A note on Poincaré recurrence in Anosov diffeomorphic transformation of discretized outline of some plant leaves

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1. Introduction: The origin of this note is in the curiosity aroused by the Arnold's cat map (Arnold and Avez, 1968), which suggests that perhaps a cat has many more lives than simply nine. The discretized Arnold's cat map, when repeatedly subjected to Anosov diffeomorphic transformation, sheared and wrapped around iteration after iteration, though apparently random or disordered intermediately, finally returns to the original image after a finite number of iterations (Dyson and Falk, 1992). In other words, repeated operation of Anosov diffeomorphic transformation on the cat map exhibits Poincaré recurrence. It has been shown (Wikipedia: Arnold's cat map) that the said cat map resumes its original shape in the 300th (is it a magic figure for the cat: 300 = $\sqrt{9}\times 10^2$?) iteration. Could we generate some more examples of this type? This inquisitiveness is the main drive behind this note.

2. The scheme of transformation: Let $X(n,2)$ be the n points in two dimensional space obtained by discretization of a closed curve (deformed ellipse) resembling the outline of a plant leaf such as given in Fig.1. If X is subjected to Anosov diffeomorphic transformation repeatedly, it is interesting to note that it exhibits a mixing behavior and then Poincaré recurrence after some iterations. This transformation is done as follows:

$X^{(t+1)} = \mathrm{mod}(X^{(t)}B, n)$, where $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, which with $\det(B) = 1$, is area-preserving.

3. Datasets and Findings: Plant leaves are extremely varied in shapes. Leaf shapes range from simple linear (or simple needle-like) to highly complicated compound ones observed in many types of fern. These shapes are described qualitatively by using geometric words such as cordate, cuneate, deltoid, elliptical, hastate, lanceolate, oblong, obovate, oval, palmate, reniform, spatulate, etc. Geometrically, all leaf shapes may be viewed as deformed and/or modified ellipse (Gielis, 2003). This deformation is a sort of modification that might have been undergone due to some forces in nature.

We obtained data by smooth (avoiding serration on the margin) tracing and then discretizing the outlines of some common plant leaves with different shapes. In what follows, we provide some findings (figures Fig.1 through Fig.8 and datasets Table.1 through Table.8) on the same.

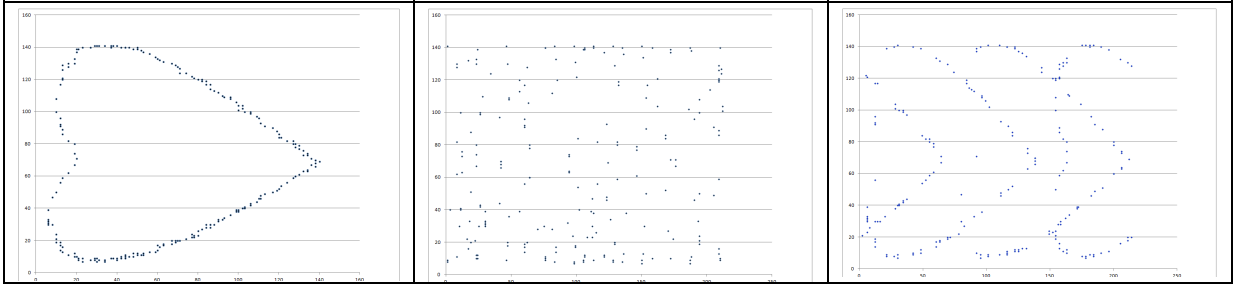
3.1. Cordate leaf dataset: The cordate leaf dataset contains 216 (=n) points. The original dataset (AX) is real, but it has been converted into an integer dataset (X in Table-1) according to the rule given below:

If(ax(i,j)-int(ax(i,j)) < 0.5) x(i,j)=int(ax(i,j).kr) else x(i,j)=int(ax(i,j).kr)+1; j=1,2; i=1,n; kr=10.

Subjected to repeated transformations, it resumes its original position in the 36th iteration.

Fig.1. Cordate Leaf (216 points)

Iteration: 0, 36	Iteration: 14	Iteration: 24



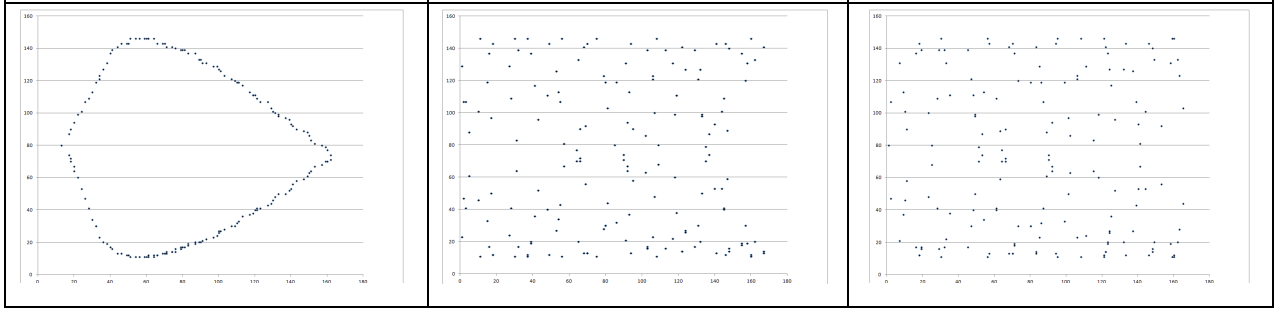
3.2. Convex-deltoid leaf dataset: The convex-deltoid leaf dataset has 168 (=n) points. The original dataset (AX) is real, but converted into an integer dataset (X in Table-2) under the rule given below:

$$\text{If}(ax(i,j)-\text{int}(ax(i,j)) < 0.5)\ x(i,j)=\text{int}(ax(i,j).kr)\ \text{else}\ x(i,j)=\text{int}(ax(i,j).kr)+1;\ j=1,2;\ i=1,n;\ kr=10.$$

Subjected to repeated transformations, it resumes its original position in the 24th iteration.

