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Abstract

This paper provides a price equilibrium existence theorem in economies where commodities may be indivisible and aggregate excess demand functions may be discontinuous. We introduce a very weak notion of continuity, called *recursive transfer lower semi-continuity*, which is weaker than transfer lower semi-continuity and in turn weaker than lower semi-continuity. It is shown that the condition, together with Walras's law, guarantees the existence of price equilibrium in economies with excess demand functions. The condition is also necessary, and thus our results generalize all the existing results on the existence of price equilibrium in economies where excess demand is a function.

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Keywords: Existence of price equilibrium; recursive transfer lower semi-continuity; discontinuity; excess demand function.

1 Introduction

This paper presents a theorem on the existence of price equilibrium in terms of excess demand function, which fully characterizes the existence of equilibrium price systems in economies where commodities may be indivisible and aggregate excess demand functions may not be continuous.

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One of the great achievements of economic theory in the last century is the general equilibrium theory. It aims at studying the behavior of demand, supply, and prices in a whole economy, by considering equilibrium in many markets simultaneously. It is a benchmark model to study market economy and also an abstraction from a real economy. It can be used for either considering equilibrium prices as long-term prices or considering actual prices as deviations from equilibrium.

A price equilibrium is defined as a state where the aggregate demand does not exceed the aggregate supply for all markets. The proof of the existence of general equilibrium is generally considered one of the most important and robust results of economic theory. While there are different ways of establishing the existence of general equilibrium, all the classic proofs use a fixed point theorem (see Debreu, 1982). It includes the 'excess demand approach' which solves this problem by showing that there is a price system at which excess demand can be non-positive. The significance of such an approach lies in the fact that supply may not be continuous or even not be necessarily derived from profit maximizing behavior of price taking firms, but is determined by prices in completely different ways. It is well known that Walrasian equilibrium precludes the existence of an equilibrium in the presence of increasing returns to scale and assumes price-taking and profit-maximizing behavior. As such, some other alternative pricing rules have been proposed such as loss-free, average cost, marginal cost, voluntary trading, and quantity-taking pricing rules in the presence of increasing returns to scale or more general types of non-convexities (cf. Beato (1982), Brown and Heal (1982), Cornet (1988, 1989), Bonnisseau (1988), Bonnisseau and Cornet (1988, 1990), Kamiya (1988), Vohra (1988), Brown (1990), and Brown, Heller, and Starr (1992)).

At the heart of the excess demand approach is a technical result known as the Gale-Nikaido-Debreu lemma. Many existence results in terms of excess demand functions or correspondences have been given. Some use Kakutani's Fixed Point Theorem, as in Debreu (1974, 1982, 1983), while some others use modifications of the excess demand functions, as in Dierker (1974), McKenzie (1954), and Neuefeind (1980).

When preferences and production sets are strictly convex, excess demand from the Walrasian pricing rule is a function rather than a correspondence. In this case, it can be defined only on the open price simplex. Some then uses a technique that simplex is exhausted by an increasing sequence of compact subsets so that the Gale-Nikaido-Debreu lemma can be applied. The resulting sequence of price systems then is shown to converge to an equilibrium price system (cf. Hildenbrand (1983) and Hildenbrand and Kirman (1988)). However, all the existing results only provide sufficient conditions for the existence of price equilibrium.

This paper fully characterizes the existence of price equilibrium in economies where commodities may be indivisible and excess demand functions may be discontinuous or do not have any structure except Walras' Law. We establish a condition, called *recursive transfer lower semi-continuity*, which is weaker than transfer lower semi-continuity and in turn weaker than lower semi-continuity, for the existence of general equilibrium in such economies. The condition is also necessary, and thus generalizes all the existing results on the existence of equilibrium in economies with aggregate excess demand functions.

The basic transfer method is systematically developed in Tian (1992a, 1993), Tian and Zhou (1992, 1995), Zhou and Tian (1992), and Baye, Tian, and Zhou (1993) for studying the maximization of binary relations that may be nontotal or nontransitive and the existence of equilibrium in games that may have discontinuous or nonquasiconcave payoffs. The notion of recursive transfer continuity extends usual transfer continuity from single transfer to allowing recursive (sequential) transfers so that it turns out to be a necessary and sufficient condition for the existence of price equilibrium in economies with excess demand functions. This method is somewhat similar to extending the weak axiom of revealed preference (WARP) to strong axiom of revealed preference (SARP) in order to fully reveal individuals' preferences.

