of the average household. Even though the average household is not always represented by one specific unique household in some states, if the average household is always represented by a household in a set of households that are very similar in preferences and other features, then the representative household assumption can be used to approximate the average household.

Suppose, for simplicity, that households are heterogeneous such that they are identical except for a particular preference. Because of the heterogeneous preference, household consumption varies. However, levels of consumption will not be distributed randomly because the distribution of consumption will correspond to the distribution of the preference. The consumption of a household that has a very different preference from the average will be very different from the average household consumption. Conversely, it is likely that the consumption of a household that has the average preference will nearly have the average consumption. In addition, the order of the degree of consumption will be almost unchanged for any static state because the order of the degree of the preference does not change for the given state.

If the order of consumption is unchanged for any given static state, it is likely that the household with consumption that is closest to the average consumption will also always be a household belonging to a group of households that have very similar preferences. Hence, it is possible to argue that, approximately, one specific unique household's consumption is always average for any static state. Of course, it is possible to show evidence that is counter to this argument, particularly in some special situations, but it is likely that this conjecture is usually true in normal situations, and the assumption that the representative household approximates the average household is acceptable in static models.

2.2 The representative household in dynamic models

In dynamic models, however, the story is more complicated. In particular, heterogeneous rates of time preference pose a serious problem. This problem is easily understood in a dynamic model with exogenous technology (i.e., a Ramsey growth model). Suppose that households are heterogeneous in time preference rate, degree of risk aversion (ε), and productivity of the labor they provide. Suppose also for simplicity that there are many "economies" in a country, and an economy consists of a household and a firm. The household provides labor to the firm in the particular economy, and the firm's level of technology (A) varies depending on the productivity of labor that the household in its economy provides. Economies trade with each other: that is, the entire economy of a country consists of many individual small economies that trade with each other.

A household maximizes its expected utility, $E\int_0^{\infty} u(c_t)\exp(-\theta t)dt$, subject to $\dot{k}_t = f(k_t) - c_t$, where $u(\bullet)$ is the utility function; $f(\bullet)$ is the production function; θ is the rate of time preference; E is the expectation operator; $y_t = \frac{Y_t}{L_t}$, $k_t = \frac{K_t}{L_t}$, and $c_t = \frac{C_t}{L_t}$; Y_t (≥ 0) is output, K_t (≥ 0) is capital input, L_t (≥ 0) is labor input, and C_t (≥ 0) is consumption in period t. The optimal consumption path of this Ramsey-type growth model is

$$\frac{\dot{c}_t}{c_t} = \varepsilon^{-1}\left(\frac{\partial y_t}{\partial k_t} - \theta\right) ,$$

and at steady state,

$$\frac{\partial y_t}{\partial k_t} = \theta \quad . \tag{1}$$

Therefore, at steady state, the heterogeneity in the degree of risk aversion (ε) is irrelevant, and the heterogeneity in productivity does not result in permanent trade imbalances among economies because $\frac{\partial y_t}{\partial k_t}$ in all economies is kept equal by market arbitrage. Hence, heterogeneity in the degree of risk aversion and productivity does not matter at steady state. Therefore, the same logic as that used for static models can be applied. Approximately, one specific unique household's consumption is always average for any time in dynamic models, even if the degree of risk aversion and the productivity are heterogeneous. Thus, the assumption of the representative household is also acceptable in dynamic models even if the degree of risk aversion and the productivity are heterogeneous.

However, equation (1) clearly indicates that heterogeneity in the rate of time preference is problematic. As Becker (1980) shows, if the rate of time preference is heterogeneous, the household that has the lowest rate of time preference will eventually possess all capital. With heterogeneous rates of time preference, there is no steady state where all households achieve all of their optimality conditions. In addition, the household with consumption that is average at present has a very different rate of time preference from the household with consumption that is average in the distant future. The consumption of a household that has the average time preference rate will initially be almost average, but in the future the household with the lowest time preference rate will be the one with consumption that is almost average. That is, the consumption path of the household that presently has average consumption is notably different from that of the household with average consumption in the future. Therefore, any individual household cannot be almost average in any period and thus cannot even approximate the average household. As a result, even if the representative household is assumed in a dynamic model, its discounted expected utility $E\int_0^\infty u(c_t)\exp(-\theta t)dt$ is meaningless, and analyses based on it are fallacious.

If we assume that the rate of time preference is identical for all households, the above problem is solved. However, this solution is still problematic because that assumption is not merely expedient for the sake of simplicity; rather, it is a critical requirement to allow for an assumed representative household. Therefore, the rationale for identical time preference rates should be validated; that is, it should be demonstrated that identical rates of time preference are actually and universally observed. The rate of time preference is, however, unquestionably not identical among households. Hence, it is difficult to accept the representative household assumption in dynamic models based on the assumption of identical time preference.

The conclusion that the representative household assumption in dynamic models is meaningless and leads to fallacious results is very important, because a huge number of studies have used the representative household assumption in dynamic models. To solve this severe problem, an alternative interpretation or definition of the representative household is needed.

Note that in an endogenous growth model the situation is even more complicated. Because a heterogeneous degree of risk aversion also matters, the assumption of the representative household is more difficult to accept, so an alternative interpretation or definition is even more important when endogenous growth models are used.

3 SUSTAINABLE HETEROGENEITY

3.1 The model

Suppose that two heterogeneous economies—economy 1 and economy 2—are identical except for their time preference rates. Households within each economy are assumed to be identical for simplicity. The population growth rate is zero. The economies are fully open to each other, and goods, services, and capital are freely transacted between them, but labor is immobilized in each economy.

Each economy can be interpreted as representing either a country (the international interpretation) or a group of identical households in a country (the national interpretation). Because the economies are fully open, they are integrated through trade and form a combined economy. The combined economy is the world economy in the international interpretation and the national economy in the national interpretation. In the following discussion, a model based on the international interpretation is called an international model and that based on the national interpretation is called a national model. Usually, the concept of the balance of payments is used only for the international transactions. However, because both national and international interpretations are possible, this concept and terminology are also used for the national models in this paper.

The rate of time preference of household in economy 1 is θ_1 and that in economy 2 is θ_2, and $\theta_1 < \theta_2$. The production function in economy 1 is $y_{1,t} = A^{\alpha} f(k_{1,t})$ and that in economy 2 is $y_{2,t} = A^{\alpha} f(k_{2,t})$, where $y_{i,t}$ and $k_{i,t}$ are, respectively, output and capital per capita in economy i in period t for i = 1, 2; A is technology; and α $(0 < \alpha < 1)$ is a constant. The population of each economy is $\frac{L}{2}$; thus, the total for both is L, which is sufficiently large. Firms operate in both economies. The current account balance in economy 1 is τ_t and that in economy 2 is $-\tau_t$. The production functions are specified as

$$y_{i,t} = A^{\alpha} k_{i,t}^{1-\alpha} \quad ;$$

thus, $Y_{i,t} = K_{i,t}^{1-\alpha} (AL)^{\alpha}$ $(i = 1,2)$. Because A is given exogenously, this model is an exogenous technology model (Ramsey growth model). The examination of sustainable heterogeneity based on an endogenous growth model is shown in Appendix.

Because both economies are fully open, returns on investments in each economy are kept equal through arbitration, such that

$$\frac{\partial y_{1,t}}{\partial k_{1,t}} = \frac{\partial y_{2,t}}{\partial k_{2,t}} \quad . \tag{2}$$

Because equation (2) always holds through arbitration, equations $k_{1,t} = k_{2,t}$, $\dot{k}_{1,t} = \dot{k}_{2,t}$, $y_{1,t} = y_{2,t}$, and $\dot{y}_{1,t} = \dot{y}_{2,t}$ also hold.

The accumulated current account balance $\int_0^t \tau_s ds$ mirrors capital flows between the two economies. The economy with current account surpluses invests them in the other economy. Because $\frac{\partial y_{1,t}}{\partial k_{1,t}} \left(= \frac{\partial y_{2,t}}{\partial k_{2,t}} \right)$ are returns on investments, $\frac{\partial y_{1,t}}{\partial k_{1,t}} \int_0^t \tau_s ds$ and $\frac{\partial y_{2,t}}{\partial k_{2,t}} \int_0^t \tau_s ds$ represent income receipts or payments on the assets that an economy owns in the other economy. Hence,

$$\tau_t - \frac{\partial y_{2,t}}{\partial k_{2,t}} \int_0^t \tau_s ds$$

is the balance on goods and services of economy 1, and

$$\frac{\partial y_{1,t}}{\partial k_{1,t}} \int_0^t \tau_s ds - \tau_t$$

is that of economy 2. Because the current account balance mirrors capital flows between the economies, the balance is a function of capital in both economies, such that

$$\tau_t = \kappa(k_{1,t}, k_{2,t}) \quad .$$

The government (or an international supranational organization) intervenes in the activities of economies 1 and 2 by transferring money from economy 1 to economy 2. The amount of transfer in period t is g_t, and it is assumed that g_t depends on capital inputs, such that

$$g_t = \bar{g} k_{1,t} \, ,$$

where $\bar{g}$ is a constant. Because $k_{1,t} = k_{2,t}$ and $\dot{k}_{1,t} = \dot{k}_{2,t}$,

$$g_t = \bar{g} k_{1,t} = \bar{g} k_{2,t} \quad .$$

Each household in economy 1 therefore maximizes its expected utility

$$E\int_0^\infty u_1(c_{1,t}) \exp(-\theta_1 t) dt \quad ,$$

subject to

$$\dot{k}_{1,t} = A^\alpha k_{1,t}^{1-\alpha} - c_{1,t} + (1-\alpha) A^\alpha k_{1,t}^{-\alpha} \int_0^t \tau_s ds - \tau_t - \bar{g} k_{1,t} \quad , \tag{3}$$

and each household in economy 2 maximizes its expected utility

$$E\int_0^\infty u_2(c_{2,t}) \exp(-\theta_2 t) dt \quad ,$$

subject to

$$\dot{k}_{2,t} = A^\alpha k_{2,t}^{1-\alpha} - c_{2,t} - (1-\alpha) A^\alpha k_{2,t}^{-\alpha} \int_0^t \tau_s ds + \tau_t + \bar{g} k_{2,t} \quad , \tag{4}$$

where $u_{i,t}$ and $c_{i,t}$, respectively, are the utility function and per capita consumption in economy i in period t for i = 1, 2; and E is the expectation operator. Equations (3) and (4) implicitly assume that each economy does not have foreign assets or debt in period t = 0.

3.2 Sustainable heterogeneity

Heterogeneity is defined as being sustainable if all of the optimality conditions of all heterogeneous households are satisfied indefinitely. First, the natures of the model when the government does not intervene (i.e., $\bar{g} = 0$) are examined. The growth rate of consumption in economy 1 is

$$\frac{\dot{c}_{1,t}}{c_{1,t}} = \varepsilon^{-1}\left\{(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha} + (1-\alpha)A^{\alpha}k_{1,t}^{-\alpha}\frac{\partial\int_0^t \tau_s ds}{\partial k_{1,t}} - \alpha(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha-1}\int_0^t \tau_s ds - \frac{\partial \tau_t}{\partial k_{1,t}} - \theta_1\right\} \cdot$$

Hence,

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \varepsilon^{-1}\lim_{t\to\infty}\left\{(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha} + (1-\alpha)A^{\alpha}k_{1,t}^{-\alpha}\frac{\partial\int_0^t \tau_s ds}{\partial k_{1,t}} - \alpha(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha-1}\int_0^t \tau_s ds - \frac{\partial \tau_t}{\partial k_{1,t}} - \theta_1\right\} = 0$$

and thereby

$$\lim_{t\to\infty}(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha}\left[1+(1-\alpha)\Psi\right] - \Xi - \theta_1 = 0 \quad ,$$

where $\Xi = \lim_{t\to\infty}\frac{\tau_t}{k_{1,t}} = \lim_{t\to\infty}\frac{\tau_t}{k_{2,t}}$ and $\Psi = \lim_{t\to\infty}\frac{\int_0^t \tau_s ds}{k_{1,t}} = \lim_{t\to\infty}\frac{\int_0^t \tau_s ds}{k_{2,t}}$ · $\lim_{t\to\infty}\frac{\dot{y}_{1,t}}{y_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}} = \lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t} = 0$, and Ψ is constant at steady state because $k_{1,t}$ and τ_t are constant; thus, $\Xi = \lim_{t\to\infty}\frac{\tau_t}{k_{1,t}}$ is constant at steady state. For Ψ to be constant at steady state, it is necessary that $\lim_{t\to\infty}\tau_t = 0$ and thus $\Xi = 0$. Therefore,

$$\lim_{t\to\infty}(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha}\left[1+(1-\alpha)\Psi\right] - \theta_1 = 0 \quad , \tag{5}$$

and

$$\lim_{t\to\infty}(1-\alpha)A^{\alpha}k_{2,t}^{-\alpha}\left[1-(1-\alpha)\Psi\right] - \theta_2 = 0 \tag{6}$$

because

$$\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \varepsilon^{-1}\lim_{t\to\infty}\left\{(1-\alpha)A^{\alpha}k_{2,t}^{-\alpha} - (1-\alpha)A^{\alpha}k_{2,t}^{-\alpha}\frac{\partial\int_0^t \tau_s ds}{\partial k_{2,t}} + \alpha(1-\alpha)A^{\alpha}k_{2,t}^{-\alpha-1}\int_0^t \tau_s ds + \frac{\partial \tau_t}{\partial k_{2,t}} - \theta_2\right\} = 0 \cdot$$