Fig.2. Convex-deltoid Leaf (168 points)

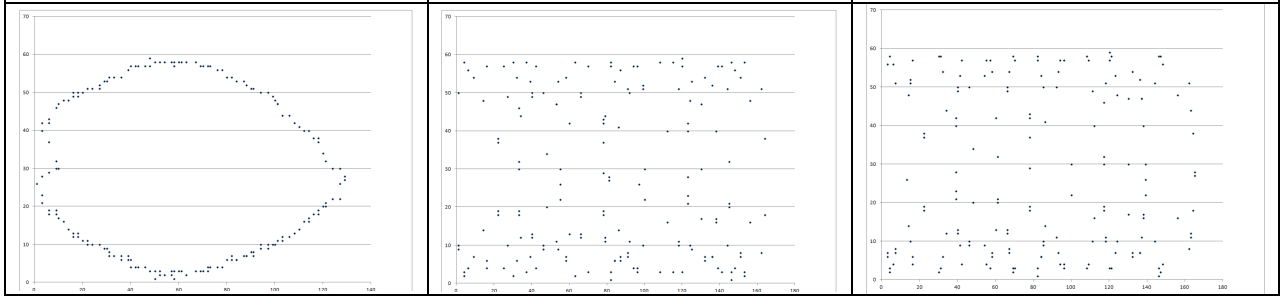
Iteration: 0, 24	Iteration: 12	Iteration: 20



3.3. Cardio-lanceolate leaf dataset: The cardio-lanceolate leaf dataset contains 168 (=n) points. The original dataset (AX) is real, but converted into an integer dataset (X in Table.3).

Fig.3. Cardio-lanceolate Leaf (168 points)

Iteration: 0, 24	Iteration: 12	Iteration: 20



The real data have been converted to integers (plotted in Fig.3) by the rule given below:

If(ax(i,j)-int(ax(i,j)) < 0.5) x(i,j)=int(ax(i,j).kr) else x(i,j)=int(ax(i,j).kr)+1; j=1,2; i=1,n; kr=10.

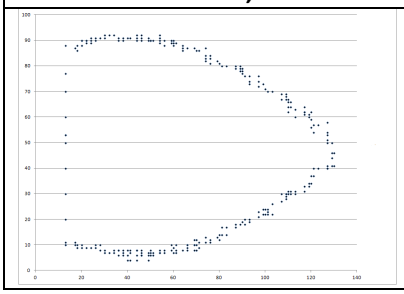
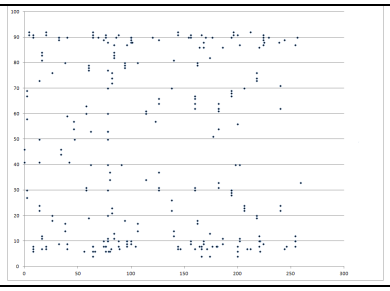
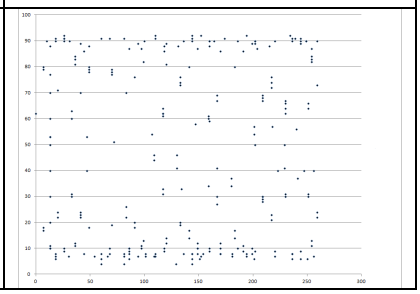
Subjected to repeated transformations, it resumes its original position in the 24th iteration.

3.4. Truncate leaf dataset: The truncate leaf dataset contains 260 (=n) points. The original dataset (AX) is real, but it has been converted into an integer dataset (X in Table-4) by the rule given below:

If(ax(i,j)-int(ax(i,j)) < 0.5) x(i,j)=int(ax(i,j).kr) else x(i,j)=int(ax(i,j).kr)+1; j=1,2; i=1,n; kr=10.

Subjected to repeated transformations, it resumes its original position in the 210th iteration.

Fig.4. Truncate Leaf (260 points)

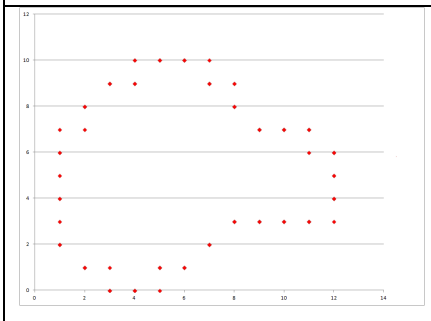
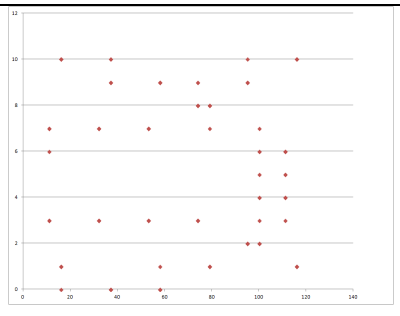
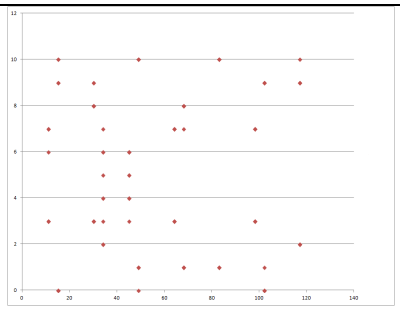
Iteration: 0, 210	Iteration: 100	Iteration: 200
		

3.5. Cucumberoid leaf dataset: The cucumberoid leaf dataset contains 121 (=n) points. The original dataset (AX) is real, but it has been converted into an integer dataset (X in Table-5) as given below:

If(ax(i,j)-int(ax(i,j)) < 0.5) x(i,j)=int(ax(i,j).kr) else x(i,j)=int(ax(i,j).kr)+1; j=1,2; i=1,n; kr=1.

Subjected to repeated transformations, it resumes its original position in the 210th iteration. It may be noted, however, that the recurrence was not observed (even for 3000 iterations) when kr=10 was used. Therefore, kr=1 was chosen. Compared to other datasets, it took many more iteration to show recurrence. The figure (fig.5) and the dataset (table-5) are based on the transformation for kr=1.

