A note on Poincaré recurrence in Anosov diffeomorphic transformation of discretized outline of some plant leaves

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1. Introduction: The origin of this note is in the curiosity aroused by the Arnold's cat map (Arnold and Avez, 1968), which suggests that perhaps a cat has many more lives than simply nine. The discretized Arnold's cat map, when repeatedly subjected to Anosov diffeomorphic transformation, sheared and wrapped around iteration after iteration, though apparently random or disordered intermediately, finally returns to the original image after a finite number of iterations (Dyson and Falk, 1992). In other words, repeated operation of Anosov diffeomorphic transformation on the cat map exhibits Poincaré recurrence. It has been shown (Wikipedia: Arnold's cat map) that the said cat map resumes its original shape in the 300th (is it a magic number for the cat: 300 = $\sqrt{9}\times 10^2$?) iteration. Could we generate some more examples of this type? This inquisitiveness is the main drive behind this note.

2. The scheme of transformation: Let $X(n,2)$ be the integer dataset of n points in two dimensional space obtained by discretization of a closed curve (deformed ellipse) resembling the outline of a plant leaf such as given in Fig.1. If X is subjected to Anosov diffeomorphic transformation repeatedly, it is interesting to note that it exhibits a mixing behavior and then Poincaré recurrence after some iterations. This transformation is done with $\kappa = \max_{i,j}(x_{ij}); j=1,2; i=1,n$ as follows:

$X^{(t+1)} = \mathrm{mod}(X^{(t)}B, \kappa)$, where $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, which with $\det(B)=1$, is area-preserving.

3. Datasets and Findings: Plant leaves are extremely varied in shapes. Leaf shapes range from simple linear (or simple needle-like) to highly complicated compound ones observed in many types of fern. These shapes are described qualitatively by using geometric words such as cordate, cuneate, deltoid, elliptical, hastate, lanceolate, oblong, obovate, oval, palmate, reniform, spatulate, bilobed, trilobed, etc. Geometrically, all leaf shapes may be viewed as deformed and/or modified ellipse (Gielis, 2003). This deformation is a sort of modification that might have been undergone due to some forces in nature.

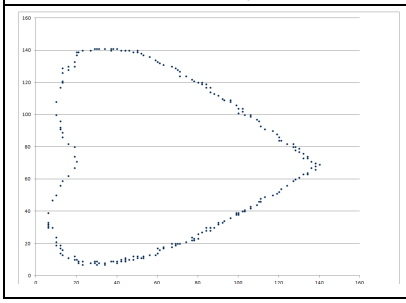
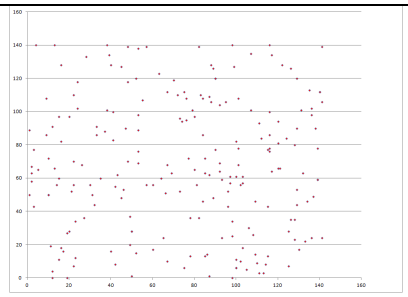
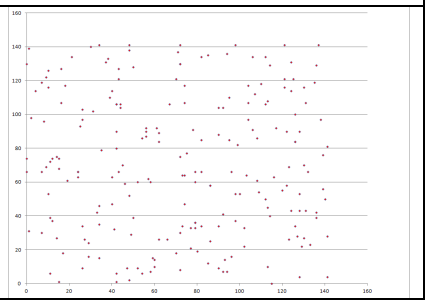
We obtained data by smooth (avoiding serration on the margin/blade) tracing and then discretizing the outlines of some common plant leaves with different shapes. In what follows, we provide some findings (figures Fig.1 through Fig.12 and datasets Table.1 through Table.12) on the same.

3.1. Cordate leaf dataset: The cordate leaf dataset contains 216 (=n) points. The original dataset (AX) is real, but it has been converted into an integer dataset (X in Table-1) according to the rule given below:

If(ax(i,j)-int(ax(i,j)) < 0.5) x(i,j)=int(ax(i,j).kr) else x(i,j)=int(ax(i,j).kr)+1; j=1,2; i=1,n; kr=1.

Subjected to repeated transformations, it resumes its original position in the 105th iteration.

Fig.1. Cordate Leaf (216 points)

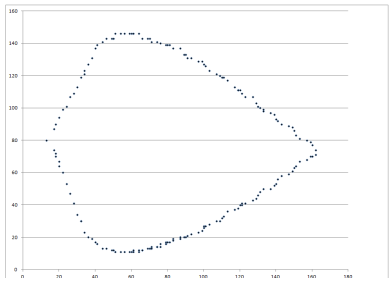
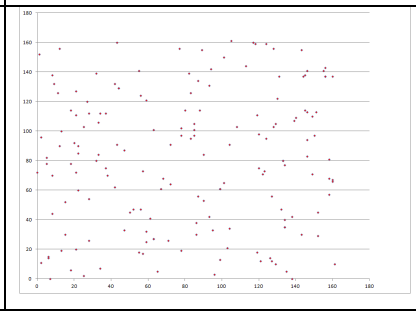
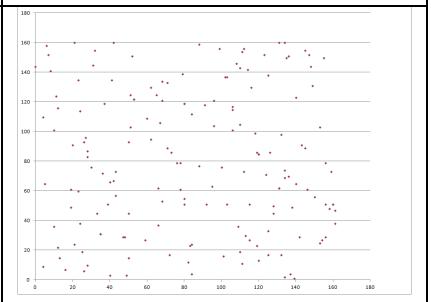
Iteration: 0, 105	Iteration: 14	Iteration: 24
		

3.2. Convex-deltoid leaf dataset: The convex-deltoid leaf dataset has 168 (=n) points. The original dataset (AX) is real, but converted into an integer dataset (X in Table-2) under the rule given below:

If(ax(i,j)-int(ax(i,j)) < 0.5) x(i,j)=int(ax(i,j).kr) else x(i,j)=int(ax(i,j).kr)+1; j=1,2 i=1,n; kr=1.

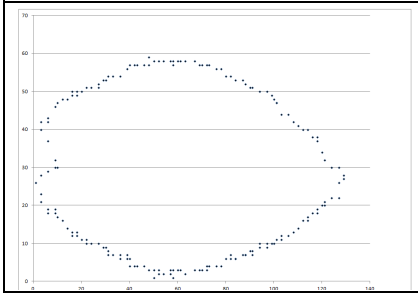
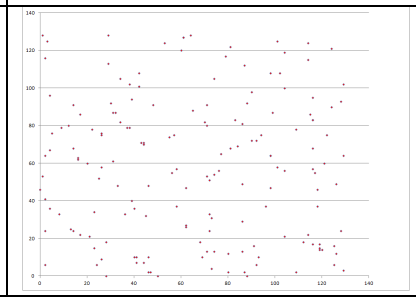
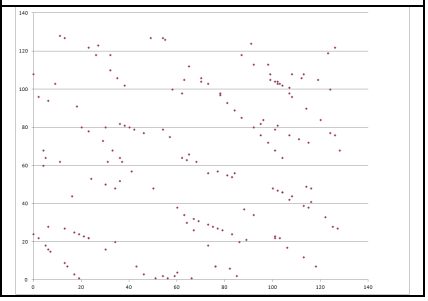
Subjected to repeated transformations, it resumes its original position in the 164th iteration.

Fig.2. Convex-deltoid Leaf (168 points)

Iteration: 0, 164	Iteration: 12	Iteration: 20
		

3.3. Cardio-lanceolate leaf dataset: The cardio-lanceolate leaf dataset contains 168 (=n) points. The original dataset (AX) is real, but converted into an integer dataset (X in Table.3).

Fig.3. Cardio-lanceolate Leaf (168 points)

Iteration: 0, 210	Iteration: 12	Iteration: 20
		

The real data have been converted to integers (plotted in Fig.3) by the rule given below:

If(ax(i,j)-int(ax(i,j)) < 0.5) x(i,j)=int(ax(i,j).kr) else x(i,j)=int(ax(i,j).kr)+1; j=1,2; i=1,n; kr=10.

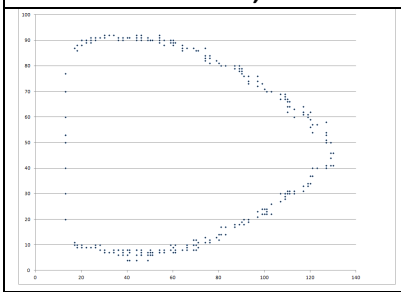
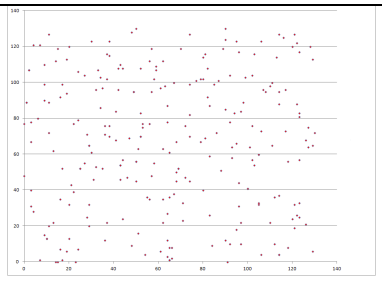
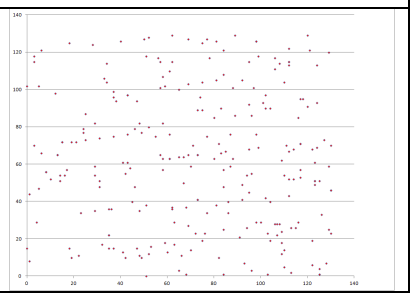
Subjected to repeated transformations, it resumes its original position in the 210th iteration.

3.4. Truncate leaf dataset: The truncate leaf dataset contains 256 (=n) points. The original dataset (AX) is real, but it has been converted into an integer dataset (X in Table-4) by the rule given below:

If(ax(i,j)-int(ax(i,j)) < 0.5) x(i,j)=int(ax(i,j).kr) else x(i,j)=int(ax(i,j).kr)+1; j=1,2; i=1,n; kr=1.

Subjected to repeated transformations, it resumes its original position in the 65th iteration.

