A Full Characterization on Fixed-Point Theorem, Minimax Inequality, Saddle Point, and KKM Theorem

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Abstract

This paper provides necessary and sufficient conditions for fixed-point theorems, minimax inequalities and some related theorems defined on arbitrary topological spaces that may be discrete, continuum, non-compact or non-convex. We establish a single condition, *γ-recursive transfer lower semicontinuity*, which fully characterizes the existence of equilibrium of minimax inequality without imposing any kind of convexity nor any restriction on topological space. The result then is employed to fully characterize fixed point theory, saddle point theory, and the FKKM theory.

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Keywords: Fixed-point theorems, minimax inequalities, saddle points, FKKM theorems, recursive transfer continuity.

1 Introduction

The Fan's minimax inequality is probably one of the most important results in mathematical sciences in general and nonlinear analysis in particular, which is mutually equivalent to many important basic mathematical theorems such as the classical Knaster Kuratowski Mazurkiewicz (KKM) lemma, Sperner's lemma, Brouwer's fixed point theorem, Kakutani fixed point theorem. It also became a crucial tool in proving many existence problems in various fields, especially in variational inequality problems, mathematical programming, partial differential equations, impulsive control, equilibrium problems in economics, various optimization problems, saddle points, fixed points, coincidence points, complementarity problems, etc.

The classical Fan's minimax inequality given by Fan [11] typically assumes *lower semicontinuity* and *quasiconcavity* for the functions, in addition to convexity and compactness in Hausdorff topological vector spaces. However, in many situations, these assumptions may not be satisfied. The function under consideration may not be lower semicontinuous and/or quasiconcave, and choice spaces may be nonconvex and/or noncompact.

Accordingly, mathematicians continually strive to seek weaker conditions that solve the equilibrium existence problems. As such the Fan minimax inequality has been followed by a large number of generalizations (cf. Allen [1], Ansari et al. [2], Ansari et al. [3], Chebbi [6], Cho et al. [7], Ding and Park [8], Ding [9], Fan [13], Iusem and Soca [18], Georgiev and Tanaka [19], Lignola [22], Lin and Chang [23], Lin and Tian [24], Nessah and Tian [25], Tian [30, 32], Yuan [38], Zhou and Chen [39] and the references therein), among which some seek to weaken the quasiconcavity/seminonintuity of function, or drop convexity/compactness of choice sets, while others seek to weaken Hausdorff topological vector spaces to be topological vector spaces, Lassonde type convex spaces, Horvath type H-spaces, generalized convex spaces, and other types of spaces.

However, all the existing results only provide sufficient conditions. Besides, they are based on relatively strong topological structure, especially like the convexity, but not based on a general topological space. In order to prove the results, they all assume some type of quasiconcavity/quasi-convexity (or transitivity/monotonicity) and semicontinuity of functions, in addition to some types of convex topological spaces. As such, the intrinsic nature of the existence of equilibrium has not been fully understood yet. Why does or does not a problem have an equilibrium? Are both lower semicontinuity and quasiconcavity (or their weaker forms) essential to the existence of equilibrium? If so, can these two conditions be merged into one single condition? One can easily find simple examples of problems that have or not have an equilibrium, but none of them can be used to reveal the existence/non-existence of equilibria in these problems.

In this paper we will provide a complete solution to the problem of minimax inequality for a general topological space. We fully characterize the existence of equilibrium of minimax inequality for an arbitrary topological space that may be discrete, continuum, non-compact or non-convex, and the function that may not be lower semicontinuous or does not have any form of quasi-concavity. We introduce the notion of γ-*recursive transfer lower semicontinuity* that fully characterizes the existence of equilibrium of minimax inequality without imposing any kind of convexity for a topological space. It is shown that the single condition, γ-recursive transfer lower semicontinuity, is necessary, and further, under compactness, sufficient for the existence of equilibrium of minimax inequalities for general topological strategy spaces and functions. We also provide a complete solution for the case of any arbitrary choice space that may be noncompact. We show that γ-recursive transfer lower semicontinuity with respect to a compact set D is necessary and sufficient for the existence of equilibrium of minimax inequalities for arbitrary (possibly noncompact or open) topological spaces and general functions.

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Since minimax inequality provides the foundation for many of the modern essential results in diverse areas of mathematical sciences, the results not only fully characterize the existence of solution to minimax inequality, but also introduce new techniques and methods for studying other optimization problems and generalize/characterize some basic mathematics results such as the FKK theorem, fixed point theorem, saddle point theorem, variational inequalities, and coincidence theorem, etc. As illustrations, we show how they can be employed to fully characterize fixed point theorem, saddle point theorem, and the FKKM theorem. The method of proof adopted to obtain our main results is also new and elementary — neither fixed-point-theorem nor KKM-theorem approach.

The basic transfer method has been systematically developed in Tian [30, 31], Tian and Zhou [35, 37], Zhou and Tian [40], and Baye et al. [4] for studying various existence problems, optimization problems and some basic mathematics results. These papers, especially Zhou and Tian [40], have developed three types of transfers: transfer continuities, transfer convexities, and transfer transitivities to study the maximization of binary relations and the existence of equilibrium in games with discontinuous and/or nonquasiconcave payoffs. Various notions of transfer continuities, transfer convexities and transfer transitivities provide complete solutions to the question of the existence of maximal elements for complete preorders and interval orders (cf. Tian [31] and Tian and Zhou [37]).

The notion of recursive transfer continuity extends transfer continuity from direct transfers to allowing indirect (called recursive or sequential) transfers so that it turns out to be a necessary and sufficient condition for the existence of equilibrium with a compact choice set. Incorporating recursive transfers into various transfer continuities allows us to obtain full characterization results for many other solution problems as shown in the application section.

The remainder of this paper is organized as follows. Section 2 states some notation and definitions. In Section 3, we generalize the Ky Fan minimax inequality by fully characterizing the existence of equilibrium of minimax inequality for an arbitrary topological space. Then in the remaining sections, we use our main results on minimax inequality to fully characterize the existence problem for other mutually equivalent theorems. Namely, we provide necessary and sufficient conditions for fixed point theorem, saddle point theorem, and FKKM theorem in Section 4-6, respectively.

2 Notation and Definitions

Before the formal discussion, we begin with some notation and definitions. Let X be a subset of a topological space T and let $D \subseteq X$. Denote the collections of all subsets, convex hull, closure, and interior of the set D by 2^D, $co\,D$, $cl\,D$, and $int\,D$, respectively. Throughout the paper all topological vector spaces are assumed to be Hausdorff and denoted by E.