Roughly speaking, an excess demand function is recursively transfer lower semi-continuous if, whenever q is not a price equilibrium, there exists another nonequilibrium price vector p^0 such that all excess demands in some neighborhood of q are not affordable at any price vector p that recursively upsets p^0. Here, a price system p upsets a price system q if q's excess demand is not affordable at price p. A non-equilibrium price vector p^0 is recursively upset by p means that there exist finite upsetting price vectors $p^1, p^2, \dots, p^m$ with $p^m = p$ such that p^0's excess demand is not affordable at p^1, p^1's excess demand is not affordable at p^2, and p^{m-1}'s excess demand is not affordable at p^m. This implies that p^0 is upset by p^1, p^1 is upset by p^2, ..., p^{m-1} is upset by p. When the number of such securing price vectors is m, the aggregate excess demand function is then called m-recursive transfer semi-continuity. Note that transfer semi-continuity introduced by Tian (1992a) implies 1-recursive transfer semi-continuity.

The remaining of the paper is organized as follows. In Section 2 we present the notion of recursive transfer lower semi-continuity. In Section 3 we prove that recursive transfer lower semi-continuity is a necessary and sufficient condition for the existence of equilibrium. Concluding remarks are offered in Section 4.

2 Notation and Definitions

Consider an economy where there are L commodities and the aggregate excess demand correspondence from a pricing rule that may be Walrasian, loss-free, average cost, marginal cost, voluntary trading, or quantity-taking pricing rule is a single-valued function. The aggregate excess demand correspondence becomes a function when preferences and production possibility sets are both strictly convex for Walrasian pricing rule.

Let Δ be the closed $L-1$ dimensional unit simplex defined by

$$\Delta = \{p \in \Re^L_+ : \sum_{l=1}^{L} p^l = 1\}, \tag{1}$$

and let $\hat{z}(\cdot) : \Delta \to \mathbb{R}^L$ denote the (aggregate) excess demand function for the economy. A very important property of excess demand function is Walras' law, which can take one of the following three forms:

(1) strong Walras' law

$$p \cdot \hat{z}(p) = 0 \quad \text{for all } p \in \Delta;$$

(2) weak Walras' law

$$p \cdot \hat{z}(p) \leq 0 \quad \text{for all } p \in \Delta;$$

(3) interior Walras' law

$$p \cdot \hat{z}(p) = 0 \quad \text{for all } p \in \text{int } \Delta,$$

where int Δ denotes the set of interior points of Δ.

The equilibrium price problem is to find a price vector p which clears the markets for all commodities (i.e., the aggregate excess demand functions $\hat{z}(p) \leq 0$ for the free disposal equilibrium price or $\hat{z}(p) = 0$) under the assumption of Walras' law.

Let X be a topological space. A function $f : X \to \mathbb{R}$ is said to be *lower semi-continuous* if for each point x', we have

$$\liminf_{x \to x'} f(x) \geq f(x'),$$

or equivalently, if its epigraph epi$f \equiv \{(x, a) \in X \times \mathbb{R} : f(x) \leq a\}$ is a closed subset of $X \times \mathbb{R}$. An excess demand function $\hat{z}(\cdot) : \Delta \to \mathbb{R}^L$ is lower semi-continuous if $\hat{z}^l(\cdot) : \Delta \to \mathbb{R}$ is lower semi-continuous for $l = 1, \ldots, L$.

The following weak notion of lower semi-continuity is introduced by Tian (1992a).

DEFINITION 2.1 An excess demand function $\hat{z}(\cdot) : \Delta \to \mathbb{R}^L$ is transfer lower semi-continuous if for all $q, p \in \Delta$, $p \cdot \hat{z}(q) > 0$ implies that there exists some point $p' \in X$ and some neighborhood $\mathcal{N}(q)$ of q such that $p' \cdot \hat{z}(q') > 0$ for all $q' \in \mathcal{N}(q)$.

Remark 2.1 The transfer lower semi-continuity of $\hat{z}(\cdot)$ means that, whenever the aggregate excess demand $\hat{z}(q)$ at price vector q is not affordable at price vector p, then there exists an other price vector p' such that $\hat{z}(q')$ are also not affordable for all price vectors p', provided q' are sufficiently close to q. Note that, since $p \geq 0$, this condition is satisfied if $\hat{z}(\cdot)$ is lower semi-continuous by letting $p' = p$.