Fig.4. Truncate Leaf (256 points)

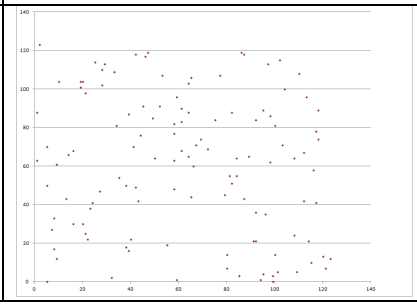
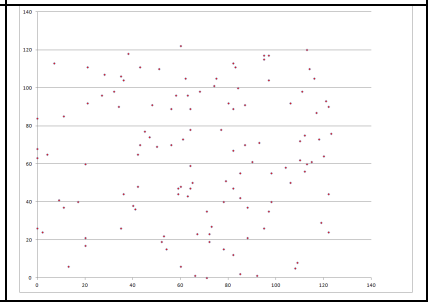
Iteration: 0, 65	Iteration: 12	Iteration: 20
		

3.5. Cucumberoid leaf dataset: The cucumberoid leaf dataset contains 121 (=n) points. The original dataset (AX) is real, but it has been converted into an integer dataset (X in Table-5) as given below:

If(ax(i,j)-int(ax(i,j)) < 0.5) x(i,j)=int(ax(i,j).kr) else x(i,j)=int(ax(i,j).kr)+1; j=1,2; i=1,n; kr=10.

Subjected to repeated transformations, it resumes its original position in the 15th iteration. It may, however, be noted that when kr=1 was used, the recurrence was observed in the 14th iteration. But a comparison of the original image (based on real measurement) with the reproduced image (based on data approximation using kr=1) revealed that the latter was not much satisfactory. Therefore, kr=10 was chosen. Also, compared to other datasets, it took a fewer number of iterations to show recurrence. The figure (fig.5) and the dataset (table-5) are based on the transformation for kr=10.

Fig.5. Cucumberoid Leaf (121 points)

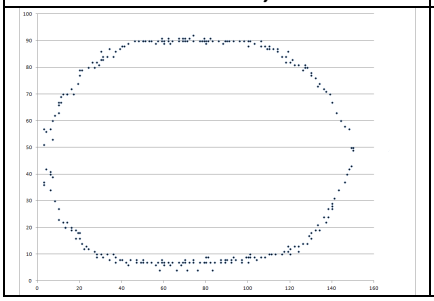
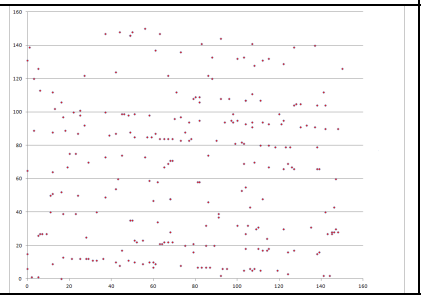
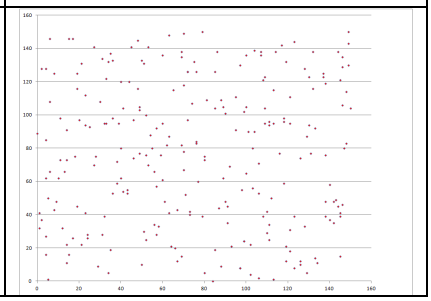
Iteration: 0, 15	Iteration: 5	Iteration: 10
		

3.6. Oblong leaf dataset: The oblong leaf dataset contains 252 (=n) points. The original dataset (AX) is real, but it has been converted into an integer dataset (X in Table-6) by the transformation given below:

If(ax(i,j)-int(ax(i,j)) < 0.5) x(i,j)=int(ax(i,j).kr) else x(i,j)=int(ax(i,j).kr)+1; j=1,2; i=1,n; kr=1.

Subjected to repeated transformations, it resumes its original position in the 25th iteration.

Fig.6. Oblong Leaf (252 points)

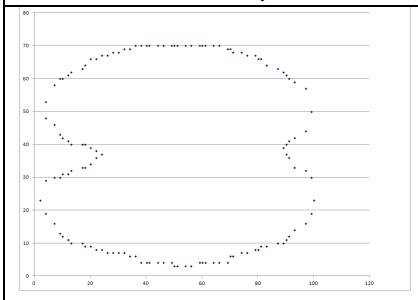
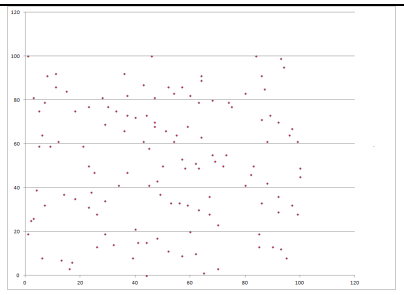
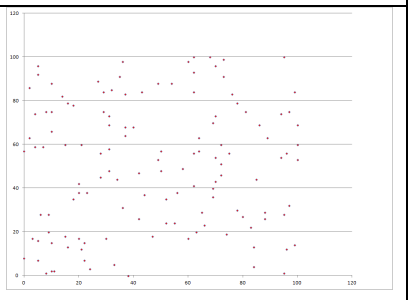
Iteration: 0, 25	Iteration: 7	Iteration: 20
		

3.7. Bilobed leaf dataset: *Bauhinia Variegata* (orchid tree, also known as Kachnar in Hindi) has bilobed leaves. In our specimen, the bilobed leaf dataset contains 129 (=n) points. The original dataset (AX) is real, but it has been converted into an integer dataset (X in Table-7) by the transformation given below:

If(ax(i,j)-int(ax(i,j)) < 0.5) x(i,j)=int(ax(i,j).kr) else x(i,j)=int(ax(i,j).kr)+1; j=1,2; i=1,n; kr=1.

Subjected to repeated transformations, it resumes its original position in the 25th iteration.

Fig.7. Bilobed Leaf (129 points)

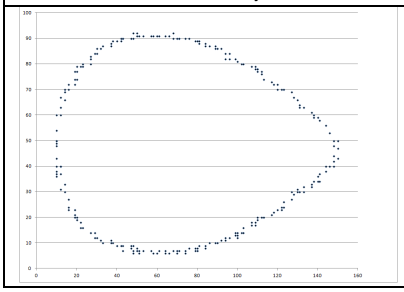
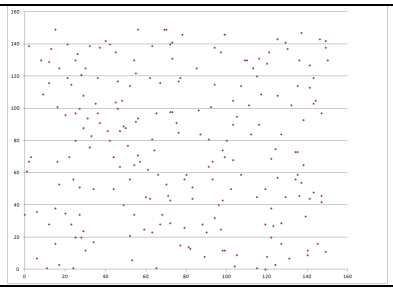
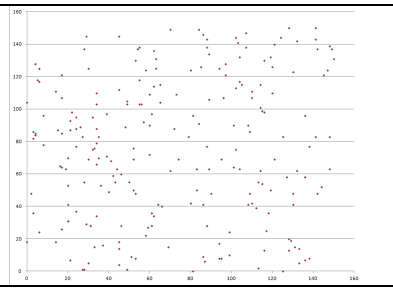
Iteration: 0, 25	Iteration: 12	Iteration: 20
		

3.8. Ovate/obovate leaf dataset: It may be noted that ovate and obovate shapes are similar except the feature as to which side of the leaf is the petiole (base) that connects it to the stem of the plant. Obovate leaves are narrower at the base. The ovate/obovate leaf dataset contains 228 (=n) points. The original dataset (AX) is real, but it has been converted into an integer dataset (X in Table-8) by the transformation given below:

If(ax(i,j)-int(ax(i,j)) < 0.5) x(i,j)=int(ax(i,j).kr) else x(i,j)=int(ax(i,j).kr)+1; j=1,2; i=1,n; kr=1.

Subjected to repeated transformations, it resumes its original position in the 25th iteration.

Fig.8. Ovate Leaf (228 points)

Iteration: 0, 25	Iteration: 12	Iteration: 20
		

3.9. Triangular leaf dataset: The triangular leaf dataset contains 309 (=n) points. The original dataset (AX) is real, but it has been converted into an integer dataset (X in Table-9) as given below:

If(ax(i,j)-int(ax(i,j)) < 0.5) x(i,j)=int(ax(i,j).kr) else x(i,j)=int(ax(i,j).kr)+1; j=1,2; i=1,n; kr=1.

Subjected to repeated transformations, it resumes its original position in the 60th iteration.

Fig.9. Triangular Leaf (309 points)

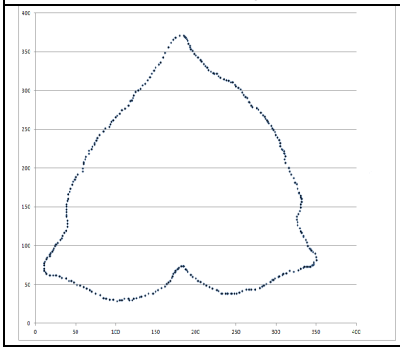
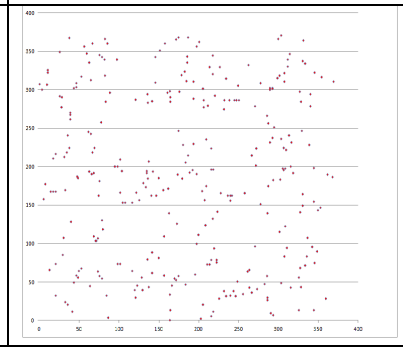
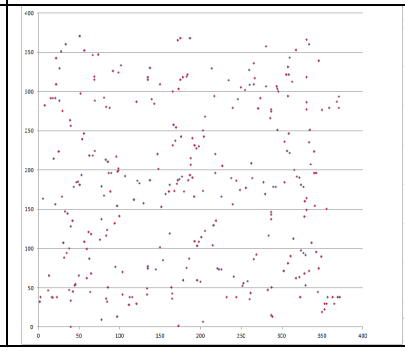
Iteration: 0, 60	Iteration: 15	Iteration: 30
		