Fig.5. Cucumberoid Leaf (121 points)

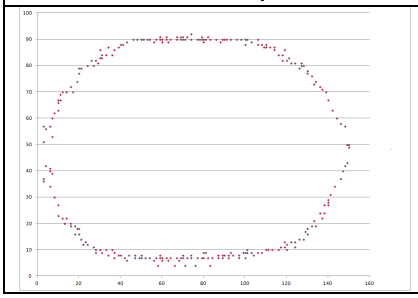
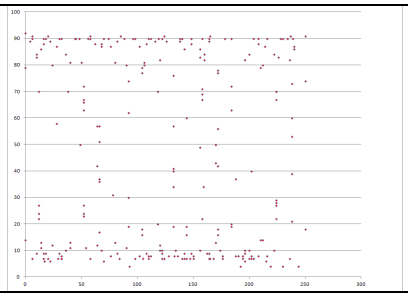
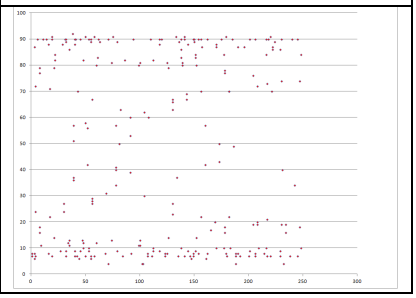
Iteration: 0, 54	Iteration: 14	Iteration: 50
		

3.6. Oblong leaf dataset: The oblong leaf dataset contains 252 (=n) points. The original dataset (AX) is real, but it has been converted into an integer dataset (X in Table-6) by the transformation given below:

If(ax(i,j)-int(ax(i,j)) < 0.5) x(i,j)=int(ax(i,j).kr) else x(i,j)=int(ax(i,j).kr)+1; j=1,2; i=1,n; kr=10.

Subjected to repeated transformations, it resumes its original position in the 24th iteration.

Fig.6. Oblong Leaf (252 points)

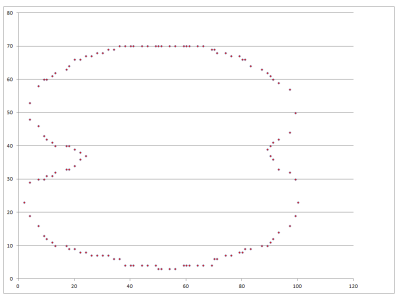
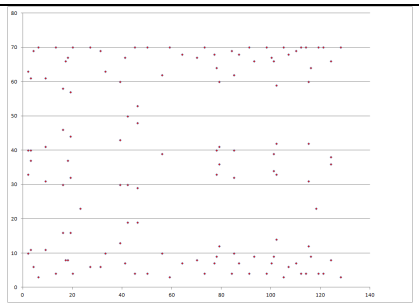
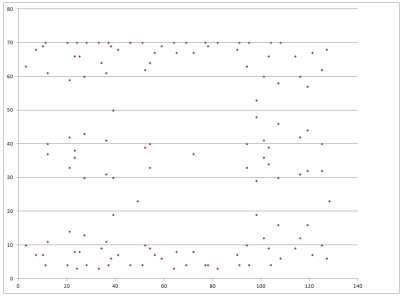
Iteration: 0, 24	Iteration: 7	Iteration: 20
		

3.7. Bilobed leaf dataset: *Bauhinia Variegata* (orchid tree, also known as Kachnar in Hindi) has bilobed leaves. In our specimen, the bilobed leaf dataset contains 129 (=n) points. The original dataset (AX) is real, but it has been converted into an integer dataset (X in Table-7) by the transformation given below:

If(ax(i,j)-int(ax(i,j)) < 0.5) x(i,j)=int(ax(i,j).kr) else x(i,j)=int(ax(i,j).kr)+1; j=1,2; i=1,n; kr=10.

Subjected to repeated transformations, it resumes its original position in the 43th iteration.

Fig.7. Bilobed Leaf (129 points)

Iteration: 0, 43	Iteration: 12	Iteration: 38
		

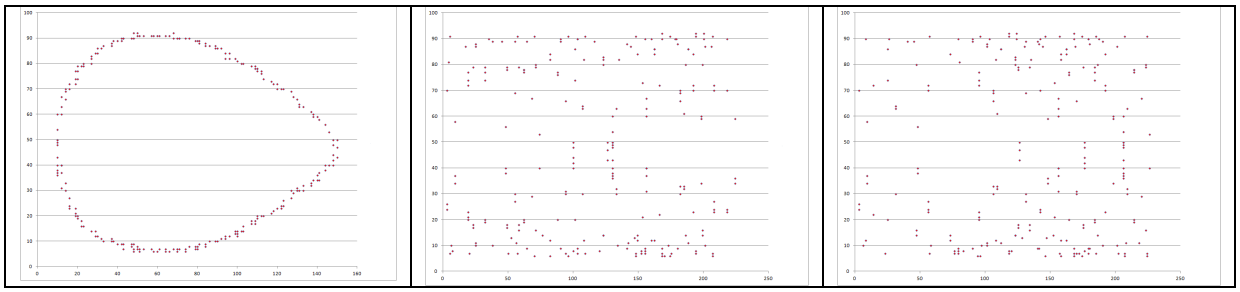
3.8. Ovate leaf dataset: The ovate leaf dataset contains 228 (=n) points. The original dataset (AX) is real, but it has been converted into an integer dataset (X in Table-8) by the transformation given below:

If(ax(i,j)-int(ax(i,j)) < 0.5) x(i,j)=int(ax(i,j).kr) else x(i,j)=int(ax(i,j).kr)+1; j=1,2; i=1,n; kr=10.

Subjected to repeated transformations, it resumes its original position in the 36th iteration.

