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Abstract

This paper studies the cost requirement for two-agent collusion-proof mechanism design. Unlike the existing results for general environments with three or more agents, it is shown that collusive behavior cannot be prevented freely in two-agent nonlinear pricing environments with correlated types. Reporting manipulation calls for distortions away from the first-best efficiency, and arbitrage calls for further distortion. Moreover, we show that the distortionary patterns are quite different for positive and negative correlations. The second-best outcome is attainable as negative correlation is vanishing, while the limit of collusion-proof efficiency is strictly lower than the second-best level as positive correlation goes to zero. Allowing arbitrage therefore breaks the continuity between correlated and uncorrelated types.

Keywords: Nonlinear pricing, collusion-proof, mechanism design, arbitrage, correlation
JEL Classification Number: D42, D62, D82

1 Introduction

The traditional principal-multiagent model assumes away collusion among agents, that is, they behave in a non-cooperative way. For economic environments with independent types, the classical result of Myerson and Satterthwaite (1983) show that there is in general no first-best outcome and only second-best outcome is achievable. But, when risk-neutral agents have correlated types and are not protected by limited liability, Crémer and McLean (hereafter CM)

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(1985, 1988) show that the principal can obtain the first-best allocation by cross-checking the agents' reports against each other and fully extracting their information rents in absence of collusion. This full surplus extraction (FSE) result holds for any degree of correlation, even if it approaches zero. However, when correlation is actually zero, the cross-checking method does not work, so only the second-best allocation is achievable, a conventional result for independent types. Therefore, a notable discontinuity occurs at zero correlation point. Another drawback of CM's FSE mechanism is its vulnerability to collusion. If the agents can coordinate their reports, it may be impossible for the principal to induce information revelation via cross-checking at no cost.

However, no collusion is a rather unrealistic assumption unless the principal can impose sufficiently large transaction costs on side-contracting so that agents are not able to collude at all. In reality, however, collusion is a widespread and noxious phenomenon. Agents often collude to increase their aggregate payoffs at the expense of the principal. This phenomenon is an important concern in mechanism design theory. Typically, collusion imposes severe limits on what can be achieved, and thus it is generally regarded as a factor that reduces the principal's payoff in addition to the asymmetric information.

The pioneering work that studies collusion in principal-multiagent setting is due to Laffont and Mortimort (hereafter LM) (1997, 2000). They offer a tractable modeling framework for analyzing the role of colluders' information asymmetry in collusion-proof mechanism design. A stark difference is found for independent types and correlated types. In procurement/public good settings with two agents, they show that the optimal outcome can be made collusion-proof at no cost to the principal if the agents' types are uncorrelated (LM, 1997), but if the types are correlated, preventing collusion entails strict cost to the principal (LM, 2000). Furthermore, the nature of the optimal incentive scheme changes continuously as correlation goes to zero. That is, allowing collusion restores continuity between the correlated and the uncorrelated environments.

In LM's procurement/public good settings, two agents may consume certain amount of goods in a non-excludable way. As such, there is no need and it is technologically impossible to split the goods between them. However, in private-goods setting, say, in monopoly pricing problem, buyers have incentive to reallocate their total purchases obtained from the principal. Thus, the mechanism designer should make optimal contractual response preventing the agents from (i) manipulating their reports, (ii) exchanging side transfers and (iii) conducting arbitrage.[1] Jeon

[1] More recently, a number of contributions, noticeably Mookherjee and Tsumagari (2004), Dequiedt (2007) and Pavlov (2008) among others, have noted that agents can coordinate not only on the way they play the grand mechanism, but also on their participation decisions.

and Menicucci (hereafter JM) (2005) extend LM's model by incorporating arbitrage. They show that collusion is preventable at no cost with uncorrelated types in a nonlinear pricing model that allows collusive consumers to arbitrage on their purchases. They do not, however, consider a more interesting case where agents' types may be correlated.

Che and Kim (hereafter CK) (2006) advance on these fronts by developing a general method for collusion-proofing a mechanism. They show that agents' collusion, including both reporting manipulation and arbitrage, is harmless to the principal, i.e., agents' collusion imposes no cost in a broad class of circumstances with more than two agents ($n \geqslant 3$) for correlated types and more than one agent ($n \geqslant 2$) for uncorrelated types.[2] This no-cost result is sharply different from Laffont and Mortimort (2000)'s result that preventing collusion entails strict cost to the principal when the types are correlated. Any payoff the principal can attain in the absence of collusion, including the second-best efficiency is attainable with uncorrelated types, and the first-best efficiency is also attainable with correlated types. More importantly, CK's result on collusion-proof implementation at no cost is rather robust for general economic environments. It does not rest on any special assumptions about preferences/technologies or type structures, nor on collusive behavior.