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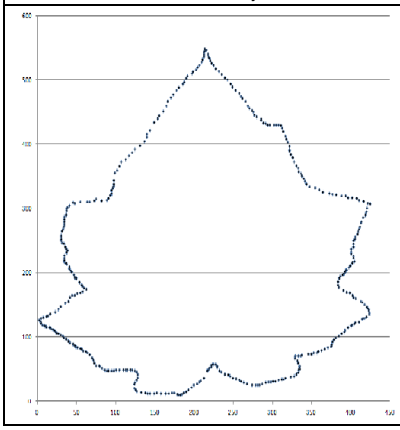
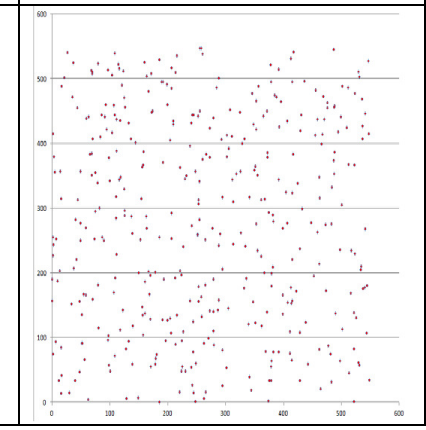
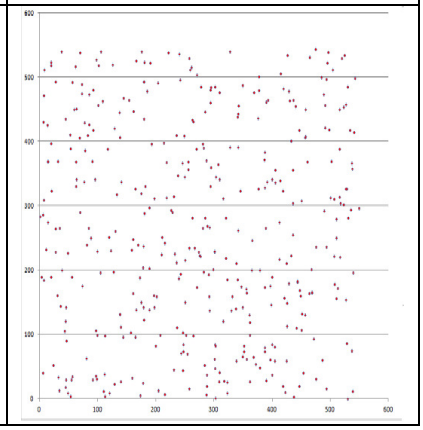
Fig.10. Nine-lobed Leaf (407 points)

Iteration: 0, 150	Iteration: 35	Iteration: 126
		

3.10. Nine-lobed leaf dataset: The nine-lobed leaf dataset contains 407 (=n) points. The original dataset (AX) is real, but it has been converted into an integer dataset (X in Table-10) as given below:

If(ax(i,j)-int(ax(i,j)) < 0.5) x(i,j)=int(ax(i,j).kr) else x(i,j)=int(ax(i,j).kr)+1; j=1,2; i=1,n; kr=1.

Subjected to repeated transformations, it resumes its original position in the 150th iteration.

3.11. Five-lobed leaf dataset: The five-lobed leaf dataset contains 270 (=n) points. The original dataset (AX) is real, but it has been converted into an integer dataset (X in Table-11) as given below:

If(ax(i,j)-int(ax(i,j)) < 0.5) x(i,j)=int(ax(i,j).kr) else x(i,j)=int(ax(i,j).kr)+1; j=1,2; i=1,n; kr=1.

Subjected to repeated transformations, it resumes its original position in the 384th iteration.

Fig.11. Five-lobed Leaf (270 points)

Iteration: 0, 384	Iteration: 160	Iteration: 260

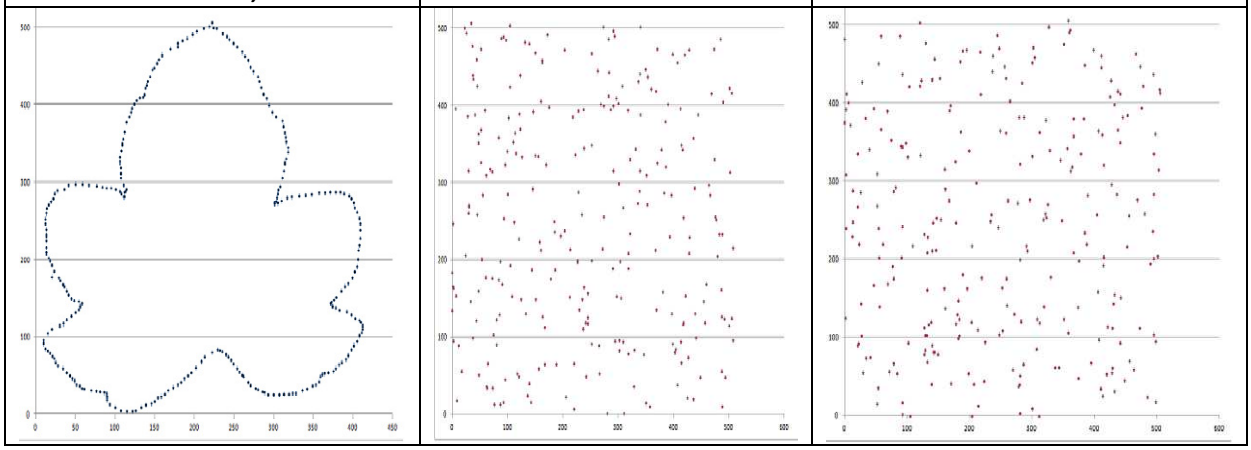


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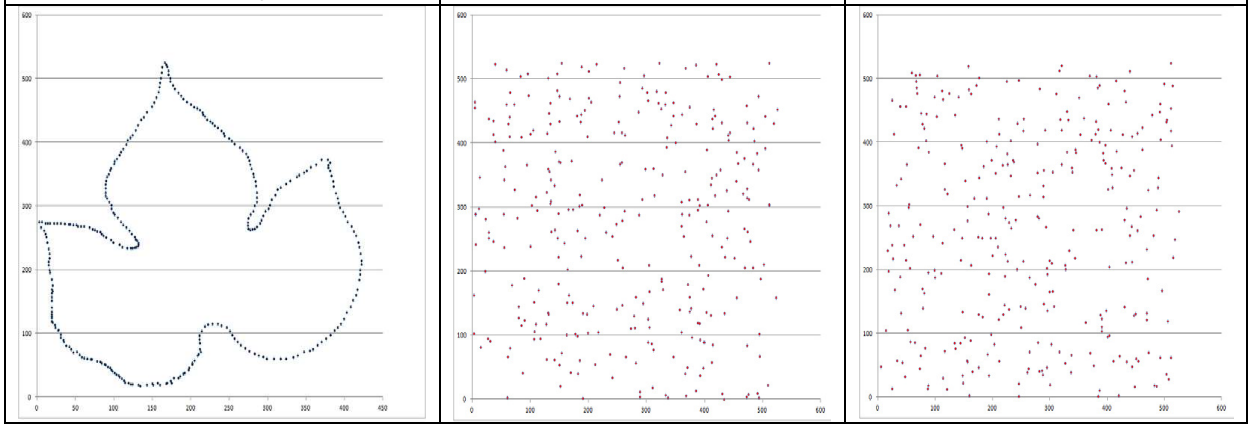
Fig.12. Three-lobed Leaf (318 points)

Iteration: 0, 264	Iteration: 50	Iteration: 100



3.12. Three-lobed leaf dataset: The three-lobed leaf dataset contains 318 (=n) points. The original dataset (AX) is real, but it has been converted into an integer dataset (X in Table-12) by the transformation given below:

If(ax(i,j)-int(ax(i,j)) < 0.5) x(i,j)=int(ax(i,j).kr) else x(i,j)=int(ax(i,j).kr)+1; j=1,2; i=1,n; kr=1.

Subjected to repeated transformations, it resumes its original position in the 264th iteration (Fig.12).

These twelve types of leaves do not exhaust the possible shapes that are extremely varied. However, they indicate that simple (not compound) leaves (that may be considered as deformed ellipses) may have similar properties as to the Poincaré recurrence under repeated Anosov diffeomorphic transformation. These examples also demonstrate that certain shapes may show up the said recurrence in a few repeated transformations, while others may take many more iterations to exhibit recurrence. It may also be noted that the number of points in the dataset is one of the determinants of the number of iterations that must be undergone in order to show up recurrence.

4. A computer program: We have carried out computation by a simple (Fortran 77) program. The code is appended. It requires the input file containing: [1] the no. of points (n) in the dataset, and [2] the data points $ax(i,1),\ ax(i,2)$ for all points in the subsequent rows. All data points must lie in the first quadrant. The input and output files are to be named by the user on the run of the program. The output begins from the original X (call it $X^{(0)}$), the intermediate X (call them $X^{(1)}, X^{(2)}, \ldots, X^{(t-1)}$) and the terminal X (call it $X^{(t)}$). The $X^{(t)} = X^{(0)}$. In most cases, $kr = 1$ suffices, but for better accuracy $kr = 10$ may be used. A larger value of kr raises the number of iterations required for showing up recurrence. One may also note that the Anosov diffeomorphic transformation operates on the square (aspect ratio of the map being unity). The loss of aspect ratio of the map may be recovered suitably.

5. Conclusion: We have provided some examples where Poincaré recurrence is observed. We have not tested for the existence of such recurrences in case of many other shapes of plant leaves. Any one, interested in finding them out for other leaf shapes, may use the computer program.

References:

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Dyson F.J. and Falk, H. (1992). Period of a Discrete Cat Mapping. *The American Mathematical Monthly* 99(7): 603–614.

Gielis, J. (2003). A Generic Geometric Transformation that unifies a Wide Range of Natural and Abstract Shapes. *American Journal of Botany* 90(3): 333–338.