Fig.8. Ovate Leaf (228 points)

Iteration: 0, 36	Iteration: 12	Iteration: 30



4. A computer program: We have carried out computation by a simple (Fortran 77) program. The code is appended. It requires the input file containing: [1] the no. of points (n) in the dataset, and [2] the data points $ax(i,1),\ ax(i,2)$ for all points in the subsequent rows. The input and output files are to be named by the user on the run of the program. The output begins from the original X (call it $X^{(0)}$), the intermediate X (call them $X^{(1)}, X^{(2)},\ \ldots,\ X^{(t-1)}$) and the terminal X (call it $X^{(t)}$). The $X^{(t)} = X^{(0)}$.

5. Conclusion: We have provided some examples where Poincaré recurrence is observed. We have not tested for the existence of such recurrences in case of many other shapes of plant leaves. Any one, interested in finding them out for other leaf shapes, may use the computer program.

References:

Arnold, V.I. and Avez, A. (1968). *Ergodic Problems in Classical Mechanics*. New York: Benjamin.

Dyson F.J. and Falk, H. (1992). Period of a Discrete Cat Mapping. *The American Mathematical Monthly* 99(7): 603–614.

Gielis, J. (2003). A Generic Geometric Transformation that unifies a Wide Range of Natural and Abstract Shapes. *American Journal of Botany* 90(3): 333–338.

Datasets

Table-1. Cordate Leaf Dataset (n=216; kr=10; Recurrence at the 36th iteration)																		
Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2
1	6	39	37	19	133	73	40	9	109	67	130	145	90	112	181	113	49
2	6	33	38	19	130	74	40	8	110	67	20	146	90	33	182	117	90
3	6	32	39	19	80	75	42	140	111	67	18	147	90	32	183	117	50
4	6	31	40	19	74	76	42	10	112	69	129	148	92	110	184	119	88
5	6	30	41	19	67	77	42	9	113	69	20	149	92	33	185	119	51
6	8	47	42	19	12	78	44	140	114	69	19	150	93	109	186	120	86
7	8	30	43	19	10	79	44	140	115	70	128	151	93	34	187	120	84
8	10	108	44	20	139	80	44	11	116	70	20	152	96	109	188	120	52
9	10	100	45	20	137	81	44	10	117	71	127	153	96	108	189	121	84
10	10	50	46	20	71	82	44	9	118	71	124	154	96	36	190	121	54
11	10	24	47	20	10	83	46	140	119	71	20	155	99	106	191	124	82
12	10	21	48	21	139	84	46	10	120	74	124	156	99	39	192	124	56
13	10	19	49	21	9	85	48	139	121	74	21	157	99	38	193	127	82
14	12	117	50	21	8	86	48	12	122	77	122	158	100	104	194	127	80
15	12	96	51	23	140	87	48	10	123	77	24	159	100	101	195	127	59
16	12	92	52	23	9	88	50	140	124	77	22	160	100	39	196	128	80
17	12	91	53	23	7	89	50	139	125	78	121	161	100	38	197	128	78
18	12	56	54	27	140	90	50	12	126	78	23	162	102	104	198	128	60
19	12	19	55	27	8	91	50	11	127	78	22	163	102	102	199	128	60
20	12	17	56	29	141	92	52	138	128	80	120	164	102	40	200	130	79
21	12	14	57	29	9	93	52	11	129	80	26	165	102	40	201	130	77
22	13	129	58	29	8	94	53	137	130	80	23	166	103	100	202	130	61
23	13	126	59	30	141	95	53	12	131	82	120	167	103	41	203	132	76
24	13	121	60	30	9	96	53	11	132	82	119	168	103	40	204	132	73
25	13	120	61	30	7	97	56	136	133	82	27	169	106	100	205	132	63
26	13	120	62	31	141	98	56	13	134	84	119	170	106	99	206	134	74
27	13	89	63	31	8	99	59	134	135	84	117	171	106	43	207	134	73
28	13	86	64	34	141	100	59	13	136	84	30	172	106	42	208	134	64
29	13	59	65	34	8	101	60	133	137	84	28	173	109	97	209	134	63
30	13	16	66	34	7	102	60	17	138	86	117	174	109	44	210	136	71
31	13	13	67	37	141	103	60	14	139	86	114	175	110	96	211	136	67
32	16	130	68	37	140	104	61	132	140	86	30	176	110	46	212	138	70
33	16	128	69	37	9	105	61	16	141	86	30	177	111	93	213	138	70
34	16	82	70	38	141	106	63	131	142	86	28	178	111	48	214	138	68
35	16	62	71	38	9	107	63	18	143	88	113	179	111	46	215	138	66
36	16	11	72	40	141	108	63	17	144	88	30	180	113	91	216	140	69

Table-2. Convex-deltoid Leaf Dataset (n=168; kr=10; Recurrence at the 24th iteration)																	
Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2
1	13	80	29	38	19	57	64	11	85	83	18	113	110	32	141	137	97
2	17	87	30	40	137	58	66	143	86	87	137	114	111	119	142	137	50
3	17	74	31	40	17	59	66	12	87	87	20	115	111	33	143	139	96
4	18	90	32	41	139	60	66	12	88	87	19	116	113	117	144	139	52
5	18	72	33	41	16	61	69	143	89	89	133	117	113	36	145	140	93
6	18	70	34	44	141	62	69	13	90	89	20	118	117	113	146	140	53
7	20	94	35	44	13	63	70	143	91	90	133	119	117	37	147	141	92
8	20	67	36	46	143	64	70	13	92	90	20	120	119	111	148	141	56
9	20	64	37	46	13	65	70	13	93	91	131	121	119	111	149	143	90
10	22	99	38	49	143	66	71	141	94	91	21	122	119	38	150	143	58
11	22	60	39	49	12	67	71	14	95	93	131	123	120	111	151	147	89
12	24	101	40	50	143	68	71	13	96	93	22	124	120	40	152	147	59
13	24	53	41	50	12	69	74	141	97	97	129	125	121	109	153	149	88
14	26	107	42	51	146	70	74	14	98	97	23	126	121	41	154	149	61
15	26	47	43	51	11	71	74	14	99	99	129	127	121	40	155	150	86
16	28	109	44	54	146	72	76	140	100	99	24	128	123	107	156	150	63
17	28	41	45	54	11	73	76	16	101	100	127	129	123	41	157	151	83
18	30	113	46	56	146	74	76	14	102	100	27	130	127	107	158	151	64
19	30	34	47	56	11	75	79	139	103	100	26	131	127	43	159	153	81
20	32	119	48	59	146	76	79	17	104	101	126	132	129	103	160	153	67
21	32	30	49	59	11	77	79	16	105	101	27	133	129	44	161	157	80
22	32	30	50	60	146	78	79	16	106	103	123	134	130	101	162	157	68
23	34	123	51	60	11	79	80	139	107	103	28	135	130	46	163	159	79
24	34	121	52	61	146	80	80	17	108	107	121	136	131	100	164	159	70
25	34	23	53	61	12	81	81	139	109	107	30	137	131	48	165	160	77
26	36	127	54	61	11	82	81	17	110	109	120	138	133	99	166	160	70
27	36	20	55	64	146	83	83	137	111	109	30	139	133	98	167	162	74
28	38	131	56	64	12	84	83	19	112	110	119	140	133	50	168	162	71

