Competitive equilibrium for indivisible objects

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2 July 2013

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July 28, 2014

Abstract

We study an exchange economy with indivisible objects that may not be substitutes for each other, and we introduce the p-substitutability condition, a relaxation of the gross substitutes condition of Kelso and Crawford (1982), in which a parameter vector p is adopted to permit complicated types of complementarity. We prove that for any economy E, there exists a corresponding vector p^E such that the p^E-substitutability condition is sufficient to guarantee the existence of a competitive equilibrium, and that the largest competitive price of each object is equal to its contribution to the social welfare. Our approach relies on a classification result that partitions economies into disjoint similarity classes such that whenever a similarity class contains an economy with an equilibrium, each economy in this class also has an equilibrium.

Keywords: Indivisibility, equilibrium, gross substitutability, p-substitutability.

1 Introduction

An essential issue for markets with heterogeneous indivisible objects and preferences that are quasi-linear in money is under which conditions an efficient allocation of objects can be supported by a system of competitive prices as an equilibrium outcome.[1] A sufficient condition for the existence of a competitive equilibrium is the *gross substitutes* condition of Kelso and Crawford (1982), which requires that objects are substitutes in the sense that the demand of each agent for an object does not decrease when prices of some other objects increase. However, in many market situations, heterogeneous objects

*Previous versions of the paper were circulated under the title "Free disposal, monotonicity, and equilibrium".

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[1] See Bikhchandani and Mamer (1997), Ma (1998), Gul and Stacchetti (1999), Sun and Yang (2006), among others.

may not be perfect substitutes for all agents. For example, a scarf and a sweater may be substitutes for one agent, but are complements for another. To analyze such markets with different types of preferences, we introduce the notion of p-substitutability, in which a parameter vector p is employed to determine the range of covered preferences and to capture partial substitutability among objects.

The p-substitutability condition extends the gross substitutes condition in three respects. First, any agent's preferences satisfy the p-substitutability condition for some proper vectors p. Hence, our framework could incorporate arbitrary patterns of complementarity. Second, p-substitutability is strictly weaker than p'-substitutability if $p \geq p'$. This suggests that for an agent i, the degree of partial substitutability among objects could be analyzed by the lower frontier of the set of vectors p such that i's preferences are p-substitutable. Finally, p-substitutability is closely linked to gross substitutability: agent i's preferences are p-substitutable for all parameter vectors p if and only if i views objects as perfect substitutes for each other.

Based on these observations, we prove that for an arbitrary exchange economy E, there exists a corresponding vector p^E such that when each agent's preferences are p^E-substitutable, the following results hold:

(i) There exists a competitive equilibrium.

(ii) The largest competitive price of each object coincides with its contribution to the social welfare.

(iii) The society's aggregate demand satisfies the gross substitutes condition.

A result by Gul and Stacchetti (1999, Theorem 2) shows that no weakening of the gross substitutes condition is sufficient for an equilibrium to exist. As in the result (i), we make a breakthrough and prove that the p-substitutability condition can guarantee the existence of an equilibrium for economy E whenever $p \geq p^E$. Another issue that concerns us is the contribution of an object to the social welfare, which is well known as an upper bound for its competitive prices. The result (ii) shows that this bound itself is a competitive price under p^E-substitutability, extending a former result by Gul and Stacchetti (1999, Theorem 5). In the final result, we consider an representative agent whose demand function coincides with the society's aggregate demand, and show that the gross substitutability of individual agents' preferences is sufficient, but not necessary for the gross substitutability of the representative agent's preferences. Hence, objects could be substitutes for the whole society even when complementarity exists among the objects for individual agents.

Our results contribute to the literature on gross substitutability. Gul and Stacchetti (1999) prove that both their single improvement condition and no complementarities condition are equivalent to the gross substitutes condition. Fujishige and Yang (2003) draw the equivalence between $M^\natural$-concavity and gross substitutability in the framework of discrete convex analysis. Hatfield and Milgrom (2005) and Hatfield and Kojima (2010) introduce various extensions of gross substitutability to the frameworks of matching with contracts, respectively.

The rest of the paper is organized as follows. We present the model and some fundamental results on competitive equilibria in Section 2. In Section 3, we introduce the p-substitutability condition and our main equilibrium results. In Section 4, we prove our main results with a structural result that partitions economies into similarity classes. Section 5 relates our analysis to an existence result by Sun and Yang (2006) and concludes.

2 The model

We consider an exchange economy with a finite set $N = \{1, \ldots, n\}$ of agents and a finite set $\Omega = \{a_1, \ldots, a_m\}$ of heterogeneous indivisible objects, and a perfectly divisible good called money. Each agent $i \in N$ has a valuation function $v_i : 2^\Omega \rightarrow \mathbb{R}$ with $v_i(\emptyset) = 0$. The valuation v_i gives rise to a quasi-linear utility function u_i such that the utility of agent i holding the set of objects $A \subseteq \Omega$ and c units of money is

$$u_i(A, c) \equiv v_i(A) + c.$$

For each coalition of agents $C \subseteq N$, the corresponding *aggregate valuation function*, $v_{i_C} : 2^\Omega \rightarrow \mathbb{R}$, is defined by

$$v_{i_C}(A) \equiv \max\left\{\sum_{i\in C} v_i(A_i) : \bigcup_{i\in C} A_i = A \text{ and } A_i \cap A_j = \emptyset \text{ for } i \neq j\right\} \text{ for } A \subseteq \Omega. \quad (1)$$

An *allocation* is a partition of objects among all agents in N, i.e., a set $\mathbf{X} = (X_1, \ldots, X_n)$ of mutually exclusive bundles that exhaust Ω, where X_i represents agent i's consumption bundle under the allocation $\mathbf{X}$. The possibility that $X_i = \emptyset$ for some i is allowed. An allocation $\mathbf{X} = (X_1, \ldots, X_n)$ is called *efficient* if it maximizes the sum of agents' values, i.e., $\sum_{i=1}^n v_i(X_i) = v_{i_N}(\Omega)$.