We say that price system p *upsets* price system q if q's excess demand is not affordable at price p, i.e., $p \cdot \hat{z}(q) > 0$.

Definition 2.2 (Recursive Upset Pricing) Let $\hat{z}(\cdot) : \Delta \rightarrow \mathbb{R}^L$be an excess demand function. We say that a price system $p^0 \in \Delta$ is *m-recursively upset by* $p \in \Delta$ if there exist m price systems $p^1, p^2, \ldots, p^m$ with $p^m = p$ such that $p \cdot \hat{z}(p^{m-1}) > 0, p^{m-1} \cdot \hat{z}(p^{m-2}) > 0$, . . ., $p^1 \cdot \hat{z}(p^0) > 0$. We say that $p^0 \in \Delta$ is *recursively upset by* $p \in \Delta$ if there is some $m \geq 1$ such that it is m-recursively upset by p.

In words, non-equilibrium price system p^0 is recursively upset by p means that there exist finite upsetting price systems $p^1, p^2, \ldots, p^m$ with $p^m = p$ such that p^0's excess demand is not affordable at p^1, p^1's excess demand is not affordable at p^2, and p^{m-1}'s excess demand is not affordable at p^m. Note that, by definition, when p^0 is recursively upset by p, it must be a non-equilibrium price system (since $p^1 \cdot \hat{z}(p^0) > 0$).

For convenience, we say p^0 is *directly upset by* p when $m = 1$, and *indirectly upset by* p when $m > 1$. Recursive upsetting says that nonequilibrium price system p^0 can be directly or indirectly upset by a price system q through sequential upsetting price systems $\{p^1, p^2, \ldots, p^{m-1}\}$ in a recursive way that p^0 is upset by p^1, p^1 is upset by p^2, . . ., and p^{m-1} is upset by p.

Definition 2.3 (m-Recursive Transfer Lower Semi-Continuity) An excess demand function $\hat{z}(\cdot) : \Delta \rightarrow \mathbb{R}^L$ is said to be *m-recursively transfer lower semi-continuous* on Δ if for $r \cdot \hat{z}(q) > 0$ with $r, q \in \Delta$, there exists $p^0 \in \Delta$ and a neighborhood $\mathcal{V}_q$ such that $p \cdot \hat{z}(\mathcal{V}_q) > 0$ whenever p^0 is m-recursively upset by $p \in \Delta$, i.e., for any sequence of price vectors $\{p^0, p^1, \ldots, p^{m-1}, p\}$, $p \cdot \hat{z}(p^{m-1}) > 0$, $p^{m-1} \cdot \hat{z}(p^{m-2}) > 0$, . . ., $p^1 \cdot \hat{z}(p^0) > 0$ for $m \geq 1$ imply that $p \cdot \hat{z}(\mathcal{V}_q) > 0$. Here $p \cdot \hat{z}(\mathcal{V}_q) > 0$ means that $p \cdot \hat{z}(q') > 0$ for all $q' \in \mathcal{V}_q$.

Definition 2.4 (Recursive Transfer Lower Semi-Continuity) An excess demand function $\hat{z}(\cdot) : \Delta \rightarrow \mathbb{R}^L$ is said to be *recursively transfer lower semi-continuous* on Δ if $\hat{z}(\cdot)$ is m-recursively transfer continuous for $m = 1, 2 \ldots$, i.e., whenever $q \in \Delta$ is not a equilibrium price system, there exists some price system $p^0 \in \Delta$ (possibly $p^0 = q$) and a neighborhood $\mathcal{V}_q$ such that $p \cdot \hat{z}(\mathcal{V}_q) > 0$ for any p that recursively upsets p^0.

In the definition of recursive transfer lower semi-continuity, q is transferred to q^0 that could be any point in Δ. Roughly speaking, recursive transfer lower semi-continuity of $\hat{z}(\cdot)$ means that, whenever q is not an equilibrium price system, there exists another nonequilibrium price system p^0 such that all excess demands in some neighborhood of q are not affordable at any price system p that recursively upsets p^0. This implies that, if an excess demand function $\hat{z}(\cdot) : \Delta \to \mathbb{R}^L$ is not recursively transfer lower semi-continuous, then there is some non-equilibrium price system q such that for every other price system p^0 and every neighborhood of q, excess demand of some price system q' in the neighborhood becomes affordable at price system p that recursively upsets p^0.