Datasets

Table-1. Cordate Leaf Dataset (n=216; kr=1; Recurrence at the 105th iteration)																		
Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2
1	6	39	37	19	133	73	40	9	109	67	130	145	90	112	181	113	49
2	6	33	38	19	130	74	40	8	110	67	20	146	90	33	182	117	90
3	6	32	39	19	80	75	42	140	111	67	18	147	90	32	183	117	50
4	6	31	40	19	74	76	42	10	112	69	129	148	92	110	184	119	88
5	6	30	41	19	67	77	42	9	113	69	20	149	92	33	185	119	51
6	8	47	42	19	12	78	44	140	114	69	19	150	93	109	186	120	86
7	8	30	43	19	10	79	44	140	115	70	128	151	93	34	187	120	84
8	10	108	44	20	139	80	44	11	116	70	20	152	96	109	188	120	52
9	10	100	45	20	137	81	44	10	117	71	127	153	96	108	189	121	84
10	10	50	46	20	71	82	44	9	118	71	124	154	96	36	190	121	54
11	10	24	47	20	10	83	46	140	119	71	20	155	99	106	191	124	82
12	10	21	48	21	139	84	46	10	120	74	124	156	99	39	192	124	56
13	10	19	49	21	9	85	48	139	121	74	21	157	99	38	193	127	82
14	12	117	50	21	8	86	48	12	122	77	122	158	100	104	194	127	80
15	12	96	51	23	140	87	48	10	123	77	24	159	100	101	195	127	59
16	12	92	52	23	9	88	50	140	124	77	22	160	100	39	196	128	80
17	12	91	53	23	7	89	50	139	125	78	121	161	100	38	197	128	78
18	12	56	54	27	140	90	50	12	126	78	23	162	102	104	198	128	60
19	12	19	55	27	8	91	50	11	127	78	22	163	102	102	199	128	60
20	12	17	56	29	141	92	52	138	128	80	120	164	102	40	200	130	79
21	12	14	57	29	9	93	52	11	129	80	26	165	102	40	201	130	77
22	13	129	58	29	8	94	53	137	130	80	23	166	103	100	202	130	61
23	13	126	59	30	141	95	53	12	131	82	120	167	103	41	203	132	76
24	13	121	60	30	9	96	53	11	132	82	119	168	103	40	204	132	73
25	13	120	61	30	7	97	56	136	133	82	27	169	106	100	205	132	63
26	13	120	62	31	141	98	56	13	134	84	119	170	106	99	206	134	74
27	13	89	63	31	8	99	59	134	135	84	117	171	106	43	207	134	73
28	13	86	64	34	141	100	59	13	136	84	30	172	106	42	208	134	64
29	13	59	65	34	8	101	60	133	137	84	28	173	109	97	209	134	63
30	13	16	66	34	7	102	60	17	138	86	117	174	109	44	210	136	71
31	13	13	67	37	141	103	60	14	139	86	114	175	110	96	211	136	67
32	16	130	68	37	140	104	61	132	140	86	30	176	110	46	212	138	70
33	16	128	69	37	9	105	61	16	141	86	30	177	111	93	213	138	70
34	16	82	70	38	141	106	63	131	142	86	28	178	111	48	214	138	68
35	16	62	71	38	9	107	63	18	143	88	113	179	111	46	215	138	66
36	16	11	72	40	141	108	63	17	144	88	30	180	113	91	216	140	69

Table-2. Convex-deltoid Leaf Dataset (n=168; kr=1; Recurrence at the 164th iteration)																	
Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2
1	13	80	29	38	19	57	64	11	85	83	18	113	110	32	141	137	97
2	17	87	30	40	137	58	66	143	86	87	137	114	111	119	142	137	50
3	17	74	31	40	17	59	66	12	87	87	20	115	111	33	143	139	96
4	18	90	32	41	139	60	66	12	88	87	19	116	113	117	144	139	52
5	18	72	33	41	16	61	69	143	89	89	133	117	113	36	145	140	93
6	18	70	34	44	141	62	69	13	90	89	20	118	117	113	146	140	53
7	20	94	35	44	13	63	70	143	91	90	133	119	117	37	147	141	92
8	20	67	36	46	143	64	70	13	92	90	20	120	119	111	148	141	56
9	20	64	37	46	13	65	70	13	93	91	131	121	119	111	149	143	90
10	22	99	38	49	143	66	71	141	94	91	21	122	119	38	150	143	58
11	22	60	39	49	12	67	71	14	95	93	131	123	120	111	151	147	89
12	24	101	40	50	143	68	71	13	96	93	22	124	120	40	152	147	59
13	24	53	41	50	12	69	74	141	97	97	129	125	121	109	153	149	88
14	26	107	42	51	146	70	74	14	98	97	23	126	121	41	154	149	61
15	26	47	43	51	11	71	74	14	99	99	129	127	121	40	155	150	86
16	28	109	44	54	146	72	76	140	100	99	24	128	123	107	156	150	63
17	28	41	45	54	11	73	76	16	101	100	127	129	123	41	157	151	83
18	30	113	46	56	146	74	76	14	102	100	27	130	127	107	158	151	64
19	30	34	47	56	11	75	79	139	103	100	26	131	127	43	159	153	81
20	32	119	48	59	146	76	79	17	104	101	126	132	129	103	160	153	67
21	32	30	49	59	11	77	79	16	105	101	27	133	129	44	161	157	80
22	32	30	50	60	146	78	79	16	106	103	123	134	130	101	162	157	68
23	34	123	51	60	11	79	80	139	107	103	28	135	130	46	163	159	79
24	34	121	52	61	146	80	80	17	108	107	121	136	131	100	164	159	70
25	34	23	53	61	12	81	81	139	109	107	30	137	131	48	165	160	77
26	36	127	54	61	11	82	81	17	110	109	120	138	133	99	166	160	70
27	36	20	55	64	146	83	83	137	111	109	30	139	133	98	167	162	74
28	38	131	56	64	12	84	83	19	112	110	119	140	133	50	168	162	71

Table-3. Cardio-lanceolate Leaf Dataset (n=168; kr=1; Recurrence at the 210th iteration)																	
Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2
1	1	26	29	18	50	57	39	6	85	60	58	113	84	6	141	106	12
2	3	42	30	18	49	58	40	57	86	60	3	114	87	53	142	108	42
3	3	40	31	18	13	59	40	6	87	61	58	115	87	7	143	108	13
4	3	28	32	18	12	60	40	4	88	61	3	116	88	52	144	110	41
5	3	23	33	20	50	61	42	57	89	63	58	117	88	7	145	110	14
6	3	21	34	20	11	62	42	4	90	63	2	118	88	7	146	112	40
7	6	43	35	22	51	63	43	57	91	67	58	119	90	51	147	112	16
8	6	42	36	22	11	64	43	4	92	67	3	120	90	8	148	114	40
9	6	37	37	22	10	65	46	57	93	69	57	121	91	51	149	114	17
10	6	29	38	24	51	66	46	4	94	69	3	122	91	8	150	114	16
11	6	19	39	24	10	67	48	59	95	70	57	123	91	7	151	116	38
12	6	18	40	27	52	68	48	57	96	70	3	124	94	50	152	116	18

13	9	46	41	27	51	69	48	3	97	72	57	125	94	10	153	118	38
14	9	32	42	27	10	70	50	58	98	72	4	126	94	9	154	118	37
15	9	30	43	29	53	71	50	3	99	72	3	127	97	50	155	118	19
16	9	19	44	29	9	72	50	1	100	73	57	128	97	10	156	118	18
17	9	18	45	30	53	73	52	58	101	73	4	129	97	9	157	120	34
18	10	47	46	30	9	74	52	3	102	76	56	130	99	49	158	120	20
19	10	30	47	31	54	75	52	2	103	76	4	131	99	10	159	121	32
20	10	17	48	31	8	76	54	58	104	78	56	132	100	48	160	121	21
21	12	48	49	31	7	77	54	2	105	78	4	133	100	10	161	121	20
22	12	16	50	33	54	78	57	58	106	80	54	134	101	47	162	124	30
23	14	48	51	33	7	79	57	3	107	80	6	135	101	47	163	124	22
24	14	14	52	36	54	80	57	2	108	82	54	136	101	11	164	127	30
25	16	50	53	36	7	81	58	58	109	82	7	137	103	44	165	127	26
26	16	49	54	36	6	82	58	57	110	82	6	138	103	12	166	127	22
27	16	13	55	39	56	83	58	3	111	84	53	139	103	11	167	129	28
28	16	12	56	39	7	84	58	1	112	84	6	140	106	44	168	129	27

Table-4. Truncate Leaf Dataset (n=256; kr=1; Recurrence at the 65th iteration)

Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2
1	13	20	44	34	8	87	50	6	130	70	86	173	91	18	216	111	31
2	13	30	45	34	7	88	51	90	131	70	12	174	93	76	217	111	30
3	13	40	46	36	91	89	51	7	132	70	10	175	93	74	218	113	63
4	13	50	47	36	90	90	51	6	133	70	8	176	93	73	219	113	60
5	13	60	48	36	8	91	54	92	134	71	86	177	93	20	220	113	31
6	13	70	49	36	8	92	54	91	135	71	11	178	93	19	221	113	30
7	13	53	50	36	6	93	54	90	136	71	9	179	97	76	222	117	64
8	13	53	51	38	91	94	54	89	137	74	87	180	97	74	223	117	62
9	13	77	52	38	90	95	54	8	138	74	84	181	97	72	224	117	62
10	17	87	53	38	8	96	54	7	139	74	83	182	97	23	225	117	61
11	17	11	54	38	7	97	54	7	140	74	82	183	97	21	226	117	33
12	17	10	55	38	6	98	56	90	141	74	13	184	99	73	227	117	31
13	18	88	56	40	91	99	56	88	142	74	11	185	99	24	228	119	61
14	18	86	57	40	8	100	56	8	143	76	84	186	99	22	229	119	60
15	18	10	58	40	6	101	56	7	144	76	83	187	100	71	230	119	34
16	18	9	59	40	4	102	59	90	145	76	81	188	100	24	231	119	33
17	20	90	60	41	91	103	59	90	146	76	12	189	100	22	232	120	62
18	20	90	61	41	8	104	59	89	147	76	11	190	101	70	233	120	59
19	20	88	62	41	7	105	59	10	148	79	82	191	101	24	234	120	56
20	20	10	63	41	4	106	59	8	149	79	13	192	101	23	235	120	37
21	20	9	64	44	92	107	60	90	150	80	81	193	101	22	236	120	34
22	22	90	65	44	91	108	60	89	151	80	14	194	103	70	237	121	57
23	22	90	66	44	90	109	60	88	152	80	12	195	103	26	238	121	54
24	22	89	67	44	8	110	60	9	153	81	80	196	103	22	239	121	40
25	22	9	68	44	8	111	60	7	154	81	17	197	107	69	240	121	37
26	24	91	69	44	6	112	61	89	155	81	14	198	107	67	241	123	57
27	24	90	70	44	4	113	61	10	156	83	80	199	107	30	242	123	40
28	24	89	71	46	92	114	61	8	157	83	17	200	107	27	243	127	58
29	24	9	72	46	91	115	61	7	158	83	14	201	109	69	244	127	54
30	26	91	73	46	90	116	64	88	159	87	80	202	109	68	245	127	53
31	26	90	74	46	8	117	64	87	160	87	79	203	109	67	246	127	51
32	26	10	75	46	7	118	64	86	161	87	18	204	109	30	247	127	50
33	26	9	76	46	6	119	64	8	162	87	17	205	109	30	248	127	41
34	28	91	77	46	6	120	66	87	163	89	80	206	109	29	249	127	40
35	28	10	78	49	91	121	66	10	164	89	79	207	109	28	250	127	40