Let X be a topological space. A function $f : X \to \mathbb{R} \cup \{\pm\infty\}$ is said to be *lower semicontinuous* on X if for each point x', we have

$$\liminf_{x \to x'} f(x) \geq f(x'),$$

or equivalently, if its epigraph epi$f \equiv \{(x, a) \in X \times \mathbb{R} : f(x) \leq a\}$ is a closed subset of $X \times \mathbb{R}$. A function $f : X \to \mathbb{R} \cup \{\pm\infty\}$ is said to be *upper semicontinuous* on X if $-f$ is lower semicontinuous on X. f is continuous on Y if f is both upper and lower semicontinuous on Y.

Let E be a topological vector space and X a convex subset of E. A function $f : X \to \mathbb{R}$ is quasiconcave on X if for any y_1, y_2 in Y and any $\theta \in [0, 1]$, $\min\{f(y_1), f(y_2)\} \leq f(\theta y_1 + (1 - \theta)y_2)$, and f is quasiconvex on X if $-f$ is quasiconcave on X. A function $f : X \times X \to \mathbb{R}$ is diagonally quasiconcave in y if for any finite points $y^1, \ldots, y^m \in Y$ and any $y \in co\{y^1, \ldots, y^m\}$, $\min_{1 \leq k \leq m} f(y, y^k) \leq f(y, y)$. A function $\phi : X \times X \to \mathbb{R}$ is γ-diagonally quasiconcave in y if for any $y^1, \ldots, y^m \in Y$ and $y \in co\{y^1, \ldots, y^m\}$, $\min_{1 \leq k \leq m} f(y, y^k) \leq \gamma$.

Let E be a topological vector space and X a convex set. A correspondence $F : X \to 2^X$ is said to be *FS convex*[1] on X if for every finite subset $\{x_1, x_2, \ldots, x_m\}$ of X

$$co\{x_1, x_2, \ldots, x_m\} \subseteq \bigcup_{j=1}^{m} F(x_j).$$

Note that $x \in F(x)$ for all $x \in X$ when F is FS convex. A correspondence $F : X \to 2^X$ is said to be *SS convex*[2] if $x \notin co\, F(x)$ for all $x \in X$. It is easily shown that a function $\phi : X \times X \to \mathbb{R} \cup \{\pm\infty\}$ is γ-DQCV in x if and only if the correspondence $F : X \to 2^X$ defined by $F(x) = \{y \in X : \phi(x_j, y) \leq \gamma\}$ for all $x \in X$ is FS convex on X.

3 Full Characterization of the Ky Fan Minimax Inequality

We begin by stating the classical minimax inequality by Fan [11].

THEOREM 3.1 (FAN MINIMAX INEQUALITY) *Let X be a compact convex set in a Hausdorff topological vector space, $\gamma \in \mathbb{R}$. Let $\phi: X \times X \to \mathbb{R}$ be a function suppose that*

(a) $\phi(x, x) \leq 0$ for all $x \in X$,

(b) ϕ is lower semicontinuous in y,

(c) ϕ is quasiconcave in x.

Then there exists a point $y^* \in X$ such that $\phi(x, y^*) \leq 0$ for all $x \in X$.

Fan minimax inequality has then been generalized by various ways. Some weaken quasi-concavity to be (γ-)diagonal quasiconcavity or transfer (γ-diagonal) quasiconcavity, some weaken lower semi-continuity to be transfer lower semi-continuity or γ-transfer lower semi-continuity, some weaken compactness to noncompactness, while others weaken Hausdorff topological vector space to be topological vector space, Lassonde type convex space, Horvath type H-space, generalized convex

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[1]The FS is for Fan [12] and Sonnenschein [27].

[2]The SS is for Shafer and Sonnenschein [26].

space, etc. (cf. Allen [1], Ansari et al. [2], Ansari et al. [3], Chebbi [6], Cho et al. [7], Ding and Park [8], Ding [9], Fan [13], Iusem and Soca [18], Georgiev and Tanaka [19], Lignola [22], Lin and Chang [23], Lin and Tian [24], Nessah and Tian [25], Tian [30, 32], Yuan [38], Zhou and Chen [39] among others). Yet so far, there is no full characterization available and the topological spaces are still relatively strong to get their results.

In this section we provide a full characterization on the Fan minimax inequality by giving a single condition that is necessary and sufficient for the existence of solution to a minimax inequality defined on an arbitrary topological space that may be discrete, continuum, non-compact or non-convex. We begin with the notion of γ-transfer lower semicontinuity introduced by Tian [30].

DEFINITION 3.1 (γ-**Transfer Lower Semicontinuity**) Let X be a topological space. A function ϕ: $X \times X \to \mathbb{R} \cup \{\pm\infty\}$ is said to be γ-transfer lower semicontinuous in y if for all $x \in X$ and $y \in Y$, $\phi(x, y) > \gamma$ implies that there exists some point $z \in X$ and some neighborhood $\mathcal{N}(y)$ of y such that $\phi(z, y') > \gamma$ for all $y' \in \mathcal{N}(y)$.

Now we define the notion of γ-recursive transfer lower semicontinuity, which fully characterizes the existence of equilibrium of a minimax inequality. To do so, we first define the notion of recursive upsetting.

DEFINITION 3.2 (**Recursive Upsetting**) A point $z^0 \in X$ is said to be γ-*recursively upset by* $z \in X$ if there exists a finite set of points $\{z^1, z^2, \ldots, z^{m-1}, z\}$ such that $\phi(z^1, z^0) > \gamma$, $\phi(z^2, z^1) > \gamma$, $\ldots$, $\phi(z, z^{m-1}) > \gamma$.

We say that a point $z^0 \in X$ is m-γ-*recursively upset by* $z \in X$ if the number of such recursive upsetting points is m. For convenience, we say z^0 is *directly upset by* z when $m = 1$, and *indirectly* γ-*upset by* z when $m > 1$. γ-recursive upsetting says that a point z^0 can be directly or indirectly γ-upset by a point z through sequential points $\{z^0, z^1, z^2, \ldots, z^{m-1}, z\}$ in a recursive way that z^0 is γ-upset by z^1, z^1 is γ-upset by z^2, $\ldots$, and z^{m-1} is γ-upset by z.

