Existence of Nash Equilibrium in Discontinuous Games

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November 2008

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April, 2014

Abstract

This paper offers an equilibrium existence theorem in discontinuous games. We introduce a new notion of continuity, called *quasi-weak transfer continuity* that guarantees the existence of pure strategy Nash equilibrium in compact and quasiconcave games. We also consider possible extensions and improvements of the main result. Our conditions are simple and easy to verify. We present applications to show that our conditions allow for economically meaningful payoff discontinuities.

Keywords: Discontinuous games; quasi-weak transfer continuity; various notions of transfer continuity; Nash equilibrium

1 Introduction

The concept of Nash equilibrium in Nash (1950, 1951) is probably the most important solution concept in game theory. It is immune from unilateral deviations, that is, each player has no incentive to deviate from his/her strategy given that other players do not deviate from theirs. Nash (1951) proved that a finite game has a Nash equilibrium in mixed strategies. Debreu (1952) then showed that games possess a pure strategy Nash equilibrium if (1) the strategy spaces are convex and compact, and (2) players have continuous and quasiconcave payoff functions. However, in many important economic models, such as those in Bertrand (1883), Hotelling (1929), Milgrom

*We wish to thank Philip Reny, Raluca Parvulescu, and two anonymous referees for helpful comments and suggestions, especially an advisory editor for constructive suggestions that significantly improved the exposition of the paper.

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(1985), Dasgupta and Maskin (1986), and Jackson (2009), payoffs are particularly discontinuous and/or non-quasiconcave.

Economists then seek weaker conditions that can guarantee the existence of equilibrium. Some seek to weaken the quasiconcavity of payoffs or substitute it with some forms of transitivity/monotonicity of payoffs (cf. McManus (1964), Roberts and Sonnenschein (1977), Nishimura and Friedman (1981), Topkis (1979), Vives (1990), and Milgrom and Roberts (1990)), some seek to weaken the continuity of payoff functions (cf. Dasgupta and Maskin (1986), Simon (1987), Simon and Zame (1990), Tian (1992a, 1992b, 1992c, 2009), Tian and Zhou (1992, 1995), Reny (1999, 2009), Bagh and Jofre (2006), Morgan and Scalzo (2007), Carmona (2009, 2011), and Nessah (2011)), while others seek to weaken both quasiconcavity and continuity (cf. Yao (1992), Baye *et al.* (1993), Tian (2009), McLennan *et al.* (2011), Prokopovych (2011, 2013), Barelli and Meneghel (2013), and Nessah and Tian (2009, 2013)).

This paper investigates the existence of pure strategy Nash equilibria in discontinuous games. We introduce a new notion of very weak continuity, called *quasi-weak transfer continuity*, which holds in a large class of discontinuous games. Roughly speaking, a game is quasi-weakly transfer continuous if for every nonequilibrium strategy x^*, there exists a player i, a neighborhood $\mathcal{N}$ and a securing strategy profile such that for every deviation strategy profile z in $\mathcal{N}$, agent i's payoff at securing strategy is strictly above the payoff at the local security level even if the others deviate slightly from z.

We establish that a compact, convex, quasiconcave and quasi-weakly transfer continuous game has a Nash equilibrium and show that it is unrelated to Reny (1999), Carmona (2009, 2011), Nessah (2011), Prokopovych (2011, 2013), and Barelli and Meneghel (2013). We provide sufficient conditions for quasi-weak transfer continuity such as weak transfer continuity, quasi-weak upper semicontinuity and payoff security, and transfer lower continuity and quasi upper semicontinuity. These conditions are satisfied in many economic games and are often simple to check. We also provide the existence theorems for symmetric games, and consider further extensions and improvements of our main result.

The remainder of the paper is organized as follows. In Section 2, we first introduce the notion of quasi-weak transfer continuity, and then provide the main existence result on pure strategy Nash equilibrium. We also provide examples illustrating the theorems as well as some sufficient conditions for quasi-weak transfer continuity. Section 3 considers the equilibrium existence for symmetric games. Section 4 gives some possible extensions and improvements. Section 5 presents some applications of interest to economists that illustrate the usefulness of our results. Section 6 concludes the paper. All the proofs are presented in the appendix.

2 Existence of Nash Equilibria

Consider a game in normal form: $G = (X_i,\ u_i)_{i \in I}$, where $I = \{1, ..., n\}$ is a finite set of players, X_i is player i's strategy space that is a nonempty subset of a Hausdorff locally convex topological vector space, and u_i is player i's payoff function from the set of strategy profiles $X = \prod_{i \in I} X_i$ to $\mathbb{R}$. For each player $i \in I$, denote by $-i$ all players rather than player i. Also denote by $X_{-i} = \prod_{j \neq i} X_j$ the set of strategies of the players in $-i$. Product sets are endowed with the product topology.

A game $G = (X_i,\ u_i)_{i \in I}$ is said to be *compact* if for all $i \in I$, u_i is bounded and X_i is compact. A game $G = (X_i,\ u_i)_{i \in I}$ is said to be *quasiconcave* if for every $i \in I$, X_i is convex and the function u_i is quasiconcave in x_i. A *pure strategy Nash equilibrium* of G is a strategy profile $x^* \in X$ such that $u_i(y_i, x^*_{-i}) \leq u_i(x^*)$ for $y_i \in X_i$ and all $i \in I$.

The following weak notion of continuity, *quasi-weak transfer continuity*, guarantees the existence of equilibrium in compact and quasiconcave games.

DEFINITION 2.1 A game $G = (X_i, u_i)_{i \in I}$ is said to be *quasi-weakly transfer continuous* if whenever $x \in X$ is not an equilibrium, there exists a player i, $\overline{y}_i \in X_i$, $\epsilon > 0$, and some neighborhood $\mathcal{N}_x$ of x such that for every $z \in \mathcal{N}_x$ and every neighborhood $\mathcal{N}_z \subseteq \mathcal{N}_x$ of z, $u_i(\overline{y}_i, z_{-i}) > u_i(z_i, z'_{-i}) + \epsilon$ for some $z' \in \mathcal{N}_z$.

Quasi-weak transfer continuity means that whenever x is not an equilibrium, some player i has a strategy $\overline{y}_i$ yielding a strictly large payoff at the local security level even if the others play slightly differently than at x.[1]

We then have the following main result.

THEOREM 2.1 If $G = (X_i, u_i)_{i \in I}$ is compact, quasiconcave, and quasi-weakly transfer continuous, then it possesses a pure strategy Nash equilibrium.

The proof of the theorem will be presented in the appendix. We here briefly explain why quasi-weak transfer continuity ensures the existence of pure strategy Nash equilibrium for compact and quasiconcave games. When a game fails to have a pure strategy Nash equilibrium, by quasi-weak transfer continuity, for every strategy profile x, some player i has a strategy $\overline{y}_i$ yielding a strictly large payoff at the local security level provided the others play slightly differently than at x. As such, the difference of payoffs at deviation strategy profile $(\overline{y}_i, x_{-i})$ and disequilibrium strategy x is uniformly positive. On the other hand, it can be shown that the resulting maximum value function Ψ_i of this difference is lower semicontinuous, and further, by quasiconcavity, the set of deviation strategies $\overline{y}_i$ that results in positive maximum value of the difference, i.e., the set

[1] The local security level at z means the value of the least favorable outcome in a neighborhood of z, given by $\underline{u}_i(z) \equiv \sup_{\mathcal{N}_z \subseteq \mathcal{N}} \inf_{z' \in \mathcal{N}_z} u_i(z_i, z'_{-i})$.

defined by $C_i(x) = \{y_i \in X_i : \Psi_i(x, y_i) > 0\}$ is convex for all $x \in X$, and its inverse set $C_i^{-1}(y_i) = \{x \in X : \Psi_i(x, y_i) > 0\}$ is an open set for each of such y_i. Then, by Yannelis and Prabhakar Theorem, there exists a strategy profile $\overline{x}$ such that the maximum value of the function is nonpositive at the deviation strategy $\overline{y}_i$, which is impossible.

Note that, contrary to the results of Reny (1999), Bagh and Jofre (2006), Carmona (2009, 2011), and Prokopovych (2011), which require verifying the closureness of the graph of the vector payoff function, quasi-weak transfer continuity is relatively easier to verify, requiring no analysis of any closures of high-dimensional objects.

EXAMPLE 2.1 Consider the following game with two players and the unit square $X_1 = X_2 = [0, 1]$. For player $i = 1, 2$ and $x = (x_1, x_2) \in X = [0, 1]^2$, let the payoff functions for the players be given by

$$u_i(x_1, x_2) = \begin{cases} x_i + 1, & \text{if } x_{-i} > \frac{1}{2} \\ 1, & \text{if } x_i > \frac{1}{2} \text{ and } x_{-i} = \frac{1}{2} \\ -1, & \text{if } x_i \leq \frac{1}{2} \text{ and } x_{-i} = \frac{1}{2} \\ x_i - 1, & \text{if } x_{-i} < \frac{1}{2}. \end{cases}$$

It can be verified that the game is not (generalized) better-reply secure so that Proposition 2.4 of Barelli and Meneghel (2013), Theorem 1 of Carmona (2011) and Theorem 3.1 in Reny (1999) cannot be applied. It is not generalized weakly transfer continuous so that Theorem 3.1 of Nessah (2011) cannot be used. It is neither weakly reciprocal upper semicontinuous. As such, Theorem 4 in Prokopovych (2011) and Corollary 2 in Carmona (2009) cannot be applied.

However, it is quasi-weakly transfer continuous. Indeed, let $x = (x_1, x_2)$ be a nonequilibrium strategy profile. If $(x_1, x_2) \neq (\frac{1}{2}, \frac{1}{2})$, then by nonequilibrium of x and continuity of payoffs at x, there exists a player i, $\overline{y}_i \in X_i$, $\epsilon > 0$, and some neighborhood $\mathcal{N}$ of x such that for all $z \in \mathcal{N}$, we have $u_i(\overline{y}_i, z_{-i}) - \sup_{\mathcal{N}_z \subseteq \mathcal{N}} \inf_{z' \in \mathcal{N}_z} u_i(z_i, z'_{-i}) > \epsilon$. Since $x_1 = x_2 = \frac{1}{2}$, for $i = 1$, sufficiently small $\epsilon > 0$, neighborhood $\mathcal{N} \subseteq (\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon)^2$ of $(\frac{1}{2}, \frac{1}{2})$, and $\overline{y}_i = 1$, we have

Case 1) If $z_{-i} > \frac{1}{2}$, then for every neighborhood $\mathcal{N}_z$ of z, there is $z' \in \mathcal{N}_z$ so that $z'_{-i} = z_{-i} > \frac{1}{2}$ and $u_i(\overline{y}_i, z_{-i}) = 2 > 1 + z_i + \epsilon = u_i(z_i, z'_{-i}) + \epsilon$.

Case 2) If $z_{-i} < \frac{1}{2}$, then for every neighborhood $\mathcal{N}_z$ of z, there is $z' \in \mathcal{N}_z$ so that $z'_{-i} = z_{-i} < \frac{1}{2}$ and $u_i(\overline{y}_i, z_{-i}) = 0 > z_i - 1 + \epsilon = u_i(z_i, z'_{-i}) + \epsilon$.

Case 3) If $z_{-i} = \frac{1}{2}$, then for every neighborhood $\mathcal{N}_z$ of z, there is $z' \in \mathcal{N}_z$ so that $z'_{-i} < \frac{1}{2}$ and $u_i(\overline{y}_i, z_{-i}) = 1 > z_i - 1 + \epsilon = u_i(z_i, z'_{-i}) + \epsilon$.

Since the game is compact and quasiconcave, by Theorem 2.1, it possesses a pure strategy Nash equilibrium.

