Efficient estimation of extreme value-at-risks for standalone structural exchange rate risk

He, Zhongfang

6 August 2014

Efficient Estimation of Extreme Value-at-Risks for Standalone Structural Exchange Rate Risk

Zhongfang He*

July 31, 2014

Abstract

The standalone structural exchange rate risk depends on the product of the future foreign currency earning and the change in the exchange rate. Its Value-at-Risk (VaR) implying an extremely high survival probability, usually exceeding 99.9%, is used in practice to determine its economic capital. This paper proposes a new conditional method to calculate such extreme VaRs that is shown to be more efficient than the conventional method of directly simulating from the joint distribution of the future foreign currency earning and the change in the exchange rate. The intuition of the proposed method is that, conditional on either the future foreign currency earning or the change in the exchange rate, the distribution of the structural exchange rate risk is usually analytically tractable. The proposed method can be implemented by solving a nonlinear equation via a simple one-dimensional numerical integration and is generally applicable under the distributional assumptions commonly employed in practice.

Key words: value-at-risk, structural exchange rate risk, extreme value

*Corporate Treasury, Royal Bank of Canada, hezhongfang2004@yahoo.com. Disclaimer: The analysis and conclusions in this paper are those of the author and do not represent the view of the Royal Bank of Canada.

1 Introduction

The structural exchange rate risk of a financial institution arises from its foreign currency denominated earnings or from the foreign currency capital deployed in its foreign branches and subsidiaries. Let E_0 be the current unhedged foreign currency exposure and F_0 be the current exchange rate expressed as domestic currency per unit of foreign one. Suppose the future foreign currency earning over the relevant horizon is X and the exchange rate at the end of the horizon is F_1. The structural exchange rate risk, denoted by Z, is measured as the change in the foreign currency exposure denominated in domestic currency, that is, $Z = (E_0 + X)\,F_1 - E_0F_0$. In practice, it is often convenient to rewrite the structural exchange rate risk as $Z = (E_0 + X)\,Y + F_0X$, where $Y = F_1 - F_0$ is the change of the exchange rate. The value-at-risk (VaR) of Z implying an extremely high survival probability, usually exceeding 99.9%, is calculated to determine the economic capital for the structural exchange rate risk (Papaioannou (2006))[1].

Statistical estimation of the joint distribution of the foreign currency earning X and the change of the exchange rate Y is usually limited by data availability in practice. Historical data on exchange rate is readily available. But a consistent history of data on a bank's foreign currency earning is usually not, either due to limitations of its internal data collection or structural changes of its foreign business. In this case, expert judgment is usually applied to determine the foreign-currency-denominated economic capital of the bank's foreign operation at a given confidence level. The distribution of the bank's foreign currency earning X is then calibrated to match the economic capital from expert judgment. Assumption of the correlation or tail dependence between X and the change in exchange rate Y is also made through expert judgment. Combining these assumptions with the distribution of Y calibrated from the history of exchange rate produces the joint distribution of X and Y. In this paper, we take these calibrated marginal and joint distributions of X and Y as given and study how to efficiently estimate the extreme VaRs

[1] The confidence level of the economic capital is usually calibrated to be consistent with the default rate of a financial institution's target credit rating (KPMG (2004)). Jackson, Perraudin, and Sapporta (2004) finds that the capital charges of most banks imply survival probabilities significantly higher than 99.9%, which is consistent with the confidence level of the standalone structural exchange rate risk commonly used among banks.

based on them.

Theoretically the estimation of the VaR for the structural exchange rate risk is straightforward if its distribution admits a closed-form expression. Once the assumptions on the distributions of the foreign currency earning X and the change of the exchange rate Y are made, the structural exchange rate risk Z can be viewed as a product of affine transformations of X and Y in the form of $Z = (E_0 + X)(F_0 + Y) - E_0 F_0$. The cumulative distribution function (CDF) of Z at the extreme probabilities can be used to solve for the VaRs. Unfortunately the distribution of such a product of two random variables X and Y is often not available in closed form when X and Y follow the distributions commonly used in practice.

For the case of X and Y being normal variables, Craig (1936) first provides the distribution of the product of two independent normal variables, whose CDF is proportional to the difference of two integrals over the domain $(0, \infty)$ and $(-\infty, 0)$ respectively. The convergence speed of numerical computation of the CDF is slow. See Seijas-Macias and Oliveira (2012) for the details. The product of two correlated Student's t variables is studied in Wallgren (1980), in which the two Student's t variables are constructed in a particular way as two correlated normal variables with the same variance divided by the same χ^2 distributed scalar. The resulting CDF is expressed as an integral and can be found in Kotz and Nadarajah (2004).