Because $\lim_{t\to\infty}(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha}\left[1+(1-\alpha)\Psi\right] = \theta_1$, $\lim_{t\to\infty}(1-\alpha)A^{\alpha}k_{2,t}^{-\alpha}\left[1-(1-\alpha)\Psi\right] = \theta_2$, and $\frac{\partial y_{1,t}}{\partial k_{1.t}} = \frac{\partial y_{2,t}}{\partial k_{2.t}} = A^{\alpha}k_{1,t}^{-\alpha} = A^{\alpha}k_{2,t}^{-\alpha}$, then

$$\Psi = \frac{\theta_1 - \theta_2}{2(1-\alpha)\lim_{t\to\infty}\frac{\partial y_{1,t}}{\partial k_{1.t}}} \quad \cdot \tag{7}$$

By equations (5) and (7),

$$\lim_{t\to\infty}\frac{\partial y_{1,t}}{\partial k_{1.t}}+\lim_{t\to\infty}\frac{\partial y_{1,t}}{\partial k_{1.t}}(1-\alpha)\Psi=\theta_1 \quad ;$$

thus,

$$\lim_{t\to\infty}\frac{\partial y_{1,t}}{\partial k_{1.t}}=\frac{\theta_1+\theta_2}{2}=\lim_{t\to\infty}\frac{\partial y_{2,t}}{\partial k_{2.t}} \quad . \tag{8}$$

If equation (8) holds, all of the optimality conditions of both economies are indefinitely satisfied. The state indicated by equation (8) is called the "multilateral steady state" or "multilateral state" in the following discussion. By procedures similar to those used for the endogenous growth model in Appendix, the condition of the multilateral steady state for H economies that are identical except for their time preference rates is shown as

$$\lim_{t\to\infty}\frac{\partial y_{i,t}}{\partial k_{i.t}}=\frac{\sum_{q=1}^{H}\theta_q}{H} \tag{9}$$

for any i, where i = 1, 2, … , H.

Because

$$\Psi=\frac{\theta_1-\theta_2}{2(1-\alpha)\lim_{t\to\infty}\frac{\partial y_{1,t}}{\partial k_{1.t}}}=\frac{\theta_1-\theta_2}{(1-\alpha)(\theta_1+\theta_2)}<0$$

by equation (8), then by $\lim_{t\to\infty}\frac{\int_0^t\tau_s ds}{k_{1,t}}=\Psi<0$,

$$\lim_{t\to\infty}\int_0^t\tau_s ds<0\,;$$

that is, economy 1 possesses accumulated debts owed to economy 2 at steady state, and economy 1 has to export goods and services to economy 2 by

$$\left|(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha}\int_0^t\tau_s ds\right|$$

in every period to pay the debts. Nevertheless, because $\lim_{t\to\infty}\tau_t=0$ and $\Xi=0$, the debts do not explode but stabilize at steady state. Because of the debts, the consumption of economy 1 is smaller than that of economy 2 at steady state under the condition of sustainable heterogeneity.

Note that many empirical studies conclude that the rate of time preference is negatively correlated with income (e.g., Lawrance, 1991; Samwick, 1998; Ventura, 2003). Suppose that, in addition to the heterogeneity in time preference rate ($\theta_1 < \theta_2$), the productivity of economy 1 is higher than that of economy 2. At steady state, the consumption of economy 1 would be larger than that of economy 2 as a result of the heterogeneity in productivity. However, as a result of the heterogeneity in the time preference rate, the consumption of economy 1 is smaller than that of economy 2 at steady state under sustainable heterogeneity. Which effect prevails will depend on differences in the degrees of heterogeneity. For example, if the difference in productivity is relatively large whereas that in time preference rate is relatively small, the effect of the productivity difference will prevail and the consumption of economy 1 will be larger than that of economy 2 at steady state under sustainable heterogeneity.

3.3 Sustainable heterogeneity with government intervention

Sustainable heterogeneity is a very different state from the one Becker (1980) described. The difference emerges because, in a multilateral state, economy 1 behaves by fully considering economy 2's conditions. The multilateral state therefore will not be naturally selected by economy 1, and the path selection may have to be decided politically (see Harashima, 2010). On the other hand, when economy 1 behaves unilaterally, the government may intervene in economic activities so as to achieve, for example, social justice.

In this section, I show that, even if economy 1 behaves unilaterally, sustainable heterogeneity can always be achieved with appropriate government intervention.

3.3.1 The two-economy model

Government intervention is first considered in the two-economy model constructed in Section 3.1. If the government intervenes (i.e., $\bar{g} > 0$),

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} \neq \lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t} = \lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds} .$$

Because $\bar{g} > 0$, equations (5) and (6) are changed to

$$\lim_{t\to\infty}(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha}\left[1+(1-\alpha)\Psi\right]-\theta_1-\bar{g}=0 , \tag{10}$$

and

$$\lim_{t\to\infty}(1-\alpha)A^{\alpha}k_{2,t}^{-\alpha}\left[1-(1-\alpha)\Psi\right]-\theta_2+\bar{g}=0 . \tag{11}$$

If economy 1 behaves unilaterally such that equation (10) is satisfied, then

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=0$$

and

$$\Psi=\frac{(\theta_1+\bar{g})\left[\lim_{t\to\infty}(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha}\right]^{-1}-1}{1-\alpha} .$$

At the same time, if economy 2 behaves unilaterally such that equation (11) is satisfied, then

$$\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=0 \quad .$$

By equations (10) and (11)

$$\bar{g}=\frac{\theta_2-\theta_1}{2}+\lim_{t\to\infty}(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha}(1-\alpha)\Psi$$

because $k_{1,t}=k_{2,t}$. In addition,

$$\lim_{t\to\infty}(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha}=\frac{\theta_1+\theta_2}{2}=\lim_{t\to\infty}\frac{\partial y_{1,t}}{\partial k_{1.t}}=\lim_{t\to\infty}\frac{\partial y_{2,t}}{\partial k_{2.t}} \quad .$$

This equation is identical to equation (8) and is satisfied at the multilateral steady state. Therefore,

$$\bar{g}=\frac{\theta_2-\theta_1}{2}+(1-\alpha)\lim_{t\to\infty}\frac{\partial y_{1,t}}{\partial k_{1.t}}\Psi=\frac{\theta_2-\theta_1}{2}+(1-\alpha)\frac{\theta_1+\theta_2}{2}\lim_{t\to\infty}\frac{\int_0^t\tau_s ds}{k_{1,t}} \quad . \tag{12}$$

If $\bar{g}$ is set equal to equation (12), all optimality conditions of both economies 1 and 2 are satisfied even though economy 1 behaves unilaterally.

There are various values of Ψ, depending on the initial consumption economy 1 sets. If economy 1 behaves in such a way as to make $\lim_{t\to\infty}\int_0^t\tau_s ds<0$, and particularly, make $\bar{g}$ = 0 such that

$$\bar{g}=\frac{\theta_2-\theta_1}{2}+(1-\alpha)\frac{\theta_1+\theta_2}{2}\lim_{t\to\infty}\frac{\int_0^t\tau_s ds}{k_{1,t}}=0 \quad ,$$

then

$$\Psi=\frac{\theta_1-\theta_2}{2(1-\alpha)\lim_{t\to\infty}\frac{\partial y_{1,t}}{\partial k_{1.t}}} \tag{13}$$

by equation (12). Equation (13) is identical to equation (7); that is, the state where equation (13) is satisfied is identical to the multilateral state with no government intervention (i.e., $\bar{g}$ = 0). On the other hand, if economy 1 behaves in such a way as to make $\lim_{t\to\infty}\int_0^t\tau_s ds=0$,

$$\bar{g}=\frac{\theta_2-\theta_1}{2}>0 \quad .$$

This condition is identical to that for sustainable heterogeneity with government intervention in the endogenous growth model shown by Harashima (2012). Furthermore, if economy 1 behaves in such a way as to make $\lim_{t\to\infty}\int_0^t \tau_s ds > 0$, $\bar{g}$ is positive and is given by equation (12).

There are various steady states, depending on the values of $\Psi = \lim_{t\to\infty}\frac{\int_0^t \tau_s ds}{k_{1,t}}$ and the initial consumption set by economy 1. Nevertheless, at any steady state that satisfies equation (13), all of the optimality conditions of economy 1 are satisfied (by government intervention, all optimality conditions of economy 2 are also satisfied). For economy 1, all steady states are equally optimal. Economy 1 selects one of the steady states (i.e., sets the initial consumption); for example, it may select the one that gives the highest expected utility, the highest steady state consumption, or some values based on other criteria. Note, however, that an overly large positive Ψ requires zero initial consumption and thus a certain upper bound of Ψ will exist.

3.3.2 The multi-economy model

In this section, for simplicity, only the case of $\Psi = \lim_{t\to\infty}\frac{\int_0^t \tau_s ds}{k_{1,t}} = 0$ is considered. It is assumed that there are H economies that are identical except for their time preference rates. If H = 2, when sustainable heterogeneity is achieved, economies 1 and 2 consist of a combined economy (economy 1+2) with twice the population and a time preference rate of $\frac{\theta_1+\theta_2}{2}$. Suppose there is a third economy with a time preference of θ_3. Because economy 1+2 has twice the population of economy 3, if

$$\bar{g} = \frac{\theta_3 - \frac{\theta_1+\theta_2}{2}}{3} ,$$

then

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \lim_{t\to\infty}\frac{\dot{c}_{3,t}}{c_{3,t}} = 0 .$$

By iterating similar procedures, if government transfer between economy H and economy 1+2+ · · · + (H – 1) is such that

$$\bar{g} = \frac{\theta_H - \frac{\sum_{q=1}^{H-1}\theta_q}{H-1}}{H} ,$$

then

$$\lim_{t\to\infty}\frac{\dot{c}_{i,t}}{c_{i,t}} = 0$$

for any i (= 1, 2, · · ·, H).

4 AN ALTERNATIVE DEFINITION OF THE REPRESENTATIVE HOUSEHOLD

4.1 The definition

Section 3 indicates that, when sustainable heterogeneity is achieved, all heterogeneous households are connected (in the sense that all households behave by considering other households' optimality) and appear to be behaving collectively as a combined supra-household that unites all households, as equations (8) and (9) indicate. The supra-household is unique and its behavior is time-consistent. Its actions always and consistently represent those of all households. Considering these natures of households under sustainable heterogeneity, I present the following alternative definition of the representative household: "the behavior of the representative household is defined as the collective behavior of all households under sustainable heterogeneity."

Even if households are heterogeneous, they can be represented by a representative household as defined above. Unlike the representative household defined as the average household, the collective representative household reaches a steady state where all households satisfy all of their optimality conditions in dynamic models. In addition, this representative household has a rate of time preference that is equal to the average rate of time preference as shown in equations (8) and (9).[1] Hence, we can assume not only a representative household but also that its rate of time preference is the average rate of all households.

4.2 Universality of sustainable heterogeneity

An important point, however, is that this alternatively defined representative household can be used in dynamic models only if sustainable heterogeneity is achieved, but this condition is not necessarily always naturally satisfied. Sustainable heterogeneity is achieved only if households with lower rates of time preference behave multilaterally or the government appropriately intervenes. Therefore, the representative household assumption is not necessarily naturally acceptable in dynamic models unless it is confirmed that sustainable heterogeneity is usually achieved in an economy.

Notwithstanding this flaw, the representative household assumption has been widely used in many macroeconomic studies that use dynamic models. Furthermore, these studies have been little criticized for using the inappropriate representative household assumption. In addition, in most economies, the dire state that Becker (1980) predicts has not been observed even though the time preference rates of households are unquestionably heterogeneous. These facts conversely indicate that sustainable heterogeneity—probably with government interventions—has been usually and universally achieved across economies and time periods. In a sense, these facts are indirect evidence that sustainable heterogeneity usually prevails in economies.