Prokopovych (2013) introduced the notion of weak single deviation property that generalizes better-reply security of Reny (1999), weak transfer quasi-continuity of Nessah and Tian (2009) and single deviation property of Reny (2009) as follows: A game $G = (X_i, u_i)_{i\in I}$ has the weak single deviation property if whenever $\overline{x} \in X$ is not an equilibrium, there exists an open neighborhood $\mathcal{N}$ of $\overline{x}$, a set of players $I(\overline{x}) \subseteq I$ and a collection of deviation strategies $\{y_i(\overline{x}) \in X_i : \ i \in I(\overline{x})\}$ such that for every $z \in \mathcal{N}$ nonequilibrium, there exists a player $j \in I(\overline{x})$ satisfying $u_j(y_j(\overline{x}), z_{-j}) > u_j(z)$. He then provided a theorem (Theorem 2) that shows under the weak single deviation property and a condition (Condition (ii) in Theorem 2), there is a pure strategy Nash equilibrium in games with compact and convex strategy spaces. Notice that Condition (ii) is unrelated to quasiconcavity. Indeed, Reny (2009) constructed a counterexample where the game $G = (X_i, u_i)_{i\in I}$ is compact, quasiconcave, and has the single deviation property which implies weak single deviation property, but it may not possess a pure strategy Nash equilibrium (also see Example 2 in Prokopovych (2013)). Moreover, the following example shows that quasi-weak transfer continuity does not imply the weak single deviation property either. As such, Theorem 2.1 is unrelated to Theorem 2 in Prokopovych (2013).

EXAMPLE 2.2 Consider the following concession game with two players and the unit square $X_1 = X_2 = [0, 1]$. For player $i = 1, 2$ and $x = (x_1, x_2) \in X = [0, 1]^2$, let the payoff functions for the players be given by

$$u_i(x_1, x_2) = \begin{cases} 1, & \text{if } x_i = 0 \text{ and } x_{-i} > 0 \\ x_{-i} - x_i + 1, & \text{if } x_i < x_{-i} \text{ and } x_i > 0 \\ 3x_i, & \text{if } x_i = x_{-i} < \frac{1}{2} \\ 0, & \text{if } x_i = x_{-i} \geq \frac{1}{2} \\ x_{-i} - x_i - 1, & \text{if } x_i > x_{-i}. \end{cases}$$

It can be verified that the game does not have the weak single deviation property. Indeed, let $\overline{x} = (\frac{1}{2}, \frac{1}{2})$ be a nonequilibrium. For an open neighborhood $\mathcal{N}$ of $(\frac{1}{2}, \frac{1}{2})$, a set of players $I(\frac{1}{2}, \frac{1}{2}) \subseteq I$ and a collection of deviation strategies $\{y_i(\overline{x}) \in X_i : \ i \in I(\overline{x})\}$, we can find a nonequilibrium strategy z in $\mathcal{N}$ so that $u_j(y_j(\overline{x}), z_{-j}) \leq u_j(z)$, for each $j \in I(\overline{x})$. To see this, consider two cases:

(1) If $I(\frac{1}{2}, \frac{1}{2}) = \{i\}$ where $i = 1, 2$. Let $z \in \mathcal{N}$ so that $z_i = z_{-i} = t \neq y_i$, $t < \frac{1}{2}$ and $t > \frac{1}{2} - \frac{y_i}{2}$ if $y_i > 0$. Then

$$\begin{aligned} u_i(y_i, z_{-i}) = & \begin{cases} 1, & \text{if } y_i = 0 \\ t - y_i + 1, & \text{if } y_i < t \text{ and } y_i > 0 \\ t - y_i - 1, & \text{if } y_i > t. \end{cases} \\ & \leq 3t = u_i(z). \end{aligned}$$

(2) If $I(\frac{1}{2}, \frac{1}{2}) = I = \{1, 2\}$. Let $z \in \mathcal{N}$ so that $z_i = z_{-i} = t \neq y_i$, for each $i = 1, 2$, $t < \frac{1}{2}$ and $t > \frac{1}{2} - \frac{y_i}{2}$ if $y_i > 0$, for each $i = 1, 2$. Therefore, $u_j(z) = 3t$ for each $j = 1, 2$.

(i) If $y_1 = y_2 = 0$, then $u_j(y_j, z_{-j}) = 1 < 3t = u_j(z)$, for each $j = 1, 2$.

(ii) If $y_i = 0$ and $y_{-i} > 0$ for each $i = 1, 2$, then $u_i(y_i, z_{-i}) = 1 < 3t = u_i(z)$ and

$$\begin{aligned} u_{-i}(y_{-i}, z_i) = & \begin{cases} t - y_{-i} + 1, & \text{if } y_{-i} < t \\ t - y_{-i} - 1, & \text{if } y_{-i} > t. \end{cases} \\ & \leq 3t = u_{-i}(z). \end{aligned}$$

(iii) If $y_1 > 0$ and $y_2 > 0$, then for each $j = 1, 2$ we have

$$\begin{aligned} u_j(y_j, z_{-j}) = & \begin{cases} t - y_j + 1, & \text{if } y_j < t \\ t - y_j - 1, & \text{if } y_j > t. \end{cases} \\ & \leq 3t = u_j(z). \end{aligned}$$

However, it is quasi-weakly transfer continuous. Indeed, $\overline{x} \neq (\frac{1}{2}, \frac{1}{2})$ is obviously quasi-weakly transfer continuous. Suppose that $\overline{x} = (\frac{1}{2}, \frac{1}{2})$. Let $\epsilon > 0$ be sufficiently small, $\mathcal{N} \subset (\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon)^2$ and $\overline{y}_i = \epsilon$, for some $i = 1, 2$. For each $z \in \mathcal{N}$, we have the following three cases:

(1) If $z_i < z_{-i}$, then for every neighborhood $\mathcal{N}_z$ of z, there is $z' \in \mathcal{N}_z$ so that $z'_{-i} > z_{-i} > z_i$ and $u_i(\overline{y}_i, z_{-i}) = 1 + z_{-i} - \epsilon > 1 + z'_{-i} - z_i + \epsilon = u_i(z_i, z'_{-i}) + \epsilon$.

(2) If $z_i > z_{-i}$, then for every neighborhood $\mathcal{N}_z$ of z, there is $z' \in \mathcal{N}_z$ so that $z'_{-i} < z_{-i} < z_i$ and $u_i(\overline{y}_i, z_{-i}) = 1 + z_{-i} - \epsilon > z'_{-i} - z_i - 1 + \epsilon = u_i(z_i, z'_{-i}) + \epsilon$.

(3) If $z_i = z_{-i}$, then for every neighborhood $\mathcal{N}_z$ of z, there is $z' \in \mathcal{N}_z$ so that $z'_{-i} < z_i$ and $u_i(\overline{y}_i, z_{-i}) = 1 + z_{-i} - \epsilon > z'_{-i} - z_i - 1 + \epsilon = u_i(z_i, z'_{-i}) + \epsilon$.

While it is somewhat simple to verify quasi-weak transfer continuity, it is sometimes even simpler to verify other conditions leading to it.

DEFINITION 2.2 A game $G = (X_i, u_i)_{i \in I}$ is said to be *weakly transfer continuous* if whenever $x \in X$ is not an equilibrium, there exists a player i, $\bar{y}_i \in X_i$, $\epsilon > 0$, and some neighborhood $\mathcal{N}_x$ of x such that $u_i(\bar{y}_i, x'_{-i}) > u_i(x') + \epsilon$ for all $x' \in \mathcal{N}_x$.

DEFINITION 2.3 A game $G = (X_i,\ u_i)_{i \in I}$ is said to be *quasi upper semicontinuous* (QUSC) if for all $i \in I$, $x \in X$, and $\epsilon > 0$, there exists a neighborhood $\mathcal{N}$ of x such that for every $z \in \mathcal{N}$ and every neighborhood $\mathcal{N}_z \subseteq \mathcal{N}$ of z, $u_i(x) \geq u_i(z_i, z'_{-i}) - \epsilon$ for some $z' \in \mathcal{N}_z$.

DEFINITION 2.4 A game $G = (X_i,\ u_i)_{i \in I}$ is said to be *quasi-weakly upper semicontinuous* (QWUSC) if whenever $x \in X$ is not an equilibrium, there exists a player i, $\hat{x}_i \in X_i$, $\epsilon > 0$, and some neighborhood $\mathcal{N}_x$ of x such that for every $z \in \mathcal{N}_x$ and every neighborhood $\mathcal{N}_z \subseteq \mathcal{N}_x$ of z, $u_i(\hat{x}_i, x_{-i}) > u_i(z_i, z'_{-i}) + \epsilon$ for some $z' \in \mathcal{N}_z$.

DEFINITION 2.5 A game $G = (X_i,\ u_i)_{i \in I}$ is said to be *weakly transfer lower semicontinuous* (WTLSC) if whenever x is not a Nash equilibrium, there exists a player i, $y_i \in X_i$, $\epsilon > 0$ and some neighborhood $\mathcal{N}_x$ of x such that $u_i(y_i, z_{-i}) > u_i(x) + \epsilon$ for all $z \in \mathcal{N}_x$.

Weak transfer continuity was independently introduced in our previously circulated Nessah and Tian (2008), which is the same as the single player deviation property introduced in Prokopovych (2013). It is obvious that (1) weak transfer continuity implies quasi-weak transfer continuity, (2) upper semicontinuity implies quasi upper semicontinuity, which in turn implies quasi-weak upper semicontinuity, and (3) lower semicontinuity implies payoff security, which in turn implies weak transfer lower semicontinuity. Also, quasi-weak upper semicontinuity and transfer lower semicontinuity, when combined with payoff security[2] and quasi upper semicontinuity respectively, imply quasi-weak transfer continuity. We then have the following proposition.

PROPOSITION 2.1 *Suppose that a game G satisfies any of the following conditions:*

(a) it is weakly transfer continuous;

(b) it is quasi-weakly upper semicontinuous and payoff secure;

(c) it is weakly transfer lower semicontinuous and quasi upper semicontinuous.

Then it is quasi-weakly transfer continuous, and consequently, there exists a pure strategy Nash equilibrium provided that it is also compact and quasiconcave.

EXAMPLE 2.3 Consider the two-player game with the following payoff functions defined on $[0,1] \times [0,1]$:

$$u_i(x_1, x_2) = \begin{cases} x_i + 1 & \text{if } x_{-i} > \frac{1}{2} \\ x_i - 1 & \text{if } x_{-i} \leq \frac{1}{2}. \end{cases}$$

This game is not (generalized) better-reply secure nor (weakly) reciprocal upper semicontinuous. As such, Corollary 3.3, Corollary 3.4 of Reny (1999), Proposition 1 of Bagh and Jofre (2006), and Theorem 4 in Prokopovych (2011) cannot be applied.

However, the game is payoff secure and quasi-weakly upper semicontinuous. To see this, let $i \in I$, $\epsilon > 0$, and $x \in X$. If $x_{-i} \neq \frac{1}{2}$, then it is clear that there exists a strategy $y_i = 1$ and some neighborhood $\mathcal{V}$ of x_{-i} such that $u_i(y_i, z_{-i}) \geq u_i(x) - \epsilon$ for each $z_{-i} \in \mathcal{V}$. If $x_{-i} = \frac{1}{2}$, then there exists a strategy $y_i = 1$ and some neighborhood $\mathcal{V}$ of x_{-i} such that $u_i(x) - \epsilon = x_i - 1 - \epsilon \leq 0 \leq u_i(y_i, z_{-i})$ for each $z_{-i} \in \mathcal{V}$. Thus, the game is payoff secure.

Also, let $x = (x_1, x_2)$ be a nonequilibrium strategy profile. Then there exists a player i such that $x_i < 1$. Let $x_i + 2\epsilon < 1$ for some $\epsilon > 0$. If $x_{-i} \neq \frac{1}{2}$, then it is clear that there exists a strategy

[2] A game is payoff secure if for every $x \in X$, every $\epsilon > 0$, and every player i, respectively, there exists $\overline{x}_i \in X_i$ such that $u_i(\overline{x}_i, z_{-i}) \geq u_i(x) - \epsilon$ for all z_{-i} in some open neighborhood of x_{-i}.