Table-3. Cardio-lanceolate Leaf Dataset (n=168; kr=10; Recurrence at the 24th iteration)																	
Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2
1	1	26	29	18	50	57	39	6	85	60	58	113	84	6	141	106	12
2	3	42	30	18	49	58	40	57	86	60	3	114	87	53	142	108	42
3	3	40	31	18	13	59	40	6	87	61	58	115	87	7	143	108	13
4	3	28	32	18	12	60	40	4	88	61	3	116	88	52	144	110	41
5	3	23	33	20	50	61	42	57	89	63	58	117	88	7	145	110	14
6	3	21	34	20	11	62	42	4	90	63	2	118	88	7	146	112	40
7	6	43	35	22	51	63	43	57	91	67	58	119	90	51	147	112	16
8	6	42	36	22	11	64	43	4	92	67	3	120	90	8	148	114	40
9	6	37	37	22	10	65	46	57	93	69	57	121	91	51	149	114	17
10	6	29	38	24	51	66	46	4	94	69	3	122	91	8	150	114	16
11	6	19	39	24	10	67	48	59	95	70	57	123	91	7	151	116	38
12	6	18	40	27	52	68	48	57	96	70	3	124	94	50	152	116	18

13	9	46	41	27	51	69	48	3	97	72	57	125	94	10	153	118	38
14	9	32	42	27	10	70	50	58	98	72	4	126	94	9	154	118	37
15	9	30	43	29	53	71	50	3	99	72	3	127	97	50	155	118	19
16	9	19	44	29	9	72	50	1	100	73	57	128	97	10	156	118	18
17	9	18	45	30	53	73	52	58	101	73	4	129	97	9	157	120	34
18	10	47	46	30	9	74	52	3	102	76	56	130	99	49	158	120	20
19	10	30	47	31	54	75	52	2	103	76	4	131	99	10	159	121	32
20	10	17	48	31	8	76	54	58	104	78	56	132	100	48	160	121	21
21	12	48	49	31	7	77	54	2	105	78	4	133	100	10	161	121	20
22	12	16	50	33	54	78	57	58	106	80	54	134	101	47	162	124	30
23	14	48	51	33	7	79	57	3	107	80	6	135	101	47	163	124	22
24	14	14	52	36	54	80	57	2	108	82	54	136	101	11	164	127	30
25	16	50	53	36	7	81	58	58	109	82	7	137	103	44	165	127	26
26	16	49	54	36	6	82	58	57	110	82	6	138	103	12	166	127	22
27	16	13	55	39	56	83	58	3	111	84	53	139	103	11	167	129	28
28	16	12	56	39	7	84	58	1	112	84	6	140	106	44	168	129	27

Table-4. Truncate Leaf Dataset (n=260; kr=10; Recurrence at the 210th iteration)

Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2
1	13	20	45	32	92	89	50	8	133	69	8	177	91	18	221	111	30
2	13	30	46	32	7	90	50	7	134	70	86	178	93	76	222	113	63
3	13	40	47	34	92	91	50	6	135	70	12	179	93	74	223	113	60
4	13	50	48	34	8	92	51	90	136	70	10	180	93	73	224	113	31
5	13	60	49	34	7	93	51	7	137	70	8	181	93	20	225	113	30
6	13	70	50	36	91	94	51	6	138	71	86	182	93	19	226	117	64
7	13	53	51	36	90	95	54	92	139	71	11	183	97	76	227	117	62
8	13	53	52	36	8	96	54	91	140	71	9	184	97	74	228	117	62
9	13	88	53	36	8	97	54	90	141	74	87	185	97	72	229	117	61
10	13	77	54	36	6	98	54	89	142	74	84	186	97	23	230	117	33
11	13	11	55	38	91	99	54	8	143	74	83	187	97	21	231	117	31
12	13	10	56	38	90	100	54	7	144	74	82	188	99	73	232	119	61
13	13	10	57	38	8	101	54	7	145	74	13	189	99	24	233	119	60
14	17	87	58	38	7	102	56	90	146	74	11	190	99	22	234	119	34
15	17	11	59	38	6	103	56	88	147	76	84	191	100	71	235	119	33
16	17	10	60	40	91	104	56	8	148	76	83	192	100	24	236	120	62
17	18	88	61	40	8	105	56	7	149	76	81	193	100	22	237	120	59
18	18	86	62	40	6	106	59	90	150	76	12	194	101	70	238	120	56
19	18	10	63	40	4	107	59	90	151	76	11	195	101	24	239	120	37
20	18	9	64	41	91	108	59	89	152	79	82	196	101	23	240	120	34
21	20	90	65	41	8	109	59	10	153	79	13	197	101	22	241	121	57
22	20	90	66	41	7	110	59	8	154	80	81	198	103	70	242	121	54
23	20	88	67	41	4	111	60	90	155	80	14	199	103	26	243	121	40
24	20	10	68	44	92	112	60	89	156	80	12	200	103	22	244	121	37
25	20	9	69	44	91	113	60	88	157	81	80	201	107	69	245	123	57
26	22	90	70	44	90	114	60	9	158	81	17	202	107	67	246	123	40
27	22	90	71	44	8	115	60	7	159	81	14	203	107	30	247	127	58
28	22	89	72	44	8	116	61	89	160	83	80	204	107	27	248	127	54
29	22	9	73	44	6	117	61	10	161	83	17	205	109	69	249	127	53
30	24	91	74	44	4	118	61	8	162	83	14	206	109	68	250	127	51
31	24	90	75	46	92	119	61	7	163	87	80	207	109	67	251	127	50
32	24	89	76	46	91	120	64	88	164	87	79	208	109	30	252	127	41
33	24	9	77	46	90	121	64	87	165	87	18	209	109	30	253	127	40
34	26	91	78	46	8	122	64	86	166	87	17	210	109	29	254	127	40
35	26	90	79	46	7	123	64	8	167	89	80	211	109	28	255	129	50