A *price vector* $p = (p_a)_{a\in\Omega} \in \mathbb{R}^{|\Omega|}$ assigns a price to each object $a \in \Omega$. For any set of objects $A \subseteq \Omega$, let $p(A)$ be a shorthand for $\sum_{a\in A} p_a$. A valuation function v_i is *additively separable* if there exists a price vector p such that $v_i(A) = p(A)$ for all $A \subseteq \Omega$.

Given two vectors $p', p'' \in \mathbb{R}^{|\Omega|}$, we write $p = p' \vee p''$ if p is the vector in $\mathbb{R}^{|\Omega|}$ satisfying $p_a = \max\{p'_a, p''_a\}$ for all $a \in \Omega$. Given a sequence of vectors $p^1, p^2, \ldots, p^r$ in $\mathbb{R}^{|\Omega|}$, we write $p = \vee_{k=1}^r p^k$ if p is the vector in $\mathbb{R}^{|\Omega|}$ satisfying $p_a = \max\{p_a^1, p_a^2, \ldots, p_a^r\}$ for all $a \in \Omega$. For any valuation function v_i, let $p^{v_i} \in \mathbb{R}^{|\Omega|}$ denote the minimal marginal value vector of v_i given by

$$p_a^{v_i} \equiv \min \{ v_i (A \cup \{a\}) - v_i (A) : A \subseteq \Omega \setminus \{a\} \} \text{ for } a \in \Omega. \tag{2}$$

A *competitive equilibrium* is a pair $\langle \mathbf{X}; p \rangle$, where $\mathbf{X} = (X_1, \ldots, X_n)$ is an allocation for E and $p \in \mathbb{R}^{|\Omega|}$ is a price vector such that for each agent $i \in N$, the bundle X_i maximizes i's utility at price level p, i.e.,

$$X_i \in D_{v_i}(p) \equiv \{ A \subseteq \Omega : v_i (A) - p(A) \geq v_i (B) - p(B) \text{ for all } B \subseteq \Omega \}.$$

In this case, $\mathbf{X}$ is called an *equilibrium allocation* and p is called an *equilibrium price vector*.

We assume that each agent $i \in N$ is initially endowed with a bundle of objects Ω_i and a sufficient amount of money c_i such that $\Omega = \cup_{i \in N} \Omega_i$ and $c_i \geq v_i (A)$ for all $A \subseteq \Omega$. Under these assumptions, the initial endowments of objects and money will be irrelevant to the competitive equilibria. Hence, we leave them unspecified and simply represent this economy by $E = \left(\Omega; (v_i)_{i \in N} \right)$.

We close this section with some fundamental observations on competitive equilibria. The (a) and (b) parts of Lemma 1, originally given by Bikhchandani and Mamer (1997) and Gul and Stacchetti (1999), show that the standard theorems of welfare economics hold for an economy with indivisible objects; and the result of (c) shows that the contribution of an object $a \in \Omega$ to the social welfare, or its social value, is an upper bound for the equilibrium prices of a. Finally, Lemma 2 shows that the formation of a coalition will not eliminate existing competitive equilibria.

Lemma 1 *Let $E = \left(\Omega; (v_i)_{i \in N} \right)$ be an economy and $\langle \mathbf{X}; p \rangle$ a competitive equilibrium for E.*

(a) *The equilibrium allocation $\mathbf{X}$ is efficient.*

(b) *For any efficient allocation $\mathbf{Y}$ for E, $\langle \mathbf{Y}; p \rangle$ is also a competitive equilibrium.*

(c) *For each object $a \in \Omega$, $p_a \leq v_{i_N} (\Omega) - v_{i_N} (\Omega \setminus \{a\})$.*

Proof. Let $\mathbf{Y} = (Y_1, \ldots, Y_n)$ be an allocation for E.

(a) Since $X_i \in D_{v_i}(p)$ for each $i \in N$, we have

$$\begin{aligned} \sum_{i=1}^{n} v_i (X_i) &= \sum_{i=1}^{n} [v_i (X_i) - p(X_i)] + p(\Omega) \\ &\geq \sum_{i=1}^{n} [v_i (Y_i) - p(Y_i)] + p(\Omega) = \sum_{i=1}^{n} v_i (Y_i). \end{aligned}$$

(b) In case $\mathbf{Y}$ is efficient, the above inequality implies $v_i(X_i) - p(X_i) = v_i(Y_i) - p(Y_i)$ for each $i \in N$, and hence $\langle \mathbf{Y}; p \rangle$ is also a competitive equilibrium for E.