However, while Che and Kim give a full answer in a broad class of environments when types are uncorrelated, they leave the two-agent correlated-type case unanswered. It is still unknown how far these transaction costs can be exploited in contract design for general two-agent economic environments when types are correlated. The two-agent case is important in the theory of mechanism design, since a variety of economic phenomena are basically bilateral. As Moore and Repullo (1990) have emphasized, two-agent model is the leading case for applications to contracting or bargaining. The results for two-agent design problem may, and in fact, as shown in the paper, be very different from its multi-agent counterpart. As such, this case needs to be considered separately.[3]

In this paper, we show that the result that agents' asymmetric information imposes no additional transaction costs on their abilities to carry out collusive arrangements for economies

[2]An additional requirement is that at least one agent has more than two types if $n = 3$ for correlated types.

[3]Notable examples are those in the literature on Nash implementation. It is well known that when $n \geqslant 3$, if a SCR satisfies *monotonicity* and *no-veto power*, it is Nash implementable. These conditions are, however, not sufficient for Nash implementability of two-person SCR. (See Maskin (1999), Moore and Repullo (1990), Dutta and Sen (1991), Denilov (1992), Sjöstrom (1991) among many others for detailed discussion.) Contrary to the case of three or more agents, it is generally impossible to Nash implement Walrasian or Lindahl correspondences with smooth and balanced mechanism in the case of two agents. (See Hurwicz and Weinberger (1984), Reichelstein (1984), Nakamura (1987), Kwan and Nakamura (1987) for detailed discussion.)

with three or more agents is no longer true for economies that include nonlinear-pricing/priviate-goods economic environments with correlation and arbitrage. Preventing collusion entails strict cost to the principal when there are only two agents and their types are correlated.

Our results depart from the existing literature as follows. First, our two-agent result complements CK's work and gives a full answer to the question whether collusion with both reporting manipulation and arbitrage is preventable at no cost. Che and Kim (2006) show that the principal can always fight off collusion for economies with three or more agents at no cost using a robustly collusion-proof grand mechanism. In contrast, our finding is that preventing collusion entails strict cost to the principal for two-agent economies with correlation. This is consistent with Olson's famous argument that small groups are more able to act in their common interest than large ones because they face lower per capita transaction costs when attempting to organize for collective action [Olson (1965)].[4]

Secondly, we extend the result of LM (2000) by considering both arbitrage and negative correlation. LM (2000) give the collusion-proof mechanism in procurement/public good environments. It is unnecessary and impossible to split the goods between consumers. In contrast, we discuss the private good problem. Consumers could conduct arbitrage on their total purchases. We find that the possibility of arbitrage may call for further distortions away from the efficiency obtained in arbitrage-free case – i.e., preventing collusion may entail an even higher cost to the principal in two-agent nonlinear pricing setting with correlated types and arbitrage. Moreover, LM (2000)'s model considers only positive correlation, while we consider negative correlation as well. In our analysis, the heterogeneous transaction costs endogenously imposed by the principal on side contracting play an important role in determining the optimal mechanism. We find that, in the presence of arbitrage, asymmetric information between agents does not generate any transaction cost in the weak positive correlation case while it generates transaction costs in weak negative correlation case. This leads to a striking discontinuity of collusion-proof mechanism at zero correlation.

Lastly, we also extend the main conclusion of JM (2005). They consider information manipulation and arbitrage with only uncorrelated types. In contrast, we consider both positive, negative and zero correlations. JM's result is therefore a special case of ours.

The rest of this paper is organized as follows. Section 2 describes the economic environments studied. Section 3 reviews as a benchmark the optimal pricing mechanism without collusion. Section 4 characterizes the coalitional incentive and no-arbitrage constraints that must be satisfied

[4] This argument is often used to explain why taxpayers often do not form an interest group while managers of an industry do.

by a weakly collusion-proof mechanism. Section 5 describes the optimal weakly collusion-proof mechanism with reporting manipulation alone. Section 6 discusses the case with both reporting manipulation and arbitrage. Section 7 gives conclusions.

2 The Model

2.1 Preferences, Information, and Mechanisms

A monopolist seller can produce any amount of a homogeneous good at constant marginal cost c and sells it to two buyers whose consumptions are $q_i, i \in \{1, 2\}$. Buyer i obtains utility $\theta_i V(q_i) - t_i$ from consuming q_i units of goods and paying t_i units of money to the seller. $V(\cdot)$ is an increasing concave function with $V'(\cdot) > 0$ and $V''(\cdot) < 0$. The consumer privately observes his own type $\theta_i \in \Theta \equiv \{\theta_L, \theta_H\}$, where $\Delta\theta \equiv \theta_H - \theta_L$. The probabilities $p(\theta_1, \theta_2)$ of each state $(\theta_1, \theta_2) \in \Theta^2$, are common knowledge prior beliefs. For simplicity, we write $p_{LL} = p(\theta_L, \theta_L), p_{LH} = p(\theta_L, \theta_H) = p(\theta_H, \theta_L), p_{HH} = p(\theta_H, \theta_H)$. We also denote by $\rho \equiv p_{LL}p_{HH} - p_{LH}^2$ the degree of correlation between the agents' types.