36	26	10	80	46	6	124	66	87	168	89	79	212	110	67	256	129	46
37	26	9	81	46	6	125	66	10	169	89	78	213	110	66	257	129	44
38	28	91	82	49	91	126	66	10	170	89	18	214	110	64	258	129	41
39	28	10	83	49	90	127	66	9	171	90	79	215	110	62	259	130	46
40	28	8	84	49	7	128	66	8	172	90	78	216	110	31	260	130	41
41	30	92	85	49	6	129	69	87	173	90	77	217	110	30	-	-	-
42	30	91	86	49	4	130	69	12	174	90	19	218	111	66	-	-	-
43	30	8	87	50	90	131	69	10	175	91	76	219	111	64	-	-	-
44	30	7	88	50	90	132	69	10	176	91	20	220	111	31	-	-	-

Table-5. Cucumeraroid Leaf Dataset (n=121; kr=1; Recurrence at the 54th iteration)

Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2
1	1	4	22	3	1	43	5	10	64	7	2	85	9	3	106	11	7
2	1	4	23	3	9	44	5	1	65	7	9	86	9	7	107	11	3
3	1	6	24	3	0	45	5	10	66	7	2	87	9	3	108	11	7
4	1	5	25	3	9	46	5	1	67	7	9	88	9	7	109	11	3
5	1	3	26	3	0	47	5	10	68	7	2	89	9	3	110	11	6
6	1	2	27	3	9	48	5	1	69	7	9	90	9	7	111	11	3
7	1	6	28	3	0	49	5	10	70	7	2	91	9	3	112	12	6
8	1	2	29	3	9	50	5	1	71	8	9	92	10	7	113	12	3
9	1	7	30	3	0	51	6	10	72	8	3	93	10	3	114	12	6
10	1	2	31	4	9	52	6	1	73	8	9	94	10	7	115	12	4
11	2	7	32	4	0	53	6	10	74	8	3	95	10	3	116	12	6
12	2	1	33	4	9	54	6	1	75	8	8	96	10	7	117	12	4
13	2	8	34	4	0	55	6	10	76	8	3	97	10	3	118	12	6
14	2	1	35	4	9	56	6	1	77	8	8	98	10	7	119	12	4
15	2	8	36	4	0	57	6	10	78	8	8	99	10	3	120	12	5
16	2	1	37	4	10	58	6	1	79	8	3	100	10	7	121	12	5
17	2	8	38	4	0	59	6	10	80	8	8	101	10	3	-	-	-
18	2	1	39	4	10	60	6	1	81	8	3	102	11	7	-		-
19	2	8	40	4	0	61	7	10	82	9	7	103	11	3	-		-
20	2	1	41	5	10	62	7	2	83	9	3	104	11	7	-		-
21	3	9	42	5	0	63	7	9	84	9	7	105	11	3	-		-

Table-6. Oblong Leaf Dataset (n=252; kr=10; Recurrence at the 24th iteration)

Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2
1	3	57	43	21	14	85	50	90	127	72	7	169	98	90	211	124	11
2	4	56	44	22	12	86	50	8	128	71	4	170	99	9	212	127	81
3	3	51	45	24	80	87	50	7	129	74	92	171	100	90	213	127	80
4	4	42	46	23	13	88	51	90	130	74	90	172	100	88	214	126	79
5	3	37	47	24	12	89	52	7	131	74	8	173	100	9	215	126	14
6	3	36	48	26	82	90	54	90	132	74	7	174	100	7	216	128	80
7	7	60	49	27	80	91	53	90	133	77	90	175	101	90	217	129	17
8	6	57	50	27	11	92	54	7	134	77	7	176	101	10	218	128	14
9	7	53	51	28	82	93	57	90	135	76	4	177	101	9	219	130	78
10	6	41	52	29	81	94	56	89	136	79	91	178	103	89	220	130	77
11	6	40	53	28	10	95	57	7	137	79	90	179	103	9	221	130	18
12	7	39	54	28	9	96	56	6	138	78	90	180	104	8	222	130	16
13	6	34	55	30	86	97	59	91	139	79	7	181	106	90	223	132	76
14	8	62	56	30	83	98	59	90	140	80	90	182	106	88	224	132	19
15	8	30	57	30	10	99	59	8	141	80	89	183	106	9	225	134	74
16	10	67	58	31	84	100	59	7	142	80	9	184	108	88	226	133	73
17	10	66	59	31	83	101	58	4	143	80	7	185	109	87	227	133	21