(c) Let $N_0 = N \cup \{0\}$, $X_0 = \emptyset$ and let $E_0 = \left(\Omega; (v_i)_{i \in N_0}\right)$ be the economy constructed from E by adding an agent 0 whose valuation function v_0 satisfies $v_0(A) = p(A)$ for $A \subseteq \Omega$. Clearly, $\langle (X_0, X_1, \ldots, X_n); p \rangle$ is a competitive equilibrium for E_0. By (a), $(X_0, X_1, \ldots, X_n)$ is an efficient allocation for E_0, and hence for each $a \in \Omega$, $v_{i_N}(\Omega) = \sum_{i=0}^{n} v_i(X_i) \geq v_0(\{a\}) + v_{i_N}(\Omega \backslash \{a\}) = p_a + v_{i_N}(\Omega \backslash \{a\})$. ■

Lemma 2 *Let $E = \left(\Omega; (v_i)_{i \in N}\right)$ be an economy. Let $C = \{1, \ldots, r\} \subseteq N$ and let $E_C \equiv (\Omega; v_{i_C}, v_{r+1}, \ldots, v_n)$ denote the corresponding economy in which each agent in C agrees to form a coalition and v_{i_C}, the valuation function of representative agent i_C, is given by* (1)*. If $\langle (X_1, \ldots, X_n); p \rangle$ is a competitive equilibrium for E and let $X_{i_C} = \bigcup_{j=1}^{r} X_j$, then $\langle (X_{i_C}, X_{r+1} \ldots, X_n); p \rangle$ is a competitive equilibrium for E_C.*

Proof. Suppose that $\langle (X_{i_C}, X_{r+1}, \ldots, X_n), p \rangle$ is not a competitive equilibrium for E_C. Then there exists $Y_{i_C} \subseteq \Omega$ such that $v_{i_C}(Y_{i_C}) - p(Y_{i_C}) > v_{i_C}(X_{i_C}) - p(X_{i_C})$. By definition there is a sequence of mutually disjoint bundles $\{Y_1, \ldots, Y_r\}$ such that $\bigcup_{j=1}^{r} Y_j = Y_{i_C}$ and $\sum_{j=1}^{r} v_j(Y_j) = v_{i_C}(Y_{i_C})$. Together with the fact that $X_j \in D_{v_j}(p)$ for $j = 1, \ldots, r$, we obtain

$$
\begin{aligned}
v_{i_C}(Y_{i_C}) - p(Y_{i_C}) &> v_{i_C}(X_{i_C}) - p(X_{i_C}) \geq \sum_{j=1}^{r} [v_j(X_j) - p(X_j)] \\
&\geq \sum_{j=1}^{r} [v_j(Y_j) - p(Y_j)] = v_{i_C}(Y_{i_C}) - p(Y_{i_C}),
\end{aligned}
$$

which is impossible. ■

3 The p-substitutability condition

A sufficient condition for the existence of a competitive equilibrium is the gross substitutes condition (Kelso and Crawford, 1982), the requirement that agents views heterogeneous objects as perfect substitutes for each other.

Definition 1 *A valuation function v_i satisfies the gross substitutes condition if for any two price vectors $p, q \in \mathbb{R}^{|\Omega|}$ with $p \leq q$, and any bundle $A \in D_{v_i}(p)$, there exists $B \in D_{v_i}(q)$ such that $\{a \in A : q_a = p_a\} \subseteq B$.*

Note that additive separability implies gross substitutability, and a result of Reijnierse et al. (2002, Theorem 8) shows that if a valuation function v_i satisfies the gross substitutes condition, then v_i has *decreasing marginal returns*, i.e., for each $a \in \Omega$,

$$a \in A \subseteq B \subseteq \Omega \Rightarrow v_i(B) - v_i(B \setminus \{a\}) \leq v_i(A) - v_i(A \setminus \{a\}).$$

However, different agents may have different types of preferences in many market situations. Consider the three-agent economy with one scarf $\{a_1\}$ and two sweaters $\{a_2, a_3\}$ given in Table I. The efficient allocation $X_1 = \{\emptyset\}, X_2 = \{a\}, X_3 = \{b, c\}$ augmented with the price vector $(8, 8, 8)$ is a competitive equilibrium, but only agent 1's valuation function satisfies the gross substitutes condition. The other two agents view a scarf and a sweater as complements in the sense that both v_2 and v_3 satisfy the *gross substitutes and complements* condition by Sun and Yang (2006).[2]

Table I

Agents' valuations

	$\emptyset$	$\{a_1\}$	$\{a_2\}$	$\{a_3\}$	$\{a_1, a_2\}$	$\{a_1, a_3\}$	$\{a_2, a_3\}$	$\{a_1, a_2, a_3\}$
v_1	0	7	7	7	13	13	12	19
v_2	0	16	3	3	22	22	5	24
v_3	0	5	11	11	17	17	20	23

To analyze such an economy with heterogeneous preferences, we introduce the notion of *p-substitutability*, a relaxation of the gross substitutes condition, in which a parameter vector $p \in \mathbb{R}^{|\Omega|}$ is employed for measuring the degree of substitutability of preferences.