The monopolist seller designs a grand sale mechanism $\mathbf{M}$ to maximize her expected profit. Considering the Revelation Principle, we can restrict our attention to direct revelation mechanism which maps any pair of reported types $(\widehat{\theta_1}, \widehat{\theta_2})$ into a combination of consumptions and payments:

$$\mathbf{M} = \left\{ q_1(\widehat{\theta_1}, \widehat{\theta_2}), q_2(\widehat{\theta_1}, \widehat{\theta_2}), t_1(\widehat{\theta_1}, \widehat{\theta_2}), t_2(\widehat{\theta_1}, \widehat{\theta_2}) \right\}, \forall (\widehat{\theta_1}, \widehat{\theta_2}) \in \Theta^2.$$

Since buyers are ex ante identical, without loss of generality, we focus on anonymous mechanism in which the consumption and payment of a buyer depend only on the reports and not on his identity. Then we denote by t_{kl} for $k, l \in \{H, L\}$ the tax paid by an agent whose report is θ_k and the other agent's report is θ_l, and q_{kl} is defined analogously. Let $\mathbf{q} = (q_{LL}, q_{LH}, q_{HL}, q_{HH}) \in \mathbb{R}_+^4$ and $\mathbf{t} = (t_{LL}, t_{LH}, t_{HL}, t_{HH}) \in \mathbb{R}^4$ denote the vectors of quantities and transfers respectively.

2.2 Coalition Formation

Applying the methodology of LM (1997, 2000), we model the buyers' coalition formation by a side-contract, denoted by $\mathbf{S}$, offered by a benevolent uninformed third party. The third party organizes the buyers into collusion in order to maximize the sum of their payoffs subject to incentive compatibility and participation constraints written with respect to the utility a buyer obtains when the grand mechanism $\mathbf{M}$ is played non-cooperatively. We study a collusive arrangement that allows the agents (i) to collectively manipulate their reports to the principal and exchange transfers in a budget-balanced way, (ii) to reallocate quantities assigned by the grand contract. The timing of the overall game of contract offer and coalition formation is the following:

- **Stage** 1: Buyers learn their respective "types".
- **Stage** 2: The seller proposes a grand sale mechanism $\mathbf{M}$. If any buyer vetoes the grand mechanism, all buyers get their reservation utility normalized exogenously at zero and the following stages do not occur.
- **Stage** 3: The third party proposes a side mechanism $\mathbf{S}$ to the buyers. If anyone refuses this side mechanism, $\mathbf{M}$ is played non-cooperatively. If both buyers accept $\mathbf{S}$, they report their types to the third party who enforces manipulation of report into $\mathbf{M}$, and commits to enforce the corresponding side transfers and reallocation within coalition.
- **Stage** 4: Reports are sent into the grand mechanism. Quantities and payments specified in $\mathbf{M}$ are enforced. Quantities reallocation and side transfers specified in $\mathbf{S}$, if any, are implemented.

Formally, a side mechanism $\mathbf{S}$ takes the following form:

$$\mathbf{S} = \left\{\phi(\widetilde{\theta_1}, \widetilde{\theta_2}), x_1(\widetilde{\theta_1}, \widetilde{\theta_2}, \phi), x_2(\widetilde{\theta_1}, \widetilde{\theta_2}, \phi), y_1(\widetilde{\theta_1}, \widetilde{\theta_2}), y_2(\widetilde{\theta_1}, \widetilde{\theta_2})\right\}, \forall(\widetilde{\theta_1}, \widetilde{\theta_2}) \in \Theta^2.$$

$\widetilde{\theta_i}$ is buyer i's report to the third party. $\phi(\cdot)$ is the manipulated report to the grand mechanism. $y_i(\widetilde{\theta_1}, \widetilde{\theta_2})$ denotes the monetary transfer from buyer i to the third party. $x_i(\widetilde{\theta_1}, \widetilde{\theta_2}, \phi)$ represents the quantity of goods buyer i receives from the third party when ϕ is reported to the seller and $(\widetilde{\theta_1}, \widetilde{\theta_2})$ are reported to the third party. Such a reallocation rule maximizes the joint surplus of the buyers subject to the total amount of the goods being allocated to all consumers. Since the third party is neither a source of goods nor money, we assume that a side mechanism should satisfy the ex post budget-balance constraints for the reallocation of goods and for the side transfers, respectively

$$\sum_{i=1}^{2} x_i(\widetilde{\theta_1}, \widetilde{\theta_2}, \phi) = 0 \text{ and } \sum_{i=1}^{2} y_i(\widetilde{\theta_1}, \widetilde{\theta_2}) = 0, \forall(\widetilde{\theta_1}, \widetilde{\theta_1}) \in \Theta^2 \text{ and } \forall\phi \in \Theta^2.$$

Let $U^M(\theta_i)$ denote the expected payoff of a θ_i type in truthful equilibrium of $\mathbf{M}$. The side mechanism must guarantee to an agent a utility level greater than what he expects from playing non-cooperatively the grand mechanism and then getting a utility $U^M(\theta_i)$.