DEFINITION 3.3 (γ-**Recursive Transfer Lower Semicontinuity**) Let X be a topological space. A function ϕ: $X \times X \to \mathbb{R} \cup \{\pm\infty\}$ is said to be γ-*recursively transfer lower semicontinuous* in y if, whenever $\phi(x, y) > \gamma$ for $x, y \in X$, there exists a point $z^0 \in X$ (possibly $z^0 = y$) and a neighborhood $\mathcal{V}_y$ of y such that $\phi(z, \mathcal{V}_y) > \gamma$ for any z that γ-recursively upsets z^0, i.e., for any sequence of points $\{z^0, z^1, \ldots, z^{m-1}, z\}$, $\phi(z, z^{m-1}) > \gamma$, $\phi(z^{m-1}, z^{m-2}) > \gamma$, $\ldots$, $\phi(z^1, z^0) > \gamma$ for $m \geq 1$ imply that $\phi(z, \mathcal{V}_y) > 0$. Here $\phi(z, \mathcal{V}_y) > 0$ means that $\phi(z, y') > 0$ for all $y' \in \mathcal{V}_y$.

In the definition of γ-recursive transfer lower semicontinuity, y is transferred to z^0 that could be any point in X. γ-recursive transfer lower semicontinuity implies that, whenever $\phi(x, y) > \gamma$, there exists a starting point z^0 such that any γ-recursive upsetting chain $\{z^0, z^1, z^2, \ldots, z^m\}$ disproves the

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possibility of an equilibrium in a sufficiently small neighborhood of y, i.e., all points in the neighborhood are γ-upset by all securing points that directly or indirectly upset z^0. This implies that, if ϕ is not γ-recursively transfer lower semicontinuous, then there is a point y such that for every $z^0 \in X$ and every neighborhood $\mathcal{V}_y$ of y, some point in the neighborhood cannot be γ-upset by a securing point z that directly or indirectly γ-upsets z^0.

REMARK 3.1 Under γ-recursive transfer lower semicontinuity, when $\phi(z, z^{m-1}) > \gamma$, $\phi(z^{m-1}, z^{m-2}) > \gamma$, ..., $\phi(z^1, z^0) > \gamma$, we have not only $\phi(z, \mathcal{V}_y) > \gamma$, but also $\phi(z^{m-1}, \mathcal{V}_y) > \gamma$, ..., $\phi(z^1, \mathcal{V}_y) > \gamma$. That is, any chain of securing points $\{z^1, z^2, \ldots, z^{m-j}\}$ obtained by truncating a γ-recursive upsetting chain $\{z^1, z^2, \ldots, z^{m-1}, z\}$ is also a γ-recursive upsetting chain, including z^1.

Similarly, we can define m-γ-recursive transfer lower semicontinuity in y. A function ϕ: $X \times X \to \mathbb{R} \cup \{\pm\infty\}$ is m-γ-*recursively transfer lower semi-continuous in y* if the phrase "for any z that γ-recursively upsets z^0" in the above definition is replaced by "for any z that m-recursively upsets z^0". Thus, a function ϕ: $X \times X \to \mathbb{R} \cup \{\pm\infty\}$ is γ-recursively transfer lower semicontinuous in y if it is m-γ-recursively transfer lower semicontinuous in y for all $m = 1, 2 \ldots$.

REMARK 3.2 It is clear that γ-transfer lower semicontinuity implies 1-γ-recursive transfer lower semicontinuity by letting $z^0 = y$, but the converse may not be true since y possibly cannot be selected as z^0. Thus, γ-transfer lower semicontinuity (thus lower semicontinuity) is in general stronger than 1-γ-recursive transfer lower semi-continuity. Also, γ-recursive transfer lower semicontinuity neither implies nor is implied by continuity. This point becomes clear when one sees γ-recursive transfer lower semi-continuity is a necessary and sufficient condition for the existence of equilibrium while continuity is neither a necessary nor a sufficient condition for the existence of equilibrium to minimax inequality.

Now we are ready to state our main result on the existence of equilibrium of minimax inequality defined on a general topological space.

THEOREM 3.2 *Let X be a compact subset of a topological space T, $\gamma \in \mathbb{R}$, and ϕ: $X \times X \to \mathbb{R} \cup \{\pm\infty\}$ be a function with $\phi(x, x) \leq \gamma$ for all $x \in X$. Then there exists a point $y^* \in X$ such that $\phi(x, y^*) \leq \gamma$ for all $x \in X$ if and only if ϕ is γ-recursively transfer lower semicontinuous in y.*

PROOF. *Sufficiency* ($\Leftarrow$). Suppose y is not an equilibrium point. Then there is an $x \in X$ such that $\phi(x, y) > \gamma$. Then, by γ-recursive transfer lower semicontinuity of $\phi(\cdot)$ in y, for each $y \in X$, there exists a point z^0 and a neighborhood $\mathcal{V}_y$ such that $\phi(z, \mathcal{V}_y) > 0$ whenever $z^0 \in X$ is γ-recursively upset by z, i.e., for any sequence of recursive points $\{z^0, z^1, \ldots, z^{m-1}, z\}$ with $\phi(z, z^{m-1}) > \gamma$, $\phi(z^{m-1}, z^{m-2}) > \gamma$, ..., $\phi(z^1, z^0) > \gamma$ for $m \geq 1$, we have $\phi(z, \mathcal{V}_y) > \gamma$. Since there is no equilibrium by the contrapositive hypothesis, z^0 is not an equilibrium and thus, by γ-recursive transfer

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lower semicontinuity in y, such a sequence of recursive points $\{z^0, z^1, \ldots, z^{m-1}, z\}$ exists for some $m \geq 1$.

Since X is compact and $X \subseteq \bigcup_{y \in X} \mathcal{V}_y$, there is a finite set $\{y^1, \ldots, y^T\}$ such that $X \subseteq \bigcup_{i=1}^T \mathcal{V}_{y^i}$. For each of such y^i, the corresponding initial point is denoted by z^{0i} so that $\phi(z^i, \mathcal{V}_{y^i}) > \gamma$ whenever z^{0i} is γ-recursively upset by z^i.

Since there is no equilibrium, for each of such z^{0i}, there exists z^i such that $\phi(z^i, z^{0i}) > \gamma$, and then, by 1-$\gamma$-recursive transfer lower semicontinuity, we have $\phi(z^i, \mathcal{V}_{y^i}) > \gamma$. Now consider the set of points $\{z^1, \ldots, z^T\}$. Then, $z^i \notin \mathcal{V}_{y^i}$; otherwise, by $\phi(z^i, \mathcal{V}_{y^i}) > \gamma$, we will have $\phi(z^i, z^i) > \gamma$, a contradiction. So we must have $z^1 \notin \mathcal{V}_{y^1}$.