Definition 2 *A valuation function v_i satisfies the p-substitutability condition for some vector $p \in \mathbb{R}^{|\Omega|}$ if the function $v_i[p]$ given by*

$$v_i[p](A) \equiv \max\{v_i(B) + p(A \setminus B) : B \subseteq A\} \text{ for } A \subseteq \Omega. \tag{3}$$

satisfies the gross substitutes condition.

Note that the function $v_i[p]$ coincides with the aggregate valuation function v_{i_C} of the coalition $C = \{i, j\}$, where j is a virtual agent who has an additively separable valuation function v_j satisfying $v_j(A) = p(A)$ for $A \subseteq \Omega$. Hence, the p-substitutability condition requires that objects are substitutes for the representative agent i_C.

Lemma 3 *Consider a sequence of valuation function $v_1, \ldots, v_r$ and let $C = \{1, \ldots, r\}$. If v_i satisfies the gross substitutes condition for $i = 1, \ldots, r$, then the aggregate valuation function v_{i_C} also satisfies the gross substitutes condition.*

[2]See the end of this section for the definition of the gross substitutes and complements condition.

Proof. Suppose that v_{i_C} fails the gross substitutes condition. Theorem 2 of Gul and Stacchetti (1999) implies that there exists an economy $E = (\Omega; v_{i_C}, v_{r+1}, \ldots, v_n)$ such that v_i satisfies the gross substitutes condition for $i = r+1, \ldots, n$, but E has no competitive equilibria. However, Theorem 2 of Kelso and Crawford (1982) implies that there exists a competitive for the economy $E' = (\Omega; v_1, \ldots, v_r, v_{r+1} \ldots, v_n)$, contradicting to the result of Lemma 2. ■

Lemma 3 improves on Theorem 6 of Gul and Stacchetti (1999), which shows that under the same conditions, the aggregate valuation function v_{i_C} has decreasing marginal returns. Moreover, since additive separability is stronger than gross substitutability, Lemma 3 implies that the p-substitutability condition is indeed a weakening of the gross substitutes condition.

In the following result, we note that for an arbitrary valuation function $v_i : 2^\Omega \to \mathbb{R}$, the set of vectors

$$\Gamma(v_i) \equiv \left\{p \in \mathbb{R}^{|\Omega|} : v_i \text{ satisfies the } p\text{-substitutability condition}\right\}$$

provide a good deal of information about v_i and the markets involves v_i. Lemma 4 (a) shows that objects are perfect substitutes for agent i if and only if $\Gamma(v_i) = \mathbb{R}^{|\Omega|}$. Lemma 4 (b) and (c) give some insights into the structure of $\Gamma(v_i)$. Moreover, Theorem 1 shows that for any economy $E = \left(\Omega; (v_i)_{i \in N}\right)$, there exists a corresponding vector $p^E \in \mathbb{R}^{|\Omega|}$ such that a number of results hold whenever p^E lies in $\Gamma(v_i)$ for all $i \in N$. The proof of Theorem 1 is given in the next section.

Lemma 4 *Let $v_i : 2^\Omega \to \mathbb{R}$ be an arbitrary valuation function.*

(a) *v_i satisfies the gross substitutes condition if and only if $\Gamma(v_i) = \mathbb{R}^{|\Omega|}$.*

(b) *$\Gamma(v_i)$ is never empty.*

(c) *If $p \in \Gamma(v_i)$ and $p \leq q \in \mathbb{R}^{|\Omega|}$, then $q \in \Gamma(v_i)$.*

Proof. (a) The "only if" part of the proof follows from the fact that p-substitutability is weaker than gross substitutability. The "if" part relies on the observation that v_i coincides with $v_i[p^{v_i}]$, where the minimal marginal value vector p^{v_i} is defined by (2).

(b) Let $p \in \mathbb{R}^{|\Omega|}$ be a vector satisfying $p(A) \geq v_i(A)$ for all $A \subseteq \Omega$. Then $v_i[p](A) = p(A)$ for all $A \subseteq \Omega$. This implies that $v_i[p]$ is additively separable, and hence $p \in \Gamma(v_i)$.

(c) Assume that v_i satisfies p-substitutability condition and $p \leq q \in \mathbb{R}^{|\Omega|}$. By definition $v_i[p]$ satisfies the gross substitutes condition. Then the result of (a) implies that $(v_i[p])[q]$ also satisfies the gross substitutes condition. Thus, it suffices to show that $v_i[q]$ coincides with $(v_i[p])[q]$. Let A be a set of objects. By definition, there exist two subsets B and B' of A such that $v_i[q](A) = v_i(B) + q(A \backslash B)$ and $(v_i[p])[q](A) =$

$v_i[p](B') + q(A \backslash B')$. Similarly, there exists $C' \subseteq B'$ such that $v_i[p](B') = v_i(C') + p(B' \backslash C')$. Then we have

$$\begin{aligned} v_i[q](A) &= v_i(B) + q(A \backslash B) \leq v_i[p](B) + q(A \backslash B) \leq (v_i[p])[q](A) \\ &= v_i[p](B') + q(A \backslash B') = v_i(C') + p(B' \backslash C') + q(A \backslash B') \\ &\leq v_i(C') + q(A \backslash C') \leq v_i[q](A), \end{aligned}$$

and hence $v_i[q](A) = (v_i[p])[q](A)$. ■

Theorem 1 *Let $E = (\Omega; (v_i)_{i \in N})$ be an economy and let $p^E \equiv \vee_{i=1}^n p^{v_i} \in \mathbb{R}^{|\Omega|}$. If each agent i's valuation function v_i satisfies the p^E-substitutability condition, then the following results hold:*

(a) *There exists a competitive equilibrium.*

(b) *The social value vector $\bar{p} = (\bar{p}_a) \in \mathbb{R}^{|\Omega|}$ defined by $\bar{p}_a = v_{i_N}(\Omega) - v_{i_N}(\Omega \backslash \{a\})$ is an equilibrium price vector.*

(c) *The social valuation function v_{i_N} satisfies the gross substitutes condition, and hence has decreasing marginal returns.*

Theorem 1 (a) and (b) contribute to the analysis of competitive equilibrium in three respects.