$y_i = 1$ and some neighborhood $\mathcal{V} \subseteq (x_i - \epsilon, x_i + \epsilon) \times [0,1]$ of x such that for every $z \in \mathcal{V}$ and every neighborhood $\mathcal{V}_z$ of z, $u_i(y_i, x_{-i}) > u_i(z_i, z'_{-i}) + \epsilon$ for some $z' \in \mathcal{V}_z$. If $x_{-i} = \frac{1}{2}$, then there exists a strategy $y_i = 1$ and some neighborhood $\mathcal{V} \subseteq (x_i - \epsilon, x_i + \epsilon) \times [0,1]$ of x such that for each $z \in \mathcal{V}$ and every neighborhood $\mathcal{V}_z$ of z, there exists $z' \in \mathcal{V}_z$ with $z'_{-i} = \frac{1}{2}$ so that $u_i(z_i, z'_{-i}) + \epsilon = z_i - 1 + \epsilon \leq x_i + 2\epsilon - 1 < u_i(y_i, x_{-i}) = 0$. Thus, it is quasi-weakly upper semicontinuous. Since the game is also compact and quasiconcave, then by Proposition 2.1.(b), it possesses a Nash equilibrium.

EXAMPLE 2.4 Consider the two-player game with the following payoff functions defined on $[-1,1] \times [-1,1]$ by

$$u_i(x_1, x_2) = \begin{cases} x_i + 1 & \text{if } x_{-i} > 0 \\ x_i & \text{if } x_{-i} = 0 \\ x_i - \frac{1}{2} & \text{if } x_{-i} < 0. \end{cases}$$

This game is not generalized better-reply secure. However, it is clearly weakly transfer lower continuous. To see that it is also quasi upper semicontinuous, consider a player i, a strategy x and $\epsilon > 0$. If $x_{-i} \neq 0$, it is obvious that the game is quasi upper semicontinuous. If $x_{-i} = 0$, then there exists a neighborhood $\mathcal{N} \subseteq (x_i - \delta, x_i + \delta] \times (-\delta, \delta)$ of x (with $\delta < \frac{1}{2}$) such that for each $z \in \mathcal{N}$ and each $\mathcal{N}_z$ as a neighborhood of z, there exists $z' \in \mathcal{N}_z$ with $z'_{-i} < 0$ so as $u_i(x) = x_i \geq (x_i + \delta) - \frac{1}{2} - \epsilon \geq u_i(z_i, z'_{-i}) - \epsilon$, which means it is also quasi upper semicontinuous at $x_{-i} = 0$. Since the game is also compact and quasiconcave, then by Proposition 2.1.(c), it possesses a Nash equilibrium.

3 Pure Strategy Symmetric Nash Equilibrium

In this section, it is assumed that $G = (X_i, u_i)_{i \in I}$ is a quasi-symmetric game, *i.e.*, $Z = X_1 = ... = X_n$ and $u_1(x, y, ..., y) = u_2(y, x, y, ..., y) = u_n(y, ..., y, x)$ for all $x, y \in Z$. Recall that a Nash equilibrium $(\overline{x}_1, ..., \overline{x}_n)$ is symmetric if $\overline{x}_1 = \overline{x}_2 = ... = \overline{x}_n$.

DEFINITION 3.1 A symmetric game $G = (X_i, u_i)_{i \in I}$ is said to be *diagonally quasi-weak transfer continuous* if whenever $(x^*, ..., x^*) \in X^n$ is not an equilibrium, there exists a player i, strategy $\overline{y} \in X$, $\epsilon > 0$, and neighborhood $\mathcal{N}$ of x^* such that for all $z^1, z^2 \in \mathcal{N}$ and every neighborhood $\mathcal{N}_{(z^1,z^2)} \subseteq \mathcal{N}$ that contains z^1 and z^2, $u_i(z^1, ...z^1, \overline{y}, z^1, ..., z^1) > u_i(z', ..., z', z^2, z', ..., z') + \epsilon$ for some $z' \in \mathcal{N}_{(z^1,z^2)}$.

We then have the following existence theorem for quasi-symmetric games.

THEOREM 3.1 Suppose that $G = (X_i, u_i)_{i \in I}$ is quasi-symmetric, compact, quasiconcave, and diagonally quasi-weak transfer continuous. Then it has a symmetric pure strategy Nash equilibrium.

The following example illustrates Theorem 3.1.

EXAMPLE 3.1 Consider a timing game between two players on the unit square $X_1 = X_2 = [0,1]$ studied by Prokopovych (2013). For player $i = 1,2$ and $x = (x_1, x_2) \in X = [0,1]^2$, let the payoff functions for the players be given by

$$u_i(x_1,x_2) = \begin{cases} 2, & \text{if } x_i < x_{-i} \\ 2, & \text{if } x_i = x_{-i} < \frac{1}{2} \\ 0, & \text{if } x_i = x_{-i} \geq \frac{1}{2} \\ -2, & \text{if } x_i > x_{-i}. \end{cases}$$

It can be verified that the game is not diagonally better-reply secure so that Theorem 4.1 in Reny (1999) cannot be applied. This game is not generalized weakly transfer continuous nor weakly reciprocal upper semicontinuous so that the results in Nessah (2011), Prokopovych (2011) and Carmona (2009) cannot be applied.

However, it is diagonally quasi-weak transfer continuous. Indeed, let (x,x) be a nonequilibrium strategy profile. By nonequilibrium of (x,x), we have $\frac{1}{2} \leq x \leq 1$. Then, there exists a player $i = 1$, $\epsilon = 1$, some neighborhood $\mathcal{N} \subseteq (\epsilon, 1)$ of x and $\overline{y} = 0$ such that for all $z^1, z^2 \in \mathcal{N}$, we have $u_i(\overline{y}, z^1) = 2$. For each neighborhood $\mathcal{N}_{(z^1,z^2)}$ that contains z^1 and z^2, there exists a $z' \in \mathcal{N}_{(z^1,z^2)}$ with $z' < z^2$ such that $u_i(z^2, z') = -2$. Thus $u_i(\overline{y}, z^1) - \sup_{\mathcal{N}_{(z^1,z^2)} \subseteq \mathcal{N}} \inf_{z' \in \mathcal{N}_{(z^1,z^2)}} u_i(z^2, z') \geq \epsilon$ for all $z^1, z^2 \in \mathcal{N}$. Since the game is also quasi-symmetric, compact and quasiconcave, by Theorem 3.1, it possesses a symmetric Nash equilibrium.

Quasiconcavity is still a strong assumption for many economic games. For instance, the classic Bertrand model typically results in nonquasiconcave and discontinuous payoffs. Thus, a general existence result for nonquasiconcave and discontinuous games is called for. In the following, we provide an existence result for general nonquasiconcave and discontinuous games. First, recall the following definition of diagonal transfer quasiconcavity introduced by Baye *et al.* (1993).

DEFINITION 3.2 A symmetric game $G = (X_i, u_i)_{i \in I}$ is said to be *diagonally transfer quasiconcave* if X is convex, and for every player i, any finite subset of $\{y^1, y^2, ..., y^m\} \subseteq X$, there is a corresponding finite subset $\{x^1, ..., x^m\} \subseteq X$ such that for any subset J of $\{1, ..., m\}$ and every $\overline{x} \in \text{co}\{x^j,\ j \in J\}$, we have

$$u_i(\overline{x}, ..., \overline{x}) \geq \min_{k \in J} u_i(\overline{x}, ..., \overline{x}, y^k, \overline{x}, ..., \overline{x}).$$

While diagonal transfer quasiconcavity is weaker than diagonal quasiconcavity,[3] and consequently weaker than quasiconcavity, the following notion of diagonal weak transfer continuity is stronger than the diagonal quasi-weak transfer continuity.

[3]A game $G = (X_i, u_i)_{i \in I}$ is said to be *diagonally quasiconcave* if X is convex, and for every player i, all $x^1, ..., x^m \in X$ and all $\overline{x} \in co\{x^1, ..., x^m\}$, $u_i(\overline{x}, ..., \overline{x}) \geq \min_{k=1,...,m} u_i(\overline{x}, ..., \overline{x}, x^k, \overline{x}, ..., \overline{x})$.

DEFINITION 3.3 A symmetric game $G = (X_i, u_i)_{i \in I}$ is said to be *diagonally weak transfer continuous* if whenever $(x^*, ..., x^*) \in X^n$ is not an equilibrium, there exists a player i, strategy $\overline{y} \in X$, $\epsilon > 0$, and neighborhood $\mathcal{N}$ of x^* such that $u_i(z, ...z, \overline{y}, z, ..., z) > u_i(z, ..., z) + \epsilon$ for all $z \in \mathcal{N}$.

We now state an existence result for nonquasiconcave and discontinuous games.

THEOREM 3.2 Suppose that $G = (X_i, u_i)_{i \in I}$ is quasi-symmetric, compact, diagonally transfer quasiconcave[4], and diagonally weak transfer continuous. Then it has a symmetric pure strategy Nash equilibrium.

As an illustration, we will use the Bertrand model to show the usefulness of Theorem 3.2 in the following example.

EXAMPLE 3.2 Consider a quasi-symmetric two-player Bertrand price competition game on the square $[0, a] \times [0, a]$ with $a > 0$. Assume that the demand function is discontinuous and is defined by

$$D_i(p_i, p_{-i}) = \begin{cases} \alpha f(p_i) & \text{if } p_i < p_{-i} \\ \beta f(p_i) & \text{if } p_i = p_{-i} \\ \gamma f(p_i) & \text{if } p_i > p_{-i} \end{cases}$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous and nonincreasing function and $\alpha > \beta > \gamma \geq 0$. Suppose that the total cost of production is zero for each firm. Then, the payoff function for each firm i becomes

$$\pi_i(p_i, p_{-i}) = \begin{cases} \alpha p_i f(p_i) & \text{if } p_i < p_{-i} \\ \beta p_i f(p_i) & \text{if } p_i = p_{-i} \\ \gamma p_i f(p_i) & \text{if } p_i > p_{-i} \end{cases}.$$

The game is quasi-symmetric and compact. Since $\alpha > \beta > \gamma$ and f is nonincreasing, it is clearly diagonally quasiconcave. Note that the set of discontinuity points is given by $A = \{(p_1, p_2) \colon p_1 = p_2\}$. Let (p, p) be any nonequilibrium strategy. Then obviously there is $q_i \neq p$ so that $\pi_i(q_i, p) > \pi_i(p, p)$ (i.e., $(q_i, p) \notin A$). Choose $\epsilon > 0$ so as $\pi_i(q_i, p) > \pi_i(p, p) + 3\epsilon$. Thus there exists a neighborhood $\mathcal{N}$ of p with $q_i \notin \mathcal{N}$ such that $\pi_i(q_i, p) - \epsilon \leq \pi_i(q_i, p')$ for all $p' \in \mathcal{N}$ by the continuity of f. We also have $\pi_i(p, p) + \epsilon \geq \pi_i(p', p')$ for all $p' \in \mathcal{N}$. Therefore, $\pi_i(q_i, p') > \pi_i(p', p') + \epsilon$ for every $p' \in \mathcal{N}$. Then, this game is diagonally weak transfer continuous and by Theorem 3.2, it possesses a symmetric pure strategy Nash equilibrium.

[4]The converse holds as well: if G has a Nash equilibrium, then G is diagonally transfer quasiconcave.

Similar to the previous section, we can provide some sets of sufficient conditions for diagonal quasi-weak transfer continuity by introducing various notions of diagonal upper/lower semicontinuity, such as diagonal quasi-weak upper semicontinuity, diagonal quasi upper semicontinuity, and diagonal weak transfer lower semicontinuity.[5]

4 Further Extensions and Improvements

We can further improve our main result.

DEFINITION 4.1 A game $G = (X_i, u_i)_{i \in I}$ is said to be *pseudo quasi-weakly transfer continuous* if whenever $x^* \in X$ is not an equilibrium, there exists an $\epsilon > 0$, a neighborhood $\mathcal{N}$ of x^*, a player i, and a strategy $\overline{y}_i \in X_i$ such that for all $z \in \mathcal{N}$, $u_i(\overline{y}_i, z_{-i}) > u_i(z_i, z'_{-i}) + \epsilon$ for some $z' \in \mathcal{N}$.