Without loss of generality, we suppose $z^1 \in \mathcal{V}_{y^2}$. Since $\phi(z^2, z^1) > \gamma$ (by noting that $z^1 \in \mathcal{V}_{y^2}$) and $\phi(z^1, z^{01}) > 0$, then, by 2-γ-recursive transfer lower semicontinuity, we have $\phi(z^2, \mathcal{V}_{y^1}) > \gamma$. Also, $\phi(z^2, \mathcal{V}_{y^2}) > 0$. Thus $\phi(z^2, \mathcal{V}_{y^1} \cup \mathcal{V}_{y^2}) > \gamma$, and consequently $z^2 \notin \mathcal{V}_{y^1} \cup \mathcal{V}_{y^2}$.

Again, without loss of generality, we suppose $z^2 \in \mathcal{V}_{y^3}$. Since $\phi(z^3, z^2) > \gamma$ by noting that $z^2 \in \mathcal{V}_{y^3}$, $\phi(z^2, z^1) > \gamma$, and $\phi(z^1, z^{01}) > \gamma$, then, by 3-$\gamma$-recursive transfer lower semicontinuity, we have $\phi(z^3, \mathcal{V}_{y^1}) > \gamma$. Also, since $\phi(z^3, z^2) > \gamma$ and $\phi(z^2, z^{02}) > \gamma$, by 2-$\gamma$-recursive transfer lower semicontinuity, we have $\phi(z^3, \mathcal{V}_{y^2}) > \gamma$. Thus, $\phi(z^3, \mathcal{V}_{y^1} \cup \mathcal{V}_{x^2} \cup \mathcal{V}_{y^3}) > \gamma$, and consequently $z^3 \notin \mathcal{V}_{y^1} \cup \mathcal{V}_{y^2} \cup \mathcal{V}_{y^3}$.

With this process going on, we can show that $z^k \notin \mathcal{V}_{y^1} \cup \mathcal{V}_{y^2} \cup \ldots, \cup \mathcal{V}_{y^k}$, i.e., z^k is not in the union of $\mathcal{V}_{y^1}, \mathcal{V}_{y^2}, \ldots, \mathcal{V}_{y^k}$ for $k = 1, 2, \ldots, T$. In particular, for $k = T$, we have $z^L \notin \mathcal{V}_{y^1} \cup \mathcal{V}_{y^2} \ldots \cup \mathcal{V}_{y^T}$ and so $z^T \notin X \subseteq \mathcal{V}_{y^1} \cup \mathcal{V}_{y^2} \ldots \cup \mathcal{V}_{y^T}$, a contradiction.

Thus, there exists $y^* \in X$ such that $(x, y^*) \leq \gamma$ for all $x \in X$, and thus y^* is an equilibrium point of the minimax inequality.

Necessity ($\Rightarrow$). Suppose y^* is an equilibrium and $\phi(x, y) > \gamma$ for $x, y \in X$. Let $z^0 = y^*$ and $\mathcal{V}_y$ be a neighborhood of y. Since $\phi(x, y^*) \leq \gamma$ for all $x \in X$, it is impossible to find any sequence of finite points $\{z^0, z^1, z^2, \ldots, z^m\}$ such that $\phi(z^1, x^0) > \gamma, \phi(z^2, z^1) > \gamma, \ldots, \phi(z^m, z^{m-1}) > 0$. Hence, the γ-recursive transfer lower semicontinuity holds trivially. ∎

Although γ-recursive transfer lower semicontinuity is necessary for the existence of solution to the problem, it may not be sufficient for the existence of equilibrium when a choice space X is noncompact. To see this, consider the following counterexample.

EXAMPLE 3.1 Let $X = (0, 1)$ and $\phi \colon X \times X \to \mathbb{R} \cup \{\pm\infty\}$ be defined by

$$\phi(x, y) = x - y.$$

The minimax inequality clearly does not possess an equilibrium. However, it is 0-recursively transfer lower semicontinuous in y.

Indeed, for any two points $x, y \in X$ with $\phi(x, y) = x - y > 0$, choose $\epsilon > 0$ such that $(y - \epsilon, y + \epsilon) \subset X$. Let $z^0 = y + \epsilon \in X$ and $\mathcal{V}_y \subseteq (y - \epsilon, y + \epsilon)$. Then, for any finite set

of points $\{z^0, z^1, z^2, \ldots, z^{m-1}, z\}$ with $\phi(z^1, z^0) = z^1 - z^0 > 0$, $\phi(z^2, z^1) = z^2 - z^1 > 0$, $\ldots$, $\phi(z, z^{m-1}) = z - z^{m-1} > 0$, i.e., $z > z^{m-1} > \ldots > z^0$, we have $\phi(z, y') = z - y' > z^0 - y' > 0$ for all $y' \in \mathcal{V}_y$. Thus, $\phi(z, \mathcal{V}_y) > 0$, which means ϕ is 0-recursively transfer lower semicontinuous in y.

The above theorem assumes that X is compact. This may still be a restrictive assumption since a choice space may not be closed or bounded. In this case, we cannot use Theorem 3.2 to fully characterize the existence of solution to a minimax inequality.

Nevertheless, Theorem 3.2 can be extended to any topological choice space. To do so, we introduce the following version of γ-recursive transfer lower semicontinuity.

DEFINITION 3.4 Let X be a set of a topological space T and $D \subseteq X$. A function ϕ: $X \times X \to \mathbb{R} \cup \{\pm\infty\}$ is said to be γ-recursively transfer lower semicontinuous in y with respect to D if, whenever $y \in X$ is not an equilibrium, there exists a point $z^0 \in X$ (possibly $z^0 = y$) and a neighborhood $\mathcal{V}_y$ such that (1) whenever z^0 is upset by a point in $X \setminus D$, it is upset by a point in D, and (2) $\phi(z, \mathcal{V}_y) > \gamma$ for any finite subset of securing points $\{z^0, z^1, \ldots, z^m\} \subseteq D$ with $z^m = z$ and $\phi(z, z^{m-1}) > \gamma$, $\phi(z^{m-1}, z^{m-2}) > \gamma$, $\ldots$, $\phi(z^1, z^0) > \gamma$ for $m \geq 1$.

Now we have the following theorem that fully characterizes the existence of solution to a minimax inequality.