In the absence of a closed-form expression for the distribution of the structural exchange rate risk, the Monte Carlo simulation method is a convenient alternative. Draws of X and Y from their joint distribution can be used to produce simulations of Z, whose tail percentiles are valid estimates of the VaRs. Nevertheless, for the extreme VaRs such as the 99.99% VaR, which is commonly used for economic capital calculation, this direct simulation method is vastly inefficient. Depending on the parameters of the joint distribution of X and Y, an impractically large number of simulations would be required to produce a stable estimate of the extreme VaRs whose numerical error across independent runs are within acceptable range.

In this paper, I propose an alternative method, termed conditional method, to calculate the extreme VaRs of the structural exchange rate risk that is shown to be more efficient than the direct simulation method. The intuition of the proposed method is to use the fact that, conditional on either the foreign currency earning X or the change of exchange rate Y, the structural exchange rate risk Z becomes an affine function of the other random variable. This conditional distribution of Z usually has a closed-form expression and can be used to eliminate the need of simulations via a simple one-dimensional numerical integration. The details of the proposed conditional method can be found in Section 2. A numerical example is provided in Section 3 to compare the efficiency of the direct sampling method and the proposed conditional method. Section 4 concludes.

Though this conditional method focuses on the standalone structural exchange rate risk, the analysis of this paper could be incorporated into an integrated structural exchange rate risk framework in which the co-movement of the exchange rate risk with other risk factors such as market, interest rate and operational risks are integrated.

2 Estimating the Extreme Value-at-Risks

Denote the marginal and joint densities of the foreign currency earning X and the change in exchange rate Y by $f_X(x)$, $f_Y(y)$ and $f_{XY}(x, y)$ respectively. For a given probability p, the corresponding VaR, denoted by z^*, satisfies $\text{Prob}(Z < z^*) = p$. Let $f_Z(z)$ be the marginal density of Z. It follows that the VaR z^* is the solution to the equation:

$$\int_{-\infty}^{z^*} f_Z(z)dz = p \tag{1}$$

Let $f_Z(z|x)$ be the density of Z conditional on $X = x$. By the law of total probability, we have the equation:

$$f_Z(z) = \int_{-\infty}^{+\infty} f_Z(z|x)f_X(x)dx \tag{2}$$

Inserting Equation (2) into Equation (1) produces $\int_{-\infty}^{z^*} \int_{-\infty}^{+\infty} f_Z(z|x)f_X(x)dxdz = p$.

Changing the order of the integrals results in the equation:

$$\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{z^*} f_Z(z|x)dz\right) f_X(x)dx = p \tag{3}$$

Since Z becomes an affine function of Y conditional on $X = x$, it follows that

$$\begin{aligned}\int_{-\infty}^{z^*} f_Z(z|x)dz &= \text{Prob}(Z < z^*|X = x) \\ &= \text{Prob}\left(F_0 x + (E_0 + x)Y < z^*|X = x\right) \\ &= \begin{cases} \text{Prob}\left(Y < \frac{z^*-F_0 x}{E_0+x}\middle| X = x\right) & \text{if } x > -E_0 \\ 1 - \text{Prob}\left(Y < \frac{z^*-F_0 x}{E_0+x}\middle| X = x\right) & \text{if } x < -E_0 \\ 1 & \text{if } x = -E_0,\ z^* > -F_0 E_0 \\ 0 & \text{if } x = -E_0,\ z^* \leq -F_0 E_0 \end{cases}\end{aligned} \tag{4}$$

Given the joint distribution $f_{XY}(x, y)$, the conditional distribution $f_Y(y|x)$ usually has a closed-form expression. For example, when X and Y are jointly normally distributed, it is well known that the conditional distribution $f_Y(y|x)$ is normal as well. On the other hand, if X and Y follow a bivariate Student's t distribution, the conditional distribution $f_Y(y|x)$ is Student's t (Roth (2013)). Therefore the conditional CDF Prob $(Y < y|\, X = x)$ is available in close form. Denote this conditional CDF by $F_Y(y|x)$ and insert it into Equations (3) and (4). We have the equation:

$$\begin{aligned}&\int_{-\infty}^{+\infty}\left(F_Y\left(\frac{z^*-F_0 x}{E_0+x}\middle| x\right) I_{\{x>-E_0\}} + \left(1 - F_Y\left(\frac{z^*-F_0 x}{E_0+x}\middle| x\right)\right) I_{\{x<-E_0\}}\right) f_X(x)dx \\ =&p\end{aligned} \tag{5}$$

where $I_{\{x>-E_0\}} = 1$ if $x > -E_0$ and 0 otherwise, $I_{\{x<-E_0\}} = 1$ if $x < -E_0$ and 0 otherwise.