3 The Optimal Grand-Mechanism without Coalition

Firstly, we study, as a benchmark, the optimal grand-mechanism without side-contracting with correlated types, i.e., $\rho \neq 0$. The seller's expected profit is

$$\Pi(\mathbf{t},\mathbf{q}) \equiv 2p_{LL}\left(t_{LL}-cq_{LL}\right)+2p_{LH}\left(t_{LH}+t_{HL}-cq_{LH}-cq_{HL}\right)+2p_{HH}\left(t_{HH}-cq_{HH}\right).$$

The following Bayesian incentive-compatibility constraints should be satisfied. For a θ_L type buyer

$$\begin{aligned} BIC_L: p_{LL}\left[\theta_L V(q_{LL})-t_{LL}\right]+p_{LH}[\theta_L V(q_{LH})-t_{LH}] \\ \geqslant p_{LL}[\theta_L V(q_{HL})-t_{HL}]+p_{LH}[\theta_L V(q_{HH})-t_{HH}]; \end{aligned} \tag{1}$$

for a θ_H type buyer

$$\begin{aligned} BIC_H: p_{LH}[\theta_H V(q_{HL})-t_{HL}]+p_{HH}[\theta_H V(q_{HH})-t_{HH}] \\ \geqslant p_{LH}[\theta_H V(q_{LL})-t_{LL}]+p_{HH}[\theta_H V(q_{LH})-t_{LH}]. \end{aligned} \tag{2}$$

The mechanism should also satisfy the following interim individual-rationality constraints. For a θ_L and θ_H type, respectively,

$$BIR_L: \quad p_{LL}[\theta_L V(q_{LL})-t_{LL}]+p_{LH}[\theta_L V(q_{LH})-t_{LH}] \quad \geqslant 0, \tag{3}$$

$$BIR_H: \quad p_{LH}[\theta_H V(q_{HL})-t_{HL}]+p_{HH}[\theta_H V(q_{HH})-t_{HH}] \quad \geqslant 0. \tag{4}$$

Then the seller maximizes her expected profit $\Pi(\mathbf{t},\mathbf{q})$ subject to constraints (1) to (4).

We look for the transfers such that the four constraints are all binding, i.e., which satisfy incentive compatibility without leaving any expected rent at the interim stage to any buyer.[5] Indeed, for $\rho \neq 0$, the equation system of (1) to (4) is invertible since the determinant is ρ^2. Thus, the transfers are determined uniquely as

$$t_{LL} = \frac{\left(p_{LL}p_{HH}\theta_L - p_{LH}^2\theta_H\right)V(q_{LL}) - p_{LH}p_{HH}\Delta\theta V(q_{LH})}{\rho}, \tag{5}$$

$$t_{LH} = \frac{\left(p_{LL}p_{HH}\theta_H - p_{LH}^2\theta_L\right)V(q_{LH}) + p_{LL}p_{LH}\Delta\theta V(q_{LL})}{\rho}, \tag{6}$$

$$t_{HL} = \frac{\left(p_{LL}p_{HH}\theta_L - p_{LH}^2\theta_H\right)V(q_{HL}) - p_{LH}p_{HH}\Delta\theta V(q_{HH})}{\rho}, \tag{7}$$

$$t_{HH} = \frac{\left(p_{LL}p_{HH}\theta_H - p_{LH}^2\theta_L\right)V(q_{HH}) + p_{LL}p_{LH}\Delta\theta V(q_{HL})}{\rho}. \tag{8}$$

Substituting these transfers into the the seller's expected profit function $\Pi(\mathbf{t},\mathbf{q})$ and then optimizing this expression yield the second-best consumptions represented as functions of correlation:

[5] CM (1988) show that incentive constraints can be slack.

$\mathbf{q}^{sb}(\rho) = (q^{sb}_{LL}(\rho), q^{sb}_{LH}(\rho), q^{sb}_{HL}(\rho), q^{sb}_{HH}(\rho))$, where

$$\theta_L V'[q^{sb}_{LL}(\rho)] = \theta_L V'[q^{sb}_{LH}(\rho)] = \theta_H V'[q^{sb}_{HL}(\rho)] = \theta_H V'[q^{sb}_{HH}(\rho)] = c. \tag{9}$$

It is easy to find that each buyer has the same quantity as with complete information, i.e., $q^{sb}_{kl}(\rho) = q^{fb}_{kl}(\rho), \forall \rho \neq 0, \forall k, l \in \{H, L\}$. From the expressions of transfers (5) to (8), we observe that, if ρ is positive and goes to zero, then t_{LL}, t_{HL} go to $-\infty$, t_{LH}, t_{HH} go to $+\infty$. The consumers' quasilinear utility function suggests that they are risk neutral on transfers. The selling mechanism exploits the risk neutrality of the agents by specifying extreme rewards and penalties. If the correlation is positive, a θ_H- agent faces, when he tells the truth, an extreme reward if the other agent is a θ_L type and extreme penalties if the other agent is a θ_H type. Similarly we can work out the result for θ_L- agent. For negative correlation, the sign of all the transfers will be altered, which implies the opposite directions of awards and penalties. Given such a mechanism, the buyers will always accept the contract and tell the truth. The weaker is the correlation, the larger penalties or awards are needed to elicit truthtelling. It may not be surprising that when buyers are very similar, it is relatively simple to extract their rents by "cross-checking" method. A more interesting point is that the above first-best result holds for any degree of correlation, even if it is infinitesimal.