The difference between pseudo quasi-weak transfer continuity and quasi-weak transfer continuity is that the former takes the neighborhood $\mathcal{N}_z$ equal to $\mathcal{N}_x$ so that quasi-weak transfer continuity implies pseudo quasi-weak transfer continuity.

For each player $i \in I$, define a function $\digamma_i : X \times X_i \to \mathbb{R}$ by

$$\digamma_i(x, y_i) = \sup_{\mathcal{N} \in \Omega(x)} \inf_{z \in \mathcal{N}} \left[u_i(y_i, z_{-i}) - \inf_{z' \in \mathcal{N}} u_i(z_i, z'_{-i}) \right]$$

where $\Omega(x)$ is the set of all open neighborhoods $\mathcal{N}$ of x. We then have the following result.

THEOREM 4.1 Suppose that the game $G = (X_i, u_i)_{i \in I}$ is compact and pseudo quasi-weakly transfer continuous. If $\digamma_i$ is quasiconcave in y_i for all i, then the game G possesses a pure strategy Nash equilibrium.

The following proposition shows that pseudo quasi-weak transfer continuity is also weaker than better-reply security, and consequently Theorem 4.1 extends Theorem 3.1 in Reny (1999) by weakening better-reply security.

PROPOSITION 4.1 *If $G = (X_i, u_i)_{i \in I}$ is better-reply secure, then it is pseudo quasi-weakly transfer continuous.*

While in terms of continuity, Theorem 4.1 is more interesting than Theorem 2.1 as well as Theorem 3.1 in Reny (1999), the quasiconcavity of $\digamma_i(x, .)$ is more complicated to check. A question is then whether quasiconcavity of $\digamma_i(x, y_i)$ in y_i can be replaced by quasiconcavity of $u_i(x_i, x_{-i})$ in x_i. Unfortunately, the answer is negative.[6]

[5] See Nessah and Tian (2009).

[6] We would like to thank an anonymous referee for Example 4.1.

EXAMPLE 4.1 Consider, on the unit square, the following game that has no pure strategy Nash equilibrium.

$$u_1(x) = \begin{cases} x_1, & \text{if } x_2 = 0 \\ 2 - x_1, & \text{otherwise} \end{cases}$$

$$u_2(x) = \begin{cases} x_2, & \text{if } x_1 = 1 \\ 2 - x_2, & \text{otherwise} \end{cases}$$

The considered game is pseudo quasi-weakly transfer continuous and quasiconcave, but in general the function $\digamma_1(x, y_1)$ is not quasiconcave in y_1. Indeed, for $x = (\frac{1}{2}, \frac{1}{2})$,

$$\digamma_1((\frac{1}{2}, \frac{1}{2}), y_1) = \max(y_1 - \frac{1}{2}, \frac{1}{2} - y_1)$$

is not quasiconcave in y_1.

Our main result can be further improved by introducing the following notions of transfer quasi-continuity and strong diagonal transfer quasiconcavity.

DEFINITION 4.2 A game $G = (X_i, u_i)_{i \in I}$ is said to be *weakly transfer quasi-continuous* if whenever $x \in X$ is not an equilibrium, there exists a $y \in X$ and a neighborhood $\mathcal{N}_x$ of x such that for every $x' \in \mathcal{N}_x$, there exists a player i satisfying $u_i(y_i, x'_{-i}) > u_i(x')$.

Weak transfer quasi-continuity, which was independently introduced in our previously circulated Nessah and Tian (2008) and also called *single-deviation property* in Reny (2009), only requires that each strategy profile in a neighborhood of x be upset by one, but not all players. Thus, it is a very weak notion of continuity so that it is a form of *quasi-continuity*.

DEFINITION 4.3 A game $G = (X_i, u_i)_{i \in I}$ is said to be *strongly diagonal transfer quasiconcave* if for any finite subset $\{y^1, ..., y^m\} \subseteq X$, there exists a corresponding finite subset $\{x^1, ..., x^m\} \subseteq X$ such that for any subset $\{x^{k^1}, x^{k^2}, ..., x^{k^s}\} \subseteq \{x^{k^1}, x^{k^2}, ..., x^{k^s}\}$, $1 \leq s \leq m$, and any $x \in co\{x^{k^1}, x^{k^2}, ..., x^{k^s}\}$, there exists $y \in \{y^{k^1}, ..., y^{k^s}\}$ so that

$$u_i(y_i, x_{-i}) \leq u_i(x) \quad \forall i \in I. \tag{4.1}$$

It is clear that a game is diagonally transfer quasiconcave if it is strongly diagonal transfer quasiconcave.[7] We then have the following result that generalizes Theorem 2 in Prokopovych (2013).

THEOREM 4.2 *Suppose that a game $G = (X_i, u_i)_{i \in I}$ is convex, compact, weakly transfer quasi-continuous. Then, the game possesses a pure strategy Nash equilibrium if and only if it is strongly diagonal transfer quasiconcave.*

[7]Indeed, summing up (4.1) and denoting $U(x, y) = \sum_{i \in I} u_i(y_i, x_i)$, we have $\min_{1 \leq l \leq s} U(x, y^{k^l}) \leq U(x, x)$, which is the condition for diagonal transfer quasiconcavity.

It may be remarked that weak transfer quasi-continuity and quasi-weak transfer continuity are not implied by nor imply each other. The game considered in Example 2.2 is quasi-weakly transfer continuous, but it does not have the weak single-deviation property which in turn does not satisfy the single-deviation property/weak transfer quasi-continuity. On the other hand, the game in Example 3.1 in Reny (2009) is weakly transfer quasi-continuous, but it is not quasi-weakly transfer continuous.

While weak transfer quasi-continuity in Theorem 4.2 is weaker than the better-reply security and diagonal transfer continuity, it requires that the game be strongly diagonal transfer quasiconcave. Can strong diagonal transfer quasiconcavity in Theorem 4.2 be replaced by conventional quasiconcavity? Unfortunately, the answer is no. Reny (2009) showed this by giving a counterexample (Example 3.1 in his paper) where a game $G = (X_i, u_i)_{i \in I}$ is compact, quasiconcave, and weakly transfer quasi-continuous, but it may not possess a pure strategy Nash equilibrium.

Thus, Theorems 3.2 and 4.2 both show that there is a trade-off between continuity condition and quasiconcavity condition.

While Theorems 4.1 and 4.2 are not strict generalization of Theorem 2.1, we now introduce a result that strictly generalizes Theorem 2.1 in games with discontinuous and nonquasiconcave payoffs.

For each set B, denote by co$B$ the convex hull of B. Let $\Omega(x)$ be the set of all open neighborhoods $\mathcal{N}$ of x. For each player $i \in I$ and every $(x, y_i) \in X \times X_i$, define the following function

$$\Psi_i(x, y_i) = \sup_{\mathcal{N} \in \Omega(x)} \inf_{z \in \mathcal{N}} \left[u_i(y_i, z_{-i}) - \sup_{\mathcal{N}_z \subseteq \mathcal{N}} \inf_{z' \in \mathcal{N}_z} u_i(z_i, z'_{-i}) \right]$$

where $\mathcal{N}_z$ is a neighborhood of z.

Definition 4.4 A game $G = (X_i, u_i)_{i \in I}$ is said to be Ψ-*correspondence transfer continuous* if whenever $\overline{x} \in X$ is not an equilibrium, there exists an open neighborhood $\mathcal{N}$ of $\overline{x}$ and a well-behaved correspondence[8] $\phi : \mathcal{N} \rightrightarrows X$ such that for every $z \in \mathcal{N}$, there exists a player j so as

$$z_j \notin co\{t_j \in X_j : \Psi_j(z, t_j) \geq \alpha_j\},$$

where $\alpha_j \leq \inf_{(x,y_j) \in \text{Graph}(\phi_j)} \Psi_j(x, y_j)$.

Remark 4.1 By the same method, we can extend Definitions 3.1, 3.3, and 4.1.

Remark 4.2 If the game G is quasiconcave, then the condition $z_j \notin co\{t_j \in X_j : \Psi_j(z, t_j) \geq \alpha_j\}$ becomes $\Psi_j(x, y_j) > \Psi_j(z, z_j)$, for each $(x, y_j) \in \text{Graph}(\phi_j)$.

[8] C is said to be a well-behaved correspondence if it is upper hemicontinuous with nonempty, convex and closed values and for each $x \in X$, $C(x)$ has the following form $C(x) = (C_1(x), ..., C_n(x))$.

We have the following theorem which is a strict generalization of Theorem 2.1.

THEOREM 4.3 Suppose that $G = (X_i, u_i)_{i \in I}$ is compact and convex, then G has a pure strategy Nash equilibrium provided it is Ψ-correspondence transfer continuous.

REMARK 4.3 The considered game in Example 2.2 is Ψ-correspondence transfer continuous and consequently by Theorem 4.3 it has a Nash equilibrium.

5 Applications

In this section we show how our main existence results are applied to some important economic games. We provide two applications: one is the shared resource games that is intensively studied by Rothstein (2007), and the other is the classic Bertrand price competition games studied first by Bertrand (1883).

5.1 The Shared Resource Games

The shared resource games that usually result in discontinuous payoffs include a wide class of games such as the canonical game of fiscal competition for mobile capital. In these games, players compete for a share of a resource that is in fixed total supply, except perhaps at certain joint strategies. Each player's payoff depends on her opponents' strategies only through the effect those strategies have on the amount of the shared resource that the player obtains. As Rothstein (2007) argued, when ad valorem taxes instead of unit taxes are adopted and the aggregate amount of mobile capital is fixed instead of variable, it will typically result in at least one, and possibly many, discontinuity points.

Formally, for such a game $G = (X_i, u_i)_{i \in I}$, each player i has a convex and compact strategy space $X_i \subset \mathbb{R}^l$ and a payoff function u_i that depends on other players' strategies only through the *sharing rule* defined by $S_i : X \to [0, \overline{s}]$ with $\overline{s} \in (0, +\infty)$. That is to say, each player has a payoff function $u_i : X \to \mathbb{R}$ with the form $u_i(x_i, x_{-i}) = F_i[x_i, S_i(x_i, x_{-i})]$ where $F_i : X_i \times [0, \overline{s}] \to \mathbb{R}$ and u_i is bounded.[9]

Let $D_i \subseteq X$ be the set of joint strategies at which S_i is discontinuous and the set $\Delta = \bigcup\limits_{i \in I} D_i$ be all of the joint strategies at which one or more of the sharing rules are discontinuous. The set $X \backslash \Delta$ is then all of the joint strategies at which the sharing rules are continuous.

Rothstein (2007) showed a shared resource game possesses a pure strategy Nash equilibrium if the following conditions are satisfied: (1) X is compact and convex; (2) u_i is continuous on X and quasiconcave in x_i; (3) S_i satisfies: (3.i) $\sum_{i=1}^{n} S_i(x) = \bar{s}$ for all $x \in X \setminus \Delta$; (3.ii) there exists $\underline{s} \in [0, \bar{s}]$ such that $\sum_{i=1}^{n} S_i(x) = \underline{s}$ for all $x \in \Delta$; (3.iii) for all i, $(x_i, x_{-i}) \in D_i$ and

[9] For more details on this model, see Rothstein (2007).

every neighborhood $\mathcal{V}(x_i)$ of x_i, there exists $x_i' \in \mathcal{V}(x_i)$ such that $(x_i', x_{-i}) \in X \setminus D_i$; (3.iv) there exists a constant $\tilde{s}_i$ satisfying $\bar{s} \geq \tilde{s}_i > \bar{s}/n$ such that for all i, all $(x_i, x_{-i}) \in \Delta$, and all $(x_i', x_{-i}) \in X \setminus D_i$, $S_i(x_i', x_{-i}) \geq \tilde{s}_i \geq S_i(x_i, x_{-i})$; (4) F_i is continuous, nondecreasing in s_i, and satisfies $\max\limits_{x_i \in X_i} F_i(x_i, s_i) > \max\limits_{x_i \in X_i} F_i(x_i, \bar{s}/n)$ for any $s_i > \bar{s}/n$.

In the following, we will give an existence result with much simpler conditions and its proof is also much easier:

Assumption 1: The game is compact and quasiconcave.