To numerically approximate the integral in Equation (5), one could use the simple rectangle rule[2]. Let $\{x_i\}_{i=0}^n$ be a grid of points on the real line. The VaR z^* can be

[2]More sophisticated methods such as the Gauss-Hermite quadrature method could be applied to approximate the integral in Equation(5) and is likely to be more numerically efficient than the simple rectangle rule.

estimated by solving the equation[3]:

$$\sum_{i=1}^{n}\left(F_Y\left(\frac{z^*-F_0x_i}{E_0+x_i}\middle| x_i\right)I_{\{x_i>-E_0\}}+\left(1-F_Y\left(\frac{z^*-F_0x_i}{E_0+x_i}\middle| x_i\right)\right)I_{\{x_i<-E_0\}}\right)f(x_i)(x_i-x_{i-1})$$
$$=p \tag{6}$$

To decide the starting point x_0 and the ending point x_n of the grid, one could apply the Chebyshev's inequality Prob $\left(|X-\mu_x|\geq\frac{\sigma_x}{\sqrt{\pi}}\right)\leq\pi$, where μ_x is the mean of X, σ_x is the standard deviation of X and π is a small number indicating the user-selected tolerance level of the approximation error. If $x_0=\mu_x-\frac{\sigma_x}{\sqrt{\pi}}$ and $x_n=\mu_x+\frac{\sigma_x}{\sqrt{\pi}}$, the Chebyshev's inequality guarantees that this grid covers more than $(1-\pi)\times 100\%$ of the possible values of X. Combined with a sufficiently small step size x_i-x_{i-1}, the grid should approximate the integral in Equation(5) reasonably well in practical applications.

In the above discussion, the foreign currency earning X is used as the conditioning variable. Alternatively, the change in exchange rate Y could be conditioned on as well. In practice, the variable that has the smaller variance should be preferred to be conditioned on. The proposed method depends crucially on the use of the conditional CDF $F_Y(y|x)$ and hence is termed "conditional" method. Compared with the direct simulation method, the conditional method avoids the randomness of simulation that usually induces large numerical error for estimates of extreme VaRs. Relative to the method of analytically deriving the distribution of the structural exchange rate risk Z, the conditional method has the advantage of being simple and general so long as the conditional distribution of the foreign currency earning X or the change of exchange rate Y is available in closed form.

3 A Numerical Example

In this section, a numerical example is studied to compare the proposed conditioning method for estimating extreme VaRs. The foreign currency earning X is assumed to

[3]Note the singular case when $x_i=-E_0$. This singular case can be handled by moving the grid point slightly above or below $-E_0$.

follow a standard normal distribution $N(0,1)(\sigma_x = 1)$. The change of exchange rate Y is assumed to follow the normal distribution $N(0, 0.12^2)(\sigma_y = 0.12)$ and is based on the annual change of historical USD per Euro from January 1999 to August 2014. Hence this example could be viewed as the 1-year structural Euro risk for a hypothetical US based bank. The correlation between X and Y is assumed to be $\rho = -0.5$ considering that appreciation of Euro is likely to be associated with higher Euro earning for the US bank. The current USD per Euro F_0 is 1.3 and the the current Euro exposure E_0 is €1 unit. Given this setup, the 0.05% and 0.01% VaRs that imply survival probabilities of 99.95% and 99.99% respectively are estimated.

Even for this simple setup in the above example, the distribution of the structural USD risk $Z = F_0 X + (E_0 + X)Y$ is rather involved[4]. We focus on the comparison of the proposed conditional method with the direct simulation method.

For the direct simulation method, we generate n simulations from the bivariate normal distribution of X and Y and calculate the resulting 0.0005-th and 0.0001-st percentiles of the resulting sample of Z as an estimate of the VaRs. We repeat this simulation independently for 1,000 times. The VaR estimates across the 1,000 independent repetitions provide an assessment of the numerical stability of the direct simulation method. We compare two different simulation sizes $n = 1$ million and 10 millions. Table 1 provides the resulting mean and 95% confidence interval of the estimated 0.05% and 0.01% VaRs across the 1,000 independent repetitions.

When the simulation size $n = 1$ million, a single run of the direct simulation method is likely to produce 0.05% VaR estimate ranging from -\$4.81 units to -\$4.73 units based on its 95% confidence interval, while the true 0.05% VaR is about -\$4.77 units. For the more extreme 0.01% VaR, the possible estimates in a single run of the direct simulation method could diverge even further, from -\$5.60 units to -\$5.42 units based on its 95% confidence interval. When the simulation size n increases to 10 millions, which is a rather

[4]To apply the result in Craig (1936) which is for the product of two independent normal variables, we can define $W = -\frac{\rho\sigma_y}{\sigma_x} X + Y$, which is independent of X by construction, and rewrite $Z = \left(F_0 + \frac{\rho\sigma_y}{\sigma_x} E_0\right) X + E_0 W + XW + \frac{\rho\sigma_y}{\sigma_x} X^2$. Even though we know the CDF of XW based on Craig (1936), the resulting CDF of Z is clearly complicated and is not available in closed form.