REMARK 2.2 By recursive transfer lower semi-continuity, when $p \cdot \hat{z}(p^{m-1}) > 0$, $p^{m-1} \cdot \hat{z}(p^{m-2}) > 0$, ..., $p^1 \cdot \hat{z}(p^0) > 0$, we have not only $p \cdot \hat{z}(\mathcal{V}_q) > 0$, but also $p^{m-1} \cdot \hat{z}(\mathcal{V}_q) > 0$, ..., $p^1 \cdot \hat{z}(\mathcal{V}_q) > 0$ since it is also k-recursively transfer lower semi-continuous for $k = 1, 2 \ldots$, m-1. That means all of the points in $\mathcal{V}_q$ are upset by $\{p^1, \ldots, p^{m-1}, p^m\}$ that directly or indirectly upset p^0.

REMARK 2.3 Recursive transfer lower semi-continuity is weaker than lower semi-continuity. Indeed, when $\hat{z}(\cdot)$ is lower semi-continuous, $p \cdot \hat{z}(\cdot)$ is also lower semi-continuous for any nonnegative vector p, and thus we have $p \cdot \hat{z}(q') > 0$ for all $q' \in \mathcal{V}_q$ and $p \in \Delta$. Then, for any finite price vectors $\{p^1, p^2, \ldots, p^m\}$, we of course have $p^k \cdot \hat{z}(q') > 0$ for all $q' \in \mathcal{V}_q$, $k = 1, \ldots, m$, which means $\hat{z}(\cdot)$ is transfer lower semi-continuous.

3 The Existence of Price Equilibrium

Before proceeding to our main result, we describe the main idea why the recursive transfer lower semi-continuity ensures the existence of price equilibria. When an economy fails to have a price equilibrium, every price vector q is upset by some price vector p^0. Then, by recursive transfer lower semi-continuity, there is some open set of candidate solutions containing q, all of which will be upset by some solution p that directly or indirectly upsets p^0. Then there are finite price vectors $\{q^1, q^2, \ldots, q^n\}$ whose neighborhoods cover Δ. Then, all of the points in a neighborhood, say $\mathcal{V}_{q^1}$, will be upset by a corresponding price vector p^1, which means p^1 cannot be a point in $\mathcal{V}_{q^1}$. If it is in some other neighborhood, say, $\mathcal{V}_{q^2}$, then it can be shown that p^2 will upset all points in the union of $\mathcal{V}_{q^1}$ and $\mathcal{V}_{q^2}$ so that p^2 is not in the union. We suppose $p^2 \in \mathcal{V}_{q^3}$, and then we can similarly show that p^3 is not in the union of $\mathcal{V}_{q^1}$, $\mathcal{V}_{q^2}$ and $\mathcal{V}_{q^3}$. Repeating such a process, we can show that price vectors in $\{p^1, \ldots, p^n\}$ will not be in Δ, which is impossible. Thus recursive transfer lower semi-continuity guarantees the existence of a price equilibrium.

Now we state our main result on the existence of price equilibrium in economies that have single-valued excess demand functions.

THEOREM 3.1 *Let Δ be the closed standard $L-1$ dimensional unit simplex. Suppose an excess demand function $\hat{z}(\cdot) : \Delta \to \mathbb{R}^L$ satisfies either the strong or the weak form of Walras' law. If $\hat{z}(\cdot)$ is recursively transfer lower semi-continuous on Δ, then there exists an equilibrium price system $p^* \in \Delta$.*

PROOF. Suppose, by way of contradiction, that there is no price equilibrium. Then, by recursive transfer lower semi-continuity of $\hat{z}(\cdot)$, for each $q \in S^{L-1}$, there exists p^0 and a neighborhood $\mathcal{V}_q$ such that $p \cdot \hat{z}(\mathcal{V}_q) > 0$ whenever $p^0 \in S^{L-1}$ is recursively upset by p, i.e., for any sequence of recursive price systems $\{p^1, \ldots, p^{m-1}, p\}$ with $p \cdot \hat{z}(p^{m-1}) > 0$, $p^{m-1} \cdot \hat{z}(p^{m-2}) > 0$, $\ldots$, $p^1 \cdot \hat{z}(p^0) > 0$ for $m \geq 1$, we have $p \cdot \hat{z}(\mathcal{V}_q) > 0$. Since there is no price equilibrium by the contrapositive hypothesis, p^0 is not a price equilibrium and thus, by recursive transfer lower semi-continuity, such a sequence of recursive price systems $\{p^1, \ldots, p^{m-1}, p\}$ exists for some $m \geq 1$.