For the case with independent types, we denote $\nu = \Pr(\theta_i = \theta_L), 1-\nu = \Pr(\theta_i = \theta_H), i = 1, 2$, then $p_{LL} = \nu^2, p_{LH} = \nu(1-\nu), p_{HH} = (1-\nu)^2$. The system of binding constraints can no more be inverted. The standard method for solving single-agent adverse selection model shows that BIC_H and BIR_L bind in the optimum. When the transfers in $\Pi(\mathbf{t}, \mathbf{q})$ are replaced with those obtained from BIC_H and BIR_L written with equality, the solution to the principal's program is characterized as $\mathbf{q}^{sb}(0)$, where

$$\begin{aligned} &\left(\theta_L - \frac{1-\nu}{\nu}\Delta\theta\right) V'[q^{sb}_{LL}(0)] = \left(\theta_L - \frac{1-\nu}{\nu}\Delta\theta\right) V'[q^{sb}_{LH}(0)] \\ =&\theta_H V'[q^{sb}_{HL}(0)] = \theta_H V'[q^{sb}_{HH}(0)] = c. \end{aligned} \tag{10}$$

From the above two results with correlated and uncorrelated types respectively one can see that a striking discontinuity occurs at $\rho = 0$. Indeed, for correlated types the seller can exploit "cross-checking" method to induce their revelation at no cost, while for uncorrelated types, he cannot do that since the report of one consumer is uninformative signal for the other consumer's type. The first-best allocation is thus not achievable when $\rho = 0$. The seller should give information rents to the θ_H buyer and, to decrease those rents, distort the quantities of the θ_L buyer downward.

4 The Third Party's Optimization Program

The above analysis shows that the agents with correlated types get zero rent from playing non-cooperatively the grand mechanism, so the optimal grand mechanism with a noncooperative behavior creates endogenously the stakes for collusive behavior. In this section, we study formally the third party's optimizing problem and derive the coalitional incentive constraints which must be satisfied in the optimal collusion-proof grand mechanism under asymmetric information.

Assume that the third-party's optimal problem is given by:

$$[PT] : \max_{\phi(\cdot), x_i(\cdot), y_i(\cdot)} \sum_{(\theta_1,\theta_2)\in\Theta^2} p(\theta_1,\theta_2)\left[U^1(\theta_1) + U^2(\theta_2)\right]$$

subject to :

$$U^i(\theta_i) = \sum_{\theta_j\in\Theta} p(\theta_j|\theta_i)\left[\theta_i V\Big(x_i\big(\theta_i,\theta_j,\phi(\theta_i,\theta_j)\big) + q_i\big(\phi(\theta_i,\theta_j)\big)\Big) + y_i(\theta_i,\theta_j) - t_i\big(\phi(\theta_i,\theta_j)\big)\right]$$

for any $\theta_i \in \Theta$ and $i, j = 1, 2$ with $i \neq j$;

$$\left(BIC_i^S\right) : U^i(\theta_i) \geqslant U^i(\widetilde{\theta}_i \mid \theta_i)$$

where

$$U^i(\widetilde{\theta}_i \mid \theta_i) = \sum_{\theta_j\in\Theta} p(\theta_j|\theta_i)\left[\theta_i V\Big(x_i\big(\widetilde{\theta}_i,\theta_j,\phi(\widetilde{\theta}_i,\theta_j)\big) + q_i\big(\phi(\widetilde{\theta}_i,\theta_j)\big)\Big) + y_i(\widetilde{\theta}_i,\theta_j) - t_i\big(\phi(\widetilde{\theta}_i,\theta_j)\big)\right]$$

for any $\left(\theta_i, \widetilde{\theta}_i\right) \in \Theta^2$ and $i, j = 1, 2$ with $i \neq j$;

$$\left(BIR_i^S\right) : U^i(\theta_i) \geqslant U^M(\theta_i)$$

for any $\theta_i \in \Theta$ and $i = 1, 2$;

$$(BB : y) : \sum_{i=1}^{2} y_i(\theta_1,\theta_2) = 0$$

$$(BB : x) : \sum_{i=1}^{2} x_i(\theta_1,\theta_2,\widetilde{\phi}) = 0$$

for any $(\theta_1, \theta_2) \in \Theta^2$ and $\widetilde{\phi} \in \Theta^2$.