First, Theorem 2 of Gul and Stacchetti (1999) shows that the p-substitutability condition, a strict weakening of the gross substitutes condition, cannot guarantee the existence of an equilibrium for generic economies. However, we make a breakthrough and prove that the p-substitutability condition is sufficient for the existence of a competitive equilibrium for economy E whenever $p \leq p^E$.

Second, we prove that the contribution of object a to the social welfare, $\bar{p}_a = v_N(\Omega) - v_N(\Omega \backslash \{a\})$, is not only an upper bound for a's competitive prices, but itself is also a competitive price under p^E-substitutability. This result generalizes Theorem 5 of Gul and Stacchetti (1999). Recall the economy given in Table I. It is not difficult to verify $p^E = (16, 6, 6)$ and that v_i satisfies p^E-substitutability condition for $i = 1, 2, 3$. Hence, the efficient allocation $X_1 = \{\emptyset\}, X_2 = \{a\}, X_3 = \{b, c\}$ can be supported by vector $\bar{p} = (16, 9, 9)$ as a competitive equilibrium.

Third, in case the market $E = (\Omega; (v_i)_{i \in N})$ under consideration has no equilibria. To generate a competitive equilibrium, the government can choose a vector $\tilde{p} \in \mathbb{R}_+^{|\Omega|}$ such that $\tilde{p} \vee p^E \in \cap_{i \in N} \Gamma(v_i)$, and then promise to purchase any set of objects at price level $\tilde{p}$. For example, the two-agent economy given in Table II has no equilibria, and each agent's valuation function violates the p^E-substitutability condition. Since $p^E = (1.5, 2, 0)$ and $(1.5, 2, 1.5) \in \Gamma(v_1) \cap \Gamma(v_2)$, the government can create a new economy E' by adding itself as the third agent who has a valuation function v_3 such that $v_3(A) = \tilde{p}(A)$ for $\tilde{p} = (0, 0, 1.5)$ and $A \subseteq \Omega$, and then yields an equilibrium for E'.

Table II

Agents' valuations

	$\emptyset$	$\{a_1\}$	$\{a_2\}$	$\{a_3\}$	$\{a_1, a_2\}$	$\{a_1, a_3\}$	$\{a_2, a_3\}$	$\{a_1, a_2, a_3\}$
v_1	0	2	2	1	2	2	2	6
v_2	0	5	5	1	7	5	5.5	7

The final part of Theorem 1 extends Lemma 2 and shows that the gross substitutability of individual agents' valuation functions is sufficient but not necessary for the gross substitutability of the social valuation function. This implies that objects could be substitutes for each other from the viewpoint of the whole society even when complementarity exists among the objects for individual agents. Consider the following economy with one table (t) and two chairs (c_1, c_2) from Sun and Yang (2006). As shown in Table III, chair c_1 complements table t and is a perfect substitute for another chair c_2, and each agent's valuation function satisfies the *gross substitutes and complements* (GSC) condition for $S_1 = \{t\}$ and $S_2 = \{c_1, c_2\}$, i.e., for any price vector $p \in \mathbb{R}^{|\Omega|}, a \in S_k, \delta \geq 0$, and $A \in D_{v_i}(p)$, there exists $B \in D_{v_i}(p + \delta e^a)$ such that $[A \cap S_k] \setminus \{a\} \subseteq B \subseteq [A \cup S_k]$, where $e^a \in \mathbb{R}^{|\Omega|}$ denotes the characteristic vector whose i-th coordinate is 1 if $a_i = a$ and 0 otherwise. Hence, there exists a competitive equilibrium by Theorem 3.1 of Sun and Yang (2006), which shows that the GSC condition is sufficient for the existence of an equilibrium.

Table III

Agents' valuations

	$\emptyset$	$\{t\}$	$\{c_1\}$	$\{c_2\}$	$\{t, c_1\}$	$\{t, c_2\}$	$\{c_1, c_2\}$	$\{t, c_1, c_2\}$
v_1	0	18	3	3	22	22	4	24
v_2	0	1	11	11	13	13	20	23
v_3	0	12	6	6	20	20	10	25

Theorem 1 gives an alternative way to analyze the economy. For this economy, $p^E = (16, 6, 6)$ and v_i satisfies p^E-substitutability for $i = 1, 2, 3$. This implies that the efficient allocation $X_1 = \{t\}, X_2 = \{c_2, c_3\}, X_3 = \emptyset$ can be supported by $\bar{p} = (18, 9, 9)$ and that the social valuation function v_{i_N} is gross substitutable.