Since S^{L-1} is compact and $S^{L-1} \subseteq \bigcup_{q \in S^{L-1}} \mathcal{V}_q$, there is a finite set $\{q^1, \ldots, q^T\}$ such that $S^{L-1} \subseteq \bigcup_{i=1}^T \mathcal{V}_{q^i}$. For each of such q^i, the corresponding initial price system is denoted by p^{0i} so that $p^i \cdot \hat{z}(\mathcal{V}_{q^i}) > 0$ whenever p^{0i} is recursively upset by p^i.

Since there is no price equilibrium, for each of such p^{0i}, there exists p^i such that $p^i \cdot \hat{z}(p^{0i}) > 0$, and then, by 1-recursive transfer lower semi-continuity, we have $p^i \cdot \hat{z}(\mathcal{V}_{q^i}) > 0$. Now consider the set of price systems $\{p^1, \ldots, p^T\}$. Then, $p^i \notin \mathcal{V}_{q^i}$; otherwise, by $p^i \cdot \hat{z}(\mathcal{V}_{q^i}) > 0$, we will have $p^i \cdot \hat{z}(p^i) > 0$, contradicting to Walras' law. So we must have $p^1 \notin \mathcal{V}_{p^1}$.

Without loss of generality, we suppose $p^1 \in \mathcal{V}_{p^2}$. Since $p^2 \cdot \hat{z}(p^1) > 0$ by noting that $p^1 \in \mathcal{V}_{q^2}$ and $p^1 \cdot \hat{z}(p^{01}) > 0$, then, by 2-recursive transfer lower semi-continuity, we have $p^2 \cdot \hat{z}(\mathcal{V}_{q^1}) > 0$. Also, $q^2 \cdot \hat{z}(\mathcal{V}_{q^2}) > 0$. Thus $p^2 \cdot \hat{z}(\mathcal{V}_{q^1} \cup \mathcal{V}_{q^2}) > 0$, and consequently $p^2 \notin \mathcal{V}_{q^1} \cup \mathcal{V}_{q^2}$.

Again, without loss of generality, we suppose $p^2 \in \mathcal{V}_{q^3}$. Since $p^3 \cdot \hat{z}(p^2) > 0$ by noting that $p^2 \in \mathcal{V}_{p^3}$, $p^2 \cdot \hat{z}(p^1) > 0$, and $p^1 \cdot \hat{z}(p^{01}) > 0$, by 3-recursive transfer lower semi-continuity, we have $p^3 \cdot \hat{z}(\mathcal{V}_{q^1}) > 0$. Also, since $p^3 \cdot \hat{z}(p^2) > 0$ and $p^2 \cdot \hat{z}(p^{02}) > 0$, by 2-recursive transfer lower semi-continuity, we have $p^3 \cdot \hat{z}(\mathcal{V}_{q^2}) > 0$. Thus, we have $p^3 \cdot \hat{z}(\mathcal{V}_{q^1} \cup \mathcal{V}_{q^2} \cup \mathcal{V}_{q^3}) > 0$, and consequently $p^3 \notin \mathcal{V}_{q^1} \cup \mathcal{V}_{q^2} \cup \mathcal{V}_{q^3}$.

With this process going on, we can show that $p^k \notin \mathcal{V}_{q^1} \cup \mathcal{V}_{q^2} \cup \ldots, \cup \mathcal{V}_{q^k}$, i.e., p^k is not in the union of $\mathcal{V}_{q^1}, \mathcal{V}_{q^2}, \ldots, \mathcal{V}_{q^k}$ for $k = 1, 2, \ldots, T$. In particular, for $k = T$, we have $p^L \notin \mathcal{V}_{q^1} \cup \mathcal{V}_{q^2} \ldots \cup \mathcal{V}_{q^T}$ and so $p^T \notin S^{L-1} \subseteq \mathcal{V}_{q^1} \cup \mathcal{V}_{q^2} \ldots \cup \mathcal{V}_{q^T}$, a contradiction.