Assumption 2: If $(y_i, x_{-i}) \in D_i$ and $F_i(y_i, S_i(y_i, x_{-i})) > F_i(x_i, S_i(x))$ for player i, then there exists some player $j \in I$ and $\bar{y}_j$ such that $(\bar{y}_j, x_{-j}) \in X \backslash D_j$ and $F_j(\bar{y}_j, S_j(\bar{y}_j, x_{-j})) > F_j(x_j, S_j(x))$.

Assumption 3: If $(y_i, x_{-i}) \in X \backslash D_i$ and $F_i(y_i, S_i(y_i, x_{-i})) > F_i(x_i, S_i(x))$ for player i, then there exists a player $j \in I$, a deviation strategy profile $\bar{y}_i$, $\epsilon > 0$, and a neighborhood $\mathcal{N}_x$ of x such that for every $z \in \mathcal{N}_x$ and every neighborhood $\mathcal{N}_z \subseteq \mathcal{N}_x$ of z, $F_j(\bar{y}_j, S_j(\bar{y}_j, z_{-j})) > F_j(z_j, S_j(z_j, z'_{-j})) + \epsilon$ for some $z' \in \mathcal{N}_z$.

Assumption 1 is standard. A well-known sufficient condition for a composite function $u_i = F_i[x_i, S_i(x_i, x_{-i})]$ to be quasiconcave is that F_i is quasiconcave and nondecreasing in s_i, and S_i is concave. Assumption 2 means that if x is not an equilibrium and can be improved at a discontinuous strategy profile (y_i, x_{-i}) when player i uses the deviation strategy y_i, then there exists a player j such that it must also be improved by a continuous strategy profile $(\bar{y}_j, x_{-j})$ when player j uses the deviation strategy $\bar{y}_j$. Assumption 3 means that if a strategy profile x is not an equilibrium and can be improved by a continuous strategy profile (y_i, x_{-i}) when player i uses a deviation strategy y_i, then there exists a securing strategy profile $\bar{y}$ and a neighborhood of x such that all points in the neighborhood cannot be equilibria. We then have the following result.

PROPOSITION 5.1 *A shared resource game possesses a pure strategy Nash equilibrium if it satisfies Assumptions 1-3.*

5.2 The Bertrand Price Competition Games

It is well known that Bertrand competition typically results in discontinuous and nonquasiconcave payoffs. It is a normal form game in which each of $n \geq 2$ firms, $i = 1, 2, ..., n$, simultaneously sets a price $p_i \in P = [0, \overline{p}]$. Under the assumption of profit maximization, the payoff to each firm i is

$$\pi_i(p_i, p_{-i}) = p_i D_i(p_i, p_{-i}) - C_i(D_i(p_i, p_{-i})),$$

where p_{-i} denotes the vector of prices charged by all firms other than i, $D_i(p_i, p_{-i})$ represents the total demand for firm i's product at prices (p_i, p_{-i}), and $C_i(D_i(p_i, p_{-i}))$ is firm i's total cost of producing the output $D_i(p_i, p_{-i})$. A Bertrand equilibrium is a Nash equilibrium of this game.

Let $A_i \subseteq P^n$ be the set of joint strategies at which π_i is discontinuous, $\Delta = \bigcup_{i \in I} A_i$ be the set of all of the joint strategies at which one or more of the payoffs are discontinuous, and $X \backslash \Delta$ be the set of all joint strategies at which all of the payoffs are continuous.

We make the following assumptions:

Assumption $1'$**:** The game is compact, convex, and quasiconcave.

Assumption $2'$**:** If $(q_i, p_{-i}) \in A_i$ and $\pi_i(q_i, p_{-i}) > \pi_i(p)$ for $i \in I$, then there exists a firm $j \in I$, and $\bar{q}_j$ such that $(\bar{q}_j, p_{-i}) \in P^n \backslash A_j$ and $\pi_j(\bar{q}_j, p_{-i}) > \pi_i(p)$.

Assumption $3'$**:** If $(q_i, p_{-i}) \in P^n \backslash A_i$ and $\pi_i(q_i, p_{-i}) > \pi_i(p)$ for player i, then there exists a player $j \in I$, $\epsilon > 0$, a deviation strategy profile $\bar{q}_j$ and a neighborhood $\mathcal{N}_p$ of p such that for every $r \in \mathcal{N}_p$, every neighborhood $\mathcal{N}_r \subseteq \mathcal{N}_p$ of r, $\pi_j(\bar{q}_j, r_{-i}) > \pi_j(r_j, r'_{-j}) + \epsilon$, for some $r' \in \mathcal{N}_r$.

We then have the following result.

PROPOSITION 5.2 *Each Bertrand price competition game has a pure strategy Nash equilibrium if it satisfies Assumptions* $1'$-$3'$.

6 Conclusion

In this paper, we investigate the existence of Nash equilibria in games that may be discontinuous and/or nonquasiconcave. We offer new existence results on Nash equilibrium for a large class of discontinuous games by introducing new notions of weak continuity, such as quasi-weak transfer continuity, pseudo quasi-weak transfer continuity, weak transfer quasi-continuity, etc. Our equilibrium existence results neither imply nor are implied by those results in Baye *et al.* (1993), Reny (1999), Carmona (2009, 2011), and Nessah (2011).

These results permit us to significantly weaken the continuity conditions for the existence of Nash equilibria. We also provide examples and economic applications where our general results are applicable. Although some work has been done for seeking necessary and sufficient conditions for the existence of equilibrium, such as those in Tian (2009), McLennan *et al.* (2011), and Barelli and Meneghel (2013), these full characterization results mainly show what is possible for the existence of equilibrium, yet the conditions are more complicated. As such, they may be harder to verify.

The approach developed in the paper can be similarly used to study the existence of mixed strategy Nash and Bayesian Nash equilibria in general discontinuous games. For details, see our earlier version of this paper (cf. Nessah and Tian (2009)).

Appendix

PROOF OF THEOREM 2.1. The proof of Theorem 2.1 is divided into three steps. In the first step, we construct for each player i an approximation function $\Psi_i(x, y_i)$ defined on $X \times X_i$, which is lower semicontinuous in x. Second step shows that if the game is quasi-weakly transfer continuous and $\sup_{y_i \in X_i} \Psi_i(\overline{x}, y_i) \leq 0$ for each i, then $\overline{x}$ is a Nash equilibrium. Therefore in the third step, we need only to find a strategy profile which satisfies the maximum of function Ψ. For this, based on function Ψ, we can construct a correspondence C and show that it is convex valued, $x \notin C(x)$ for each $x \in X$, and C has the lower open section. Then, by Yannelis and Prabhakar Theorem, there exists $\overline{x} \in X$ such that $\sup_{y_i \in X_i} \Psi_i(\overline{x}, y_i) \leq 0$.

Step I: Construction. Let $\Omega(x)$ be the set of all open neighborhoods $\mathcal{N}$ of x. For each player $i \in I$ and every $(x, y_i) \in X \times X_i$, define the following function

$$\Psi_i(x, y_i) = \sup_{\mathcal{N} \in \Omega(x)} \inf_{z \in \mathcal{N}} \left[u_i(y_i, z_{-i}) - \sup_{\mathcal{N}_z \subseteq \mathcal{N}} \inf_{z' \in \mathcal{N}_z} u_i(z_i, z'_{-i}) \right]$$

where $\mathcal{N}_z$ is a neighborhood of z. For each $x \in X$, we have $\Psi_i(x, x_i) \leq 0$. Indeed, if $\Psi_i(x, x_i) > 0$ for some $i \in I$ and $x \in X$, choose $\epsilon > 0$ with $\Psi_i(x, x_i) > 2\epsilon$, then there exists a neighborhood $\mathcal{N}$ of x such that for each $z = (x_i, z_{-i}) \in \mathcal{N}$, we have $u_i(x_i, z_{-i}) > \sup_{\mathcal{N}_z \subseteq \mathcal{N}} \inf_{z' \in \mathcal{N}_z} u_i(x_i, z'_{-i}) + \epsilon$. Then, for $\mathcal{N}_z = \mathcal{N}$, we have $u_i(x_i, z_{-i}) > \inf_{z' \in \mathcal{N}} u_i(x_i, z'_{-i}) + \epsilon$ for each $z = (x_i, z_{-i}) \in \mathcal{N}$. Moreover, there exists $z \in \mathcal{N}$ such that $u_i(x_i, z_{-i}) \leq \inf_{z' \in \mathcal{N}} u_i(x_i, z'_{-i}) + \frac{\epsilon}{2}$. Therefore, $\inf_{z' \in \mathcal{N}} u_i(x_i, z'_{-i}) + \epsilon < u_i(x_i, z_{-i}) \leq \inf_{z' \in \mathcal{N}} u_i(x_i, z'_{-i}) + \frac{\epsilon}{2}$; *i.e.*, $\epsilon < 0$, which is impossible.

For each i and every $y_i \in X_i$, the function $\Psi_i(., y_i)$ is real-valued by boundedness of payoff function. Let us consider the following function $g^i_{\mathcal{N}}(., y_i)$ defined by

$$g^i_{\mathcal{N}}(x, y_i) = \begin{cases} \inf_{z \in \mathcal{N}} [u_i(y_i, z_{-i}) - \sup_{\mathcal{N}_z \subseteq \mathcal{N}} \inf_{z' \in \mathcal{N}_z} u_i(z_i, z'_{-i})] & \text{if } x \in \mathcal{N} \\ -\infty & \text{if } x \notin \mathcal{N} \end{cases}$$

where $i \in I$, $y_i \in X_i$ and $\mathcal{N}$ is an open neighborhood. The function $g^i_{\mathcal{N}}(., y_i)$ is lower semicontinuous on X. Since the function $\Psi_i(., y_i)$ is the pointwise supremum of a collection of lower semicontinuous functions on X, then $\Psi_i(., y_i)$ is lower semicontinuous on X (Lemma 2.41, page 43 in Aliprantis and Border (2006)).

Step II: Detection of Nash Equilibria. If there exists a point $\overline{x} \in X$ such that

$$\Psi_i(\overline{x}, y_i) \leq 0, \text{ for all } i \in I \text{ and } y_i \in X_i, \tag{6.1}$$

then $\overline{x}$ is a Nash equilibrium. Indeed, if $\overline{x}$ is not a Nash equilibrium and since the game G is quasi-weakly transfer continuous, then there exists a player i, strategy $\overline{y}_i$, $\epsilon > 0$, and neighborhood $\mathcal{N}$ of $\overline{x}$ such that for every $z \in \mathcal{N}$ and every neighborhood $\mathcal{N}_z \subseteq \mathcal{N}$ of z, $u_i(\overline{y}_i, z_{-i}) > u_i(z_i, z'_{-i}) + \epsilon$

for some $z' \in \mathcal{N}_z$. Then, for each $z \in \mathcal{N}$, we have $u_i(\overline{y}_i, z_{-i}) - \sup_{\mathcal{N}_z \subseteq \mathcal{N}} \inf_{z' \in \mathcal{N}_z} u_i(z_i, z'_{-i}) > \epsilon$. Thus, $\Psi_i(\overline{x}, \overline{y}_i) > \epsilon$, which contradicts (6.1). Therefore, $\overline{x}$ is a Nash equilibrium.