Table 1: Results of Direct Simulation Method

Panel (a): Simulation Size = 1 Million		
	0.05% VaR	0.01% VaR
Mean	-4.7713	-5.5117
95% Confidence Interval	(-4.8109, -4.7309)	(-5.6007, -5.4241)
Panel (b): Simulation Size = 10 Million		
	0.05% VaR	0.01% VaR
Mean	-4.7723	-5.5113
95% Confidence Interval	(-4.7852, -4.7593)	(-5.5405, -5.4840)

large simulation sample rarely used in practice, the 0.01% VaR estimate could still show about \$0.06 units of variation based on its 95% confidence interval.

For the proposed conditional method, we select the change of exchange rate Y as the conditioning variable since its variance is smaller than that of the foreign currency earning X. It follows that $X|Y = y \sim N\left(\mu_{x|y}, \sigma^2_{x|y}\right)$, where $\mu_{x|y} = \frac{\rho\sigma_x}{\sigma_y} y$ and $\sigma_{x|y} = \sqrt{1-\rho^2}\sigma_x$. Armed with the conditional distribution of X, Equation (5) can be adaptd to solve for the VaRs. Nevertheless, in the case of a bivariate normal distribution for X and Y, the distribution of the structural exchange rate risk Z conditional on $Y = y$ is directly available as $N\left(\mu_{z|y}, \sigma^2_{z|y}\right)$, where $\mu_{z|y} = E_0 y + (F_0 + y)\frac{\rho\sigma_x}{\sigma_y} y$ and $\sigma_{z|y} = |F_0 + y|\sqrt{1-\rho^2}\sigma_x$. This conditional distribution of Z can be directly inserted into Equation(3) to estimate the VaRs.

To solve Equation(6), we set the convergence criterion of the equation solver to be that the left side of Equation(6) is within the range of $p\pm$1e-6. The grid of the numerical integration is selected to be from -12 to 12, which results in $\pi = 0.01\%$. That is, more than $1 - \pi = 99.99\%$ of the possible values of Y is covered by this grid. The grid step size is 0.01, which results 2,401 grid points in the numerical integration. The resulting estimate of the 0.05% and 0.01% VaRs is -\$4.7723 units and -\$5.5118 units respectively, very close to the corresponding average VaR estimates across 1,000 independent repetitions by the direct simulation method. Changing the grid step size to be 0.001 or the grid coverage probability to be $\pi = 0.001\%$ has little impact on the VaR estimates, resulting in differences within \$1e-4 units.

4 Conclusion

Given the distributions of the foreign currency earning X and the change of exchange rate Y, this paper proposes a conditional method to estimate VaRs of the structural exchange rate risk that imply extremely high survival probabilities of usually more than 99.9%. Compared with the direct simulation method that directly calculates the tail percentiles of simulations from the joint distribution of X and Y, the conditional method produces numerically more stable estimate of the extreme VaRs by solving a non-linear equation via a simple one-dimensional numerical integration. The proposed conditional method is generally applicable as long as the CDF of either X or Y conditional on the other variable has a closed-form expression. The bivariate normal and Student's t distributions fall into this category.

References

Craig, C. (1936): "On the frequency function of xy," *Annals of Mathematical Statistics*, 7, 1–15.

Jackson, P., W. Perraudin, and V. Sapporta (2004): "Regulatory and economic solvency standards for internationally active banks," *Journal of Banking and Finance*, 26, 953–976.

Kotz, S., and S. Nadarajah (2004): *Multivariate t distribution and their applications.* Cambridge University Press.

KPMG (2004): "Basel II - a closer look: managing economic capital," Discussion paper, KPMG International.

Papaioannou, M. (2006): "Exchange risk measurement and management: issues and approaches for firms," Discussion paper, International Monetary Fund, Working Paper No. WP/06/255.

Roth, M. (2013): "On the Multivariate t Distribution," Discussion paper, Department of Electrical Engineering, Linkopings Universitet, Sweden, Report No. LiTH-ISY-R-3059.

Seijas-Macias, A., and A. Oliveira (2012): "An approach to distribution of the product of two normal variables," *Discussiones Mathematicae - Probability and Statistics*, 32, 87–99.

Wallgren, C. (1980): "The distribution of the product of two correlated t variates," *Journal of the American Statistical Association*, 75, 996–1000.