Thus, there exists $p^* \in S^{L-1}$ such that $p \cdot \hat{z}(p^*) \leq 0$ for all $p \in S^{L-1}$. Letting $p^1 = (1, 0, \ldots, 0)$, $p^2 = (0, 1, 0, \ldots, 0)$, and $p^L = (0, 0, \ldots, 0, 1)$, we have $\hat{z}^l(p^*) \leq 0$ for

$l = 1, \ldots, L$ and thus p^* is a price equilibrium. ■

REMARK 3.1 Recursive transfer lower semi-continuity is, in fact, also a necessary condition for the existence of price equilibrium. Indeed, suppose p^* is a price equilibrium and $p \cdot \hat{z}(q) > 0$ for $q, p \in S^{L-1}$. Let $p^0 = p^*$ and $\mathcal{V}_q$ be a neighborhood of q. Since $p \cdot \hat{z}(p^*) \leq 0$ for all $p \in S^{L-1}$, it is impossible to find any sequence of finite price vectors $\{p^1, p^2, \ldots, p^m\}$ such that $p^1 \cdot \hat{z}(p^0) > 0, p^2 \cdot \hat{z}(p^1) > 0, \ldots, p^m \cdot \hat{z}(p^{m-1}) > 0$. Hence, the recursive transfer lower semi-continuity holds trivially.

REMARK 3.2 Although recursive transfer lower semi-continuity is necessary for the existence of price equilibrium, but it may not be sufficient for the existence of price equilibrium when an excess demand function is well-defined only on the set of positive price vectors. As such, unlike people might think, recursive transfer lower semi-continuity cannot be regarded as being equivalent to the definition of price equilibrium. To see this, consider the following counterexample.

EXAMPLE 3.1 Consider a two-commodity economy and let $p = (p_1, 1 - p_1)$ denote the price vector where $p_1 \in [0, 1]$. Let $\hat{z}(\cdot)$ be a function defined on intΔ such that

$$\hat{z}(p) = \left(\frac{1}{p_1}, -\frac{1}{1 - p_1}\right).$$

This excess demand function clearly satisfies Walras' law and does not have an competitive equilibrium. However, it is recursively transfer lower semi-continuous on intΔ. To see this, observe that for prices $p = (p_1, 1 - p_1)$ and $q = (q_1, 1 - q_1)$, $p \cdot \hat{z}(q) = \frac{p_1}{q_1} - (1 - p_1)\frac{1}{1-q_1} > 0$ if and only if $p_1 > q_1 > 0$.[1] As such, $\hat{z}(q)$ is upset by p if and only if $p_1 > q_1$ for $q_1 \neq 1$.

Now for any two price vectors $r, q \in$ intΔ with $r \cdot \hat{z}(q) > 0$, choose $\epsilon > 0$ such that $(q_1 - \epsilon, q_1 + \epsilon) \times (q_2 - \epsilon, q_2 + \epsilon) \subset$ intΔ. Let $p^0 = (q_1 + \epsilon, q_2 + \epsilon) \in$ intΔ and $\mathcal{V}_q \subseteq (q_1 - \epsilon, q_1 + \epsilon) \times (q_2 - \epsilon, q_2 + \epsilon)$. Then, for any sequence of price vectors $\{p^0, p^1, p^2, \ldots, p^{m-1}, p\}$ with $p \cdot \hat{z}(p^{m-1}) > 0, \ldots, p^1 \cdot \hat{z}(p^0) > 0$ (as such we must have $p_1^0 < p_1^1 < p_1^2 < \ldots < p_1^{m-1} < p_1$ by the above observation), we have $p \cdot \hat{z}(q') > 0$ for all $q' \in \mathcal{V}_q$, which means the excess demand function is recursively transfer lower semi-continuous on Δ.

Thus, Theorem 3.1 assumes that the excess demand function is well defined for all prices in the closed unit simplex Δ, including zero prices. However, when preferences are strictly monotone, excess demand functions are not well defined on the boundary of Δ. Then, some boundary

[1]Note that, even for any $p_1 < 0$, we still have $p \cdot \hat{z}(q) = \frac{p_1}{q_1} - (1 - p_1)\frac{1}{1-q_1} > 0$ as q_1 goes to 1. Thus the necessity part is not true when $q_1 = 1$.

conditions have been used to show the existence of price equilibrium in the case of an open set of price systems for which excess demand is defined —cf. Neuefeind (1980). In this case, we naturally cannot use Theorem 3.1 to fully characterize the existence of price equilibrium.

Nevertheless, Theorem 3.1 can be extended to the case of any set, especially the positive price open set, of price systems for which excess demand is defined. To do so, we introduce the following version of recursive transfer lower semi-continuity.