36	28	8	79	49	90	122	66	10	165	89	78	208	110	67	251	129	50
37	30	92	80	49	7	123	66	9	166	89	18	209	110	66	252	129	46
38	30	91	81	49	6	124	66	8	167	90	79	210	110	64	253	129	44
39	30	8	82	49	4	125	69	87	168	90	78	211	110	62	254	129	41
40	30	7	83	50	90	126	69	12	169	90	77	212	110	31	255	130	46
41	32	92	84	50	90	127	69	10	170	90	19	213	110	30	256	130	41
42	32	7	85	50	8	128	69	10	171	91	76	214	111	66	-	-	-
43	34	92	86	50	7	129	69	8	172	91	20	215	111	64	-	-	-

Table-5. Cucumeraroid Leaf Dataset (n=121; kr=10; Recurrence at the 15th iteration)

Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2
1	9	44	22	26	6	43	49	97	64	69	17	85	89	33	106	110	66
2	9	36	23	28	88	44	49	7	65	70	93	86	90	70	107	110	30
3	10	58	24	28	4	45	50	97	66	70	20	87	90	32	108	111	66
4	10	54	25	30	89	46	50	7	67	71	92	88	91	68	109	111	31
5	10	27	26	30	4	47	51	98	68	71	22	89	91	34	110	113	64
6	10	22	27	32	90	48	51	8	69	74	91	90	93	67	111	113	33
7	12	63	28	32	4	49	54	98	70	74	23	91	93	34	112	117	63
8	12	21	29	34	91	50	54	8	71	76	90	92	97	67	113	117	34
9	13	70	30	34	4	51	56	98	72	76	26	93	97	34	114	119	61
10	13	18	31	36	91	52	56	9	73	79	88	94	99	67	115	119	37
11	17	73	32	36	3	53	59	98	74	79	27	95	99	33	116	120	60
12	17	13	33	38	93	54	59	10	75	80	84	96	100	67	117	120	40
13	18	77	34	38	3	55	60	98	76	80	29	97	100	31	118	121	56
14	18	11	35	40	94	56	60	11	77	81	81	98	101	68	119	121	44
15	20	80	36	40	3	57	61	97	78	81	80	99	101	30	120	123	51
16	20	9	37	41	96	58	61	12	79	81	30	100	103	67	121	123	49
17	22	82	38	41	3	59	64	96	80	83	78	101	103	30	-	-	-
18	22	8	39	44	96	60	64	13	81	83	30	102	107	67	-	-	-
19	24	83	40	44	4	61	66	96	82	87	74	103	107	29	-	-	-
20	24	7	41	46	97	62	66	16	83	87	32	104	109	67	-	-	-
21	26	86	42	46	4	63	69	94	84	89	71	105	109	30	-	-	-

Table-6. Oblong Leaf Dataset (n=252; kr=1; Recurrence at the 25th iteration)

Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2
1	3	57	43	21	14	85	50	90	127	72	7	169	98	90	211	124	11
2	4	56	44	22	12	86	50	8	128	71	4	170	99	9	212	127	81
3	3	51	45	24	80	87	50	7	129	74	92	171	100	90	213	127	80
4	4	42	46	23	13	88	51	90	130	74	90	172	100	88	214	126	79
5	3	37	47	24	12	89	52	7	131	74	8	173	100	9	215	126	14
6	3	36	48	26	82	90	54	90	132	74	7	174	100	7	216	128	80
7	7	60	49	27	80	91	53	90	133	77	90	175	101	90	217	129	17
8	6	57	50	27	11	92	54	7	134	77	7	176	101	10	218	128	14
9	7	53	51	28	82	93	57	90	135	76	4	177	101	9	219	130	78
10	6	41	52	29	81	94	56	89	136	79	91	178	103	89	220	130	77
11	6	40	53	28	10	95	57	7	137	79	90	179	103	9	221	130	18
12	7	39	54	28	9	96	56	6	138	78	90	180	104	8	222	130	16
13	6	34	55	30	86	97	59	91	139	79	7	181	106	90	223	132	76
14	8	62	56	30	83	98	59	90	140	80	90	182	106	88	224	132	19
15	8	30	57	30	10	99	59	8	141	80	89	183	106	9	225	134	74
16	10	67	58	31	84	100	59	7	142	80	9	184	108	88	226	133	73
17	10	66	59	31	83	101	58	4	143	80	7	185	109	87	227	133	21
18	10	63	60	31	9	102	60	90	144	82	91	186	108	9	228	134	19

19	10	27	61	34	87	103	60	89	145	81	90	187	110	88	229	136	72
20	10	23	62	33	84	104	60	7	146	81	9	188	110	87	230	137	71
21	12	70	63	33	10	105	60	6	147	82	7	189	110	10	231	136	24
22	11	69	64	34	8	106	62	91	148	83	90	190	112	87	232	137	22
23	11	67	65	37	86	107	62	90	149	84	7	191	111	10	233	139	70
24	12	22	66	36	84	108	62	7	150	83	4	192	114	87	234	138	27
25	14	70	67	36	10	109	64	90	151	86	90	193	114	86	235	138	24
26	14	70	68	37	9	110	63	89	152	87	8	194	113	10	236	140	67
27	14	22	69	37	7	111	64	7	153	87	7	195	116	84	237	140	29
28	13	20	70	39	87	112	63	6	154	89	90	196	117	11	238	140	28
29	13	20	71	39	8	113	67	91	155	88	89	197	116	10	239	140	27
30	16	72	72	40	88	114	67	90	156	89	8	198	119	86	240	142	63
31	17	70	73	40	8	115	67	7	157	89	7	199	118	84	241	141	31
32	16	20	74	41	88	116	66	4	158	90	90	200	118	82	242	144	60
33	16	19	75	42	7	117	69	91	159	90	8	201	119	13	243	143	34
34	19	74	76	43	89	118	69	90	160	91	90	202	119	11	244	146	58
35	18	19	77	44	8	119	69	7	161	92	8	203	120	82	245	147	40
36	19	18	78	43	6	120	68	6	162	92	7	204	120	12	246	146	37
37	18	16	79	46	90	121	70	91	163	93	90	205	120	10	247	148	57
38	20	79	80	47	8	122	70	90	164	93	9	206	121	83	248	149	50
39	20	77	81	47	7	123	70	8	165	94	8	207	122	81	249	149	43
40	20	18	82	48	90	124	70	7	166	96	90	208	122	13	250	148	42
41	20	16	83	49	7	125	72	91	167	97	8	209	124	81	251	150	50
42	21	79	84	50	90	126	71	90	168	97	7	210	124	13	252	150	49

Table-7. Bilobed Leaf Dataset (n=129; kr=10; Recurrence at the 25th iteration)

Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2
1	2	23	23	13	40	45	26	67	67	49	70	89	70	69	111	90	40
2	4	53	24	13	32	46	26	7	68	49	4	90	70	6	112	90	37
3	4	48	25	13	10	47	28	68	69	50	70	91	71	68	113	90	11
4	4	29	26	17	63	48	28	7	70	50	3	92	71	6	114	91	60
5	4	19	27	17	40	49	30	68	71	51	70	93	74	68	115	91	41
6	7	58	28	17	33	50	30	7	72	51	3	94	74	7	116	91	36
7	7	46	29	17	10	51	32	69	73	54	70	95	76	67	117	91	12
8	7	30	30	18	64	52	32	7	74	54	3	96	76	7	118	93	59
9	7	16	31	18	40	53	34	69	75	56	70	97	79	67	119	93	42
10	9	60	32	18	33	54	34	6	76	56	3	98	79	8	120	93	33
11	9	43	33	18	9	55	36	70	77	59	70	99	80	66	121	93	14
12	9	30	34	20	66	56	36	6	78	59	4	100	80	8	122	97	57
13	9	13	35	20	39	57	38	70	79	60	70	101	81	66	123	97	44
14	10	60	36	20	34	58	38	4	80	60	4	102	81	9	124	97	32
15	10	42	37	20	9	59	40	70	81	61	70	103	83	64	125	97	16
16	10	31	38	22	66	60	40	4	82	61	4	104	83	9	126	99	50
17	10	12	39	22	38	61	41	70	83	64	70	105	87	63	127	99	30
18	12	61	40	22	36	62	41	4	84	64	4	106	87	10	128	99	19
19	12	41	41	22	8	63	44	70	85	66	70	107	89	62	129	100	23
20	12	31	42	24	67	64	44	4	86	66	4	108	89	39	-	-	-
21	12	11	43	24	37	65	46	70	87	69	69	109	89	10	-	-	-
22	13	62	44	24	8	66	46	4	88	69	4	110	90	61	-	-	-

Table-8. Ovate Leaf Dataset (n=228; kr=1; Recurrence at the 25th iteration)

Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2
1	10	60	39	20	19	77	47	8	115	74	7	153	100	13	191	127	30
2	10	54	40	22	79	78	48	92	116	74	6	154	100	12	192	127	27
3	10	50	41	22	18	79	48	90	117	76	90	155	102	80	193	128	67
4	10	50	42	22	16	80	48	9	118	76	8	156	102	14	194	128	29
5	10	49	43	23	80	81	48	7	119	79	89	157	103	80	195	130	66
6	10	48	44	23	79	82	48	6	120	79	8	158	103	16	196	130	31
7	10	43	45	23	16	83	50	92	121	79	7	159	103	14	197	130	30
8	10	40	46	27	83	84	50	91	122	80	89	160	107	79	198	131	64
9	10	38	47	27	82	85	50	8	123	80	8	161	107	18	199	131	63
10	10	37	48	27	80	86	50	7	124	80	7	162	107	17	200	131	30
11	10	36	49	27	14	87	51	91	125	81	89	163	109	79	201	133	63
12	12	67	50	29	84	88	51	7	126	81	88	164	109	78	202	133	32
13	12	63	51	29	14	89	51	6	127	81	9	165	109	18	203	133	30
14	12	60	52	29	12	90	53	91	128	84	88	166	109	17	204	137	61
15	12	40	53	30	86	91	53	7	129	84	87	167	110	78	205	137	33
16	12	40	54	30	84	92	57	91	130	84	9	168	110	78	206	137	32
17	12	37	55	30	12	93	57	7	131	84	8	169	110	77	207	138	60
18	12	31	56	32	86	94	58	91	132	86	87	170	110	20	208	138	59
19	14	70	57	32	11	95	58	6	133	86	10	171	110	19	209	138	34
20	14	69	58	33	87	96	60	91	134	89	87	172	112	77	210	140	59
21	14	66	59	33	10	97	60	6	135	89	86	173	112	76	211	140	36
22	14	33	60	37	88	98	61	91	136	89	10	174	112	20	212	140	34
23	14	30	61	37	87	99	61	7	137	90	86	175	113	74	213	141	58
24	16	72	62	37	11	100	64	91	138	90	10	176	113	20	214	141	37
25	16	70	63	37	10	101	64	7	139	92	86	177	117	73	215	141	34
26	16	27	64	38	89	102	64	6	140	92	11	178	117	21	216	144	56
27	16	24	65	38	10	103	66	91	141	94	84	179	118	72	217	144	40
28	16	23	66	40	89	104	66	6	142	94	82	180	118	22	218	144	38
29	19	77	67	40	9	105	68	92	143	94	12	181	120	72	219	146	53
30	19	74	68	42	90	106	68	90	144	94	11	182	120	70	220	146	40
31	19	72	69	42	89	107	68	7	145	96	84	183	120	23	221	148	50
32	19	23	70	42	9	108	70	90	146	96	82	184	122	70	222	148	48
33	19	21	71	43	90	109	70	7	147	96	12	185	122	24	223	148	44
34	19	20	72	43	90	110	70	6	148	99	82	186	122	23	224	148	42
35	20	79	73	43	9	111	71	90	149	99	14	187	123	70	225	148	40
36	20	77	74	43	7	112	71	90	150	99	13	188	123	26	226	150	50
37	20	74	75	47	90	113	71	7	151	100	81	189	123	24	227	150	47
38	20	20	76	47	9	114	74	90	152	100	14	190	127	69	228	150	43

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Table-9. Triangular Leaf Dataset (n=309; kr=1; Recurrence at the 60th iteration)

Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2
1	12	66	53	181	74	105	339	73	157	306	225	209	193	354	261	69	225
2	13	64	54	183	74	106	341	73	158	304	229	210	192	356	262	66	223
3	17	62	55	184	74	107	342	73	159	304	233	211	192	358	263	66	219
4	21	62	56	186	70	108	344	75	160	303	237	212	190	361	264	62	215
5	25	62	57	188	69	109	345	75	161	301	240	213	189	365	265	61	212
6	29	61	58	189	67	110	346	76	162	299	243	214	188	367	266	59	208
7	32	59	59	193	63	111	346	77	163	298	247	215	186	369	267	59	206
8	37	58	60	195	60	112	346	79	164	296	250	216	185	370	268	58	200
9	41	55	61	199	58	113	350	82	165	294	252	217	184	371	269	58	196
10	44	55	62	202	55	114	349	86	166	293	255	218	179	371	270	52	192
11	47	53	63	205	54	115	348	90	167	290	258	219	174	369	271	49	189
12	51	50	64	209	51	116	345	92	168	289	260	220	171	366	272	48	186

13	55	49	65	212	49	117	343	94	169	288	262	221	168	362	273	46	183
14	59	47	66	216	47	118	342	96	170	286	264	222	165	356	274	45	179
15	63	45	67	220	45	119	341	97	171	285	267	223	161	349	275	43	174
16	67	43	68	225	43	120	339	100	172	283	269	224	158	343	276	42	170
17	69	41	69	227	42	121	338	105	173	280	272	225	155	337	277	40	167
18	74	39	70	231	39	122	336	108	174	277	276	226	153	334	278	40	161
19	80	37	71	232	39	123	333	112	175	275	278	227	149	330	279	39	158
20	84	33	72	238	39	124	333	113	176	270	279	228	147	326	280	38	154
21	87	32	73	236	39	125	331	117	177	268	281	229	144	322	281	38	151
22	92	31	74	239	39	126	330	119	178	266	285	230	142	318	282	38	147
23	96	31	75	243	39	127	329	123	179	265	286	231	139	314	283	38	144
24	101	29	76	244	39	128	326	127	180	263	291	232	136	309	284	38	140
25	104	30	77	247	39	129	326	131	181	261	292	233	133	307	285	38	138
26	107	30	78	250	39	130	325	134	182	259	295	234	130	303	286	39	133
27	110	32	79	253	40	131	325	138	183	257	298	235	127	301	287	39	129
28	115	32	80	257	42	132	327	141	184	255	302	236	124	299	288	39	125
29	116	30	81	262	44	133	329	145	185	252	304	237	122	294	289	37	120
30	120	30	82	265	44	134	330	150	186	249	306	238	121	291	290	35	117
31	122	32	83	268	44	135	330	154	187	248	308	239	120	288	291	33	113
32	125	33	84	273	44	136	331	157	188	245	311	240	119	288	292	32	110
33	129	34	85	279	46	137	331	161	189	243	311	241	117	287	293	30	108
34	131	35	86	282	48	138	330	163	190	240	313	242	117	287	294	27	104
35	136	36	87	284	49	139	330	165	191	237	314	243	115	281	295	25	102
36	140	39	88	287	51	140	328	168	192	234	315	244	111	277	296	24	100
37	146	39	89	292	54	141	328	169	193	230	317	245	107	275	297	22	97
38	149	41	90	294	56	142	326	174	194	227	319	246	104	271	298	21	95
39	152	43	91	296	56	143	325	180	195	225	322	247	101	268	299	20	93
40	156	46	92	298	58	144	323	182	196	222	322	248	99	266	300	18	91
41	161	48	93	302	60	145	321	188	197	221	323	249	96	262	301	17	89
42	163	51	94	303	61	146	318	192	198	218	325	250	94	260	302	17	87
43	165	53	95	307	63	147	316	196	199	215	327	251	93	257	303	14	85
44	167	55	96	308	64	148	315	201	200	213	330	252	91	254	304	13	83
45	169	59	97	312	65	149	311	207	201	211	332	253	86	252	305	11	80
46	170	61	98	316	68	150	310	212	202	209	335	254	84	248	306	10	78
47	171	64	99	321	67	151	311	216	203	206	338	255	79	243	307	10	75
48	173	66	100	327	69	152	310	219	204	205	340	256	76	239	308	10	71
49	174	68	101	330	71	153	309	220	205	202	343	257	75	236	309	10	68
50	176	70	102	333	72	154	309	222	206	199	346	258	73	233	-	-	-
51	177	71	103	334	73	155	307	224	207	197	348	259	73	231	-	-	-
52	178	72	104	337	73	156	306	225	208	195	352	260	70	228	-	-	-

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Table-10. Nine-lobed Leaf Dataset (n=407; kr=1; Recurrence at the 150th iteration)

Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2
1	212	38	69	33	100	137	33	273	205	213	547	273	421	309	341	384	102
2	208	34	70	32	102	138	34	277	206	213	549	274	424	308	342	380	100
3	206	32	71	29	104	139	34	281	207	215	548	275	420	301	343	378	97
4	203	29	72	26	106	140	34	285	208	217	542	276	419	296	344	376	94
5	199	27	73	24	107	141	34	289	209	218	538	277	418	292	345	375	90
6	196	23	74	23	109	142	36	292	210	219	536	278	415	288	346	374	87
7	192	19	75	21	111	143	37	298	211	220	532	279	413	281	347	370	86
8	189	16	76	19	113	144	37	301	212	222	528	280	412	279	348	366	83
9	186	14	77	17	114	145	38	302	213	224	525	281	411	273	349	362	81
10	184	12	78	15	116	146	40	305	214	227	519	282	410	269	350	358	79
11	182	11	79	15	116	147	45	310	215	231	515	283	409	265	351	356	77
12	180	11	80	11	117	148	47	309	216	235	510	284	408	261	352	352	77