18	10	63	60	31	9	102	60	90	144	82	91	186	108	9	228	134	19
19	10	27	61	34	87	103	60	89	145	81	90	187	110	88	229	136	72
20	10	23	62	33	84	104	60	7	146	81	9	188	110	87	230	137	71
21	12	70	63	33	10	105	60	6	147	82	7	189	110	10	231	136	24
22	11	69	64	34	8	106	62	91	148	83	90	190	112	87	232	137	22
23	11	67	65	37	86	107	62	90	149	84	7	191	111	10	233	139	70
24	12	22	66	36	84	108	62	7	150	83	4	192	114	87	234	138	27
25	14	70	67	36	10	109	64	90	151	86	90	193	114	86	235	138	24
26	14	70	68	37	9	110	63	89	152	87	8	194	113	10	236	140	67
27	14	22	69	37	7	111	64	7	153	87	7	195	116	84	237	140	29
28	13	20	70	39	87	112	63	6	154	89	90	196	117	11	238	140	28
29	13	20	71	39	8	113	67	91	155	88	89	197	116	10	239	140	27
30	16	72	72	40	88	114	67	90	156	89	8	198	119	86	240	142	63
31	17	70	73	40	8	115	67	7	157	89	7	199	118	84	241	141	31
32	16	20	74	41	88	116	66	4	158	90	90	200	118	82	242	144	60
33	16	19	75	42	7	117	69	91	159	90	8	201	119	13	243	143	34
34	19	74	76	43	89	118	69	90	160	91	90	202	119	11	244	146	58
35	18	19	77	44	8	119	69	7	161	92	8	203	120	82	245	147	40
36	19	18	78	43	6	120	68	6	162	92	7	204	120	12	246	146	37
37	18	16	79	46	90	121	70	91	163	93	90	205	120	10	247	148	57
38	20	79	80	47	8	122	70	90	164	93	9	206	121	83	248	149	50
39	20	77	81	47	7	123	70	8	165	94	8	207	122	81	249	149	43
40	20	18	82	48	90	124	70	7	166	96	90	208	122	13	250	148	42
41	20	16	83	49	7	125	72	91	167	97	8	209	124	81	251	150	50
42	21	79	84	50	90	126	71	90	168	97	7	210	124	13	252	150	49

Table-7. Bilobed Leaf Dataset (n=129; kr=10; Recurrence at the 43th iteration)

Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2
1	2	23	23	13	40	45	26	67	67	49	70	89	70	69	111	90	40
2	4	53	24	13	32	46	26	7	68	49	4	90	70	6	112	90	37
3	4	48	25	13	10	47	28	68	69	50	70	91	71	68	113	90	11
4	4	29	26	17	63	48	28	7	70	50	3	92	71	6	114	91	60
5	4	19	27	17	40	49	30	68	71	51	70	93	74	68	115	91	41
6	7	58	28	17	33	50	30	7	72	51	3	94	74	7	116	91	36
7	7	46	29	17	10	51	32	69	73	54	70	95	76	67	117	91	12
8	7	30	30	18	64	52	32	7	74	54	3	96	76	7	118	93	59
9	7	16	31	18	40	53	34	69	75	56	70	97	79	67	119	93	42
10	9	60	32	18	33	54	34	6	76	56	3	98	79	8	120	93	33
11	9	43	33	18	9	55	36	70	77	59	70	99	80	66	121	93	14
12	9	30	34	20	66	56	36	6	78	59	4	100	80	8	122	97	57
13	9	13	35	20	39	57	38	70	79	60	70	101	81	66	123	97	44
14	10	60	36	20	34	58	38	4	80	60	4	102	81	9	124	97	32
15	10	42	37	20	9	59	40	70	81	61	70	103	83	64	125	97	16
16	10	31	38	22	66	60	40	4	82	61	4	104	83	9	126	99	50
17	10	12	39	22	38	61	41	70	83	64	70	105	87	63	127	99	30
18	12	61	40	22	36	62	41	4	84	64	4	106	87	10	128	99	19
19	12	41	41	22	8	63	44	70	85	66	70	107	89	62	129	100	23
20	12	31	42	24	67	64	44	4	86	66	4	108	89	39	-	-	-
21	12	11	43	24	37	65	46	70	87	69	69	109	89	10	-	-	-
22	13	62	44	24	8	66	46	4	88	69	4	110	90	61	-	-	-