DEFINITION 1 A side mechanism

$$\mathbf{S} = \left\{\phi(\widetilde{\theta}_1,\widetilde{\theta}_2), x_1(\widetilde{\theta}_1,\widetilde{\theta}_2,\phi), x_2(\widetilde{\theta}_1,\widetilde{\theta}_2,\phi), y_1(\widetilde{\theta}_1,\widetilde{\theta}_2), y_2(\widetilde{\theta}_1,\widetilde{\theta}_2)\right\} \forall(\widetilde{\theta}_1,\widetilde{\theta}_2) \in \Theta^2$$

is coalition-interim-efficient with respect to an incentive-compatible grand mechanism $\mathbf{M}$ providing the reservation utilities $U^M(\theta)$[6] if and only if it solves the above third party program.

Let $\mathbf{S}^0 \equiv \Big\{\phi(\cdot) = Id(\cdot), x_1(\cdot) = x_2(\cdot) = 0, y_1(\cdot) = y_2(\cdot) = 0\Big\}$ denote the null contract that implements no manipulation of reports, no reallocation of quantities, and no side transfers, then we have the following definition.

DEFINITION 2 An incentive-compatible grand mechanism $\mathbf{M}$ is weakly [7] collusion-proof if and only if it is a truthtelling direct mechanism and the null side mechanism $\mathbf{S}^0$ is coalition-interim-efficient with respect to $\mathbf{M}$.

PROPOSITION 1 (Weak Collusion-Proofness Principle, WCP). Any Bayesian perfect equilibrium of the two-stage game of contract offer and collusion contract offer $\mathbf{M}\circ\mathbf{S}$ can be achieved by a weakly collusion-proof mechanism.

PROOF. The proof is omitted since it is a straightforward adaptation of proof in Proposition 3 of LM (2000). ■

The next proposition characterizes the coalitional incentive constraints which must be satisfied in the weakly collusion-proof grand mechanism.

PROPOSITION 2 A symmetric Bayesian incentive compatible grand mechanism $\mathbf{M}$ such that the $L-$ type's incentive constraints are not binding is weakly collusion-proof if and only if there exists $\epsilon \in [0, 1)$ such that:

- The following coalitional incentive constraints are satisfied: for a (θ_L, θ_L) coalition,

$$\begin{aligned}
CIC_{LL;LH}: \quad & 2\left(\theta_L - \frac{p_{LH}^2 \epsilon \Delta\theta}{p_{LL}p_{LH} + \rho\epsilon}\right) V(q_{LL}) - 2t_{LL} \\
& \geqslant 2\left(\theta_L - \frac{p_{LH}^2 \epsilon \Delta\theta}{p_{LL}p_{LH} + \rho\epsilon}\right) V\left(\frac{q_{LH} + q_{HL}}{2}\right) - t_{LH} - t_{HL} \qquad (11) \\
CIC_{LL;HH}: \quad & 2\left(\theta_L - \frac{p_{LH}^2 \epsilon \Delta\theta}{p_{LL}p_{LH} + \rho\epsilon}\right) V(q_{LL}) - 2t_{LL} \\
& \geqslant 2\left(\theta_L - \frac{p_{LH}^2 \epsilon \Delta\theta}{p_{LL}p_{LH} + \rho\epsilon}\right) V(q_{HH}) - 2t_{HH} \qquad (12)
\end{aligned}$$

[6] We assume here that, if buyer i vetoes $\mathbf{S}$, then the other buyer still has prior beliefs about θ_i. Therefore, if we denote by $U^M(\theta_i)$ the expected payoff of an i type in the truthful equilibrium of $\mathbf{M}$, the reservation utility for an i type when deciding whether to accept $\mathbf{S}$ or not is also $U^M(\theta_i)$ (see LM (2000) for more general analysis).

[7] The qualifier "weakly" comes from the assumption made in footnote 6.

for a (θ_L, θ_H) coalition,

$$\begin{aligned}
&CIC_{LH;LL}: \\
&\left(\theta_L - \frac{p_{HH}\epsilon\Delta\theta}{p_{LH}}\right) V\big(\varphi_1(q_{LH}+q_{HL})\big) + \theta_H V\big(\varphi_2(q_{LH}+q_{HL})\big) - t_{LH} - t_{HL} \\
&\geqslant \left(\theta_L - \frac{p_{HH}\epsilon\Delta\theta}{p_{LH}}\right) V\big(\varphi_1(2q_{LL})\big) + \theta_H V\big(\varphi_2(2q_{LL})\big) - 2t_{LL} \qquad (13) \\
&CIC_{LH;HH}: \\
&\left(\theta_L - \frac{p_{HH}\epsilon\Delta\theta}{p_{LH}}\right) V\big(\varphi_1(q_{LH}+q_{HL})\big) + \theta_H V\big(\varphi_2(q_{LH}+q_{HL})\big) - t_{LH} - t_{HL} \\
&\geqslant \left(\theta_L - \frac{p_{HH}\epsilon\Delta\theta}{p_{LH}}\right) V\big(\varphi_1(2q_{HH})\big) + \theta_H V\big(\varphi_2(2q_{HH}\big) - 2t_{HH} \qquad (14)
\end{aligned}$$