Note that because the representative household's behavior in dynamic models is represented by the collective behavior of all households under sustainable heterogeneity, the time preference rate of the representative household is not intrinsically known to households, but they do need to have an expected rate. Each household intrinsically knows its own preferences, but it does not intrinsically know the collective preference of all households. Therefore, in dynamic models, it must be assumed that all households do not *ex ante* know the time preference rate of the representative household, but households estimate it from

[1] If sustainable heterogeneity is achieved with the help of the government's intervention, the time preference rate of the representative household will not be exactly equal to the average rate of time preference.

information on the behaviors of other households and the government.

5 CONCLUDING REMARKS

In dynamic models it is hard to accept the representative household as an approximation of the average household, because if the time preference rates of households are heterogeneous there is no steady state where all of the optimality conditions of the heterogeneous households are satisfied. Therefore, macroeconomic studies using dynamic models are fallacious if the average household is used as the representative household. I offer an alternative definition of the representative household that can be used in dynamic models based on the concept of sustainable heterogeneity. If sustainable heterogeneity is achieved, all heterogeneous households look like they are behaving collectively as a combined supra-household that unites all heterogeneous households. The supra-household's behavior is time-consistent. Therefore, the supra-household can be seen as always and consistently representing all heterogeneous households. If the representative household is defined as such a supra-household under the condition of sustainable heterogeneity, it can even be used in dynamic models.

APPENDIX

Sustainable heterogeneity in an endogenous growth model

A.1 The model

A.1.1 The base model

Most endogenous growth models commonly have problems with scale effects or the influence of population growth (e.g., Jones, 1995a, b). Hence, this paper uses the model presented by Harashima (2013), which is free from both problems (see also Jones, 1995a; Aghion and Howitt, 1998; Peretto and Smulders, 2002). The production function is $Y_t = F(A_t, K_t, L_t)$, and the accumulation of capital is

$$\dot{K}_t = Y_t - C_t - v\dot{A}_t \quad , \tag{A1}$$

where Y_t is outputs, A_t is technology, K_t is capital inputs, L_t is labor inputs, C_t is consumption, $v(>0)$ is a constant, and a unit of K_t and $\frac{1}{v}$ of a unit of A_t are equivalent: that is, they are produced using the same quantities of inputs. All firms are identical and have the same size, and for any period,

$$m = \frac{M_t}{L_t} \quad , \tag{A2}$$

where M_t is the number of firms, and $m(>0)$ is a constant. In addition,

$$\frac{\partial Y_t}{\partial K_t} = \frac{\varpi}{M_t} \frac{\partial Y_t}{\partial (vA_t)} \quad ; \tag{A3}$$

thus,

$$\frac{\partial y_t}{\partial k_t} = \frac{\varpi}{mv} \frac{\partial y_t}{\partial A_t} \tag{A4}$$

is always kept, where y_t is output per capita, k_t is capital per capita, and $\varpi(>1)$ is a constant. For simplicity, the period of patent is assumed to be indefinite, and no capital depreciation is assumed. ϖ indicates the effect of patent protection. With patents, the income is distributed to not only capitals and labors but technologies. Equation (A2) indicates that population and number of firms are positively correlated. Equations (A3) and (A4) indicate that returns on investing in K_t and in A_t are kept equal and that a firm that produces a new technology cannot obtain all the returns on an investment in A_t. This means that investing in A_t increases Y_t, but the investing firm's return on the investment in A_t is only a fraction of the increase of Y_t, such that $\frac{\varpi}{M_t} \frac{\partial Y_t}{\partial (vA_t)} = \frac{\varpi}{mL_t} \frac{\partial Y_t}{\partial (vA_t)}$ because of uncompensated knowledge spillovers to other firms and complementarity of technologies.

A part of the knowledge generated as a result of an investment made by a firm spills over to other firms. Researchers in firms as well as universities and research institutions could not effectively generate innovations if they were isolated from other researchers. They contact and stimulate each other. Probably, mutual partial knowledge spillovers among researchers and firms give each other reciprocal benefits. Researchers take hints on their researches in exchange for spilled knowledge. Therefore, even though the investing firm wishes to keep its knowledge secret, some parts of it will spill over. In addition, many uncompensated knowledge spillovers occur because many technologies are regarded as so minor that they are not applied for patents and left unprotected by patents. Nevertheless, even if a technology that was generated as a byproduct is completely useless for the investing firm, it may be a treasure for firms in a different industry. A_t includes all these technologies, and an investment in technology generates many technologies that the investing firm cannot protect by patents.

Broadly speaking, there are two types of uncompensated knowledge spillovers: intra-sectoral knowledge spillovers (i.e., Marshall-Arrow-Romer [MAR] externalities; Marshall, 1890; Arrow, 1962; Romer, 1986) and inter-sectoral knowledge spillovers (i.e., Jacobs externalities; Jacobs, 1969). MAR theory assumes that knowledge spillovers between homogenous firms work out most effectively and that spillovers will therefore primarily emerge within one sector. As a result, uncompensated knowledge spillovers will be more active if the number of firms within a sector is larger. On the other hand, Jacobs (1969) argues that knowledge spillovers are most effective among firms that practice different activities and that diversification (i.e., a variety of sectors) is important for spillovers. As a result, uncompensated knowledge spillovers will be more active if the number of sectors in the economy is larger. Nevertheless, if all sectors have the same number of firms, an increase in the number of firms in the economy results in more active knowledge spillovers in any case, owing to either MAR externalities or Jacobs externalities.

Furthermore, as the volume of uncompensated knowledge spillovers increases, the investing firm's returns on the investment in A_t decrease. $\frac{\partial Y_t}{\partial A_t}$ indicates the total increase in Y_t in the economy by an increase in A_t, which consists of increases in both outputs in the firm that invested in the new technologies and outputs in other firms that utilize the newly invented technologies, whether the firms obtained the technologies by compensating the originating firm or by using uncompensated knowledge spillovers. If the number of firms becomes larger and uncompensated knowledge spillovers occur more actively, the compensated fraction in $\frac{\partial Y_t}{\partial A_t}$ that the investing firm can obtain becomes smaller, and the investing firm's returns on the investment in A_t also become smaller.

Complementarity of technologies also reduces the fraction of $\frac{\partial Y_t}{\partial A_t}$ that the investing firm can obtain. If a new technology is effective only if it is combined with some particular technologies, the return on the investment in technology will belong not only to the investing firm but to the firms that hold these particular technologies. For example, an innovation in software technology generated by a software company increases the sales and profits of computer hardware companies. The economy's productivity increases because of the innovation but the increased incomes are attributed not only to the firm that generated the innovation but also to the firms that hold complementary technologies. A part of $\frac{\partial Y_t}{\partial A_t}$ leaks to these firms. For them, the leaked income is a kind of rent revenue unexpectedly become obtainable thanks to the innovation. Most new technologies will have complementary technologies. In addition, as the number of firms increases, the number of firms that holds complementary technologies will also increase, and thereby these leaks will also increase.

Because of the uncompensated knowledge spillovers and the complementarity of technologies, therefore, the fraction of $\frac{\partial Y_t}{\partial A_t}$ that an investing firm can obtain on average will be comparatively small, i.e., ϖ will be far smaller than M_t except that M_t is very small,[2] and the fraction will decrease as M_t increases.

The production function is specified as $Y_t = A_t^{\alpha} f(K_t, L_t)$ where $\alpha\,(0 < \alpha < 1)$ is a constant. Let $y_t = \frac{Y_t}{L_t}$, $k_t = \frac{K_t}{L_t}$, $c_t = \frac{C_t}{L_t}$, and $n_t = \frac{\dot{L}_t}{L_t}$, and assume that $f(K_t, L_t)$ is homogenous of degree one. Thus $y_t = A_t^{\alpha} f(k_t)$ and $\dot{k}_t = y_t - c_t - \frac{v\dot{A}_t}{L_t} - n_t k_t$. By equation (A4), $A_t = \frac{\varpi\alpha f(k_t)}{mvf'(k_t)}$ because $\frac{\varpi\partial y_t}{mv\partial A_t} = \frac{\partial y_t}{\partial k_t} \Leftrightarrow \frac{\varpi\alpha}{mv} A_t^{\alpha-1} f(k_t) = A_t^{\alpha} f'(k_t)$.

A.1.2 Models with heterogeneous households

Three heterogeneities—heterogeneous time preference, risk aversion, and productivity—are examined in endogenous growth models, which are modified versions of the model shown in Section A.1.1. First, suppose that there are two economies— economy 1 and economy 2—that are identical except for time preference, risk aversion, or productivity. The population growth rate is zero (i.e., $n_t = 0$). The economies are fully open to each other, and goods, services, and capital are freely transacted between them, but labor is immobilized in each economy. Because both national and international interpretations are possible, the concept of the balance of payments is also used for the national models in Appendix.

A.1.2.1 Heterogeneous time preference model

First, a model in which the two economies are identical except for time preference is constructed. The rate of time preference of the representative household in economy 1 is θ_1 and that in economy 2 is θ_2, and $\theta_1 < \theta_2$. The production function in economy 1 is $y_{1,t} = A_t^{\alpha} f(k_{1,t})$ and that in economy 2 is $y_{2,t} = A_t^{\alpha} f(k_{2,t})$, where $y_{i,t}$ and $k_{i,t}$ are, respectively, output and capital per capita in economy i in period t for i = 1, 2. The population of each economy is $\frac{L_t}{2}$; thus, the total for both is L_t, which is sufficiently large. Firms operate in both economies, and the number of firms is M_t. The current account balance in economy 1 is τ_t and that in economy 2 is - τ_t. Because a balanced growth path requires Harrod neutral technological progress, the production functions are further specified as

$$y_{i,t} = A_t^{\alpha} k_{i,t}^{1-\alpha} \quad ;$$

thus, $Y_{i,t} = K_{i,t}^{1-\alpha}(A_t L_t)^{\alpha}\ (i = 1,2)$.

Because both economies are fully open, returns on investments in each economy are

[2] If M_t is very small, the value of ϖ will be far smaller than that for sufficiently large M_t, because the number of firms that can benefit from an innovation is constrained owing to very small M_t. The very small number of firms indicates that the economy is not sufficiently sophisticated, and thereby the benefit of an innovation cannot be fully realized in the economy. This constraint can be modeled as $\varpi = \tilde{\varpi}\left[1 - \left(1 - \tilde{\varpi}^{-1}\right)^{M_t}\right]$ where $\tilde{\varpi}(\geq 1)$ is a constant. Nevertheless, for sufficiently large M_t (i.e., in sufficiently sophisticated economies), the constraint is removed such that $\lim_{M_t \to \infty} \tilde{\varpi}\left[1 - \left(1 - \tilde{\varpi}^{-1}\right)^{M_t}\right] = \tilde{\varpi} = \varpi$.

kept equal through arbitration such that

$$\frac{\partial y_{1,t}}{\partial k_{1,t}} = \frac{\varpi}{2mv} \frac{\partial\left(y_{1,t} + y_{2,t}\right)}{\partial A_t} = \frac{\partial y_{2,t}}{\partial k_{2,t}} \quad . \tag{A5}$$

Equation (A5) indicates that an increase in A_t enhances outputs in both economies such that $\frac{\partial Y_{i,t}}{\partial K_{i,t}} = \frac{\varpi}{M_t} \frac{\partial\left(Y_{1,t} + Y_{2,t}\right)}{\partial(vA_t)}$, and because the population is equal ($\frac{L_t}{2}$), $\frac{\partial Y_{i,t}}{\partial K_{i,t}} = \frac{\partial y_{i,t}}{\partial k_{i,t}} = \frac{\varpi}{M_t} \frac{\partial\left(Y_{1,t} + Y_{2,t}\right)}{\partial(vA_t)} = \frac{\varpi}{mL_t} \frac{\partial\left(y_{1,t} + y_{2,t}\right)}{\partial(vA_t)} \frac{L_t}{2} = \frac{\varpi}{2mv} \frac{\partial\left(y_{1,t} + y_{2,t}\right)}{\partial A_t}$. Therefore,

$$A_t = \frac{\varpi\alpha\left[f\left(k_{1,t}\right) + f\left(k_{2,t}\right)\right]}{2mvf'\left(k_{1,t}\right)} = \frac{\varpi\alpha\left[f\left(k_{1,t}\right) + f\left(k_{2,t}\right)\right]}{2mvf'\left(k_{2,t}\right)} \quad .$$

Because equation (A5) is always held through arbitration, equations $k_{1,t} = k_{2,t}$, $\dot{k}_{1,t} = \dot{k}_{2,t}$, $y_{1,t} = y_{2,t}$ and $\dot{y}_{1,t} = \dot{y}_{2,t}$ are also held. Hence,

$$A_t = \frac{\varpi\alpha f\left(k_{1,t}\right)}{mvf'\left(k_{1,t}\right)} = \frac{\varpi\alpha f\left(k_{2,t}\right)}{mvf'\left(k_{2,t}\right)} \quad .$$

In addition, because $\frac{\partial\left(y_{1,t} + y_{2,t}\right)}{\partial A_{1,t}} = \frac{\partial\left(y_{1,t} + y_{2,t}\right)}{\partial A_{2,t}}$ through arbitration, then $\dot{A}_{1,t} = \dot{A}_{2,t}$ is held.