THEOREM 3.3 *Let X be a set of a topological space T, $\gamma \in \mathbb{R}$, and ϕ: $X \times X \to \mathbb{R} \cup \{\pm\infty\}$ be a function. Suppose $\phi(x, x) \leq \gamma$ for all $x \in X$. Then there is a point $y^* \in X$ such that $\phi(x, y^*) \leq \gamma$ for all $x \in X$ if and only if there exists a compact subset $D \subseteq X$ such that ϕ is γ-recursively transfer lower semicontinuous in y with respect to D.*

PROOF. *Sufficiency* ($\Leftarrow$). The proof of sufficiency is essentially the same as that of sufficiency in Theorem 3.2 and we just outline the proof here. To show the existence of an equilibrium on X, it suffices to show that there exists an equilibrium y^* in D if it is γ-recursively transfer lower semicontinuous in y with respect to D. Suppose, by way of contradiction, that there is no equilibrium in D. Then, since ϕ is γ-recursively transfer lower semicontinuous in y with respect to D, for each $y \in D$, there exists z^0 and a neighborhood $\mathcal{V}_y$ such that (1) whenever z^0 is γ-upset by a point in $X \setminus D$, it is γ-upset by a point in D and (2) $\phi(z, \mathcal{V}_y) > \gamma$ for any finite subset of points $\{z^0, z^1, \ldots, z^m\} \subseteq D$ with $z^m = z$ and $\phi(z, z^{m-1}) > \gamma$, $\phi(z^{m-1}, z^{m-2}) > \gamma$, $\ldots$, $\phi(z^1, z^0) > \gamma$ for $m \geq 1$. Since there is no equilibrium by the contrapositive hypothesis, z^0 is not an equilibrium point and thus, by γ-recursive transfer lower semicontinuity in y with respect to D, such a sequence of recursive securing points $\{z^0, z^1, \ldots, z^{m-1}, y\}$ exists for some $m \geq 1$.

Since D is compact and $D \subseteq \bigcup_{y \in X} \mathcal{V}_y$, there is a finite set $\{y^1, \ldots, y^T\} \subseteq D$ such that $D \subseteq \bigcup_{i=1}^{T} \mathcal{V}_{y^i}$. For each of such y^i, the corresponding initial point is denoted by z^{0i} so that

$\phi(z^i, \mathcal{V}_{x^i}) > 0$ whenever z^{0i} is recursively upset by z^i through any finite subset of securing points $\{z^{0i}, z^{1i}, \ldots, z^{mi}\} \subseteq D$ with $z^{mi} = z^i$. Then, by the same argument as in the proof of Theorem 3.2, we will obtain that z^k is not in the union of $\mathcal{V}_{y^1}, \mathcal{V}_{y^2}, \ldots, \mathcal{V}_{y^k}$ for $k = 1, 2, \ldots, T$. For $k = T$, we have $z^T \notin \mathcal{V}_{y^1} \cup \mathcal{V}_{y^2} \ldots \cup \mathcal{V}_{y^T}$ and so $z^T \notin D \subseteq \bigcup_{i=1}^{T} \mathcal{V}_{y^i}$, which contradicts that z^T is an upsetting point in D.

Thus, there exists a point $y^* \in X$ such that $\phi(x, y^*) \leq \gamma$ for all $x \in X$.

Necessity ($\Rightarrow$). Suppose y^* is an equilibrium. Let $D = \{y^*\}$. Then, the set D is clearly compact. Now, for any disequilibrium point $q \in \Delta$, let $z^0 = y^*$ and $\mathcal{V}_y$ be a neighborhood of y. Since $(x, y^*) \leq \gamma$ for all $x \in X$ and $z^0 = y^*$ is a unique element in D, there is no other γ-upsetting point z^1 such that $\phi(x, z^0) > \gamma$. Hence, ϕ is γ-recursively transfer continuous in y with respect to D. ∎

COROLLARY 3.1 (GENERALIZED KY FAN'S MINIMAX INEQUALITY) *Let X be a subset of a topological space T, ϕ: $X \times X \to \mathbb{R} \cup \{\pm\infty\}$ be a function, and $\gamma = \sup_{y \in X} \phi(y, y)$. Then there is a point $y^* \in X$ such that $\phi(x, y^*) \leq \sup_{y \in X} \phi(y, y)$ for all $x \in X$ if and only if there exists a compact subset $D \subseteq X$ such that ϕ is γ-recursively transfer lower semicontinuous in y with respect to D.*

Theorem 3.3 and Corollary 3.1 thus strictly generalize many existing results on the minimax inequality such as those in Allen [1], Ansari et al. [2], Ansari et al. [3], Chebbi [6], Cho et al. [7], Ding and Park [8], Ding [9], Fan [10, 11, 13], Lignola [22], Lin and Chang [23], Nessah and Tian [25], Tian [30], Yuan [38], Zhou and Chen [39].

The following example about game theory shows that, although the strategy space of a game is an open unit interval and the payoff function is highly discontinuous and nonquasiconcave, we can use Theorem 3.3 to argue the existence of Nash equilibrium.

EXAMPLE 3.2 [Tian 33] Consider a game with $n = 2$, $X = X_1 \times X_2 = (0, 1) \times (0, 1)$ that is an open unit interval set, and the payoff functions are defined by

$$u_i(x_1, x_2) = \begin{cases} 1 & \text{if } (x_1, x_2) \in \mathbb{Q} \times \mathbb{Q} \\ 0 & \text{otherwise} \end{cases} \quad i = 1, 2,$$

where $\mathbb{Q} = \{x \in (0, 1) : x \text{ is a rational number}\}$.

Let $U(x, y) = u_1(y_1, x_2) + u_2(x_1, y_2)$ and then define a function ϕ: $(0, 1) \times (0, 1) \to \mathbb{R}$ by

$$\phi(x, y) = u_1(y_1, x_2) + u_2(x_1, y_2) - u_1(y_1, y_2) - u_2(y_1, y_2).$$

Then ϕ is neither γ-(transfer) lower semicontinuous in y nor γ-(transfer) quasiconcave in x. However, it is 0-recursively transfer lower semicontinuous in y on X. Indeed, suppose $\phi(y, x) > 0$ for $x = (x_1, x_2) \in X$ and $y = (y_1, y_2) \in X$. Let z^0 be a vector with rational numbers, $B = \{z^0\}$, and $\mathcal{V}_y$ be a neighborhood of y. Since $\phi(x, z^0) \leq 0$ for all $x \in X$, it is impossible to find any securing

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strategy profile z^1 such that $\phi(z^1, y^0) > 0$. Hence, ϕ is 0-recursively transfer lower semicontinuous in y on X with respect to B. Therefore, by Theorem 3.3, there exists $\bar{y} \in X$ such that

$$\phi(x, \bar{y}) \leq 0$$

for all $x \in X$. In particular, letting $x_1 = \bar{y}_1$ and keeping x_2 vary leads to

$$u_2(\bar{y}_1, x_2) \leq u_2(\bar{y}_1, \bar{y}_2) \qquad \forall x_2 \in X_2,$$

and letting $x_2 = \bar{y}_2$ and keeping x_1 vary leads to

$$u_1(x_1, \bar{y}_2) \leq u_1(\bar{y}_1, \bar{y}_2) \qquad \forall x_1 \in X_1.$$

Hence, this game possesses a Nash equilibrium. In fact, the set of Nash equilibria consists of all rational numbers on $(0, 1)$.