for a (θ_H, θ_H) coalition,

$$CIC_{HH;LL}: 2\theta_H V(q_{HH}) - 2t_{HH} \geqslant 2\theta_H V(q_{LL}) - 2t_{LL} \qquad (15)$$

$$CIC_{HH;LH}: 2\theta_H V(q_{HH}) - 2t_{HH} \geqslant 2\theta_H V\Big(\frac{q_{LH}+q_{HL}}{2}\Big) - t_{LH} - t_{HL}, \qquad (16)$$

where

$$(\varphi_1(x), \varphi_2(x)) = \underset{x_1, x_2 : x_1 + x_2 = x}{\arg\max} \left[\left(\theta_L - \frac{p_{HH}\epsilon\Delta\theta}{p_{LH}}\right) V(x_1) + \theta_H V(x_2)\right] \qquad (17)$$

is the optimal splitting rule within a heterogenous coalition.

- The following no-arbitrage-constraint (NAC) is satisfied:

$$\left(\theta_L - \frac{p_{HH}\epsilon}{p_{LH}}\Delta\theta\right) V'(q_{LH}) = \theta_H V'(q_{HL}) \qquad (18)$$

- The $H-$ type's incentive compatibility constraint is binding in the side-contract if and only if $\epsilon > 0$.

PROOF. See appendix. ■

The coalitional incentive constraints under asymmetric information are obtained by expressing the fact that the third party has no incentive to manipulate the agents' reports. For instance, if $CIC_{LL;LH}$ is satisfied, a (θ_L, θ_L) coalition prefers truthtelling to report (θ_L, θ_H). Each coalitional incentive constraint takes into account the possibility of reallocation: if both agents report the same types to the third party, each of them receives half of the total quantities available; otherwise, the total quantities are split in accordance with a profit-maximizing rule. The symmetric assumption $q_1(\theta_i, \theta_i) = q_2(\theta_i, \theta_i)$, for all $i \in \{H, L\}$ guarantees that there is no reallocation within homogenous coalitions made of a pair of agents of the same types when they truthfully report their types to the principal. In heterogeneous coalitions, however, the third party may have incentive to reallocate the goods bought from the seller unless the no-arbitrage constraint (18) is satisfied. Therefore, conditions (11) to (16) and (18) characterize the weakly collusion-proof mechanisms.

The variable ϵ that enters coalitional incentive constraints can be interpreted as a transaction cost in side contracting due to asymmetric information. If the θ_H type's incentive compatibility constraint is binding in the third party's program, the principal has flexibility in choosing it; if the constraint is slack, it is zero. The colluding partners usually cannot fully trust and share their private information with each other, then the third party has to face the same incentive problem faced by the principal and thus some transaction cost may arise. The seller has some degree of control over this transaction cost of side contracting through the design of an appropriate grand-mechanism. Suppose that collusion is organized under complete information, then coalitional incentive constraints would be written with $\epsilon = 0$. Taking into account the transaction cost of side contracting, true valuations must be replaced by virtual valuations in the coalitional incentive constraints. The virtual valuation of a $H-$type is equal to the true valuation θ_H, while the virtual valuation of a $L-$type is distorted downward to take into account the rent the principal has to give and its value is $\theta^v_{L,1} \equiv \theta_L - \frac{p^2_{LH}\epsilon\Delta\theta}{p_{LL}p_{LH}+\rho\epsilon}$ in a homogeneous LL coalition, $\theta^v_{L,2} \equiv \theta_L - \frac{p_{HH}\epsilon\Delta\theta}{p_{LH}}$ in a heterogeneous LH coalition.

Collusion imposes that the principal behaves as if she were facing a composite bidder (the coalition) who has two dimensional preferences (virtual valuations) over the consumptions of individual agents. In this sense, collusion is an issue that transforms the multi-agent unidimensional mechanism design problem into a single-agent multidimensional mechanism design problem with the additional subtlety that the willingness to pay of this single agent is endogenous and influenced by the design of the grand-mechanism.

Notice that it is easier for the principal to detect arbitrage than reporting manipulation and side transfers. For instance, sellers of software and e-books often prohibit the buyers from reselling their goods via license limit, real-name registration, etc. We need to consider it as a special case when the principal has control over reallocations between the colluding agents. In this case, the buyers could only manipulate their reports and exchange side transfers but have no power conducting arbitrage on the goods; then the above coalitional incentive constraints (11)-(16) simplify to the following form. [8]

COROLLARY 1 If the principal has direct control over reallocations, the coalitional incentive constraints are:

[8] "CICW" stands for "coalitional incentive constrain without arbitrage".