Step III: Sufficiency for Existence. Define a correspondence $C : X \rightrightarrows X$ by $C(x) = \prod_{i \in I} C_i(x)$ such that $C_i : X \rightrightarrows X_i$ and $C_i(x) = \{y_i \in X_i : \ \Psi_i(x, y_i) > 0\}$. For each $x \in X$, $C(x)$ is convex in X. To see this, let $x \in X$, y^1, y^2 be two elements of $C(x)$ and $\theta \in [0,1]$. Since y^1 and y^2 are in $C(x)$, then for each i and some $\epsilon > 0$, we have $\Psi_i(x, y_i^1) > 2\epsilon$ and $\Psi_i(x, y_i^2) > 2\epsilon$. For $\epsilon > 0$ and $j = 1, 2$, there exists a neighborhood $\mathcal{N}^j$ of x such that for all $z \in \mathcal{N}^j$, we have

$$u_i(y_i^j, z_{-i}) - \sup_{\mathcal{N}_z \subseteq \mathcal{N}^j} \inf_{z' \in \mathcal{N}_z} u_i(z_i, z'_{-i}) \geq \Psi_i(x, y_i^j) - \epsilon > \epsilon. \tag{6.2}$$

Let $\widetilde{\mathcal{N}} = \mathcal{N}^1 \cap \mathcal{N}^2$. Suppose $\sup_{\mathcal{N}_z \subseteq \widetilde{\mathcal{N}}} \inf_{z' \in \mathcal{N}_z} u_i(z_i, z'_{-i}) > \sup_{\mathcal{N}_z \subseteq \mathcal{N}^j} \inf_{z' \in \mathcal{N}_z} u_i(z_i, z'_{-i})$ for some $z \in \widetilde{\mathcal{N}}$. Then, for some $\delta > 0$, $\sup_{\mathcal{N}_z \subseteq \widetilde{\mathcal{N}}} \inf_{z' \in \mathcal{N}_z} u_i(z_i, z'_{-i}) > \sup_{\mathcal{N}_z \subseteq \mathcal{N}^j} \inf_{z' \in \mathcal{N}_z} u_i(z_i, z'_{-i}) + 2\delta$. Therefore, there exists $\widetilde{\mathcal{N}}_z \subseteq \widetilde{\mathcal{N}} \subseteq \mathcal{N}^j$ such that $\inf_{z' \in \widetilde{\mathcal{N}}_z} u_i(z_i, z'_{-i}) > \inf_{z' \in \mathcal{N}_z} u_i(z_i, z'_{-i}) + \delta$ for each $\mathcal{N}_z \subseteq \mathcal{N}^j$. Hence for $\mathcal{N}_z = \widetilde{\mathcal{N}}_z \subseteq \widetilde{\mathcal{N}} \subseteq \mathcal{N}^j$, $\inf_{z' \in \widetilde{\mathcal{N}}_z} u_i(z_i, z'_{-i}) > \inf_{z' \in \widetilde{\mathcal{N}}_z} u_i(z_i, z'_{-i}) + \delta$, *i.e.*, $\delta < 0$, which is impossible. Thus, for each $z \in \widetilde{\mathcal{N}}$ and $j = 1, 2$ we must have

$$\sup_{\mathcal{N}_z \subseteq \widetilde{\mathcal{N}}} \inf_{z' \in \mathcal{N}_z} u_i(z_i, z'_{-i}) \leq \sup_{\mathcal{N}_z \subseteq \mathcal{N}^j} \inf_{z' \in \mathcal{N}_z} u_i(z_i, z'_{-i}). \tag{6.3}$$

By (6.2) and (6.3), for each $j = 1, 2$ and each $z \in \widetilde{\mathcal{N}}$, we have

$$u_i(y_i^j, z_{-i}) > \sup_{\mathcal{N}_z \subseteq \mathcal{N}^j} \inf_{z' \in \mathcal{N}_z} u_i(z_i, z'_{-i}) + \epsilon \geq \sup_{\mathcal{N}_z \subseteq \widetilde{\mathcal{N}}} \inf_{z' \in \mathcal{N}_z} u_i(z_i, z'_{-i}) + \epsilon.$$

Therefore, for each $z \in \widetilde{\mathcal{N}}$, we have $\min\{u_i(y_i^1, z_{-i}), u_i(y_i^2, z_{-i})\} \geq \sup_{\mathcal{N}_z \subseteq \widetilde{\mathcal{N}}} \inf_{z' \in \mathcal{N}_z} u_i(z_i, z'_{-i}) + \epsilon$. Since the game G is quasiconcave, then $\min\{u_i(y_i^1, z_{-i}), u_i(y_i^2, z_{-i})\} \leq u_i(\theta y_i^1 + (1-\theta) y_i^2, z_{-i})$ for all z. Hence, for each $z \in \widetilde{\mathcal{N}}$, we have $u_i(\theta y_i^1 + (1-\theta) y_i^2, z_{-i}) \geq \sup_{\mathcal{N}_z \subseteq \widetilde{\mathcal{N}}} \inf_{z' \in \mathcal{N}_z} u_i(z_i, z'_{-i}) + \epsilon$, *i.e.*, $\Psi_i(x, \theta y_i^1 + (1-\theta) y_i^2) > 0$ and then $\theta y_i^1 + (1-\theta) y_i^2 \in C_i(x)$ for all $i \in I$. Thus, $\theta y^1 + (1-\theta) y^2 \in C(x)$.

Since $\Psi_i(., y_i)$ is lower semicontinuous on X, the set $\{x \in X : y_i \in C_i(x)\}$ is open in X for all $y_i \in X_i$. For each $y \in X$, we have $C^{-1}(y) = \{x \in X : y \in C(x)\} = \{x \in X : y_i \in C_i(x), \ \forall i \in I\} = \bigcap_{i \in I} \{x \in X : y_i \in C_i(x)\} = \bigcap_{i \in I} \{x \in X : \ \Psi(x, y_i) > 0\}$. Then $C^{-1}(y)$ is open in X for every $y \in X$. By the convexity of $C(x)$ and $\Psi_i(x, x_i) \leq 0$ for $i \in I$ and $x \in X$, it follows that $x \notin \text{con}C(x) = C(x)$ for each $x \in X$. Then, by Yannelis and Prabhakar Theorem (Yannelis and Prabhakar (1983)), there exists $\overline{x} \in X$ such that $C(\overline{x}) = \emptyset$. Therefore, for each $i \in I$ and each $y_i \in X_i$, $\Psi_i(\overline{x}, y_i) \leq 0$, which proves (6.1). ■

PROOF OF PROPOSITION 2.1. The conclusion of 2.1.(a) is clearly true. We only need to show that the conclusions of 2.1.(b) and 2.1.(c) are also true.

2.1.(b) Suppose that G is *QWUSC* and payoff secure. If $\overline{x} \in X$ is not a Nash equilibrium, then by quasi-weak upper semicontinuity, some player i has a strategy $\hat{x}_i \in X_i$, $\epsilon > 0$ and a neighborhood $\mathcal{N}^1$ of $\overline{x}$ such that for every $z \in \mathcal{N}^1$ and every neighborhood $\mathcal{N}_z \subseteq \mathcal{N}^1$ of z, $u_i(\hat{x}_i, \overline{x}_{-i}) > u_i(z_i, z'_{-i}) + \epsilon$ for some $z' \in \mathcal{N}_z$. The payoff security of G implies that there exists a strategy y_i and a neighborhood $\mathcal{N}^2$ of $\overline{x}$ such that $u_i(y_i, z_{-i}) \geq u_i(\hat{x}_i, \overline{x}_{-i}) - \frac{\epsilon}{2}$ for all $z \in \mathcal{N}^2$. Thus, for every $z \in \mathcal{N} = \mathcal{N}^1 \cap \mathcal{N}^2$ and its neighborhood $\mathcal{N}_z \subseteq \mathcal{N}$, there exists $z' \in \mathcal{N}_z$ such that $u_i(y_i, z_{-i}) > u_i(z_i, z'_{-i}) + \frac{\epsilon}{2}$.

2.1.(c) Suppose that G is *WTLSC* and *QUSC*. Then, if $\overline{x} \in X$ is not a Nash equilibrium, by *WTLSC*, there exists a player i, $y_i \in X_i$, $\epsilon > 0$ and a neighborhood $\mathcal{N}^1$ of $\overline{x}$ such that $u_i(y_i, z_{-i}) > u_i(\overline{x}) + \epsilon$ for all $z \in \mathcal{N}^1$. The QUSC implies that for $i \in I$, $\overline{x} \in X$, and $\epsilon > 0$, there exists a neighborhood $\mathcal{N}^2$ of $\overline{x}$ such that for every $z \in \mathcal{N}^2$ and every neighborhood $\mathcal{N}_z \subseteq \mathcal{N}^2$ of z, $u_i(\overline{x}) \geq u_i(z_i, z'_{-i}) - \frac{\epsilon}{2}$ for some $z' \in \mathcal{N}_z$. Thus, for every $z \in \mathcal{N} = \mathcal{N}^1 \cap \mathcal{N}^2$ and its neighborhood $\mathcal{N}_z \subseteq \mathcal{N}$, there exists $z' \in \mathcal{N}_z$ such that $u_i(y_i, z_{-i}) > u_i(z_i, z'_{-i}) + \frac{\epsilon}{2}$.

Thus, by Theorem 2.1, the game possesses a pure strategy Nash equilibrium. ■

PROOF OF THEOREM 3.1. For notational convenience, let $i = 1$. Define the following function by, for each $(x, y) \in X \times X$,

$$w(x,y) = \sup_{\mathcal{N} \in \Omega(x)} \inf_{\{z^1, z^2\} \subseteq \mathcal{N}} \left[u_1(y, z^1, ..., z^1) - \sup_{\mathcal{N}_{(z^1,z^2)} \subseteq \mathcal{N}} \inf_{z' \in \mathcal{N}_{(z^1,z^2)}} u_1(z^2, z', ..., z') \right].$$

We first show that $w(x, x) \leq 0$ for each $x \in X$. Indeed, suppose, by way of contradiction, that $w(x, x) > 0$ for some $x \in X$. Choose $\epsilon > 0$ with $w(x, x) > 2\epsilon$. Then there exists a neighborhood $\mathcal{N}$ of x such that for all z^1 and z^2 in $\mathcal{N}$, we have $u_1(x, z^1, ..., z^1) > \sup_{\mathcal{N}_{(z^1,z^2)} \subseteq \mathcal{N}} \inf_{z' \in \mathcal{N}_{(z^1,z^2)}} u_1(z^2, z', ..., z') + \epsilon$. Then, for every $z \in \mathcal{N}$ and letting $z^2 = x$, we have $u_1(x, z, ..., z) > \sup_{\mathcal{N}_z \subseteq \mathcal{N}} \inf_{z' \in \mathcal{N}_z} u_1(x, z', ..., z') + \epsilon$. In particular, taking $\mathcal{N}_z = \mathcal{N}$, we have $u_1(x, z, ..., z) > \inf_{z' \in \mathcal{N}} u_1(x, z', ..., z') + \epsilon$ for all $z \in \mathcal{N}$. By definition of inf, for $\epsilon > 0$, there exists $z \in \mathcal{N}$ such that $u_1(x, z, ..., z) \leq \inf_{z' \in \mathcal{N}} u_1(x, z', ..., z') + \frac{\epsilon}{2}$. Finally, we obtain $\inf_{z' \in \mathcal{N}} u_1(x, z', ..., z') + \epsilon < u_1(x, z, ..., z) \leq \inf_{z' \in \mathcal{N}} u_1(x, z', ..., z') + \frac{\epsilon}{2}$, and consequently, we must have $\epsilon < 0$, a contradiction.

Also, for each $y \in X$, $w(., y)$ is real-valued and lower semicontinuous on X from the proof of Theorem 2.1. Thus, $H(y) = \{x \in X : \ w(x, y) \leq 0\}$ is closed in X, *i.e.*, cl $H(y) = H(y)$ for all $y \in X$. Then, G is transfer closed-valued.[10]

[10]A correspondence $H : X \to 2^X$ is transfer closed-valued on X if for every $y \in X$, $x \notin H(y)$ implies that there exists a point $y' \in X$ such that $x \notin clH(y)$.