13	178	12	81	10	118	149	54	311	217	239	504	285	405	257	353	349	75
14	174	14	82	8	119	150	63	311	218	242	501	286	405	254	354	345	75
15	171	14	83	6	120	151	66	311	219	246	495	287	404	252	355	337	73
16	167	13	84	4	122	152	72	313	220	248	491	288	403	251	356	335	72
17	163	13	85	4	123	153	74	316	221	252	486	289	403	246	357	333	72
18	157	13	86	3	126	154	81	313	222	257	481	290	403	242	358	332	72
19	152	14	87	2	128	155	88	314	223	259	476	291	400	239	359	329	72
20	148	13	88	4	131	156	90	317	224	261	473	292	400	235	360	328	71
21	143	13	89	7	132	157	92	322	225	265	468	293	400	232	361	328	68
22	140	13	90	11	134	158	94	324	226	268	461	294	403	228	362	331	64
23	137	14	91	13	135	159	94	324	227	270	457	295	403	225	363	332	61
24	132	15	92	16	138	160	94	325	228	273	453	296	404	219	364	332	60
25	126	17	93	22	141	161	94	329	229	275	451	297	402	218	365	334	55
26	125	19	94	26	145	162	94	330	230	276	447	298	401	216	366	334	53
27	123	22	95	29	149	163	95	333	231	279	444	299	399	212	367	334	51
28	123	28	96	35	154	164	96	334	232	285	440	300	397	209	368	333	50
29	124	31	97	39	156	165	97	337	233	288	435	301	394	205	369	332	46
30	126	35	98	41	163	166	97	340	234	291	434	302	394	205	370	330	44
31	127	39	99	44	166	167	97	345	235	293	432	303	394	205	371	327	41
32	127	44	100	47	166	168	98	356	236	296	432	304	393	205	372	323	40
33	126	47	101	50	169	169	101	360	237	304	432	305	392	202	373	316	38
34	124	49	102	54	170	170	104	367	238	301	432	306	391	200	374	313	36
35	120	50	103	57	172	171	106	374	239	307	432	307	390	199	375	311	36
36	118	50	104	62	176	172	112	378	240	310	431	308	389	198	376	307	35
37	114	50	105	59	178	173	116	384	241	312	427	309	386	196	377	303	33
38	112	50	106	57	181	174	120	389	242	313	421	310	385	193	378	300	33
39	109	50	107	56	183	175	124	394	243	316	414	311	383	188	379	297	32
40	104	50	108	53	187	176	130	399	244	317	409	312	383	183	380	294	32
41	100	50	109	49	193	177	136	405	245	319	404	313	384	179	381	291	31
42	97	49	110	47	194	178	139	412	246	321	398	314	390	174	382	286	29
43	94	49	111	46	198	179	139	417	247	321	392	315	394	171	383	283	27
44	90	49	112	43	202	180	143	423	248	323	386	316	398	170	384	281	27
45	88	49	113	42	204	181	148	436	249	325	381	317	402	166	385	278	27
46	85	50	114	40	207	182	153	442	250	327	374	318	403	163	386	273	27
47	82	52	115	40	210	183	155	446	251	330	368	319	406	161	387	267	29
48	79	55	116	37	214	184	160	452	252	332	364	320	412	156	388	263	31
49	78	56	117	36	216	185	164	461	253	333	358	321	414	155	389	259	33
50	74	56	118	34	218	186	166	468	254	336	355	322	417	151	390	257	35
51	72	60	119	34	219	187	170	474	255	337	351	323	420	148	391	253	37
52	71	64	120	34	221	188	173	479	256	339	348	324	421	146	392	249	38
53	70	67	121	34	225	189	177	486	257	341	344	325	422	144	393	245	41
54	69	69	122	34	229	190	180	490	258	343	340	326	423	140	394	243	43
55	66	74	123	37	233	191	185	495	259	347	336	327	423	137	395	239	44
56	63	77	124	37	234	192	187	498	260	354	334	328	419	133	396	234	47
57	61	79	125	38	237	193	189	503	261	359	331	329	417	131	397	232	49
58	58	80	126	36	239	194	191	507	262	364	327	330	415	131	398	229	53
59	56	83	127	36	240	195	197	511	263	372	325	331	414	130	399	229	55
60	54	84	128	35	242	196	198	514	264	376	324	332	410	125	400	227	58
61	51	86	129	33	247	197	202	517	265	381	323	333	405	124	401	224	58
62	49	87	130	31	248	198	204	521	266	388	321	334	404	122	402	223	58
63	48	87	131	30	251	199	206	525	267	393	321	335	401	120	403	221	57
64	47	89	132	30	253	200	209	530	268	396	318	336	400	119	404	221	54
65	45	91	133	30	257	201	210	534	269	401	318	337	396	116	405	219	52
66	41	93	134	30	259	202	212	538	270	407	317	338	392	112	406	218	50
67	39	95	135	31	266	203	212	542	271	410	314	339	391	109	407	216	49
68	36	98	136	32	269	204	212	545	272	416	312	340	388	106	-	-	-

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Table-11. Five-lobed Leaf Dataset (n=270; kr=1; Recurrence at the 384th iteration)

Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2
1	222	81	46	18	80	91	50	299	136	179	481	181	301	273	226	412	111
2	213	75	47	15	83	92	59	299	137	185	484	182	305	274	227	411	107
3	209	69	48	12	86	93	67	298	138	191	487	183	310	277	228	408	103
4	202	61	49	10	93	94	77	297	139	198	492	184	317	281	229	405	95
5	196	55	50	10	97	95	84	295	140	203	496	185	323	282	230	401	90
6	192	51	51	15	106	96	89	294	141	212	500	186	332	283	231	398	85
7	187	48	52	20	111	97	95	294	142	219	503	187	340	284	232	394	77
8	183	43	53	30	115	98	101	292	143	222	507	188	348	285	233	391	72
9	179	40	54	35	118	99	106	289	144	224	501	189	357	287	234	386	68
10	173	36	55	39	123	100	108	286	145	228	500	190	365	287	235	382	65
11	167	32	56	45	128	101	111	282	146	236	493	191	371	288	236	378	60
12	163	27	57	50	133	102	112	282	147	241	490	192	378	288	237	375	57
13	158	24	58	54	137	103	112	288	148	244	486	193	384	288	238	369	55
14	154	23	59	56	141	104	113	289	149	249	481	194	389	286	239	364	54
15	150	18	60	58	144	105	114	292	150	254	476	195	395	283	240	361	53
16	146	15	61	54	147	106	111	298	151	260	469	196	398	280	241	359	50
17	139	13	62	48	148	107	107	307	152	264	464	197	401	274	242	357	44
18	133	10	63	45	151	108	107	312	153	268	458	198	402	268	243	353	40
19	126	6	64	41	156	109	107	320	154	271	450	199	405	261	244	349	38
20	122	5	65	39	160	110	106	327	155	274	444	200	408	255	245	342	34
21	115	5	66	36	166	111	106	334	156	280	438	201	409	248	246	336	32
22	107	6	67	34	168	112	108	342	157	282	431	202	409	239	247	328	28
23	100	10	68	29	178	113	109	351	158	284	423	203	409	229	248	324	28
24	93	14	69	31	174	114	110	357	159	287	415	204	408	219	249	319	27
25	90	20	70	20	179	115	112	366	160	291	409	205	407	212	250	313	27
26	90	23	71	23	186	116	114	378	161	294	401	206	406	202	251	307	26
27	90	25	72	20	194	117	118	386	162	298	392	207	405	192	252	300	26
28	89	28	73	18	200	118	120	392	163	303	386	208	402	185	253	293	26
29	86	29	74	16	204	119	121	397	164	311	380	209	399	178	254	288	29
30	81	30	75	13	209	120	124	402	165	312	373	210	394	170	255	286	30
31	76	31	76	13	216	121	127	407	166	314	365	211	390	166	256	280	33
32	70	32	77	13	225	122	131	410	167	315	359	212	385	160	257	277	35
33	61	34	78	13	232	123	135	411	168	316	353	213	379	153	258	271	39
34	58	35	79	13	240	124	137	414	169	318	347	214	376	149	259	268	43
35	55	38	80	12	248	125	138	422	170	318	340	215	373	146	260	262	50
36	51	40	81	12	254	126	140	426	171	316	331	216	372	144	261	260	52
37	49	45	82	13	260	127	141	431	172	314	326	217	374	141	262	257	56
38	46	48	83	14	267	128	144	438	173	313	315	218	378	139	263	254	60
39	43	51	84	16	272	129	147	446	174	311	307	219	383	136	264	249	66
40	39	55	85	19	276	130	149	449	175	308	296	220	392	134	265	246	71
41	34	60	86	22	280	131	153	455	176	306	291	221	398	130	266	243	75
42	28	64	87	24	286	132	155	459	177	305	286	222	403	125	267	240	78
43	25	68	88	27	290	133	159	465	178	304	281	223	408	122	268	235	82
44	23	73	89	36	292	134	170	473	179	303	276	224	410	119	269	232	84
45	21	76	90	41	298	135	179	476	180	301	275	225	412	115	270	229	85

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Table-12. Three-lobed Leaf Dataset (n=318; kr=1; Recurrence at the 264th iteration)

Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2	Sl	X_1	X_2
1	3	276	54	347	72	107	310	318	160	187	474	213	116	267	266	19	185
2	12	252	55	355	74	108	305	311	161	184	478	214	120	262	267	20	176
3	20	167	56	359	76	109	303	302	162	179	484	215	123	257	268	19	170
4	19	120	57	365	82	110	301	296	163	177	490	216	127	252	269	19	164