The accumulated current account balance $\int_0^t \tau_s ds$ mirrors capital flows between the two economies. The economy with current account surpluses invests them in the other economy. Since $\frac{\partial y_{1,t}}{\partial k_{1,t}} \left(= \frac{\partial y_{2,t}}{\partial k_{2,t}} \right)$ are returns on investments, $\frac{\partial y_{1,t}}{\partial k_{1,t}} \int_0^t \tau_s ds$ and $\frac{\partial y_{2,t}}{\partial k_{2,t}} \int_0^t \tau_s ds$ represent income receipts or payments on the assets that an economy owns in the other economy. Hence,

$$\tau_t - \frac{\partial y_{2,t}}{\partial k_{2,t}} \int_0^t \tau_s ds$$

is the balance on goods and services of economy 1, and

$$\frac{\partial y_{1,t}}{\partial k_{1,t}} \int_0^t \tau_s ds - \tau_t$$

is that of economy 2. Because the current account balance mirrors capital flows between the economies, the balance is a function of capital in both economies such that

$$\tau_t = \kappa\left(k_{1,t}, k_{2,t}\right) \quad .$$

The government (or an international supranational organization) intervenes in activities of economies 1 and 2 by transferring money from economy 1 to economy 2. The amount of transfer in period t is g_t and it is assumed that g_t depends on capitals such that

$$g_t = \bar{g} k_{1,t}$$

where $\bar{g}$ is a constant. Because $k_{1,t} = k_{2,t}$ and $\dot{k}_{1,t} = \dot{k}_{2,t}$,

$$g_t = \bar{g} k_{1,t} = \bar{g} k_{2,t} \quad .$$

The representative household in economy 1 maximizes its expected utility

$$E \int_0^\infty u_1(c_{1,t}) \exp(-\theta_1 t) dt \quad ,$$

subject to

$$\begin{aligned} \dot{k}_{1,t} &= y_{1,t} + \frac{\partial y_{2,t}}{\partial k_{2,t}} \int_0^t \tau_s \, ds - \tau_t - c_{1,t} - g_t - v \dot{A}_{1,t} \left(\frac{L_t}{2} \right)^{-1} \\ &= y_{1,t} + \frac{\partial y_{2,t}}{\partial k_{2,t}} \int_0^t \tau_s \, ds - \tau_t - c_{1,t} - \bar{g} k_{1,t} - v \dot{A}_{1,t} \left(\frac{L_t}{2} \right)^{-1} \quad , \end{aligned} \tag{A6}$$

and the representative household in economy 2 maximizes its expected utility

$$E \int_0^\infty u_2(c_{2,t}) \exp(-\theta_2 t) dt \quad ,$$

subject to

$$\begin{aligned} \dot{k}_{2,t} &= y_{2,t} - \frac{\partial y_{1,t}}{\partial k_{1,t}} \int_0^t \tau_s ds + \tau_t - c_{2,t} + g_t - v \dot{A}_{2,t} \left(\frac{L_t}{2} \right)^{-1} \\ &= y_{2,t} - \frac{\partial y_{1,t}}{\partial k_{1,t}} \int_0^t \tau_s ds + \tau_t - c_{2,t} + \bar{g} k_{2,t} - v \dot{A}_{2,t} \left(\frac{L_t}{2} \right)^{-1} \quad , \end{aligned} \tag{A7}$$

where $u_{i,t}$, $c_{i,t}$, and $\dot{A}_{i,t}$, respectively, are the utility function, per capita consumption, and the increase in A_t by R&D activities in economy i in period t for i = 1, 2; E is the expectation operator; and $\dot{A}_t = \dot{A}_{1,t} + \dot{A}_{2,t}$. Equations (A6) and (A7) implicitly assume that each economy does not have foreign assets or debt in period t = 0.

Because the production function is Harrod neutral and because $A_t = \frac{\varpi \alpha f(k_{1,t})}{mvf'(k_{1,t})} = \frac{\varpi \alpha f(k_{2,t})}{mvf'(k_{2,t})}$ and $f = k_{i,t}^{1-\alpha}$, then

$$A_t = \frac{\varpi \alpha}{mv(1-\alpha)} k_{i,t}$$

and

$$\frac{\partial y_{i,t}}{\partial k_{i,t}} = \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha} \quad .$$

Since $\dot{A}_{1,t} = \dot{A}_{2,t}$ and $\frac{\partial y_{1,t}}{\partial k_{1,t}} = \frac{\partial y_{2,t}}{\partial k_{2,t}}$, then

$$\dot{k}_{1,t} = y_{1,t} + \frac{\partial y_{1,t}}{\partial k_{1,t}}\int_0^t \tau_s ds - \tau_t - c_{1,t} - \bar{g}k_{1,t} - \frac{v\dot{A}_t}{2}\left(\frac{L_t}{2}\right)^{-1}$$
$$= \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}k_{1,t} + \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds - \tau_t - c_{1,t} - \bar{g}k_{1,t} - \frac{\varpi\alpha}{mL_t(1-\alpha)}\dot{k}_{1,t}$$

and

$$\dot{k}_{1,t} = \frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\varpi\alpha}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}k_{1,t} + \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds - \tau_t - c_{1,t} - \bar{g}k_{1,t}\right] \quad .$$

Because L_t is sufficiently large and ϖ is far smaller than M_t, the problem of scale effects vanishes and thereby $\frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\varpi\alpha} = 1$.

Putting the above elements together, the optimization problem of economy 1 can be rewritten as

$$Max\ E\int_0^{\infty} u_1(c_{1,t})\exp(-\theta_1 t)dt \quad ,$$

subject to

$$\dot{k}_{1,t} = \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}k_{1,t} + \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds - \tau_t - c_{1,t} - \bar{g}k_{1,t} \quad .$$

Similarly, that of economy 2 can be rewritten as

$$Max\ E\int_0^{\infty} u_2(c_{2,t})\exp(-\theta_2 t)dt \quad ,$$

subject to

$$\dot{k}_{2,t} = \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}k_{2,t} - \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds + \tau_t - c_{2,t} + \bar{g}k_{2,t} \quad .$$

A.1.2.2 Heterogeneous risk aversion model

The basic structure of the model with heterogeneous risk aversion is the same as that of heterogeneous time preference. The two economies are identical except in regard to risk aversion. The degree of relative risk aversion of economy 1 is $\varepsilon_1 = -\frac{c_{1,t}\, u_1''}{u_1'}$ and that of economy 2 is $\varepsilon_2 = -\frac{c_{2,t}\, u_2''}{u_2'}$, which are constant, and $\varepsilon_1 < \varepsilon_2$. The optimization problem of economy 1 is

$$Max\ E\int_0^\infty u_1(c_{1,t})\exp(-\theta t)dt \quad ,$$

subject to

$$\dot{k}_{1,t} = \left(\frac{\varpi\alpha}{mv}\right)^\alpha (1-\alpha)^{-\alpha} k_{1,t} + \left(\frac{\varpi\alpha}{mv}\right)^\alpha (1-\alpha)^{1-\alpha} \int_0^t \tau_s ds - \tau_t - c_{1,t} - \bar{g}k_{1,t} \quad ,$$

and that of economy 2 is

$$Max\ E\int_0^\infty u_2(c_{2,t})\exp(-\theta t)dt \quad ,$$

subject to

$$\dot{k}_{2,t} = \left(\frac{\varpi\alpha}{mv}\right)^\alpha (1-\alpha)^{-\alpha} k_{2,t} - \left(\frac{\varpi\alpha}{mv}\right)^\alpha (1-\alpha)^{1-\alpha} \int_0^t \tau_s ds + \tau_t - c_{2,t} + \bar{g}\, k_{2,t} \quad .$$

A.1.2.3 Heterogeneous productivity model

With heterogeneous productivity, the production function is heterogeneous, not the utility function. Because technology A_t is common to both economies, a heterogeneous production function requires heterogeneity in elements other than technology. Prescott (1998) argues that unknown factors other than technology have made total factor productivity (TFP) heterogeneous across countries. Harashima (2009) argues that average workers' innovative activities are an essential element of productivity and make TFP heterogeneous across workers, firms, and economies. Since average workers are human and capable of creative intellectual activities, they can create innovations even if their innovations are minor. It is rational for firms to exploit all the opportunities that these ordinary workers' innovative activities offer. Furthermore, innovations created by ordinary workers are indispensable for efficient production. A production function incorporating average workers' innovations has been shown to have a Cobb-Douglas functional form with a labor share of about 70% (Harashima 2009), such that

$$Y_t = \bar{\sigma}\omega_A\omega_L A_t^\alpha K_t^{1-\alpha} L_t^\alpha \quad , \tag{A8}$$

where ω_A and ω_L are positive constant parameters with regard to average workers' creative activities, and $\bar{\sigma}$ is a parameter that represents a worker's accessibility limit to capital with regard to location. The parameters ω_A and ω_L are independent of A_t but are dependent on the creative activities of average workers. Thereby, unlike with technology A_t, these parameters can be heterogeneous across workers, firms, and economies.

In this model of heterogeneous productivity, it is assumed that workers whose households belong to different economies have different values of ω_A and ω_L. In addition, only productivity that is represented by $\bar{\sigma}\omega_A\omega_L A_t^\alpha$ in equation (A8) is heterogeneous between the two economies. The production function of economy 1 is $y_{1,t} = \omega_1^\alpha A_t^\alpha f(k_{1,t})$ and that of economy 2 is $y_{2,t} = \omega_2^\alpha A_t^\alpha f(k_{2,t})$, where $\omega_1\,(0 < \omega_1 \le 1)$ and $\omega_2\,(0 < \omega_2 \le 1)$ are constants and $\omega_2 < \omega_1$. Since $\frac{\partial Y_{i,t}}{\partial K_{i,t}} = \frac{\partial y_{i,t}}{\partial k_{i,t}} = M_t^{-1}\frac{\partial(Y_{1,t}+Y_{2,t})}{\partial(vA_t)} = \frac{\varpi}{mL_t}\frac{\partial(y_{1,t}+y_{2,t})}{\partial(vA_t)}\frac{L_t}{2} = \frac{\varpi}{2mL_t}\frac{\partial(y_{1,t}+y_{2,t})}{\partial A_t}$ by equation (A5), then

$$A_t = \frac{\varpi\alpha\left[\omega_1^\alpha f(k_{1,t}) + \omega_2^\alpha f(k_{2,t})\right]}{2mv\,\omega_1^\alpha f'(k_{1,t})} = \frac{\varpi\alpha\left[\omega_1^\alpha f(k_{1,t}) + \omega_2^\alpha f(k_{2,t})\right]}{2mv\,\omega_2^\alpha f'(k_{2,t})} \quad . \qquad \text{(A9)}$$