Moreover, since the game G is quasiconcave, H is transfer FS-convex.[11] To show this, suppose by way of contradiction that H is not transfer FS-convex. Then, there exists a finite subset of $\{y^1, y^2, ..., y^m\} \subseteq X$, a corresponding finite subset $\{x^l, ..., x^m\} \subseteq X$, and a subset J of $\{1, ..., m\}$ such that $\text{co}\{x^j,\ j \in J\} \nsubseteq \bigcup_{j\in J} H(y^j)$. Thus, there exists $\overline{x} \in \text{co}\{x^j,\ j \in J\}$ such that for each $j \in J$, $w(\overline{x}, y^j) > 0$. Choose $\epsilon > 0$ such that for each $j \in J$, $w(\overline{x}, y^j) > 2\epsilon$. Then, there exists a neighborhood $\mathcal{N}^j$ of $\overline{x}$ such that for all z^1 and z^2 in $\mathcal{N}^j$, we have

$$u_1(y^j, z^1, ..., z^1) - \sup_{\mathcal{N}_{(z^1,z^2)} \subseteq \mathcal{N}^j} \inf_{z' \in \mathcal{N}_{(z^1,z^2)}} u_1(z^2, z', ..., z') \geq w(\overline{x}, y^j) - \epsilon > \epsilon. \tag{6.4}$$

Let $\widetilde{\mathcal{N}} = \bigcap_{j\in J} \mathcal{N}^j$. By the same arguments as in the proof of Theorem 2.1, we obtain that for each $z^2 \in \widetilde{\mathcal{N}}$ and $j \in J$, we have

$$\sup_{\mathcal{N}_{z^2} \subseteq \widetilde{\mathcal{N}}} \inf_{z' \in \mathcal{N}_{(z^1,z^2)}} u_1(z^2, z', ..., z') \leq \sup_{\mathcal{N}_{z^2} \subseteq \mathcal{N}^j} \inf_{z' \in \mathcal{N}_{z^2}} u_1(z^2, z', ..., z'). \tag{6.5}$$

By (6.4) and (6.5), for each $(z^1, z^2) \in \widetilde{\mathcal{N}}$ and each $j \in J$, we have $u_1(y^j, z^1, ..., z^1) > \sup_{\mathcal{N}_{(z^1,z^2)} \subseteq \mathcal{N}^j} \inf_{z' \in \mathcal{N}_{(z^1,z^2)}} u_1(z^2, z', ..., z') + \epsilon \geq \sup_{\mathcal{N}_{(z^1,z^2)} \subseteq \widetilde{\mathcal{N}}} \inf_{z' \in \mathcal{N}_{(z^1,z^2)}} u_1(z^2, z', ..., z') + \epsilon$. Therefore, for each $(z^1, z^2) \in \widetilde{\mathcal{N}}$, we have $\min_{j\in J} u_1(y^j, z^1, ..., z^1) \geq \sup_{\mathcal{N}_{(z^1,z^2)} \subseteq \widetilde{\mathcal{N}}} \inf_{z' \in \mathcal{N}_{(z^1,z^2)}} u_1(z^2, z', ..., z') + \epsilon$. Since the game G is quasiconcave, then $u_1(\overline{x}, z^1, ..., z^1) \geq \sup_{\mathcal{N}_{(z^1,z^2)} \subseteq \widetilde{\mathcal{N}}} \inf_{z' \in \mathcal{N}_{(z^1,z^2)}} u_1(z^2, z', ...z') + \epsilon$ for all $\{z^1, z^2\} \in \widetilde{\mathcal{N}}$. Hence,

$$\inf_{\{z^1,z^2\} \in \widetilde{\mathcal{N}}} \left[u_1(\overline{x}, z^1, ..., z^1) - \sup_{\mathcal{N}_{(z^1,z^2)} \subseteq \widetilde{\mathcal{N}}} \inf_{z' \in \mathcal{N}_{(z^1,z^2)}} u_1(z^2, z', ..., z') \right] \geq \epsilon,$$

which means that $w(\overline{x}, \overline{x}) > 0$, contradicting to $w(x, x) \leq 0$ for all $x \in X$. Therefore, H must be transfer FS-convex.

Therefore, by Lemma 1 of Tian (1993), $\bigcap_{y\in X} H(y)$ is nonempty and compact. Thus, there exists a point $\overline{x} \in X$ such that $w(\overline{x}, y) \leq 0$ for all $y \in X$. If $\overline{x}$ is not a symmetric Nash equilibrium, by diagonal quasi-weak transfer continuity, there exists a player i, say, $i = 1$, $\overline{y} \in X$, $\epsilon > 0$, and a neighborhood $\mathcal{N}$ of $\overline{x}$ such that for every $(z^1, z^2) \in \mathcal{N}$ and every neighborhood $\mathcal{N}_{(z^1,z^2)} \subseteq \mathcal{N}$ of (z^1, z^2), we have $u_1(\overline{y}, z^1, ..., z^1) > u_1(z^2, z', ..., z') + \epsilon$ for some $z' \in \mathcal{N}_{(z^1,z^2)}$. Then, for each $(z^1, z^2) \in \mathcal{N}$, $u_1(\overline{y}, z^1, ..., z^1) - \sup_{\mathcal{N}_{(z^1,z^2)} \subseteq \mathcal{N}} \inf_{z' \in \mathcal{N}_{(z^1,z^2)}} u_1(z^2, z', ..., z') > \epsilon$. Thus, $w(\overline{x}, \overline{y}) > \epsilon$, a contradiction. Therefore, $\overline{x}$ is a Nash equilibrium. ■

[11] A correspondence $H : X \to 2^X$ is transfer FS-convex if for any finite subset $\{y^1, ..., y^m\} \subseteq X$, there exists a corresponding finite subset $\{x^1, ..., x^m\} \subseteq X$ such that for each $J \subseteq \{1, ..., m\}$, we have $co\{x^j, j \in J\} \subseteq \bigcup_{j\in J} H(y^j)$.

PROOF OF THEOREM 3.2. Define the following function by, for each $(x, y) \in X \times X$,

$$\varphi(x, y) = \sup_{\mathcal{N} \in \Omega(x)} \inf_{z \in \mathcal{N}} \left[u_1(y, z, ..., z) - u_1(z, z, ..., z) \right].$$

For each $y \in X$, the function $\varphi(., y)$ is real-valued and lower semicontinuous over X from the proof of Theorem 2.1. Thus, $K(y) = \{x \in X : \varphi(x, y) \leq 0\}$ is closed in X, and therefore K is transfer closed-valued.

K is also transfer FS-convex. Suppose not. Then, there exists a finite subset of $\{y^1, y^2, ..., y^m\} \subseteq X$, a corresponding finite subset $\{x^l, ..., x^m\} \subseteq X$, and a subset J of $\{1, ..., m\}$ such that $\text{co}\{x^j,\ j \in J\} \nsubseteq \bigcup_{j \in J} K(y^j)$. Thus, there exists $\overline{x} \in \text{co}\{x^j,\ j \in J\}$ such that for each $j \in J$, $\varphi(\overline{x}, y^j) > 0$. Choose $\epsilon > 0$ such that for each $j \in J$, $\varphi(\overline{x}, y^j) > 2\epsilon$. Then, there exists a neighborhood $\mathcal{N}^j$ of $\overline{x}$ such that for all z in $\mathcal{N}^j$, we have

$$u_1(y^j, z, ..., z) - u_1(z, z, ..., z) \geq \varphi(\overline{x}, y^j) - \epsilon > \epsilon. \tag{6.6}$$

Let $z = \overline{x} \in \mathcal{N}^j$ for each $j \in J$. Then (6.6) becomes that

$$u_1(y^j, \overline{x}, ..., \overline{x}) \geq u_1(\overline{x}, \overline{x}, ..., \overline{x}) + \epsilon, \text{ for each } j \in J. \tag{6.7}$$

Then $\min_{j \in J} u_1(y^j, \overline{x}, ..., \overline{x}) \geq u_1(\overline{x}, \overline{x}, ..., \overline{x}) + \epsilon$. Since the game G is diagonally transfer quasiconcave, then $u_1(\overline{x}, \overline{x}, ..., \overline{x}) \geq u_1(\overline{x}, \overline{x}, ..., \overline{x}) + \epsilon$. Hence, $\epsilon < 0$, which is a contradiction. Therefore, K must be transfer FS-convex.

Then, by Lemma 1 of Tian (1993), $\bigcap_{y \in X} K(y)$ is nonempty and compact. Thus, there exists a point $\overline{x} \in X$ such that $\varphi(\overline{x}, y) \leq 0$ for all $y \in X$. If $\overline{x}$ is not a symmetric Nash equilibrium, by diagonal weak transfer continuity, there exists a player $i = 1$, $\overline{y} \in X$, $\epsilon > 0$, and a neighborhood $\mathcal{N}$ of $\overline{x}$ such that $u_i(\overline{y}, z, ..., z) > u_i(z, z, ..., z) + \epsilon$ for every $z \in \mathcal{N}$. Thus, $\varphi(\overline{x}, \overline{y}) > \epsilon$, a contradiction. Therefore, $\overline{x}$ is a symmetric Nash equilibrium. ■

PROOF OF THEOREM 4.1. For each $x \in X$, we have $F_i(x, x_i) \leq 0$ by the same proof as in Theorem 2.1. For each i and every $x_i \in X_i$, the function $F_i(., y_i)$ is lower semicontinuous on X. Consider the following correspondence $C : X \rightrightarrows X$ defined by $C(x) = \prod_{i \in I} C_i(x)$ such that $C_i : X \rightrightarrows X_i$ and $C_i(x) = \{y_i \in X_i : F_i(x, y_i) > 0\}$. Since the function $F_i(x, .)$ is quasiconcave, then $C(x)$ is convex in X. For each $x \in X$, by lower semicontinuity of $F_i(., y_i)$, $C^{-1}(y)$ is open in X for each $y \in X$. Then, by Yannelis and Prabhakar Theorem, there exists $\overline{x} \in X$ such that $C(\overline{x}) = \emptyset$. Therefore, for each $i \in I$ and each $y_i \in X_i$, $F_i(\overline{x}, y_i) \leq 0$. If $\overline{x}$ is not a Nash equilibrium, then by pseudo quasi-weak transfer continuity, there exists a player i, strategy $\overline{y}_i$, $\epsilon > 0$, and a neighborhood $\mathcal{N}$ of $\overline{x}$ such that for each $z \in \mathcal{N}$, $u_i(\overline{y}_i, z_{-i}) > u_i(z_i, z'_{-i}) + \epsilon$ for some $z' \in \mathcal{N}$. Then, for each $z \in \mathcal{N}$, we have $u_i(\overline{y}_i, z_{-i}) - \inf_{z' \in \mathcal{N}} u_i(z_i, z'_{-i}) > \epsilon$, and thus,

$F_i(\overline{x}, \overline{y}_i) > \epsilon$, a contradiction. Therefore, $\overline{x}$ is a Nash equilibrium. ■

PROOF OF PROPOSITION 4.1. Let $G = (X_i, u_i)_{i \in I}$ be better-reply secure. Suppose, by way of contradiction, that the game is not pseudo quasi-weakly transfer continuous. Then, there exists a nonequilibrium $x^* \in X$ such that for all player j, $\epsilon > 0$, every neighborhood $\mathcal{N}$ of x^*, and all y_j, there exists $x' \in \mathcal{N}$ satisfying

$$u_j(y_j, x'_{-j}) \leq u_j(x'_j, x''_{-j}) + \epsilon, \quad \text{for all } x'' \in \mathcal{N}.$$

Letting $\overline{u}$ be the limit of the vector of payoffs corresponding to some sequence of strategies converging to x^*, and U^* be the set of all such points, which is a compact set by the boundedness of payoffs, we have $(x^*, \overline{u}) \in \text{cl}(\Gamma)$ for all $\overline{u} \in U^*$. Then, for each $(x^*, \overline{u}) \in \text{cl}(\Gamma)$ with $\overline{u} \in U^*$, there exists a player i, a strategy $\overline{y}_i$, $\epsilon > 0$ and a neighborhood $\overline{\mathcal{N}}$ of x^* such that $u_i(\overline{y}_i, x'_{-i}) > \overline{u}_i + \epsilon$ for all $x' \in \overline{\mathcal{N}}$. Then $\inf\limits_{x' \in \overline{\mathcal{N}}} u_i(\overline{y}_i, x'_{-i}) \geq \overline{u}_i + \epsilon$. Let U_i^* be the projection of U^* to coordinate i and

$$u_i^* = \sup\{\overline{u}_i \in U_i^* : \inf_{x' \in \overline{\mathcal{N}}} u_i(\overline{y}_i, x'_{-i}) \geq \overline{u}_i + \epsilon\}.$$