4 Proof of Theorem 1

The proof of Theorem 1 relies on a classification result, Lemma 5, which partitions the set of economies into disjoint similarity classes such that whenever a similarity class contains an economy with a competitive equilibrium, each economy in this class also has a competitive equilibrium. Lemma 5 will be proved in the Appendix.

Definition 3 *Two economies E' and E'' are directly similar, denoted by $E' \sim E''$, if there exist an economy $E = (\Omega; v_1, \dots, v_n)$ and a vector $q \in \mathbb{R}^{|\Omega|}$ such that $E' = (\Omega; v_1, \dots, v_j [q], \dots, v_n)$ for some $j \in N$ and $E'' = (\Omega; v_0, v_1, \dots, v_n)$, where v_0 is the the valuation function such that $v_0 (A) = q (A)$ for $A \subseteq \Omega$. Moreover, we say that two economies E' and E'' are similar if there exists a sequence of economies, $E_0, E_1, \dots, E_r$, such that $E' = E_0, E'' = E_r$, and $E_{k-1} \sim E_k$ for $k = 1, \dots, r$.*

Lemma 5 *Let $q \in \mathbb{R}^{|\Omega|}$ and let $E' = (\Omega; v_1 [q], v_2, \dots, v_n)$ and $E'' = (\Omega; v_0, v_1, \dots, v_n)$ be two directly similar economies such that v_0 is the valuation function satisfying $v_0 (A) = q (A)$ for $A \subseteq \Omega$. Then E' has a competitive equilibrium if and only if E'' has a competitive equilibrium.*

We are now ready to prove Theorem 1 with the aid of Lemmas 5. Assume that $v_i \left[p^E\right]$ satisfies the gross substitutes condition for all $i \in N$.

(a) By Theorem 2 of Kelso and Crawford (1982) and Lemma 5, it suffices to show that $E = \left(\Omega; (v_i)_{i \in N}\right)$ is similar to the economy $\left(\Omega; \left(v_i \left[p^E\right]\right)_{i \in N}\right)$. We first note that $v_i [p^{v_i}] = v_i$ for each $i \in N$, and hence $E = (\Omega; v_1 [p^{v_1}], \dots, v_n [p^{v_n}])$.

For each $i \in N$, let v_{0i} be the valuation function defined by $v_{0i} (A) = p^{v_i} (A)$ for $A \subseteq \Omega$. It is not difficult to see that E is similar to $E_1 = (\Omega; v_{01}, \dots, v_{0n}, v_1, \dots, v_n)$. Let $E_2 = (\Omega; v_{02} [p^{v_1}], v_{03}, \dots, v_{0n}, v_1, \dots, v_n)$ and let

$$E_j \equiv \left(\Omega; v_{0j} \left[\vee_{k=1}^{j-1} p^{v_k}\right], v_{0(j+1)}, \dots, v_{0n}, v_1, \dots, v_n\right) \text{ for } j = 3, \dots, n.$$

Since $v_{0j} \left[\vee_{k=1}^{j-1} p^{v_k}\right] (A) = \left(\vee_{k=1}^{j} p^{v_k}\right) (A)$ for $A \subseteq \Omega$ and for $j = 2, \dots, n$, it follows that E and $E_n = \left(\Omega; v_{0n} \left[\vee_{k=1}^{n-1} p^{v_k}\right], v_1, \dots, v_n\right)$ are similar. Let $v_0 = v_{0n} \left[\vee_{k=1}^{n-1} p^{v_k}\right]$. Then $v_0 (A) = (\vee_{k=1}^{n} p^{v_k}) (A) = p^E (A)$ for $A \subseteq \Omega$ and hence we may write $E_n = (\Omega; v_0, v_1, \dots, v_n)$. Finally, since $v_0 = v_0 \left[p^E\right]$ and $v_i \left[p^E\right] = \left(v_i \left[p^E\right]\right) \left[p^E\right]$ for $i = 1, \dots, n$, it follows that

$$E_n \sim \left(\Omega; v_1 \left[p^E\right], v_2, \dots, v_n\right) \sim \left(\Omega; v_1 \left[p^E\right], v_2 \left[p^E\right], v_3, \dots, v_n\right) \sim \cdots \sim \left(\Omega; \left(v_i \left[p^E\right]\right)_{i \in N}\right).$$

(c) Suppose, to the contrary, that the social valuation function v_{i_N} violates the gross substitutes condition. By Theorem 2 of Gul and Stacchetti (1999), there exists an economy $E' = (\Omega; v_{i_N}, v_{n+1}, \dots, v_{n'})$ such that v_i satisfies the gross substitutes condition for $i = n+1, \dots, n'$ but E' has no competitive equilibrium. We now consider the economy $E'' = (\Omega; v_1, \dots, v_n, v_{n+1} \dots, v_{n'})$. Note that $p^{E''} = \vee_{i=1}^{n'} p^{v_i} = \left(\vee_{i=n+1}^{n'} p^{v_i}\right) \vee p^E \geq p^E$. By Lemma 4 (a) and (c), we see that in the economy E'', each agent's valuation function satisfies the $p^{E''}$-substitutability condition. Then the combination of

(a) and Lemma 2 implies that E'' has a competitive equilibrium, and so does E'. This is impossible.