Because equation (A5) is always held through arbitration, equations $k_{1,t} = \frac{\omega_1}{\omega_2}k_{2,t}$, $\dot{k}_{1,t} = \frac{\omega_1}{\omega_2}\dot{k}_{2,t}$, $y_{1,t} = \frac{\omega_1}{\omega_2}y_{2,t}$, and $\dot{y}_{1,t} = \frac{\omega_1}{\omega_2}\dot{y}_{2,t}$ are also held. In addition, since $\frac{\partial(y_{1,t}+y_{2,t})}{\partial A_{1,t}} = \frac{\partial(y_{1,t}+y_{2,t})}{\partial A_{2,t}}$ by arbitration, $\dot{A}_{1,t} = \frac{\omega_1}{\omega_2}\dot{A}_{2,t}$ is held. Because of equation (A9) and $f = \omega_i^\alpha k_{i,t}^{1-\alpha}$, then $A_t = \frac{\varpi\alpha}{2mv(1-\alpha)\omega_1^\alpha}\left(\omega_1^\alpha k_1 + \omega_2^\alpha k_1^\alpha k_2^{1-\alpha}\right) = \frac{\varpi\alpha}{2mv(1-\alpha)\omega_2^\alpha}\left(\omega_1^\alpha k_1^{1-\alpha}k_2^\alpha + \omega_2^\alpha k_2\right)$, $\frac{\omega_1^\alpha k_1 + \omega_2^\alpha k_1^\alpha k_2^{1-\alpha}}{\omega_1^\alpha} = \frac{\omega_1^\alpha k_1^{1-\alpha}k_2^\alpha + \omega_2^\alpha k_2}{\omega_2^\alpha}$, and $\frac{\partial y_{i,t}}{\partial k_{i,t}} = \left(\frac{\varpi\alpha}{2mv}\right)^\alpha (1-\alpha)^{1-\alpha}\left(\omega_1^\alpha k_1 + \omega_2^\alpha k_1^\alpha k_2^{1-\alpha}\right)^\alpha k_1^{-\alpha} = \left(\frac{\varpi\alpha}{2mv}\right)^\alpha (1-\alpha)^{1-\alpha}\left(\omega_1^\alpha k_1^{1-\alpha}k_2^\alpha + \omega_2^\alpha k_2\right)^\alpha k_2^{-\alpha}$. Since $\frac{\omega_2}{\omega_1}k_{1,t} = k_{2,t}$, then $\frac{\omega_1^\alpha k_1 + \omega_2^\alpha k_1^\alpha k_2^{1-\alpha}}{\omega_1^\alpha} = \frac{\omega_1^\alpha k_1 + \omega_2^\alpha k_1^\alpha\left(\frac{\omega_2}{\omega_1}\right)^{1-\alpha}k_1^{1-\alpha}}{\omega_1^\alpha} = k_1\left(1+\omega_1^{-1}\omega_2\right)$ and $\frac{\omega_1^\alpha k_1^{1-\alpha}k_2^\alpha + \omega_2^\alpha k_2}{\omega_2^\alpha} = \frac{\omega_1^\alpha k_1^{1-\alpha}\left(\frac{\omega_2}{\omega_1}\right)^\alpha k_1^\alpha + \omega_2^\alpha\frac{\omega_2}{\omega_1}k_1}{\omega_2^\alpha} = k_1 + \frac{\omega_2}{\omega_1}k_1 = k_1\left(1+\omega_1^{-1}\omega_2\right) = k_2\left(1+\omega_1\omega_2^{-1}\right)$. Hence,

$$A_t = k_1\frac{\varpi\alpha\left(1+\omega_1^{-1}\omega_2\right)}{2mv(1-\alpha)} = k_2\frac{\varpi\alpha\left(1+\omega_1\omega_2^{-1}\right)}{2mv(1-\alpha)} \quad ,$$

and

$$\frac{\partial y_{i,t}}{\partial k_{i,t}} = \left(\frac{\omega_1+\omega_2}{2}\right)^\alpha\left(\frac{\varpi\alpha}{mv}\right)^\alpha(1-\alpha)^{1-\alpha}$$

for i = 1, 2. Because $\dot{A}_{1,t} = \left(\frac{\omega_2}{\omega_1}\right)^{-\frac{1}{\alpha}}\dot{A}_{2,t}$ (i.e., $\dot{A}_t = \dot{A}_{1,t} + \dot{A}_{2,t} = \left(1+\omega_1^{-1}\omega_2\right)\dot{A}_{1,t}$) and $\frac{\partial y_{1,t}}{\partial k_{1,t}} = \frac{\partial y_{2,t}}{\partial k_{2,t}}$, then

$$\dot{k}_{1,t} = y_{1,t} + \frac{\partial y_{1,t}}{\partial k_{1,t}} \int_0^t \tau_s ds - \tau_t - c_{1,t} - g_t - v\dot{A}_{1,t}\left(\frac{L_t}{2}\right)^{-1}$$

$$= y_{1,t} + \frac{\partial y_{1,t}}{\partial k_{1,t}} \int_0^t \tau_s ds - \tau_t - c_{1,t} - g_t - v\dot{A}_t\left(1+\omega_1^{-1}\omega_2\right)^{-1}\left(\frac{L_t}{2}\right)^{-1}$$

$$= \omega_1^{\alpha}\left[\frac{\left(1+\omega_1^{-1}\omega_2\right)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha} k_{1,t} + \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds - \tau_t - c_{1,t} - g_t - \frac{\varpi\alpha}{mL_t(1-\alpha)}\dot{k}_{1,t} \quad ,$$

and

$$\dot{k}_{1,t} = \frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\varpi\alpha}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha} k_{1,t} + \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds - \tau_t - c_{1,t} - g_t\right\} \quad .$$

Because L_t is sufficiently large and ϖ is far smaller than M_t and thus $\frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\varpi\alpha} = 1$, the optimization problem of economy 1 is

$$Max\ E\int_0^{\infty} u_1\left(c_{1,t}\right)\exp(-\theta t)dt \quad ,$$

subject to

$$\dot{k}_{1,t} = \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha} k_{1,t} + \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds - \tau_t - c_{1,t} - g_t \quad ,$$

and similarly, that of economy 2 is

$$Max\ E\int_0^{\infty} u_2\left(c_{2,t}\right)\exp(-\theta t)dt \quad ,$$

subject to

$$\dot{k}_{2,t} = \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha} k_{2,t} - \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds + \tau_t - c_{2,t} + g_t \quad .$$

A.2 The multilateral path

In this section, the growth path that makes heterogeneity sustainable is examined. First, the basic natures of the models presented in Section 2.1 when the government does not intervene, i.e., $\bar{g} = 0$ are examined.

A.2.1 Sustainability

Because balanced growth is the focal point for the growth path analysis, the following analyses focus on the steady state such that $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}$, $\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}$, $\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}}$, $\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}$, and $\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}$ are constants.

A.2.1.1 Heterogeneous time preference model

The balanced growth path in the heterogeneous time preference model has the following properties.

Lemma A1-1: In the model of heterogeneous time preference, if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} =$ constant, then

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}} = \lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}} = \lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t} = \lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds} .$$

Proof: See Harashima (2010)

Proposition A1-1: In the model of heterogeneous time preference, if and only if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} =$ constant, all the optimality conditions of both economies are satisfied at steady state.

The path on which $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} =$ constant has the following properties.

Proof: See Harashima (2010)

Corollary A1-1: In the model of heterogeneous time preference, if and only if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} =$ constant, then

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}} = \lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}} = \lim_{t\to\infty}\frac{\dot{y}_{1,t}}{y_{1,t}} = \lim_{t\to\infty}\frac{\dot{y}_{2,t}}{y_{2,t}} = \lim_{t\to\infty}\frac{\dot{A}_t}{A_t} = \text{constant.}$$

Proof: See Harashima (2010)

Note that the limit of the growth rate on this path is

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \varepsilon^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \frac{\theta_1+\theta_2}{2}\right] . \tag{A10}$$ [3]

Corollary A2-1: In the model of heterogeneous time preference, if and only if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} =$

[3] See Harashima (2010)

$\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \text{constant},$

$$\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t} = \lim_{t\to\infty}\frac{\frac{d\int_0^t \tau_s ds}{dt}}{\int_0^t \tau_s ds} = \lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}} = \lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}$$

$$= \lim_{t\to\infty}\frac{\dot{y}_{1,t}}{y_{1,t}} = \lim_{t\to\infty}\frac{\dot{y}_{2,t}}{y_{2,t}} = \lim_{t\to\infty}\frac{\dot{A}_t}{A_t} = \text{constant.}$$

Proof: See Harashima (2010)

Because current account imbalances eventually grow at the same rate as output, consumption, and capital on the multilateral path, the ratios of the current account balance to output, consumption, and capital do not explode, but they stabilize as shown in the proof of Proposition A1-1; that is, $\lim_{t\to\infty}\frac{\tau_t}{k_{1,t}} = \lim_{t\to\infty}\frac{\tau_t}{k_{2,t}} = \Xi$.

On the balanced growth path satisfying Proposition A1-1 and Corollaries A1-1 and A2-1, heterogeneity in time preference is sustainable by definition because all the optimality conditions of the two economies are indefinitely satisfied. The balanced growth path satisfying Proposition A1-1 and Corollaries A1-1 and A2-1 is called the "multilateral balanced growth path" or (more briefly) the "multilateral path" in the following discussion. The term "multilateral" is used even though there are only two economies, because the two-economy models shown can easily be extended to the multi-economy models shown in Section A.2.4.

Because technology will not decrease persistently (i.e., $\lim_{t\to\infty}\frac{\dot{A}_t}{A_t} > 0$), only the case such that $\lim_{t\to\infty}\frac{\dot{A}_t}{A_t} > 0$ (i.e., $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} > 0$ on the multilateral path by Corollary A1-1) is examined in the following discussion.

A.2.1.2 Heterogeneous risk aversion model

On the multilateral path in the heterogeneous risk aversion model, the same Proposition, Lemma, and Corollaries are proved by arguments similar to those shown in Section A.2.1.1.

Lemma A1-2: In the model of heterogeneous risk aversion, if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} =$ constant,

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}} = \lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}} = \lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t} = \lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds} .$$

Proposition A1-2: In the model of heterogeneous risk aversion, if and only if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} =$

$\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=$ constant, all the optimality conditions of both economies are satisfied at steady state.

Corollary A1-2: In the model of heterogeneous risk aversion, if and only if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=$ $\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=$ constant,

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}}=\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}=\lim_{t\to\infty}\frac{\dot{y}_{1,t}}{y_{1,t}}\lim_{t\to\infty}\frac{\dot{y}_{2,t}}{y_{2,t}}=\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}=\text{constant.}$$

Note that the limit of the growth rate on this path is

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}-\theta\right]\ . \quad \text{(A11)}$$
[4]

Corollary A2-2: In the model of heterogeneous risk aversion, if and only if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=$ $\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=$ constant,

$$\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}=\lim_{t\to\infty}\frac{\frac{d\int_0^t\tau_s ds}{dt}}{\int_0^t\tau_s ds}=\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}}=\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}$$
$$=\lim_{t\to\infty}\frac{\dot{y}_{1,t}}{y_{1,t}}=\lim_{t\to\infty}\frac{\dot{y}_{2,t}}{y_{2,t}}=\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}=\text{constant.}$$

On the balanced growth path satisfying Proposition A1-2 and Corollaries A1-2 and A2-2, heterogeneity in risk aversion is also sustainable by definition because all the optimality conditions of the two economies are indefinitely satisfied, and this path is the multilateral path.

A.2.1.3 Heterogeneous productivity model

Similar Proposition, Lemma, and Corollaries also hold in the heterogeneous productivity model. However, unlike heterogeneous preferences, $\lim_{t\to\infty}\tau_t=0$ and $\lim_{t\to\infty}\int_0^t\tau_s ds=0$ are possible even if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}$ as shown in Harashima (2010). Therefore, the case of $\lim_{t\to\infty}\tau_t=0$ and $\lim_{t\to\infty}\int_0^t\tau_s ds=0$ will be dealt with separately from the case of $\lim_{t\to\infty}\tau_t\neq 0$ and $\lim_{t\to\infty}\int_0^t\tau_s ds\neq 0$ if necessary.

[4] See Harashima (2010)

Lemma A1-3: In the model of heterogeneous productivity, if $\lim_{t\to\infty}\tau_t = 0$ and $\lim_{t\to\infty}\int_0^t \tau_s ds = 0$, then if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} =$ constant,

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}} = \lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}} \quad ,$$

and if $\lim_{t\to\infty}\tau_t \neq 0$ and $\lim_{t\to\infty}\int_0^t \tau_s ds \neq 0$, then if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} =$ constant,

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}} = \lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}$$

and

$$\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t} = \lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds} = \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha} \quad .$$

By Lemma A1-3, if all the optimality conditions of both economies are satisfied, either

$$\lim_{t\to\infty}\frac{\tau_t}{k_{1,t}} = \lim_{t\to\infty}\frac{\int_0^t \tau_s ds}{k_{1,t}} = 0 \tag{A12}$$

or

$$\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t} = \lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds} = \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha} \quad . \tag{A13}$$

Proposition A1-3: If and only if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} =$ constant, all the optimality conditions of both economies are satisfied at steady state.

Corollary A1-3: In the model of heterogeneous productivity, if and only if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} =$ $\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} =$ constant,

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}}=\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}=\lim_{t\to\infty}\frac{\dot{y}_{1,t}}{y_{1,t}}=\lim_{t\to\infty}\frac{\dot{y}_{2,t}}{y_{2,t}}=\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}=\text{constant.}$$

Corollary A2-3: In the model of heterogeneous productivity, if $\lim_{t\to\infty}\tau_t \neq 0$ and $\lim_{t\to\infty}\int_0^t \tau_s ds \neq 0$, then if and only if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=$ constant,

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}}=\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}=\lim_{t\to\infty}\frac{\dot{y}_{1,t}}{y_{1,t}}=\lim_{t\to\infty}\frac{\dot{y}_{2,t}}{y_{2,t}}=\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}=\text{constant}$$

and

$$\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}=\lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds}=\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha} \quad .$$

On the two balanced growth paths satisfying Proposition A1-3 and Corollaries A1-3 and A2-3, heterogeneity in productivity is sustainable by definition because all the optimality conditions of the two economies are indefinitely satisfied.