5	27	105	58	370	87	111	297	292	164	173	496	217	128	248	270	19	158
6	38	81	59	376	92	112	294	285	165	173	501	218	130	246	271	19	152
7	58	63	60	382	98	113	292	278	166	171	507	219	132	242	272	19	141
8	78	57	61	387	105	114	290	274	167	171	513	220	132	240	273	19	134
9	70	60	62	391	110	115	286	268	168	169	520	221	131	237	274	19	130
10	47	71	63	395	116	116	283	265	169	168	523	222	129	236	275	19	127
11	88	52	64	399	124	117	280	264	170	166	525	223	127	235	276	19	125
12	97	43	65	403	133	118	277	263	171	163	519	224	124	234	277	20	118
13	106	34	66	408	142	119	275	265	172	162	510	225	122	234	278	23	114
14	113	27	67	411	153	120	274	270	173	161	501	226	117	234	279	24	110
15	122	21	68	415	161	121	274	275	174	160	494	227	111	236	280	28	106
16	134	18	69	417	170	122	275	281	175	159	487	228	107	240	281	31	99
17	143	20	70	419	184	123	278	287	176	158	479	229	99	243	282	31	94
18	150	22	71	420	193	124	281	292	177	155	471	230	96	247	283	34	90
19	157	23	72	420	200	125	283	301	178	152	463	231	90	251	284	35	87
20	169	23	73	422	209	126	286	307	179	149	457	232	86	254	285	42	79
21	175	23	74	422	215	127	286	314	180	143	448	233	84	257	286	44	76
22	184	30	75	421	226	128	286	322	181	139	440	234	82	260	287	52	72
23	190	37	76	419	236	129	285	332	182	135	436	235	78	261	288	55	67
24	197	43	77	418	243	130	284	339	183	131	427	236	75	263	289	61	62
25	202	54	78	415	254	131	282	346	184	128	420	237	72	264	290	65	61
26	209	66	79	412	261	132	281	351	185	124	412	238	68	267	291	71	60
27	212	76	80	409	269	133	279	359	186	121	408	239	61	269	292	82	56
28	212	85	81	404	276	134	277	366	187	118	404	240	53	269	293	84	54
29	211	91	82	401	284	135	276	372	188	116	397	241	57	269	294	89	51
30	211	97	83	398	291	136	274	377	189	112	390	242	51	271	295	92	48
31	214	104	84	394	298	137	272	382	190	109	385	243	47	271	296	95	46
32	218	112	85	392	305	138	267	387	191	107	380	244	43	272	297	98	41
33	222	115	86	391	313	139	264	392	192	104	375	245	41	272	298	101	38
34	228	116	87	389	321	140	257	398	193	101	371	246	36	273	299	103	36
35	235	116	88	386	331	141	253	403	194	100	367	247	31	274	300	107	32
36	240	113	89	385	339	142	250	407	195	96	360	248	26	274	301	110	29
37	247	110	90	383	348	143	244	411	196	95	357	249	22	274	302	114	24
38	251	104	91	381	356	144	243	415	197	93	350	250	17	274	303	119	23
39	254	98	92	379	361	145	240	419	198	91	346	251	14	274	304	124	21
40	257	92	93	379	365	146	235	424	199	90	340	252	11	274	305	132	20
41	261	87	94	381	369	147	231	426	200	89	334	253	6	276	306	143	20
42	265	82	95	378	374	148	228	430	201	89	328	254	5	267	307	151	20
43	271	77	96	370	374	149	225	433	202	89	322	255	9	262	308	160	20
44	276	73	97	363	366	150	221	436	203	89	317	256	11	257	309	168	22
45	283	69	98	356	362	151	218	441	204	91	313	257	13	243	310	174	26
46	288	66	99	351	356	152	216	444	205	94	307	258	15	233	311	178	30
47	294	64	100	347	353	153	215	447	206	97	302	259	16	226	312	186	33
48	299	61	101	341	349	154	212	451	207	97	296	260	16	219	313	192	38
49	308	61	102	338	346	155	208	454	208	99	290	261	15	213	314	198	46
50	317	61	103	329	340	156	204	456	209	101	287	262	15	208	315	202	52
51	324	61	104	324	335	157	200	460	210	105	282	263	16	200	316	205	58
52	333	63	105	317	329	158	195	464	211	109	276	264	17	196	317	210	68
53	339	66	106	313	326	159	190	469	212	112	271	265	19	185	318	213	72

```
      PARAMETER(MAXITER=30000,KR=1)! MAX ITERATIONS
      ! KR IS THE TRUNCATION COEFFICIENT. SHOULD NOT EXCEED 100.
      ! LARGER THE KR, LARGER IS THE NO. OF ITERATIONS FOR RECURRENCE
      ! ------------------------------------------------------------
      PARAMETER(NMAX=1000,M=2)! NMAX IS THE MAX POINTS ALLOWED
      ! IT MAY BE INCRESED/DECREASED BY THE USER.
      INTEGER X(NMAX,M),Z(NMAX,M)
      DIMENSION AX(NMAX,M)
      CHARACTER *70 INPUT_FILE, OUTPUT_FILE
      DOUBLE PRECISION R(MAXITER),R0,CORRLN
      COMMON MAXV
      ! ------------------------------------------------------------
      WRITE(*,*)'WHAT ARE THE INPUT AND THE OUTPUT FILES ?'
      READ(*,*) INPUT_FILE, OUTPUT_FILE

      OPEN(7,FILE=INPUT_FILE) ! INPUT FILE
      READ(7,*) N ! NO. OF POINTS IN THE DATASET (X1, X2)
      DO I=1,N
      READ(7,*) AX(I,1),AX(I,2) ! AX IS REAL DATASET POINTS
      ENDDO
      CLOSE(7)
      ! GET INTEGER X AND PRESERVE THE ORIGINAL DATA IN Z
      DO I=1,N
      IF(AX(I,1)-INT(AX(I,1)).LT.0.5) X(I,1)=INT(AX(I,1)*KR)
      IF(AX(I,1)-INT(AX(I,1)).GE.0.5) X(I,1)=INT(AX(I,1)*KR)+1
      IF(AX(I,2)-INT(AX(I,2)).LT.0.5) X(I,2)=INT(AX(I,2)*KR)
      IF(AX(I,2)-INT(AX(I,2)).GE.0.5) X(I,2)=INT(AX(I,2)*KR)+1
      ! PRESERVE X IN Z
      Z(I,1)=X(I,1)
      Z(I,2)=X(I,2)
      ! -------------------------------
      ENDDO

      ! FIND THE MAXIMUM VALUE IN DATA X1, X2 FOR ALL I=1,N
      MAXXY=X(1,1)
      DO J=1,M
      DO I=1,N
      IF(MAXXY.LT.X(I,J)) MAXXY=X(I,J)
      ENDDO
      ENDDO
      ! ------------------------------------------------------
      MAXV=MAXXY+1 ! NEEDED IN SUBROUTINE ARNOLD
      ! ------------------------------------------------------
      OPEN(8,FILE=OUTPUT_FILE) ! OUTPUT FILE
      DO I=1,N
      WRITE(8,1) I, X(I,1),X(I,2)
      ENDDO
      CALL R_PEARSON(X,N,CORRLN)
      R0 = CORRLN
    1 FORMAT(I4,2I10)
    2 FORMAT(I4,F12.8)
      DO K=1,MAXITER
       WRITE(8,*)'------- ITERATION =',K,' ----------------------'
       CALL ARNOLD(X,N)
       DO I=1,N
       WRITE(8,1) I, X(I,1),X(I,2)
       ENDDO ! FOR I
      ! COMPARE WITH THE ORIGINAL DATA
         ! CHECK FOR RESTORATION
         CALL R_PEARSON(X,N,CORRLN)
         R(K)=CORRLN
         KCHECKX=0
         KCHECKY=0
         DO I=1,N
         IF(X(I,1).EQ.Z(I,1)) KCHECKX=KCHECKX+1
         IF(X(I,2).EQ.Z(I,2)) KCHECKY=KCHECKY+1
         ENDDO ! FOR I
```

```
68:          !IF(100*ABS(R0-R(K)).LE.0.0001) THEN
69:          IF(KCHECKX.EQ.N.AND.KCHECKY.EQ.N.AND.K.GT.1) THEN
70:          WRITE(*,*)'ORIGINAL DATA RESTORED IN',K,' ITERATIONS'
71:          WRITE(*,*)'TRANSFORMED DATA STORED IN ', OUTPUT_FILE
72:          WRITE(*,*) 'END. THANK YOU'
73:          GOTO 100
74:          ENDIF
75:       ENDDO ! FOR K
76:
77:       WRITE(*,*)'DID NOT CONVERGE.'
78:       WRITE(*,*)'TRANSFORMED DATA STORED IN ', OUTPUT_FILE
79:       WRITE(*,*)'END'
80:   100 WRITE(8,*)'CORRELATAION(X(1),X(2)) IN DIFFERENT ITERATIONS'
81:       IT0=0
82:       WRITE(8,2) IT0,R0
83:       DO I=1,K
84:       WRITE(8,2) I,R(I)
85:       ENDDO
86:       CLOSE(8)
87:       END
88:       ! ---------------------------------------------------------------
89:       SUBROUTINE ARNOLD(X,N)! SUBROUTINE FOR TRANSFORMATION
90:       PARAMETER (NMAX=1000,M=2)
91:       INTEGER X,Z,B
92:       DIMENSION X(NMAX,M),Z(NMAX,M), B(M,M)
93:       COMMON MAXV
94:       DATA ((B(I,J),J=1,M),I=1,M) /2, 1, 1, 1/ ! TRANSFORMATION MATRIX
95:       ! ---------------------------------------------------------------
96:       ! PRESERVE X IN Z
97:       DO I=1,N
98:       DO J=1,M
99:       Z(I,J)=X(I,J)
100:      ENDDO
101:      ENDDO
102:      ! ------- TRANSFORMATION ---------------------------------
103:      DO I=1,N
104:      DO J=1,M
105:      KS=0
106:      DO K=1,M
107:      KS = KS +B(J,K)* Z(I,K)
108:      ENDDO
109:      X(I,J)=MOD(KS,MAXV) ! MAXV IS THE LARGEST VALUE IN THE DATA + 1
110:      ENDDO
111:      ENDDO
112:      ! ---------------------------------------------------------------
113:      RETURN
114:      END
115:
116:      ! ---------------------------------------------------------------
117:      SUBROUTINE R_PEARSON(Z,N,R)
118:      PARAMETER (MN=1000)
119:      ! FIND CORRELATION BETWEEN SCORES (SCX AND SCY)
120:      IMPLICIT DOUBLE PRECISION (A-H,O-Y)
121:      INTEGER Z
122:      DIMENSION Z(MN,2),X(MN),Y(MN)
123:      DO I=1,N
124:      X(I)=Z(I,1)
125:      Y(I)=Z(I,2)
126:      ENDDO
127:
128:      SX=0.D0
129:      SY=0.D0
130:      SXY=0.D0
131:      SXX=0.D0
132:      SYY=0.D0
133:      DO I=1,N
134:      SX=SX + X(I)
```

```
135:        SY=SY + Y(I)
136:        SXX=SXX + X(I)**2
137:        SYY=SYY + Y(I)**2
138:        SXY=SXY + X(I)*Y(I)
139:        ENDDO
140:        SXY=SXY/N
141:        SXX=SXX/N
142:        SYY=SYY/N
143:        SX=SX/N
144:        SY=SY/N
145:        SXY=(SXY - SX*SY)
146:        SXX=(SXX - SX**2)
147:        SYY=(SYY - SY**2)
148:        R=SXY/DSQRT(SXX*SYY)
149:        RETURN
150:        END
```