(b) Let $\mathbf{Y} = (Y_1, \ldots, Y_n)$ be an efficient allocation for E. We are going to show that $\langle \mathbf{Y}, \bar{p} \rangle$ is a competitive equilibrium for E. Consider the economy $E_{\bar{0}} = (\Omega; v_{\bar{0}}, v_1, \ldots, v_n)$ constructed from E by adding an agent $\bar{0}$ with the valuation function $v_{\bar{0}}$ given by $v_{\bar{0}}(A) = \bar{p}(A)$ for $A \subseteq \Omega$ and let $N_{\bar{0}} = \{\bar{0}, 1, \ldots, n\}$. Since $p^{E_{\bar{0}}} \geq p^E$, Lemma 4 (c) implies that in economy $E_{\bar{0}}$, each agent's valuation function satisfies the $p^{E_{\bar{0}}}$-substitutability condition. By (a), there exists an equilibrium $\langle (X_0, X_1, \ldots, X_n), p \rangle$ for $E_{\bar{0}}$. Without loss of generality, we may assume that $X_0 = \{a_1, \ldots, a_r\} \subseteq \Omega$ and let $A_0 = \emptyset$, $A_j = \{a_1, \ldots, a_j\}$ for $j = 1, \ldots, r$. Note that $(X_0, X_1, \ldots, X_n)$ is an efficient allocation for $E_{\bar{0}}$ and the result of (c) implies that the social valuation function v_{i_N} has decreasing marginal returns. It follows that

$$\begin{aligned} 0 &\geq \sum_{j=1}^{r} \left[v_{i_N}(\Omega) - v_{i_N}(\Omega \backslash \{a_j\}) \right] - \sum_{j=1}^{r} \left[v_{i_N}(\Omega \backslash A_{j-1}) - v_{i_N}(\Omega \backslash A_j) \right] \qquad (4) \\ &= v_{\bar{0}}(X_0) + v_{i_N}(\Omega \backslash X_0) - v_{i_N}(\Omega) \geq v_{\bar{0}}(X_0) + \sum_{j=1}^{n} v_j(X_j) - v_{i_N}(\Omega) \\ &= v_{i_{N_{\bar{0}}}}(\Omega) - v_{i_N}(\Omega). \end{aligned}$$

Together with the fact that $v_{i_N}(\Omega) = \sum_{i=1}^{n} v_i(Y_i) = v_{\bar{0}}(\emptyset) + \sum_{i=1}^{n} v_i(Y_i) \leq v_{i_{N_{\bar{0}}}}(\Omega)$, we have $v_{i_N}(\Omega) = v_{\bar{0}}(\emptyset) + \sum_{i=1}^{n} v_i(Y_i) = v_{i_{N_{\bar{0}}}}(\Omega)$. Let $Y_0 = \emptyset$. By Lemma 1 (b), $\langle (Y_0, Y_1, \ldots, Y_n), p \rangle$ is also a competitive equilibrium for $E_{\bar{0}}$. This implies $Y_i \in D_{v_i}(p)$ for $i \in N$ and

$$0 \geq v_{\bar{0}}(\{a\}) - p_a = \bar{p}_a - p_a \text{ for all } a \in \Omega.$$

Hence, $\langle (Y_1, \ldots, Y_n), p \rangle$ is a competitive equilibrium for E and $p_a \geq \bar{p}_a$ for all $a \in \Omega$. Together with the result of Lemma 1 (c), we obtain that $\bar{p} = p$ is an equilibrium price vector for E.

5 Concluding remarks

This paper contributes to the literature on markets with indivisible objects. We introduce the notion of p-substitutability to extend the scope of gross substitutability, analyze economies with arbitrary types of preferences, and then prove a number of existence results for an economy E under p^E-substitutability. Our approach relies on a classification result given in Section 4, Lemma 5, which partitions economies into similarity classes and can be further applied to generalize each sufficient condition on preferences, giving new existence results. For example, with a proof similar to that of Theorem 1, it is not difficult to obtain the following result. The proof is omitted for brevity.

Theorem 2 *Let $E = \left(\Omega; (v_i)_{i \in N}\right)$ be an economy. If for each agent $i \in N$, there exists $p^i \in \mathbb{R}^{|\Omega|}$ such that $p^i \leq p^E$ and $v_i [p^i]$ satisfies the GSC condition, then*

(a) *there exists a competitive equilibrium; and*

(b) *the social value vector $\bar{p} = (\bar{p}_a) \in \mathbb{R}^{|\Omega|}$ defined by $\bar{p}_a = v_N (\Omega) - v_N (\Omega \backslash \{a\})$ is an equilibrium price vector whenever the social valuation function v_{i_N} has decreasing marginal returns.*

6 Appendix. Proof of Lemma 5

($\Rightarrow$) Assume that $\langle \mathbf{X}; p \rangle$ is a competitive equilibrium for E' and let $p' = p \vee q$.