As shown in Harashima (2010), the limit of the growth rate on these sustainable paths is

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\varepsilon^{-1}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha}-\theta\right\} \quad .$$

A.2.2 The balance of payments

A.2.2.1 Heterogeneous time preference model

As shown in the proof of Proposition A1-1, $\lim_{t\to\infty}\frac{\tau_t}{k_{1,t}}=\lim_{t\to\infty}\frac{\tau_t}{k_{2,t}}=\Xi$ and $\lim_{t\to\infty}\frac{\int_0^t \tau_s ds}{k_{1,t}}=\lim_{t\to\infty}\frac{\int_0^t \tau_s ds}{k_{2,t}}=\Xi\left(\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}\right)^{-1}$ on the multilateral path. Because $k_{i,t}$ is positive, if the sign of Ξ is negative, the current account of economy 1 will eventually show permanent deficits and vice versa.

Lemma A2-1: In the model of heterogeneous time preference,

$$\Xi=\frac{\theta_1-\theta_2}{2}\left\{\varepsilon\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}-\frac{\theta_1+\theta_2}{2}\right]^{-1}-1\right\}^{-1} \quad .$$

Proof: See Harashima (2010)

Lemma A2-1 indicates that the value of Ξ is uniquely determined on the multilateral path, and the sign of Ξ is also therefore uniquely determined.

Proposition A2-1: In the model of heterogeneous time preference, $\Xi < 0$ if $\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}[1-(1-\alpha)\varepsilon] < \frac{\theta_1+\theta_2}{2}$.

Proof: See Harashima (2010)

Proposition A2-1 indicates that the current account deficit of economy 1 and the current account surplus of economy 2 continue indefinitely on the multilateral path. The condition $\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}[1-(1-\alpha)\varepsilon] < \frac{\theta_1+\theta_2}{2}$ is generally satisfied for reasonable parameter values.

Conversely, the opposite is true for the trade balance.

Corollary A3-1: In the model of heterogeneous time preference, $\lim_{t\to\infty}\left(\tau_t - \frac{\partial y_{2,t}}{\partial k_{2,t}}\int_0^t \tau_s ds\right) > 0$ if $\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}[1-(1-\alpha)\varepsilon] < \frac{\theta_1+\theta_2}{2}$.

Proof: See Harashima (2010)

Corollary A3-1 indicates that, on the multilateral path, the trade surpluses of economy 1 continue indefinitely and vice versa. That is, goods and services are transferred from economy 1 to economy 2 in each period indefinitely in exchange for the returns on the accumulated current account deficits (i.e., debts) of economy 1.

Nevertheless, the trade balance of economy 1 is not a surplus from the beginning. Before Corollary A3-1 is satisfied, negative $\int_0^t \tau_s ds$ should be accumulated. In the early periods, when $\int_0^t \tau_s ds$ is small, the balance on goods and services of economy 1 ($\tau_t - \frac{\partial y_{2,t}}{\partial k_{2,t}}\int_0^t \tau_s ds$) continues to be a deficit. After a sufficient negative amount of $\int_0^t \tau_s ds$ is accumulated, the trade balances of economy 1 shift to surpluses.

Current account deficit of economy 1 means for example that a firm that is owned by economy 1 borrows money from a bank in which economy 2 deposits money. Economy 1 indirectly borrows money from economy 2. This situation can be easily understood if you see the current account deficit of the United States.

A.2.2.2 Heterogeneous risk aversion model

Similarly, the value of Ξ in the heterogeneous risk aversion model is uniquely determined on the multilateral path.

Lemma A2-2: In the model of heterogeneous risk aversion,

$$\Xi = \frac{(\varepsilon_1 - \varepsilon_2)\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \theta\right]}{(\varepsilon_1 + \varepsilon_2)\left\{\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\left(\frac{\varepsilon_1 + \varepsilon_2}{2}\right)\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \theta\right]^{-1} - 1\right\}} .$$

Proposition A2-2: In the model of heterogeneous risk aversion, $\Xi < 0$ if $1 - \theta\left(\frac{\varpi\alpha}{mv}\right)^{-\alpha}(1-\alpha)^{-1+\alpha} < \frac{\varepsilon_1 + \varepsilon_2}{2}$.

The condition $1 - \theta\left(\frac{\varpi\alpha}{mv}\right)^{-\alpha}(1-\alpha)^{-1+\alpha} < \frac{\varepsilon_1 + \varepsilon_2}{2}$ is generally satisfied for reasonable parameter values.

Corollary A3-2: In the model of heterogeneous risk aversion, $\lim_{t\to\infty}\left(\tau_t - \frac{\partial y_{2,t}}{\partial k_{2,t}}\int_0^t \tau_s ds\right) > 0$.

As shown in Harashima (2010), the limit of the growth rate on the multilateral path is

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \left(\frac{\varepsilon_1 + \varepsilon_2}{2}\right)^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \theta\right] .$$

A.2.2.3 Heterogeneous productivity model

As Lemma A1-3 shows, on the multilateral path, either $\lim_{t\to\infty}\tau_t = 0$ and $\lim_{t\to\infty}\int_0^t \tau_s ds = 0$ or $\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t} = \lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds} = \left[\frac{(\omega_1 + \omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}$. On the former path, $\Xi = 0$ and heterogeneous productivity does not result in permanent trade imbalances. However, on the latter path, trade imbalances usually grow at a higher rate than consumption, because usually $\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t} = \lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds} = \left[\frac{(\omega_1 + \omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha} > \varepsilon^{-1}\left\{\left[\frac{(\omega_1 + \omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha} - \theta\right\} =$ $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}$; thus, Ξ explodes to infinity. Hence, the latter path will generally not be selected. The question of which path is selected is examined in detail in the Section 2.3.3.

A.2.3 A model with heterogeneities in multiple elements

The three heterogeneities are not exclusive. It is particularly likely that heterogeneities in time preference and productivity coexist. Many empirical studies conclude that the rate of time preference is negatively correlated with income (e.g., Lawrance, 1991; Samwick, 1998; Ventura, 2003); this indicates that the economy with the higher productivity has a lower rate of time preference and vice versa. In this section, the models are extended to include heterogeneity in multiple elements.

Suppose that economies 1 and 2 are identical except for time preference, risk aversion, and productivity. The Hamiltonian for economy 1 is

$$H_1 = u_1(c_{1,t})\exp(-\theta_1 t) + \lambda_{1t}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha} k_{1,t} + \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds - \tau_t - c_{1,t}\right\} ,$$

and that for economy 2 is

$$H_2 = u_2(c_{2,t})\exp(-\theta_2 t) + \lambda_{2,t}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha} k_{2,t} - \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds + \tau_t - c_{2,t}\right\} .$$

The growth rates are

$$\frac{\dot{c}_{1,t}}{c_{1,t}} = \varepsilon_1^{-1}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha} + \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\frac{\partial\int_0^t \tau_s ds}{\partial k_{1,t}} - \frac{\partial \tau_t}{\partial k_{1,t}} - \theta_1\right\} ,$$

and

$$\frac{\dot{c}_{2,t}}{c_{2,t}} = \varepsilon_2^{-1}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha} - \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\frac{\partial\int_0^t \tau_s ds}{\partial k_{2,t}} + \frac{\partial \tau_t}{\partial k_{2,t}} - \theta_2\right\} .$$

Here, $\lim_{t\to\infty}\frac{\tau_t}{k_{1,t}} = \Xi$, $\lim_{t\to\infty}\frac{\tau_t}{k_{2,t}} = \frac{\omega_1}{\omega_2}\Xi$, $\lim_{t\to\infty}\frac{\int_0^t \tau_s ds}{k_{1,t}} = \Xi\left(\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}\right)^{-1}$, and $\lim_{t\to\infty}\frac{\int_0^t \tau_s ds}{k_{2,t}} = \frac{\omega_1}{\omega_2}\Xi\left(\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}\right)^{-1}$ at steady state. Thus, $\Xi = \dfrac{\frac{\varepsilon_2}{\varepsilon_1}\theta_1 - \theta_2 - \left(\frac{\varepsilon_2}{\varepsilon_1} - 1\right)\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha}}{\left(\frac{\varepsilon_2}{\varepsilon_1} + \frac{\omega_1}{\omega_2}\right)\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\left(\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}\right)^{-1} - 1\right\}}$,

and the limit of the growth rate on the multilateral path is

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \left(\frac{\varepsilon_1\omega_1 + \varepsilon_2\omega_2}{\omega_1+\omega_2}\right)^{-1}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha} - \frac{\theta_1\omega_1+\theta_2\omega_2}{\omega_1+\omega_2}\right\} . \quad \text{(A14)}$$

Clearly, if $\varepsilon_1 = \varepsilon_2$ and $\omega_1 = \omega_2 = 1$, then $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \varepsilon_1^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \frac{\theta_1+\theta_2}{2}\right]$; if $\theta_1 = \theta_2$ and $\omega_1 = \omega_2 = 1$, then $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \theta_1\right]$; and if

$\theta_1 = \theta_2$ and $\varepsilon_1 = \varepsilon_2$, then $\lim_{t\to\infty} \frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty} \frac{\dot{c}_{2,t}}{c_{2,t}} = \varepsilon_1^{-1} \left\{ \left[\frac{(\omega_1 + \omega_2)\varpi\alpha}{2mv(1-\alpha)} \right]^{\alpha} - \theta_1 \right\}$ as shown in Sections A.2.1 and A.2.2.

The sign of Ξ on the multilateral path depends on the relative values between θ_1 and θ_2, ε_1 and ε_2, and ω_1 and ω_2. Nevertheless, if the rate of time preference and productivity are negatively correlated, as argued above (i.e., if $\theta_1 < \theta_2$ and $\omega_1 > \omega_2$ while $\varepsilon_1 = \varepsilon_2$), then by similar proofs as those presented for Proposition A2-1 and Corollary A3-1, if $\left[\frac{(\omega_1 + \omega_2)\varpi\alpha}{2mv} \right]^{\alpha} \left[1 - (1-\alpha)^{1-\alpha} \varepsilon_1 \right] < \frac{\omega_1\theta_1 + \omega_2\theta_2}{\omega_1 + \omega_2}$, then $\Xi < 0$ and $\lim_{t\to\infty} \left(\tau_t - \frac{\partial y_{2,t}}{\partial k_{2,t}} \int_0^t \tau_s ds \right) > 0$ on the multilateral path; that is, the current account deficits and trade surpluses of economy 1 continue indefinitely. The condition $\left[\frac{(\omega_1 + \omega_2)\varpi\alpha}{2mv} \right]^{\alpha} \left[1 - (1-\alpha)^{1-\alpha} \varepsilon_1 \right] < \frac{\omega_1\theta_1 + \omega_2\theta_2}{\omega_1 + \omega_2}$ is generally satisfied for reasonable parameter values.

A.2.4 Multi-economy models

The two-economy models can be extended to include numerous economies that have differing degrees of heterogeneity.

A.2.4.1 Heterogeneous time preference model

Suppose that there are H economies that are identical except for time preference. Let θ_i be the rate of time preference of economy i and $\tau_{i,j,t}$ be the current account balance of economy i with economy j, where $i = 1, 2, \ldots, H$, $j = 1, 2, \ldots, H$, and $i \neq j$. Because the total population is L_t, the population in each economy is $\frac{L_t}{H}$. The representative household of economy i maximizes its expected utility

$$E \int_0^{\infty} u_i(c_{i,t}) \exp(-\theta_i t) dt \quad ,$$

subject to

$$\dot{k}_{i,t} = y_{i,t} + \sum_{j=1}^{H} \frac{\partial y_{j,t}}{\partial k_{j,t}} \int_0^t \tau_{i,j,s} ds - \sum_{j=1}^{H} \tau_{i,j,t} - c_{i,t} - v\dot{A}_{i,t} \left(\frac{L_t}{H} \right)^{-1}$$

for $i \neq j$.