DEFINITION 3.1 Let D be a subset of intΔ. An excess demand function $\hat{z}(\cdot) : \text{int}\Delta \to \mathbb{R}^L$ is said to be *recursively transfer lower semi-continuous* on int Δ with respect to D if, whenever $q \in \text{int}\Delta$ is not an equilibrium price system, there exists some price system $p^0 \in \text{int}\Delta$ (possibly $p^0 = q$) and a neighborhood $\mathcal{V}_q$ such that (1) whenever p^0 is upset by a price system in int$\Delta \setminus D$, it is upset by a price system in D, and (2) $p \cdot \hat{z}(\mathcal{V}_q) > 0$ for any finite subset of price vectors $\{p^1, \ldots, p^m\} \subset D$ with $p^m = p$ and $p \cdot \hat{z}(p^{m-1}) > 0$, $p^{m-1} \cdot \hat{z}(p^{m-2}) > 0$, $p^1 \cdot \hat{z}(p^0) > 0$ for $m \geq 1$.

Condition (1) in the above definition ensures that if q is not an equilibrium price vector, it must not be an equilibrium price vector when intΔ is constrained to be D. Note that, while $\{p^1, \ldots, p^{m-1}, p\}$ are required to be in D, p^0 is not necessarily in D but can be any point in intΔ. Also, when $D = \Delta$, recursive transfer lower semi-continuity on Δ with respect to D reduces to recursive transfer lower semi-continuity on Δ.

We then have the following theorem that fully characterizes the existence of price equilibrium in economies with possibly indivisible commodity spaces and discontinuous excess demand functions.

THEOREM 3.2 *Suppose an excess demand function $\hat{z}(\cdot) : int\Delta \to \mathbb{R}^L$ satisfies Walras' law: $p \cdot \hat{z}(p) = 0$ for all $p \in int\Delta$. Then there exists a compact subset $D \subseteq int\Delta$ such that $\hat{z}(\cdot)$ is recursively transfer lower semi-continuous on intΔ with respect to D if and only if there exists a price equilibrium $p^* \in int\Delta$.*

PROOF. The proof of necessity is essentially the same as that of sufficiency in Theorem 3.1 and we just outline the proof here. To show the existence of a price equilibrium on Δ, it suffices to show that there exists a price equilibrium p^* in D if it is recursively transfer lower semi-continuous on Δ with respect to D. Suppose, by way of contradiction, that there is no price equilibrium in D. Then, since $\hat{z}$ is recursively transfer lower semi-continuous on Δ with respect to D, for each $q \in D$, there exists p^0 and a neighborhood $\mathcal{V}_q$ such that (1) whenever p^0 is upset by a price system in $\Delta \setminus D$, it is upset by a price system in D and (2) $p \cdot \hat{z}(\mathcal{V}_q) > 0$ for any finite subset of price systems $\{p^1, \ldots, p^m\} \subset D$ with $p^m = p$ and $p \cdot \hat{z}(p^{m-1}) > 0$, $p^{m-1} \cdot \hat{z}(p^{m-2}) > 0$, …, $p^1 \cdot \hat{z}(p^0) > 0$ for $m \geq 1$. Since there is no price equilibrium by the contrapositive hypothesis, p^0 is not a price equilibrium and thus, by recursive transfer lower semi-continuity on Δ with respect to D, such a sequence of recursive securing price systems $\{p^1, \ldots, p^{m-1}, p\}$ exists for some $m \geq 1$.

Since D is compact and $D \subseteq \bigcup_{q \in \Delta} \mathcal{V}_q$, there is a finite set $\{q^1, \ldots, q^T\} \subseteq D$ such that $D \subseteq \bigcup_{i=1}^{T} \mathcal{V}_{q^i}$. For each of such q^i, the corresponding initial deviation price system is denoted by p^{0i} so that $p^i \cdot \hat{z}(\mathcal{V}_{q^i}) > 0$ whenever p^{0i} is recursively upset by p^i through any finite subset of securing price systems $\{p^{1i}, \ldots, p^{mi}\} \subset D$ with $p^{mi} = p^i$. Then, by the same argument as in the proof of Theorem 3.1, we will obtain that z^k is not in the union of $\mathcal{V}_{q^1}, \mathcal{V}_{q^2}, \ldots, \mathcal{V}_{q^k}$ for $k = 1, 2, \ldots, T$. For $k = T$, we have $p^T \notin \mathcal{V}_{q^1} \cup \mathcal{V}_{q^2} \ldots \cup \mathcal{V}_{q^T}$ and so $p^T \notin D \subseteq \bigcup_{i=1}^{T} \mathcal{V}_{q^i}$, which contradicts that p^T is an upsetting price in D.