We first prove that $\langle \mathbf{X}; p' \rangle$ is also a competitive equilibrium for E'. It is not difficult to prove that $\hat{A} = \{a \in \Omega : p_a < q_a\}$ is a subset of X_1. By definition there exists $Y_1 \subseteq X_1$ such that $v_1 [q] (X_1) = v_1 (Y_1) + q (X_1 \backslash Y_1)$ and $v_1 [q] (Y_1) = v_1 (Y_1)$. In case there exists $\hat{a} \in \hat{A} \backslash X_1$, we have

$$\begin{aligned} v_1 [q] (X_1 \cup \{\hat{a}\}) - p (X_1 \cup \{\hat{a}\}) &\geq [v_1 (Y_1) + q ((X_1 \cup \{\hat{a}\}) \backslash Y_1)] - p (X_1 \cup \{\hat{a}\}) \\ &= v_1 [q] (X_1) + q_{\hat{a}} - p (X_1 \cup \{\hat{a}\}) > v_1 [q] (X_1) - p (X_1), \end{aligned}$$

which contradicts to the fact $X_1 \in D_{v_1[q]} (p)$. Note that $\hat{A} \subseteq X_1$ implies $p'_a = p_a$ for all $a \in \Omega \backslash X_1$and hence $X_i \in D_{v_i} (p')$ for $i = 2, \ldots, n$. Moreover, since $X_1 \in D_{v_1[q]} (p)$, it follows that for each bundle $A \subseteq \Omega$, we have

$$\begin{aligned} v_1 [q] (X_1) - p' (X_1) &= v_1 [q] (X_1) - p (X_1) + p \left(\hat{A}\right) - q \left(\hat{A}\right) \\ &\geq v_1 [q] \left(A \cup \hat{A}\right) - p \left(A \cup \hat{A}\right) + p \left(\hat{A}\right) - q \left(\hat{A}\right) \\ &= v_1 [q] \left(A \cup \hat{A}\right) - p' \left(A \cup \hat{A}\right) \\ &\geq v_1 [q] (A) + q \left(\hat{A} \backslash A\right) - p' \left(A \cup \hat{A}\right) = v_1 [q] (A) - p' (A), \end{aligned}$$

i.e., $X_1 \in D_{v_1[q]} (p')$.

We are now ready to construct an equilibrium $\langle \mathbf{Y}, p' \rangle$ for E''. Let $\mathbf{Y} = (Y_0, Y_1, \ldots, Y_n)$ be the allocation given by $Y_0 = X_1 \backslash Y_1$ and $Y_i = X_i$ for $i = 2, \ldots, n$. Since $p' \geq q$ and $X_1 \in D_{v_1[q]} (p')$, it follows that

$$\begin{aligned} v_1 [q] (Y_1) - p' (Y_1) &= v_1 (Y_1) - p' (Y_1) = v_1 [q] (X_1) - q (X_1 \backslash Y_1) - p' (Y_1) \\ &\geq v_1 [q] (X_1) - p' (X_1) \geq v_1 [q] (Y_1) - p' (Y_1). \end{aligned}$$

This implies $Y_1 \in D_{v_1[q]} (p')$ and $Y_0 \in D_{v_0} (p')$.

($\Leftarrow$) Assume that $\langle \mathbf{X}; p \rangle$ is a competitive equilibrium for E'' and let $p' = p \vee q$. We are going to show that the pair $\langle \mathbf{Y}, p' \rangle$ such that $Y_1 = X_0 \cup X_1$ and $Y_i = X_i$ for $i = 2, \ldots, n$ is a competitive equilibrium for $E^{'}$.

We note that $\hat{A} = \{a \in \Omega : p_a < q_a\}$ is a subset of X_0. In case there exists $\hat{a} \in \hat{A} \backslash X_0$, then $v_0 (X_0 \cup \{\hat{a}\}) - p (X_0 \cup \{\hat{a}\}) = v_0 (X_0) + (q_{\hat{a}} - p_{\hat{a}}) - p (X_0) > v_0 (X_0) - p (X_0)$, which contradicts to the fact $X_0 \in D_{v_0} (p)$. This implies

$$p'_a = p_a \text{ for all } a \in \Omega \backslash X_0 \tag{5}$$

and hence $Y_i = X_i \in D_{v_i} (p')$ for $i = 2, \ldots, n$. On the other hand, in case there exists $\hat{b} \in X_0$ such that $p_{\hat{b}} > q_{\hat{b}}$, we have $v_0 \left(X_0 \backslash \left\{ \hat{b} \right\} \right) - p \left(X_0 \backslash \left\{ \hat{b} \right\} \right) = v_0 (X_0) + (p_{\hat{b}} - q_{\hat{b}}) - p (X_0) > v_0 (X_0) - p (X_0)$, which contradicts to the fact $X_0 \in D_{v_0} (p)$ again. This implies that for all $a \in X_0$, $p_a \leq q_a$ and hence $p'_a = q_a$. Let $A \subseteq \Omega$ be an arbitrary bundle. Then there exists $A' \subseteq A$ such that $v_1 [q] (A) = v_1 (A') + q (A \backslash A')$. Together with (5) and the facts $X_1 \in D_{v_1} (p)$ and $p' = p \vee q$, we have

$$\begin{aligned} v_1 [q] (Y_1) - p' (Y_1) &= v_1 [q] (X_0 \cup X_1) - p' (X_0 \cup X_1) \geq v_1 (X_1) + q (X_0) - p' (X_1) - p' (X_0) \\ &= v_1 (X_1) - p (X_1) \geq v_1 (A') - p (A') = v_1 [q] (A) - q (A \backslash A') - p (A') \\ &\geq v_1 [q] (A) - p' (A) . \end{aligned}$$

This implies $Y_1 \in D_{v_1 [q]} (p')$ and completes the proof.

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