4   Full Characterization of Fixed Point

This section provides necessary and sufficient conditions for the existence of fixed point of a function defined on a set that may be finite, continuum, nonconvex, or noncompact.

Let T be a topological space, and X be a subset of T. A correspondence $F : X \to 2^T$ has a fixed point $x \in X$ if $x \in F(x)$. If F is a single-valued function, then a fixed point x of F is characterized by $x = F(x)$.

We first recall the notion of diagonal transfer continuity introduced by Baye et al. [4].

DEFINITION 4.1 A function $\varphi : X \to \mathbb{R} \cup \{\pm\infty\}$ is diagonally transfer continuous in y if, whenever $\varphi(x, y) > \varphi(y, y)$ for $x, y \in X$, there exists a point $z \in X$ and a neighborhood $\mathcal{V}_y \subset X$ of y such that $\varphi(z, y') > \varphi(y', y')$ for all $y' \in \mathcal{V}_y$.

Similarly, we can define the notion of recursive diagonal transfer continuity.

DEFINITION 4.2 (**Recursive Diagonal Transfer Continuity**) Let X be a subset of a topological space T. A function $\varphi \colon X \times X \to \mathbb{R} \cup \{\pm\infty\}$ is said to be *recursively diagonally transfer continuous in y* if, whenever $\varphi(x, y) > U(y, y)$ for $x, y \in X$, there exists a point $z^0 \in X$ (possibly $z^0 = y$) and a neighborhood $\mathcal{V}_y$ of y such that $\varphi(z, \mathcal{V}_y) > \varphi(\mathcal{V}_y, \mathcal{V}_y)$ for any z that recursively upsets z^0.

THEOREM 4.1 *(Fixed Point Theorem) Let X be a nonempty and compact subset of a metric space (E, d) and $f : X \to X$ be a function. Then, f has a fixed point if and only if the function $\varphi \colon X \times X \to \mathbb{R} \cup \{\pm\infty\}$, defined by*

$$\varphi(x, y) = -d(x, f(y))$$

is recursively diagonally transfer continuous in y.

Proof. Define $\phi : X \times X \to \mathbb{R}$ by

$$\phi(x, y) = d(y, f(y)) - d(x, f(y)).$$

Then $\phi(x, x) = 0$ for all $x \in X$. Also, we can easily see ϕ is 0-recursively transfer lower semicontinuous in y if and only if φ is recursively diagonally transfer continuous in y. Then, by Theorem 3.2, there exists $\bar{y}$ such that

$$\phi(x, \bar{y}) \leq 0 \quad \forall x \in X$$

or equivalently

$$d(\bar{y}, f(\bar{y})) \leq d(x, f(\bar{y})) \quad \forall x \in X$$

if and only if φ is recursively diagonally transfer continuous in y. Letting $f(\bar{y}) = \bar{x}$, we have

$$d(\bar{y}, f(\bar{y})) \leq d(\bar{x}, f(\bar{y})) = 0$$

and thus $\bar{y} = f(\bar{y})$. Therefore, f has a fixed point if and only if the function $-d(x, f(y))$ is recursively diagonally transfer continuous in y.

Theorem 4.1 can be generalized by relaxing the compactness of X.

THEOREM 4.2 *Let X be a nonempty subset of a metric space (E, d) and $f : X \to X$ be a function. Then, f has a fixed point if and only if there exists a compact set $D \subseteq X$ such that $-d(x, f(y))$ is recursively diagonally transfer continuous in y with respect to D.*

Proof. The proof is the same as in Theorem 3.3, and it is omitted here.

Theorem 3.3 and Corollary 3.1 strictly generalize many existing fixed point theorems in the literature, including those well-known theorems such as Browder fixed point theorem, Tarski fixed point theorem in [28], and other fixed point theorems such as those in Fan [10, 11, 12, 13], Halpern [14, 15], Halpern and Bergman [16], Istrăţescu [17] and the references therein.

5 Full Characterization of Saddle Point

The saddle point theorem is an important tool in variational problems and game theory. Much work has been dedicated to the problem of weakening its existence conditions. However, almost all these results assume that a function is defined on convex set. In this section, we present some existence theorems on saddle point without imposing any form of convexity conditions.

DEFINITION 5.1 A pair $(\bar{x}, \bar{y})$ in $X \times X$ is called a saddle point of f in $X \times X$, iff,

$$\phi(\bar{x}, y) \leq \phi(\bar{x}, \bar{y}) \leq \phi(x, \bar{y}) \text{ for all } x \in X \text{ and } y \in X.$$

This definition reflects the fact that each player is individualistic.

DEFINITION 5.2 (γ-**Recursive Transfer Upper Semicontinuity**) Let X be a subset of a topological space T. A function $\phi\colon X \times X \to \mathbb{R} \cup \{\pm\infty\}$ is said to be γ-recursively transfer upper semicontinuous in x if, whenever $\phi(x,y) < \gamma$ for $x,y \in X$, there exists a point $z^0 \in X$ (possibly $z^0 = x$) and a neighborhood $\mathcal{V}_x$ of x such that $\phi(\mathcal{V}_x, z) < \gamma$ for any sequence of points $\{z^0, z^1, \ldots, z^{m-1}, z\}$, $\phi(z^{m-1}, z) < \gamma$, $\phi(z^{m-2}, z^{m-1}) < \gamma$, …, $\phi(z^0, z^1) < \gamma$ for $m \geq 1$ implies that $\phi(\mathcal{V}_x, z) < 0$. We can similarly define γ-recursive transfer upper semicontinuity in x with respect to $D \subseteq X$.

Before giving our new results, we state the classical result on saddle point.

THEOREM 5.1 *(von Neumann Theorem). Let X be nonempty, compact and convex subsets in a Hausdorff locally convex vector space E, and $\phi\colon X \times X \to \mathbb{R} \cup \{\pm\infty\}$ be a function. Suppose that*

(a) ϕ is lower semicontinuous and quasiconvex in y,

(b) ϕ is upper semicontinuous and quasiconcave in x.