Thus, there exists a price system $p^* \in \Delta$ such that $p \cdot \hat{z}(p^*) \leq 0$ for all $p \in \text{int}\Delta$. We want to show that p^* in fact is a price equilibrium. Note that intΔ is open and D is a compact subset of intΔ. One can always find a sequence of price vector $\{q_n^l\} \subseteq \text{int}\Delta \setminus D$ such that $q_n^l \to p^l$ as $n \to \infty$, where $p^l = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the unit vector that has only one argument - the lth argument - with value 1 and others with value 0. Since $p \cdot \hat{z}(q)$ is continuous in p, we have $\hat{z}^l(p^*) \leq 0$ for $l = 1, \ldots, L$ and thus p^* is a price equilibrium.

Similarly, we can show that recursive transfer lower semi-continuity on intΔ with respect to the existence of a compact D is also a necessary condition for the existence of price equilibrium.

■

EXAMPLE 3.2 Consider again Example 3.1. We know that, although the excess demand function is recursively transfer lower semi-continuous on intΔ, it does not exist a price equilibrium. In fact, the necessity part of the above theorem implies the excess demand function is not recursively transfer lower semi-continuous on intΔ with respect to any compact subset D of intΔ. In other wors, there does not exist any compact set $D \subset \text{int}\Delta$ such that the game is recursively transfer lower semi-continuous on intΔ with respect to D.

We can versify this directly. Indeed, for a given compact set D, let $\bar{r}_1$ be the maximum of r_1 for all $(r_1, r - q_1) \in D$. Choosing $q \in \text{int}\Delta \setminus D$ with $q_1 > \bar{r}_1$, we have $p \cdot \hat{z}(q) < 0$ for all $p \in D$. Also, for any p wit $p_1 > q_1$, we have $p \cdot \hat{z}(q) > 0$. Now we show that one cannot find any price vector $p^0 \in \text{int}\Delta$ and a neighborhood $\mathcal{V}_q$ of q such that (1) whenever p^0 is upset by a price vector in int$\Delta \setminus D$, it is upset by a price vector in D and (2) $p \cdot \hat{z}(\mathcal{V}_q) > 0$ for every price vector $p \in D$ that upsets directly or indirectly p^0. We show this by considering two cases.

Case 1. $p_1^0 \geq \bar{r}_1$. p^0 can be upset by a price vector profile $p' \in \text{int}\Delta \setminus D$ with $p'_1 > p_1^0$, but it cannot be upset by any price vector in D.

Case 2: $p_1^0 < \bar{r}_1$. For any price vector $p \in D$ that upsets directly or indirectly p^0, we have $p \cdot \hat{z}(\mathcal{V}_q) < 0$, but not $p \cdot \hat{z}(\mathcal{V}_q) > 0$.

Thus, we cannot find any price system $p^0 \in \text{int}\Delta \setminus D$ and a neighborhood $\mathcal{V}_q$ of q such that such that (1) whenever p^0 is upset by a price system in int$\Delta \setminus D$, it is upset by a price system in D and (2) $p \cdot \hat{z}(\mathcal{V}_q) > 0$ for every price system p that recursively upsets p^0. Hence, the excess demand function is not recursively transfer lower semi-continuous on intΔ with respect to D.

Thus, Theorem 3.2 generalizes all the existing results on the existence of price equilibrium in economies with single-valued excess demand functions, such as those in Gale (1955), Nikaido (1956), Debreu (1970, 1982), Hildenbrand (1974), Hildenbrand and Kirman (1975), Grandmont (1977), Neuefeind (1980), Aliprantis and Brown (1983), Hüsseinov (1999), Momi(2003), and Quah (2008).

4 Conclusion

The existing results only give sufficient conditions for the existence of equilibrium, and no characterization result has been given in the literature. This paper fills this gap by providing a necessary and sufficient condition for the existence of price equilibrium. We establish a condition, called *recursive transfer lower semi-continuity*, which full characterizes the existence of price equilibrium. As such, it strictly generalizes all the existing theorems on the existence of price equilibrium. The recursive transfer continuity is a useful tool, which can also be used to characterize the existence of equilibrium in games with general strategy spaces and payoffs (cf. Tian, 2010).

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