Proposition A3-1: In the multi-economy model of heterogeneous time preference, if and only if

$$\lim_{t\to\infty} \frac{\dot{c}_{i,t}}{c_{i,t}} = \varepsilon^{-1} \left[\left(\frac{\varpi\alpha}{mv} \right)^{\alpha} (1-\alpha)^{-\alpha} - \frac{\sum_{q=1}^{H} \theta_q}{H} \right] \tag{A15}$$

for any i, all the optimality conditions of all heterogeneous economies are satisfied at steady state, and

$$\lim_{t\to\infty}\frac{\dot{c}_{i,t}}{c_{i,t}}=\lim_{t\to\infty}\frac{\dot{k}_{i,t}}{k_{i,t}}=\lim_{t\to\infty}\frac{\dot{y}_{i,t}}{y_{i,t}}=\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}=\lim_{t\to\infty}\frac{\dot{\tau}_{i,j,t}}{\tau_{i,j,t}}=\lim_{t\to\infty}\frac{\frac{d\int_0^t \tau_{i,j,s}ds}{dt}}{\int_0^t \tau_{i,j,s}ds}$$

for any *i* and *j* ($i \neq j$).
Proof: See Harashima (2010)

A.2.4.2 Heterogeneous risk aversion model

The heterogeneous risk aversion model can be extended to the multi-economy model by a proof similar to that for Proposition A3-1. Suppose that *H* economies are identical except for risk aversion, and their degrees of risk aversion are ε_i (i = 1, 2, … , H).

Proposition A3-2: In the multi-economy model of heterogeneous risk aversion, if and only if

$$\lim_{t\to\infty}\frac{\dot{c}_{i,t}}{c_{i,t}}=\left(\frac{\sum_{q=1}^{H}\varepsilon_q}{H}\right)^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}-\theta\right] \tag{A16}$$

for any *i*, all the optimality conditions of all heterogeneous economies are satisfied at steady state, and

$$\lim_{t\to\infty}\frac{\dot{c}_{i,t}}{c_{i,t}}=\lim_{t\to\infty}\frac{\dot{k}_{i,t}}{k_{i,t}}=\lim_{t\to\infty}\frac{\dot{y}_{i,t}}{y_{i,t}}=\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}=\lim_{t\to\infty}\frac{\dot{\tau}_{i,j,t}}{\tau_{i,j,t}}=\lim_{t\to\infty}\frac{\frac{d\int_0^t \tau_{i,j,s}ds}{dt}}{\int_0^t \tau_{i,j,s}ds}$$

for any *i* and *j* ($i \neq j$).

A.2.4.3 Heterogeneous productivity model

The heterogeneous productivity model can also be extended by a proof similar to that for Proposition A3-1. Suppose that *H* economies are identical except for productivity, and their productivities are ω_i (i = 1, 2, … , H). Note that, because $k_{1+2,t}=k_{1,t}+k_{2,t}=k_{2,t}\left[\frac{\omega_1}{\omega_2}+1\right]$, the productivity of economy 1+2 is $y_{1+2,t}=A_t^{\alpha}\left(\omega_1^{\alpha}k_{1,t}^{1-\alpha}+\omega_2^{\alpha}k_{2,t}^{1-\alpha}\right)=\left(\omega_1+\omega_2\right)^{\alpha}A_t^{\alpha}k_{1+2,t}^{1-\alpha}$.

Proposition A3-3: In the multi-economy model of heterogeneous productivity, if and only if

$$\lim_{t\to\infty}\frac{\dot{c}_{i,t}}{c_{i,t}}=\varepsilon^{-1}\left\{\left[\frac{\left(\sum_{q=1}^{H}\omega_q\right)\varpi\alpha}{Hmv(1-\alpha)}\right]^{\alpha}-\theta\right\}$$

for any i, all the optimality conditions of all heterogeneous economies are satisfied at steady state, and

$$\lim_{t\to\infty}\frac{\dot{c}_{i,t}}{c_{i,t}}=\lim_{t\to\infty}\frac{\dot{k}_{i,t}}{k_{i,t}}=\lim_{t\to\infty}\frac{\dot{y}_{i,t}}{y_{i,t}}=\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}$$

for any i and j ($i \neq j$).

A.2.4.4 Heterogeneity in multiple elements

Similarly, the multi-economy model can be extended to heterogeneity in multiple elements, as follows.

Proposition A3-4: In the multi-economy model of heterogeneity in multiple elements, if and only if

$$\lim_{t\to\infty}\frac{\dot{c}_{i,t}}{c_{i,t}}=\left(\frac{\sum_{q=1}^{H}\varepsilon_q\omega_q}{\sum_{q=1}^{H}\omega_q}\right)^{-1}\left\{\left[\frac{\varpi\alpha\sum_{q=1}^{H}\omega_q}{Hm\nu(1-\alpha)}\right]^{\alpha}-\frac{\sum_{q=1}^{H}\theta_q\omega_q}{\sum_{q=1}^{H}\omega_q}\right\} \tag{A17}$$

for any i (= 1, 2, … , H), all the optimality conditions of all heterogeneous economies are satisfied at steady state, and

$$\lim_{t\to\infty}\frac{\dot{c}_{i,t}}{c_{i,t}}=\lim_{t\to\infty}\frac{\dot{k}_{i,t}}{k_{i,t}}=\lim_{t\to\infty}\frac{\dot{y}_{i,t}}{y_{i,t}}=\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}=\lim_{t\to\infty}\frac{\dot{\tau}_{i,j,t}}{\tau_{i,j,t}}=\lim_{t\to\infty}\frac{\frac{d\int_0^t\tau_{i,j,s}ds}{dt}}{\int_0^t\tau_{i,j,s}ds}$$

for any i and j ($i \neq j$).

Proposition A3-4 implies that the concept of the representative household in a heterogeneous population implicitly assumes that all households are on the multilateral path.

A.3 The unilateral path

The multilateral path satisfies all the optimality conditions, but that does not mean that the two economies naturally select the multilateral path. Ghiglino (2002) predicts that it is likely that, under appropriate assumptions, the results of Becker (1980) still hold in endogenous growth models. Farmer and Lahiri (2005) show that balanced growth equilibria do not exist in a multi-agent economy in general, except in the special case that all agents have the same constant rate of time preference. How the economies behave in the environments described in Sections 2 and 3 when the government does not intervene, i.e., $\bar{g}=0$. is examined in this section.

A.3.1 Heterogeneous time preference model

The multilateral path is not the only path on which all the optimality conditions of economy 1 are satisfied. Even if economy 1 behaves unilaterally, it can achieve optimality, but economy 2 cannot.

Lemma A3-1: In the heterogeneous time preference model, if each economy sets τ_t without regarding the other economy's optimality conditions, then it is not possible to satisfy all the optimality conditions of both economies.
Proof: See Harashima (2010)

Since $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \varepsilon^{-1}\left\{\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} + \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\lim_{t\to\infty}\frac{\tau_t}{k_{1,t}}\left(\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}\right)^{-1} - \lim_{t\to\infty}\frac{\tau_t}{k_{1,t}} - \theta_1\right\}$

at steady state, all the optimality conditions of economy 1 can be satisfied only if either

$$\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t} = \lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds} = \lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} \tag{A18}$$

or

$$\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t} = \lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds} = \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha} \quad . \tag{A19}$$

That is, $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}$ can be constant only when either equation (A18) or (A19) is satisfied. Conversely, economy 1 has two paths on which all its optimality conditions are satisfied. Equation (A18) indicates that $\lim_{t\to\infty}\frac{\tau_t}{k_{1,t}}$ = constant, and equation (A19) indicates that $\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\left(\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}\right)^{-1} - 1 = 0$ for any $\lim_{t\to\infty}\frac{\tau_t}{k_{1,t}}$. Equation (A18) corresponds to the multilateral path. On the path satisfying equation (A19), $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} \neq \lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t} = \lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds}$,

and $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} > \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}$. Here, by equations (A6) and (A7),

$$c_{1,t} - c_{2,t} = 2\left(\frac{\partial y_{1,t}}{\partial k_{1,t}}\int_0^t \tau_s ds - \tau_t\right) = 2\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds - \tau_t\right] \quad ,$$

and

$$\lim_{t\to\infty}(c_{1,t} - c_{2,t}) = 0$$

is required because $\lim_{t\to\infty}\frac{\tau_t}{\int_0^t \tau_s ds}=\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}$. However, because $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}>\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}$, economy 2 must initially set consumption such that $c_{2,0}=\infty$, which violates the optimality condition of economy 2. Therefore, unlike with the multilateral path, all the optimality conditions of economy 2 cannot be satisfied on the path satisfying equation (A19) even though those of economy 1 can. Hence, economy 2 has only one path on which all its optimality conditions can be satisfied—the multilateral path. The path satisfying equation (A19) is called the "unilateral balanced growth path" or the "unilateral path" in the following discussion. Clearly, heterogeneity in time preference is not sustainable on the unilateral path.

How should economy 2 respond to the unilateral behavior of economy 1? Possibly, both economies negotiate for the trade between them, and some agreements may be reached. If no agreement is reached, however, and economy 1 never regards economy 2's optimality conditions, economy 2 generally will fall into the following unfavorable situation.

Remark 1-1: In the model of heterogeneous time preference, if economy 1 does not regard the optimality conditions of economy 2, the ratio of economy 2's debts (owed to economy 1) to its consumption explodes to infinity while all the optimality conditions of economy 1 are satisfied.

The reasoning behind Remark 1-1 is as follows. When economy 1 selects the unilateral path and sets $c_{1,0}$ so as to achieve this path, there are two options for economy 2. The first option is for economy 2 to also pursue its own optimality without regarding economy 1: that is, to select its own unilateral path. The second option is to adapt to the behavior of economy 1 as a follower. If economy 2 takes the first option, it sets $c_{2,0}$ without regarding $c_{1,0}$. As the proof of Lemma A3-1 indicates, unilaterally optimal growth rates are different between the two economies and $\frac{\dot{c}_{1,t}}{c_{1,t}}>\frac{\dot{c}_{2,t}}{c_{2,t}}$; thus, the initial consumption should be set as $c_{1,0}<c_{2,0}$. Because $\frac{\partial y_{1,t}}{\partial k_{1,t}}=\frac{\varpi}{2mv}\frac{\partial(y_{1,t}+y_{2,t})}{\partial A_t}=\frac{\partial y_{2,t}}{\partial k_{2,t}}$ and $k_{1,t}=k_{2,t}$ must be kept, capital and technology are equal and grow at the same rate in both economies. Hence, because $c_{1,0}<c_{2,0}$, more capital is initially produced in economy 1 than in economy 2 and some of it will need to be exported to economy 2. As a result, $\frac{\dot{c}_{1,t}}{c_{1,t}}>\frac{\dot{k}_{1,t}}{k_{1,t}}=\frac{\dot{k}_{2,t}}{k_{2,t}}>\frac{\dot{c}_{2,t}}{c_{2,t}}$, which means that all the optimality conditions of both economies cannot be satisfied. Since $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}>\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}}=\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}>\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}$, capital soon becomes abundant in economy 2, and excess goods and services are produced in that economy. These excess products are exported to and utilized in economy 1. This process escalates as time passes because $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}>\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}}=\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}>\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}$, and eventually almost all consumer goods and services produced in economy 2 are consumed by households in economy 1. These consequences will be unfavorable for economy 2.

If economy 2 takes the second option, it should set $c_{2,0}=\infty$ to satisfy all its optimality conditions, as the proof of Lemma A3-1 indicates. Setting $c_{2,0}=\infty$ is impossible, but economy 2 as the follower will initially set $c_{2,t}$ as large as possible. This action gives economy 2 a higher expected utility than that of the first option, because consumption in economy 2 in the second case is always higher. As a result, economy 2 imports as many goods and services as possible from economy 1, and the trade deficit of economy 2 continues until $\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds = \tau_t$ is achieved; this is, $\frac{\dot{\tau}_t}{\tau_t} = \frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds}$ is achieved. The current account deficits and the accumulated debts of economy 2 will continue to increase indefinitely. Furthermore, they will increase more rapidly than the growth rate of outputs ($\lim_{t\to\infty}\frac{\dot{y}_{2,t}}{y_{2,t}}$) because, in general, $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} < \lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}$; that is, $(1-\varepsilon)\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha} < \theta_1(<\theta_2)$. If no disturbance occurs, the expansion of debts may be sustained forever, but economy 2 becomes extremely vulnerable to even a very tiny negative disturbance. If such a disturbance occurs, economy 2 will lose all its capital and will no longer be able to repay its debts. This result corresponds to the state shown by Becker (1980), and it will also be unfavorable for economy 2. Because $\lim_{t\to\infty}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds - \tau_t\right] = 0$, inequality (27) holds, and the transversality condition for economy 1 is satisfied. Thus, all the optimality conditions of economy 1 are satisfied if economy 2 takes the second option.