Then, ϕ has a saddle point.

Also, a lot of work has been done by weakening the conditions of semi-continuity and/or quasi-concavity/quasiconvexity of von Neumann Theorem. Here, we give a theorem that fully characterizes the existence of saddle point for a general topological space without assuming any kind of quasiconvexity or quasiconcavity.

THEOREM 5.2 *Let X be a compact subset of a topological space T, $\gamma \in \mathbb{R}$, and $\phi\colon X \times X \to \mathbb{R} \cup \{\pm\infty\}$ be a function with $\phi(x,x) \leq \gamma$ for all $x \in X$. Then there exists a saddle point $(\bar{x}, \bar{y}) \in X \times X$ if and only if ϕ is γ-recursively transfer upper semicontinuous in x and γ-recursively transfer lower semicontinuous in y.*

Proof. Applying Theorem 3.2 to $\phi(x,y)$, we have the existence of $\bar{y} \in X$ such that

$$\phi(x, \bar{y}) \leq \gamma, \quad \forall x \in X. \tag{1}$$

Let $\psi(x,y) = -\phi(y,x)$. Since ϕ is γ-recursively transfer upper semicontinuous in x, ψ is $-\gamma$-recursively transfer lower semicontinuous in x.

Applying Theorem 3.2 again to $\psi(x,y)$, we have the existence of $\bar{x} \in X$ such that

$$\phi(\bar{x}, y) \geq \gamma, \quad \forall y \in X. \tag{2}$$

By (1) and (2), we have $\phi(\bar{x}, \bar{y}) \leq \gamma$ and $\phi(\bar{x}, \bar{y}) \geq \gamma$, respectively, and therefore $\phi(\bar{x}, \bar{y}) = \gamma$. Thus, $(\bar{x}, \bar{y})$ is a saddle point satisfying

$$\phi(\overline{x}, y) \leq \phi(\overline{x}, \overline{y}) \leq \phi(x, \overline{y}) \text{ for all } x \in X \text{ and } y \in X.$$

Theorem 5.2 can also be generalized by relaxing the compactness of X.

THEOREM 5.3 *Let X be a subset of a topological space T, $\gamma \in \mathbb{R}$, and $\phi\colon X \times X \to \mathbb{R} \cup \{\pm\infty\}$ be a function with $\phi(x, x) \leq \gamma$ for all $x \in X$. Then there exists a saddle point $(\bar{x}, \bar{y}) \in X \times X$ if and only if there exist two compact sets D_1 and D_2 in X such that ϕ is γ-recursively transfer upper semicontinuous in x with respect to D_1 and γ-recursively transfer lower semicontinuous in y with respect to D_2.*

Proof. The proof is the same as in Theorem 3.3, and it is omitted here.

6   Full Characterization of FKKM Theorem

In this section, we use Theorems 3.2 and 3.3 to generalize the FKKM theorem, which provide sufficient conditions on noncompact and nonconvex sets.

We begin by stating the FKKM theorem due to Fan [12, 13].

THEOREM 6.1 (FKKM Theorem) *In a Hausdorff topological vector space, let Y be a convex set and $\emptyset \neq X \subset Y$. Let $F : X \to 2^Y$ be a correspondence such that*

> *(a) for each $x \in X$, $F(x)$ is a relatively closed subset of Y;*
>
> *(b) F is FS-convex on X;*
>
> *(c) there is a nonempty subset X_0 of X such that the intersection $\bigcap_{x \in X_0} F(x)$ is compact and X_0 is contained in a compact convex subset of Y.*

Then $\bigcap_{x \in X} F(x) \neq \emptyset$.

Also, this theorem has been generalized in various forms in the literature. In the following, we provide a full characterization of FKKM theorem in a special form where F is a correspondence mapping from X to X.

THEOREM 6.2 *Let X be a nonempty compact subset of a topological space T and $F : X \to 2^X$ be a correspondence such that $x \in F(x)$ for all $x \in X$. Then $\prod_{x \in X} F(x) \neq \emptyset$ if and only if the function $\phi\colon X \times X \to \mathbb{R} \cup \{\pm\infty\}$ defined by*

$$\phi(x, y) = \begin{cases} \gamma & \text{if } (x, y) \in G \\ +\infty & \text{otherwise} \end{cases},$$

where $\gamma \in \mathbb{R}$ and $G = \{(x, y) \in X \times Y : y \in F(x)\}$, is γ-recursively transfer lower semicontinuous in y.

Proof. Since $x \in F(x)$ for all $x \in X$, we have $\phi(x, x) \leq \gamma$ for all $x \in X$. Then, by Theorem 3.2, there exists a point $y^* \in X$ such that $\phi(x, y^*) \leq \gamma$ for all $y \in X$ if and only if ϕ is γ-recursively transfer lower semicontinuous in y. However, by construction, $\prod_{x \in X} F(X) \neq \emptyset$ if and only if there exists a point $x^* \in X$ such that $\phi(x, x^*) \leq \gamma$ for all $y \in X$. ∎

Similarly, we can drop the compactness of X, and have the following theorem.

THEOREM 6.3 *Let X be a nonempty subset of a topological space T and $F : X \to 2^X$ be a correspondence such that $x \in F(x)$ for all $x \in X$. Then $\prod_{x \in X} F(x) \neq \emptyset$ if and only if there exists a compact subset $D \subseteq X$ such that $\phi: X \times X \to \mathbb{R} \cup \{\pm\infty\}$ defined by*

$$\phi(x, y) = \begin{cases} \gamma & \text{if } (x, y) \in G \\ +\infty & \text{otherwise} \end{cases} \quad,$$

where $\gamma \in \mathbb{R}$ and $G = \{(x, y) \in X \times Y : y \in F(x)\}$, is γ-recursively transfer lower semicontinuous in y with respect to D.

References

1. Allen, G., "Variational inequalities, complementarity problems, and duality theorems," *Journal of Mathematical Analysis and Applications* 58 (1977), 1-10.

2. Ansari, Q.H., Lin, Y.C., Yao, J.C., "General KKM Theorem with Applications to Minimax and Variational Inequalities," *Journal of Optimization Theory and Applications*, 104 (2000), 41-57.

3. Ansari, Q.H., Wong, N.C., Yao, J.C.: "The Existence of Nonlinear Inequalities," *Applied Mathematics Letters*, 12 (1999), 89-92.