Then, for $\epsilon/2 > 0$, there is a neighborhood $\mathcal{N}^{i,*}$ of x^* and a strategy y_i^* such that

$$\inf_{x' \in \mathcal{N}^{i,*}} u_i(y_i^*, x'_{-i}) \geq (u_i^* + \epsilon) - \epsilon/2 = u_i^* + \epsilon/2. \tag{6.8}$$

Now, since the game is not pseudo quasi-weakly transfer continuous, then for a directed system of neighborhoods $\{\mathcal{N}^k\}_k$ of x^*, a sequence $\{\epsilon^k\}_k$ converging to 0, and every $j \in I$, there exists a sequence $\{x^{j,k}\}_k$ with $x^{j,k} \in \mathcal{N}^k$ so that $\{x^{j,k}\}_k$ converges to x^* and

$$u_j(y_j^*, x^{j,k}_{-j}) \leq u_j(x^{j,k}_j, x'_{-j}) + \epsilon^k, \quad \text{for each } x' \in \mathcal{N}^k. \tag{6.9}$$

Consider the following sequence: for each k, let $\widetilde{x}^k = (x_1^{1,k}, ..., x_n^{n,k})$. Since for each $j \in I$, $x^{j,k} \in \mathcal{N}^k$ and $\{x^{j,k}\}_k$ converges to x^*, then $\widetilde{x}^k \in \mathcal{N}^k$ and the sequence $\{\widetilde{x}^k\}_k$ converges to x^*. Therefore, inequality (6.9) becomes

$$u_j(y_j^*, x^{j,k}_{-j}) \leq u_j(x^{j,k}_j, \widetilde{x}^k_{-j}) = u_j(\widetilde{x}^k) + \epsilon^k, \quad \text{for each } k, \ j \in I. \tag{6.10}$$

Assume that $\{u(\widetilde{x}^k)\}_k$ converges and $\widetilde{u} = \lim\limits_{k \to \infty} u(\widetilde{x}^k)$. Hence, $(x^*, \widetilde{u}) \in \text{cl}(\Gamma)$ with $\widetilde{u} \in U^*$, then there exists a player $i \in I$ such that $\widetilde{u}_i \leq u_i^*$. Thus, for $\epsilon/3 > 0$, there exists k_1 such that whenever $k > k_1$, we have $u_i(y_i^*, x^{i,k}_{-i}) \leq u_i^* + \epsilon/3 \leq \inf\limits_{x' \in \mathcal{N}^{i,*}} u_i(y_i^*, x'_{-i}) - \epsilon/6$. Then for $k > k_1$, we obtain

$$u_i(y_i^*, x^{i,k}_{-i}) \leq u_i(y_i^*, x'_{-i}) - \epsilon/6, \text{ for each } x' \in \mathcal{N}^{i,*}. \tag{6.11}$$

Since the sequence $\{x^{i,k}\}_k$ converges to x^*, then for $\mathcal{N}^{i,*}$, there exists k_2 such that for $k > k_2$, we have $x^{i,k} \in \mathcal{N}^{i,*}$. Thus, by (6.11) for $k > \max(k_1, k_2)$, we have $u_i(y_i^*, x^{i,k}_{-i}) \leq u_i(y_i^*, x^{i,k}_{-i}) - \epsilon/6$, which is impossible. Hence, the game must be pseudo quasi-weakly transfer continuous. ■

PROOF OF THEOREM 4.2. **Sufficiency**. For each $y \in X$, let

$$F(y) = \{x \in X : u_i(y_i, x_{-i}) \leq u_i(x),\ \forall i \in I\}.$$

It is clear that G is weakly transfer quasi-continuous if and only if F is transfer closed-valued.

For $y \in X$, let $\bar{F}(y) = \text{cl}\, F(y)$. Then $\bar{F}(y)$ is closed, and by the strong diagonal transfer quasiconcavity, it is also transfer FS-convex. By Lemma 1 in Tian (1993), we know that $\bigcap_{y\in X} F(y) = \bigcap_{y\in X} \bar{F}(y) \neq \emptyset$. Thus, there exists a strategy profile $\overline{x} \in X$ such that

$$u_i(y_i, \overline{x}_{-i}) \leq u_i(\overline{x}), \text{ for all } y \in X \text{ and } i \in I.$$

Thus $\overline{x}$ is a pure strategy Nash equilibrium of the game G.

Necessity: Suppose the game Γ has a pure strategy Nash equilibrium $x^* \in X$. We want to show that it is strongly diagonal transfer quasiconcave in y. Indeed, for any finite subset $\{y^1, \ldots, y^m\} \subset X$, let the corresponding finite subset $X^m = \{x^1, \ldots, x^m\} = \{x^*\}$. Then for any subset $\{x^{k^1}, x^{k^2}, \ldots, x^{k^s}\} \subset X^m = \{x^*\}$, $1 \leq s \leq m$, $x \in co\{x^{k^1}, x^{k^2}, \ldots, x^{k^s}\} = \{x^*\}$, and $y \in \{y^{k^1}, y^{k^2}, \ldots, y^{k^s}\}$, we have

$$u_i(y_i, x_{-i}) = u_i(y_i, x^*_{-i}) \leq u_i(x^*_i, x^*_{-i}) = u_i(x_i, x_{-i}).$$

Hence U is strongly diagonal transfer quasiconcave in x. ■

PROOF OF THEOREM 4.3. Define a surrogate game $G^0 = (\tilde{X}_i, \tilde{u}_i)_{i\in I_0}$ as follows: $I_0 = I \cup \{0\}$, $\tilde{X}_i = X$ if $i = 0$ and $\tilde{X}_i = X_i$ if $i \in I$, and $\tilde{u}_i : \tilde{X} = X \times X \to \mathbb{R}$ by $\tilde{u}_i(x, y) = \Psi_i(x, y_i)$ if $i \in I$ and $\tilde{u}_0(x, y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$

We first show that G has a Nash equilibrium if G^0 has an equilibrium. Indeed, let $(\overline{x}, \overline{y})$ be a Nash equilibrium of surrogate game G^0. Then, by definition of the payoff $\tilde{u}_0$, we must have $\overline{x} = \overline{y}$, otherwise it cannot be a Nash equilibrium. Then it is clear that $\overline{x}$ is a Nash equilibrium of G.

Now we show that the surrogate game G^0 has an equilibrium. Fix any nonequilibrium strategy $\tilde{x} = (x, y)$ of G^0. Then we need distinguish two cases.

Case 1) $x \neq y$. Then choose a neighborhood $\mathcal{V}_{(x,y)} \subset X \times X$ such that for each $(z^1, z^2) \in \mathcal{V}_{(x,y)}$ with $z^1 \neq z^2$ and a well-behaved correspondence $\phi_0 : \mathcal{V}_{(x,y)} \rightrightarrows X$ defined by $\phi_0(z^1, z^2) = \{z^1\}$, we have $\tilde{u}_0(z^1, z^1) = 1 > 0 = \tilde{u}_0(z^1, z^2)$, for each $(z^1, z^2) \in \mathcal{V}_{(x,y)}$.

Case 2) $x = y$. Then, by the Ψ-correspondence transfer continuity of G, the surrogate game G^0 is continuously secure in (x, x)[12] (see Barelli and Meneghel (2013)).

Thus, in either case, the surrogate game G^0 is continuously secure, and consequently, by Theorem 2.2 of Barelli and Meneghel (2013), it has a Nash equilibrium $(\overline{x}, \overline{x})$, which implies that $\overline{x}$ is a Nash equilibrium of G. ■

PROOF OF PROPOSITION 5.1. Suppose x is not an equilibrium. Then some player i has a strategy y_i such that $u_i(y_i, x_{-i}) > u_i(x)$, i.e., $F_i(y_i, S_i(y_i, x_{-i})) > F_i(x_i, S_i(x))$. If $(y_i, x_{-i}) \in X \backslash D_i$, then by Assumption 3, there exists a player $j \in I$, a deviation strategy profile $\bar{y}$, $\epsilon > 0$, and a neighborhood $\mathcal{V}(x)$ of x such that for every $z \in \mathcal{V}(x)$ and every neighborhood $\mathcal{N}_z \subseteq \mathcal{N}_x$ of z, $F_j(\bar{y}_j, S_j(\bar{y}_j, z_{-j})) > F_j(z_j, S_j(z_j, z'_{-j})) + \epsilon$ for some $z' \in \mathcal{N}_z$, *i.e.,* $u_j(y'_j, z_{-j}) > u_j(z_j, z'_{-j}) + \epsilon$ for some $z' \in \mathcal{N}_z$. If $(y_i, x_{-i}) \in D_i$, then by Assumption 2, there exists a player $j \in I$ and $\bar{y}_j$ such that $(\bar{y}_j, x_{-j}) \in X \backslash D_j$ and $F_j(\bar{y}_j, S_j(\bar{y}_j, x_{-j})) > F_j(x_j, S_j(x))$. Thus, by Assumption 3, there exists a player $k \in I$, a deviation strategy profile $\tilde{y}$, $\epsilon > 0$, and a neighborhood $\mathcal{V}(x)$ of x such that for every $z \in \mathcal{V}(x)$ and every neighborhood $\mathcal{N}_z \subseteq \mathcal{N}_x$ of z, $F_k(\tilde{y}_k, S_k(\tilde{y}_k, z_{-k})) > F_j(z_k, S_k(z_k, z'_{-k})) + \epsilon$ for some $z' \in \mathcal{N}_z$, *i.e.,* $u_k(\tilde{y}_k, z_{-k}) > u_k(z_k, z'_{-k})$ for some $z' \in \mathcal{N}_z$. Therefore, the game is quasi-weakly transfer continuous. Since it is also compact and quasiconcave, by Theorem 2.1, it has a pure strategy Nash equilibrium. ■

PROOF OF PROPOSITION 5.2. Suppose p is not an equilibrium. Then some player i has a strategy q_i such that $\pi_i(q_i, p_{-i}) > \pi_i(p)$. If $(q_i, p_{-i}) \in P^n \backslash A_i$, then by Assumption 3', there exists a player $j \in I$, a deviation strategy profile $\bar{q}_j$ and a neighborhood $\mathcal{N}_p$ of p such that for every $r \in \mathcal{N}_p$, every neighborhood $\mathcal{N}_r \subseteq \mathcal{N}_p$ of r, $\pi_j(\bar{q}_j, r_{-i}) > \pi_j(r_j, r'_{-j}) + \epsilon$, for some $r' \in \mathcal{N}_r$. If $(q_i, p_{-i}) \in A_i$, then by Assumption 2', there exists a firm $j \in I$, and $\bar{q}_j$ such that $(\bar{q}_j, p_{-i}) \in P^n \backslash A_j$ and $\pi_j(\bar{q}_j, p_{-i}) > \pi_i(p)$. Thus, by Assumption 3, there exists a player $j \in I$, a deviation strategy profile $\bar{q}_j$ and a neighborhood $\mathcal{N}_p$ of p such that for every $r \in \mathcal{N}_p$, every neighborhood $\mathcal{N}_r \subseteq \mathcal{N}_p$ of r, $\pi_j(\bar{q}_j, r_{-i}) > \pi_j(r_j, r'_{-j}) + \epsilon$, for some $r' \in \mathcal{N}_r$. Therefore, the game is quasi-weakly transfer continuous. Since the game is also compact, convex, and quasiconcave, by Theorem 2.1, it has a pure strategy Nash equilibrium. ■

[12] $G = (X_i, u_i)_{i \in I}$ is continuously secure at x where x is not an equilibrium. Then there is a neighborhood $\mathcal{N}$ of x, $\alpha \in \mathbb{R}^n$, and a well-behaved correspondence $\phi_x : \mathcal{N} \twoheadrightarrow X$ so that

(1) for each $t \in \mathcal{N}$ and $i \in I$, we have $\phi_{x,i}(t) \subseteq B_i(t, \alpha_i)$,

(2) for each $z \in \mathcal{N}$, there exists a player j for whom $z_j \notin coB_j(z, \alpha_j)$,

where $B_i(x, \alpha_i) = \{y_i \in X_i \text{ such that } u_i(y_i, x_{-i}) \geq \alpha_i\}$.

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