As a result, all the optimality conditions of economy 2 cannot be satisfied in any case if economy 1 takes the unilateral path. Both options to counter the unilateral behavior of economy 1 are unfavorable for economy 2. However, the expected utility of economy 2 is higher if it takes the second option rather than the first, and economy 2 will choose the second option. Hence, if economy 1 does not regard economy 2's optimality conditions, the debts owed by economy 2 to economy 1 increase indefinitely at a higher rate than consumption.

A.3.2 Heterogeneous risk aversion model

The same consequences are observed in this model.

Lemma A3-2: In the model of heterogeneous risk aversion, if each economy sets τ_t without regard for the other economy's optimality conditions, then all the optimality conditions of both economies cannot be satisfied.

Therefore, heterogeneity in risk aversion is not sustainable on the unilateral path.

Remark 1-2: In the model of heterogeneous risk aversion, if economy 1 does not regard economy 2's optimality conditions, the ratio of economy 2's debts (owed to economy 1) to its consumption explodes to infinity while all the optimality conditions of economy 1 are satisfied.

A.3.3 Heterogeneous productivity model

Unlike the heterogeneous preferences shown in Sections A.3.1 and A.3.2, heterogeneity in productivity can be sustainable even on the unilateral path.

Lemma A3-3: In the heterogeneous productivity model, even if each economy sets τ_t without regard for the other economy's optimality conditions, it is possible that all the optimality conditions of both economies are satisfied if

$$\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}=\lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds}=\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha} \quad .$$

Proof: See Harashima (2010)

All the optimality conditions of economy 1 can be satisfied only if either equation (A12) or (A13) holds, because $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}$ can be constant only when equation (A12) or (A13) holds. Equation (A12) corresponds to the multilateral path, and equation (A13) corresponds to the unilateral path. Unlike the heterogeneity in preferences, Lemma A3-3 shows that, even on the unilateral path, all the optimality conditions of both economies are satisfied because the limit of both economies' growth rates is identical on the path of either equation (A12) or (A13), such that $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\varepsilon^{-1}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha}-\theta\right\}$. Therefore, heterogeneity in productivity is sustainable even on the unilateral path.

Nevertheless, on the unilateral path, current account imbalances generally grow steadily at a higher rate than consumption; this is not the case on the multilateral path. How does economy 1 set τ? If economy 1 imports as many goods and services as possible before reaching the steady state at which $\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}=\lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds}=\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}$ (i.e., if it initially sets τ_i as $\tau_t < 0$ and $\tau_t-\frac{\partial y_{2,t}}{\partial k_{2,t}}\int_0^t \tau_s ds<0$), the expected utility of economy 1 will be higher than it is in either case where $\tau_t > 0$ or in the multilateral path. However, the debts economy 1 owes to economy 2 will grow indefinitely at a higher rate than consumption, and the ratio of debt to consumption explodes to infinity. If there is no disturbance, this situation will be sustained forever, but economy 1 will become extremely vulnerable to even a very tiny negative disturbance. Hence, the unilateral path will not necessarily be favorable for economy 1 although all its optimality conditions are satisfied on this path, and economy 1 will prefer the multilateral path.

Remark 1-3: In the heterogeneous productivity model, even though economy 1 does not regard economy 2's optimality conditions, the multilateral balanced growth path will be selected.

Hence, the state shown by Becker (1980) will not be observed in the case of heterogeneous productivity.

A.3.4 Doom of the less advantaged economies

Remarks 1-1 and 1-2 indicate that economy 2's ratio of debt to consumption continues to increase indefinitely on the unilateral path. Such an indefinitely increasing ratio may not matter if there is no shock or disturbance. However, if even a very tribunal negative shock occurs, economy 2 will be ruined because the huge amount of accumulated debts cannot be refinanced. In this case, "ruin" means that economy 2 will go bankrupt or be exterminated because its consumption has to be zero unless the authority intervenes to some extent (e.g., debt relief after personal bankruptcy). Even if economy 2 continues to exist by the mercy of economy 1, it will fall into a slave-like state indefinitely without the authority's intervention.

A.4 Sustainable heterogeneity with government intervention

Sustainable heterogeneity, as described in this paper, is a very different state from what Becker (1980) described. The difference emerges because, on a multilateral path, economy 1 behaves fully considering economy 2's situation. The multilateral path therefore will not be naturally selected by economy 1, and the path selection may have to be decided politically (Harashima, 2010). On the other hand, when economy 1 behaves unilaterally, the government may intervene in economic activities so as to achieve, for example, social justice.

In this section, I show that even if economy 1 behaves unilaterally, sustainable heterogeneity can always be achieved with appropriate government intervention.

A.4.1 Heterogeneous time preference model

Government intervention was first considered in the two-economy model constructed in Section 2. If the government intervenes (i.e., $\bar{g} > 0$),

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} \neq \lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t} = \lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds}$$

and

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \varepsilon^{-1}\left\{\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \theta_1 - \bar{g}\right\}$$

on the path satisfying equation (A19). At the same time,

$$\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \varepsilon^{-1}\left\{\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \theta_2 + \bar{g}\right\} .$$

Therefore, if

$$\bar{g} = \frac{\theta_2 - \theta_1}{2} ,$$

then

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \varepsilon^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \frac{\theta_1+\theta_2}{2}\right] . \quad \text{(A20)}$$

Equation (A20) is identical to equation (A10). The government's appropriate redistribution from economy 1 to economy 2 by $\bar{g}$ leads to the same consequence with a multilateral path.

Note that if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} > \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}$ or $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} < \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}$, neither economy can achieve optimality. In this sense, the only appropriate amount of government intervention is $\bar{g} = \frac{\theta_2 - \theta_1}{2}$.

A.4.2 Heterogeneous risk aversion model

Similarly, all of the optimality conditions of economy 1 can be satisfied only if either equation (A18) or (A19) is satisfied, and if

$$\bar{g} = \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1 + \varepsilon_2}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \theta\right] ,$$

then

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \left(\frac{\varepsilon_1 + \varepsilon_2}{2}\right)^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \theta\right] . \quad \text{(A21)}$$

Equation (A21) is identical to equation (A11). Similar to the case with a heterogeneous time preference, the government's appropriate redistribution by $\bar{g} = \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1 + \varepsilon_2}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \theta\right]$ leads to the same consequence with a multilateral path.

A.4.3 Heterogeneous productivity model

Heterogeneity in productivity can be sustainable even on the unilateral path. Hence, government intervention is not necessary; that is, even if $\bar{g} = 0$, the unilateral path is sustainable.

A.4.4 A model with heterogeneities in multiple elements

By similar procedures as those used in Sections A.4.1 and A.4.2, if

$$\bar{g} = \left\{(\varepsilon_2 - \varepsilon_1)\left[\frac{(\omega_1 + \omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha} + \varepsilon_1\theta_2 - \varepsilon_2\theta_1\right\}\left(\varepsilon_1\frac{\omega_1}{\omega_2} + \varepsilon_2\right)^{-1} ,$$

then

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \left(\frac{\omega_1\varepsilon_1 + \omega_2\varepsilon_2}{\omega_1 + \omega_2}\right)^{-1}\left\{\left[\frac{(\omega_1 + \omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha} - \frac{\omega_1\theta_1 + \omega_2\theta_2}{\omega_1 + \omega_2}\right\} , \quad \text{(A22)}$$

and equation (A22) is identical to equation (A14).

A.4.5 Multi-economy models

A.4.5.1 Heterogeneous time preference model

If $H = 2$, when sustainable heterogeneity is achieved, economies 1 and 2 consist of a combined economy (economy 1+2) with twice the population and a rate of time preference of $\frac{\theta_1+\theta_2}{2}$. Suppose there is a third economy with a time preference of θ_3. Because economy 1+2 has twice the population of economy 3, if

$$\bar{g} = \frac{\theta_3 - \frac{\theta_1+\theta_2}{2}}{3} \quad ,$$

then

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \lim_{t\to\infty}\frac{\dot{c}_{3,t}}{c_{3,t}} = \varepsilon^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \frac{\sum_{q=1}^{3}\theta_q}{3}\right] .$$

By iterating similar procedures, if the government's transfers between economy H and economy $1+2+\cdots+(H-1)$ is such that

$$\bar{g} = \frac{\theta_H - \frac{\sum_{q=1}^{H-1}\theta_q}{H-1}}{H} \quad ,$$

then

$$\lim_{t\to\infty}\frac{\dot{c}_{i,t}}{c_{i,t}} = \varepsilon^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \frac{\sum_{q=1}^{H}\theta_q}{H}\right] \tag{A23}$$

for any i (= 1, 2, · · ·, H). Equation (A23) is the same as equation (A15).

A.4.5.2 Heterogeneous risk aversion model

By a similar procedure as that used for heterogeneous time preference, if the sum of the government transfers between economy H and economy $1+2+\cdots+(H-1)$ is such that

$$\bar{g} = \frac{\varepsilon_H - \frac{\sum_{q=1}^{H-1}\varepsilon_q}{H-1}}{\sum_{q=1}^{H}\varepsilon_q}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \theta\right] \quad ,$$

then

$$\lim_{t\to\infty}\frac{\dot{c}_{i,t}}{c_{i,t}} = \left(\frac{\sum_{q=1}^{H}\varepsilon_q}{H}\right)^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}-\theta\right] \quad \text{(A24)}$$

for any i (= 1, 2, · · ·, H). Equation (A24) is the same as equation (A16).

A.4.5.3 Heterogeneous productivity model

As discussed in Section A.4.3, even if government transfers between economy H and economy 1+2+ · · · + (H – 1) is nil (i.e., $\bar{g}=0$), the unilateral path is sustainable.

A.4.5.4 Heterogeneity in multiple elements

By combining the procedures and results presented in Section A.4.4, A.4.5.1 and A.4.5.2, it can be shown that, if the sum of government transfers between economy H and economy 1+2+ · · · + (H – 1) is such that

$$\bar{g} = \left[\frac{\sum_{q=1}^{H}\varepsilon_q\omega_q}{\omega_H}\right]^{-1}\left\{\frac{\left(\varepsilon_H\sum_{q=1}^{H}\omega_q-\sum_{q=1}^{H}\varepsilon_q\omega_q\right)}{\sum_{q=1}^{H-1}\omega_q}\left[\frac{\varpi\alpha\sum_{q=1}^{H}\omega_q}{Hmv(1-\alpha)}\right]^{\alpha}-\frac{\varepsilon_H\sum_{q=1}^{H}\theta_q\omega_q-\theta_H\sum_{q=1}^{H}\varepsilon_q\omega_q}{\sum_{q=1}^{H-1}\omega_q}\right\}, \quad \text{(A25)}$$

then

$$\lim_{t\to\infty}\frac{\dot{c}_{i,t}}{c_{i,t}} = \left(\frac{\sum_{q=1}^{H}\varepsilon_q\omega_q}{\sum_{q=1}^{H}\omega_q}\right)^{-1}\left\{\left[\frac{\varpi\alpha\sum_{q=1}^{H}\omega_q}{Hmv(1-\alpha)}\right]^{\alpha}-\frac{\sum_{q=1}^{H}\theta_q\omega_q}{\sum_{q=1}^{H}\omega_q}\right\}. \quad \text{(A26)}$$

for any i (= 1, 2, · · ·, H). Equation (A26) is the same as equation (A17).

A.4.6 Models of partially unilateral behaviors

Here, suppose that economy 1 undertakes partly unilateral and partly multilateral behaviors such that ψ of $k_{1,0}$ is allocated to the unilateral path and (1 − ψ) of $k_{1,0}$ is allocated to the multilateral path, where 0 < ψ < 1. In this case, if an appropriate value of $\bar{g}$ is set, both the unilateral and multilateral parts of $k_{1,t}$ achieve sustainable heterogeneity because the unilateral part of $k_{1,t}$ is forced on a path of sustainable heterogeneity by appropriate government intervention, whereas the multilateral part of $k_{1,t}$ naturally takes a path of sustainable heterogeneity. Therefore, even though economy 1 behaves partly unilaterally and partly multilaterally, if an appropriate value of $\bar{g}$ is set, the combined path can be sustainable. I call this a "sustainable partly unilateral path."

Corresponding to different values of ψ, sustainable partly unilateral paths are different and will fall somewhere between the multilateral path and the fully unilateral path with appropriate government transfers described by equation (A26). In addition, sustainable partly unilateral paths will move continuously as the value of ψ continuously moves.

Note that this paper assumes that government intervention can be represented only by $\bar{g}$, but many other types of interventions are actually possible. For example, debt relief after personal bankruptcy would work as a measure to achieve sustainable heterogeneity, but the paths of the less advantaged economies may not be continuous in that case.

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