Table-8. Ovate Leaf Dataset (n=228; kr=10; Recurrence at the 36th iteration)																	
Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2
1	10	60	39	20	19	77	47	8	115	74	7	153	100	13	191	127	30
2	10	54	40	22	79	78	48	92	116	74	6	154	100	12	192	127	27
3	10	50	41	22	18	79	48	90	117	76	90	155	102	80	193	128	67
4	10	50	42	22	16	80	48	9	118	76	8	156	102	14	194	128	29
5	10	49	43	23	80	81	48	7	119	79	89	157	103	80	195	130	66
6	10	48	44	23	79	82	48	6	120	79	8	158	103	16	196	130	31
7	10	43	45	23	16	83	50	92	121	79	7	159	103	14	197	130	30
8	10	40	46	27	83	84	50	91	122	80	89	160	107	79	198	131	64
9	10	38	47	27	82	85	50	8	123	80	8	161	107	18	199	131	63
10	10	37	48	27	80	86	50	7	124	80	7	162	107	17	200	131	30
11	10	36	49	27	14	87	51	91	125	81	89	163	109	79	201	133	63
12	12	67	50	29	84	88	51	7	126	81	88	164	109	78	202	133	32
13	12	63	51	29	14	89	51	6	127	81	9	165	109	18	203	133	30
14	12	60	52	29	12	90	53	91	128	84	88	166	109	17	204	137	61
15	12	40	53	30	86	91	53	7	129	84	87	167	110	78	205	137	33
16	12	40	54	30	84	92	57	91	130	84	9	168	110	78	206	137	32
17	12	37	55	30	12	93	57	7	131	84	8	169	110	77	207	138	60
18	12	31	56	32	86	94	58	91	132	86	87	170	110	20	208	138	59
19	14	70	57	32	11	95	58	6	133	86	10	171	110	19	209	138	34
20	14	69	58	33	87	96	60	91	134	89	87	172	112	77	210	140	59
21	14	66	59	33	10	97	60	6	135	89	86	173	112	76	211	140	36
22	14	33	60	37	88	98	61	91	136	89	10	174	112	20	212	140	34
23	14	30	61	37	87	99	61	7	137	90	86	175	113	74	213	141	58
24	16	72	62	37	11	100	64	91	138	90	10	176	113	20	214	141	37
25	16	70	63	37	10	101	64	7	139	92	86	177	117	73	215	141	34
26	16	27	64	38	89	102	64	6	140	92	11	178	117	21	216	144	56
27	16	24	65	38	10	103	66	91	141	94	84	179	118	72	217	144	40
28	16	23	66	40	89	104	66	6	142	94	82	180	118	22	218	144	38
29	19	77	67	40	9	105	68	92	143	94	12	181	120	72	219	146	53
30	19	74	68	42	90	106	68	90	144	94	11	182	120	70	220	146	40
31	19	72	69	42	89	107	68	7	145	96	84	183	120	23	221	148	50
32	19	23	70	42	9	108	70	90	146	96	82	184	122	70	222	148	48
33	19	21	71	43	90	109	70	7	147	96	12	185	122	24	223	148	44
34	19	20	72	43	90	110	70	6	148	99	82	186	122	23	224	148	42
35	20	79	73	43	9	111	71	90	149	99	14	187	123	70	225	148	40
36	20	77	74	43	7	112	71	90	150	99	13	188	123	26	226	150	50
37	20	74	75	47	90	113	71	7	151	100	81	189	123	24	227	150	47
38	20	20	76	47	9	114	74	90	152	100	14	190	127	69	228	150	43

```
      PARAMETER(MAXITER=3000,KR=10)! MAX ITERATIONS
      ! KR IS THE INTEGER-TRUNCATION COEFFICIENT
      ! ---------------------------------------------------------------
      PARAMETER(NMAX=1000,M=2)! NMAX IS THE MAX POINTS ALLOWED
      ! IT MAY BE INCRESED/DECREASED BY THE USER.
      INTEGER X(NMAX,M),Z(NMAX,M)
      DIMENSION AX(NMAX,M)
      CHARACTER *70 INPUT_FILE, OUTPUT_FILE
      ! ---------------------------------------------------------------
      WRITE(*,*)'WHAT ARE THE INPUT AND THE OUTPUT FILES ?'
      READ(*,*) INPUT_FILE, OUTPUT_FILE

      OPEN(7,FILE=INPUT_FILE) ! INPUT FILE
      READ(7,*) N ! NO. OF POINTS IN THE DATASET (X1, X2)
      DO I=1,N
      READ(7,*) AX(I,1),AX(I,2) ! AX IS REAL DATASET POINTS

      ! GET INTEGER X AND PRESERVE THE ORIGINAL DATA IN Z
      IF(AX(I,1)-INT(AX(I,1)).LT.0.5) X(I,1)=INT(AX(I,1)*KR)
      IF(AX(I,1)-INT(AX(I,1)).GE.0.5) X(I,1)=INT(AX(I,1)*KR)+1
      IF(AX(I,2)-INT(AX(I,2)).LT.0.5) X(I,2)=INT(AX(I,2)*KR)
      IF(AX(I,2)-INT(AX(I,2)).GE.0.5) X(I,2)=INT(AX(I,2)*KR)+1
      ! PRESERVE X IN Z
      Z(I,1)=X(I,1)
      Z(I,2)=X(I,2)
      ! -------------------------------
      ENDDO
      CLOSE(7)
      OPEN(8,FILE=OUTPUT_FILE) ! OUTPUT FILE
      DO I=1,N
      WRITE(8,1) I, X(I,1),X(I,2)
      ENDDO
    1 FORMAT(I4,2I10)
      DO K=1,MAXITER
       WRITE(8,*)'------- ITERATION =',K,' ----------------------'
       CALL ARNOLD(X,N)
       DO I=1,N
       WRITE(8,1) I, X(I,1),X(I,2)
       ENDDO ! FOR I
      ! COMPARE WITH THE ORIGINAL DATA
         ! CHECK FOR RESTORATION
         KCHECKX=0
         KCHECKY=0
         DO I=1,N
         IF(X(I,1).EQ.Z(I,1)) KCHECKX=KCHECKX+1
         IF(X(I,2).EQ.Z(I,2)) KCHECKY=KCHECKY+1
         ENDDO ! FOR I
         IF(KCHECKX.EQ.N.AND.KCHECKY.EQ.N.AND.K.GT.3) THEN
         WRITE(*,*)'ORIGINAL DATA RESTORED IN',K,' ITERATIONS'
         CLOSE(8)
         WRITE(*,*)'TRANSFORMED DATA STORED IN ', OUTPUT_FILE
         WRITE(*,*) 'END. THANK YOU'
         STOP
         ENDIF

      ENDDO ! FOR K
      CLOSE(8)
      WRITE(*,*)'DID NOT CONVERGE.'
      WRITE(*,*)'TRANSFORMED DATA STORED IN ', OUTPUT_FILE
      WRITE(*,*)'END'
      END
      ! ---------------------------------------------------------------
      SUBROUTINE ARNOLD(X,N)! SUBROUTINE FOR TRANSFORMATION
      PARAMETER (M=2)
      INTEGER X,Z,B
      DIMENSION X(N,M),Z(N,M), B(M,M)
      DATA ((B(I,J),J=1,M),I=1,M) /2, 1, 1, 1/ ! TRANSFORMATION MATRIX
```

```
68:       ! ---------------------------------------------------------------
69:       ! PRESERVE X IN Z
70:       DO I=1,N
71:       DO J=1,M
72:       Z(I,J)=X(I,J)
73:       ENDDO
74:       ENDDO
75:       ! ------- TRANSFORMATION --------------------------------
76:       DO I=1,N
77:       DO J=1,M
78:       KS=0
79:       DO K=1,M
80:       KS = KS + Z(I,K)*B(K,J)
81:       ENDDO
82:       X(I,J)=MOD(KS,N) ! N is the no. of points in the dataset
83:       ENDDO
84:       ENDDO
85:       ! ---------------------------------------------------------------
86:       RETURN
87:       END
```