4. Baye, M. B., Tian, G., and Zhou, J., "Characterizations of the Existence of Equilibria in Games with Discontinuous and Nonquasiconcave Payoffs," *Review of Economic Studies*, 60 (1993), 935-948.

5. Border, K. C., Fixed Point Theorems with Applications to Economics and Game Theory, Cambridge University Press, 1985.

6. Chebbi, S., "Minimax Inequality and Equilibria with a Generalized Coercivity," *Journal of Applied Analysis* 12 (2006), 117-125.

7. Cho, Y. J., Delavar, M. R., Mohammadzadeh, S. A., and Roohi, M., "Coincidence theorems and minimax inequalities in abstract convex spaces," *Journal of Inequalities and Applications*, 2011, 2011:126.

8. Ding, X.P., Park, S., "Existence of solutions for nonlinear inequalities in G-convex spaces," *Applied Mathematics Letters*, **15** (1998), 735–741.

9. Ding, X.P., Tan, K.K., "A minimax inequality with application to existence of equilibrium points and fixed point theorems," *Colloque Mathematics* **63** (1992), 233–274.

10. Fan, K., "Minimax theorem," *Proc. Nat. Acad. Sci*. 39 (1953), 42-47.

11. Fan, K., Minimax inequality and applications, *in: O. Shisha (Ed.), Inequality, vol. III, Academic Press, New York*, 103–113, 1972.

12. Fan, K., Fixed point and related theorems for non compact sets, in "Game theory and related Topics," (ed.) by O. Moeschlin and D. Pallaschke, North Holland, Amsterdam, 151-156, 1979.

13. Fan, K., "Some properties of convex sets related to fixed points theorems," *Math. Ann.* 266 (1984), 519-537.

14. Halpern, B., Fixed-point theorems for outward maps, doctoral Thesis, U.C.L.A., (1965)

15. Halpern, B., "Fixed point theorems for set-valued maps in infinite dimensional spaces," *Math. Ann.* 189 (1970), 87-98.

16. Halpern B., Bergman, G., "A fixed-point theorem for inward and outward maps," *Trans. amer. Math. Soc.* 130 (1968), No. 2, 353-358.

17. Istrăţescu, V. I., "Fixed point theory," D. Reidel Publishing Company, 1981.

18. Iusem, A. N. Soca, W., "New existence results for equilibrium problems," *Nonlinear Analysis* **54** (2003), 621-635.

19. Georgiev, P. G., Tanaka, T., "Vector-valued set-valued variants of Ky Fan's inequality," *Journal of Nonlinear and Convex Analysis* **1** (2000), 245–254.

20. Karamardian, S., "Generalized complementarity problem," *Journal of Optimization Theory and Applications* 8 (1971), 416-427.

21. Knaster, B., Kuratowski, C., Mazurkiewicz, S., "ein beweis des fixpunktsatze n demensionale simpliexe," *Fund. Math.* 14 (1929), 132-137.

22. Lignola, M. B., "Ky Fan Inequalities and Nash Equilibrium Points without Semicontinuity and Compactness," *Journal of Optimization Theory and Applications* **94** (1997), 137-145.

23. Lin, L. J., Chang, T. H., "S-KKM theorems, saddle points and minimax inequalities," *Nonlinear Analysis* **34** (1998), 86-73.

24. Lin J., Tian, G., "Minimax Inequality Equivalent to the Fan-Knaster-Kuratowski-Mazurkiewicz Theorem," *Applied Mathematics and Optimization*, 28 (1993), 173-179.

25. Nessah R., Tian, G., "Existence of Solution of Minimax Inequalities, Equilibria in Games and Fixed Points without Convexity and Compactness Assumptions," *Journal of Optimization Theory and Applications*, 157 (2013): 75-95.

26. Shafer, W., Sonnenschein, H., "Equilibrium in abstract economies without ordered preferences," *Journal of Mathematical Economics* 2 (1975), 345-348.

27. Sonnenschein, H., Demand theory without transitive preferences, with application to the theory of competitive equilibrium, in *Preferences, Utility, and Demand*, (ed.) by J. S. Chipman, L. Hurwicz, M. K. Richter, and H. Sonnenschein, New York: Harcourt Brace Jovanovich, 1971.

28. Tarski, A., "A Lattice-Theoretical Fixpoint Theorem and Its Applications," *Pacific J. Math.* 5 (1955), 285-309.

29. Tian, G., "Fixed points theorems for mappings with non compact and non convex domains," *Journal of Mathematical Analysis and Applications*, 158 (1991), 161-167.

30. Tian, G., "Generalizations of the FKKM Theorem and Ky-Fan Minimax Inequality, with Applications to Maximal Elements, Price Equilibrium, and Complementarity," *Journal of Mathematical Analysis and Applications*, 170 (1992), 457-471.

31. Tian, G., "Necessary and Sufficient Conditions for Maximization of a Class of Preference Relations," *Review of Economic Studies*, 60 (1993), 949-958.

32. Tian, G., "Generalized KKM Theorem and Minimax Inequalities and Their Applications," *Journal of Optimization Theory and Applications*, 83 (1994), 375-389.

33. Tian, G., "Existence of Equilibria in Games with Arbitrary Strategy Spaces and Preferences," working paper.

34. Tian, G., Zhou, J., "Quasi-Variational Inequalities with Non-Compact Sets," *Journal of Mathematical Analysis and Applications*, 160 (1991), 583-595.

35. Tian, G., Zhou, J., "The Maximum Theorem and the Existence of Nash Equilibrium of (Generalized) Games without Lower Semicontinuities," *Journal of Mathematical Analysis and Applications*, 166 (1992), 351-364.

36. Tian, G., Zhou, Z., "Quasi-inequalities without the concavity assumption," *Journal of Mathematical Analysis and Applications*, **172** (1993), 289-299.

37. Tian, G., Zhou, J., "Transfer Continuities, Generalizations of the Weierstrass Theorem and Maximum Theorem–A Full Characterization," *Journal of Mathematical Economics*, 24 (1995), 281-303.

38. Yuan, X. Z., "KKM principal, Ky Fan minimax inequalities and fixed point theorems," *Nonlinear World*, **2** (1995), 131-169.

39. Zhou, J., Chen, G., "Diagonal convexity conditions for problems in convex analysis and quasi variational inequalities," *Journal of Mathematical Analysis and Applications*, 132 (1988), 213-225.

40. Zhou, J. X., Tian, G., "Transfer Method for Characterizing the Existence of Maximal Elements of Binary Relations on Compact or Noncompact Sets," *SIAM Journal on Optimization*, 2 (1992), 360–375.