- for a (θ_L, θ_L) coalition

$$\begin{aligned}
CICW_{LL;LH}: \quad & 2\left(\theta_L - \frac{p_{LH}^2 \epsilon \Delta\theta}{p_{LL}p_{LH} + \rho\epsilon}\right) V(q_{LL}) - 2t_{LL} \\
& \geqslant \left(\theta_L - \frac{p_{LH}^2 \epsilon \Delta\theta}{p_{LL}p_{LH} + \rho\epsilon}\right) [V(q_{LH}) + V(q_{HL})] - t_{LH} - t_{HL} \qquad (19) \\
CICW_{LL;HH}: \quad & 2\left(\theta_L - \frac{p_{LH}^2 \epsilon \Delta\theta}{p_{LL}p_{LH} + \rho\epsilon}\right) V(q_{LL}) - 2t_{LL} \\
& \geqslant 2\left(\theta_L - \frac{p_{LH}^2 \epsilon \Delta\theta}{p_{LL}p_{LH} + \rho\epsilon}\right) V(q_{HH}) - 2t_{HH}; \qquad (20)
\end{aligned}$$

- for a (θ_L, θ_H) coalition

$$\begin{aligned}
CICW_{LH;LL}: \quad & \left(\theta_L - \frac{p_{HH} \epsilon \Delta\theta}{p_{LH}}\right) V(q_{LH}) + \theta_H V(q_{HL}) - t_{LH} - t_{HL} \\
& \geqslant \left(\theta_L - \frac{p_{HH} \epsilon \Delta\theta}{p_{LH}}\right) V(q_{LL}) + \theta_H V(q_{LL}) - 2t_{LL} \qquad (21) \\
CICW_{LH;HH}: \quad & \left(\theta_L - \frac{p_{HH} \epsilon \Delta\theta}{p_{LH}}\right) V(q_{LH}) + \theta_H V(q_{HL}) - t_{LH} - t_{HL} \\
& \geqslant \left(\theta_L - \frac{p_{HH} \epsilon \Delta\theta}{p_{LH}}\right) V(q_{HH}) + \theta_H V(q_{HH}) - 2t_{HH}; \qquad (22)
\end{aligned}$$

- for a (θ_H, θ_H) coalition

$$CICW_{HH;LL}: 2\theta_H V(q_{HH}) - 2t_{HH} \geqslant 2\theta_H V(q_{LL}) - 2t_{LL} \qquad (23)$$

$$CICW_{HH;LH}: 2\theta_H V(q_{HH}) - 2t_{HH} \geqslant \theta_H [V(q_{LH}) + V(q_{HL})] - t_{LH} - t_{HL}. \qquad (24)$$

PROOF. See appendix. ■

For the sake of simplicity, we introduce some new notations before proceeding with our analysis. Let

$$\begin{aligned}
\mathbf{P}^f: \quad & \Pi^{fb}(\rho) = \max_{\{\mathbf{t},\mathbf{q}\}} \Pi(\mathbf{t}, \mathbf{q}|\rho), s.t: \text{BIRs}[(3), (4)], \\
\mathbf{P}^s: \quad & \Pi^{sb}(\rho) = \max_{\{\mathbf{t},\mathbf{q}\}} \Pi(\mathbf{t}, \mathbf{q}|\rho), s.t: \text{BIRs, BICs}[(1), (2), (3), (4)], \\
\mathbf{P}^w: \quad & \Pi^{w}(\rho) = \max_{\{\mathbf{t},\mathbf{q},\epsilon\in[0,1)\}} \Pi(\mathbf{t}, \mathbf{q}, \epsilon|\rho), \\
& s.t: \text{BIRs, BICs, CICWs}[(1), (2), (3), (4); (19) - (24)], \\
\mathbf{P}^a: \quad & \Pi^{a}(\rho) = \max_{\{\mathbf{t},\mathbf{q},\epsilon\in[0,1)\}} \Pi(\mathbf{t}, \mathbf{q}, \epsilon|\rho), \\
& s.t: \text{BIRs, BICs, CICs, NAC}[(1), (2), (3), (4); (11) - (16), (18)].
\end{aligned}$$

$\Pi^i(\rho), i = fb, sb, w, a$ denote as functions of ρ the seller's optimal profits obtained in the cases with complete information, asymmetric information but no collusion, information manipulation but no arbitrage, both reporting manipulation and arbitrage, respectively. $\mathbf{M}^i(\rho) =$

$\{\mathbf{t}^i(\rho), \mathbf{q}^i(\rho)\}$ are the corresponding optimal sale mechanisms in these circumstances.[9] The following proposition gives a result concerning ranking of the principal's payoffs in different contexts.

PROPOSITION 3 The seller's payoffs in different environments satisfy:

$$\Pi^{fb}(\rho) \geqslant \Pi^{sb}(\rho) \geqslant \Pi^{w}(\rho) \geqslant \Pi^{a}(\rho),$$

the first inequality holds strictly if and only if $\rho = 0$.

PROOF. See appendix. ■

The intuition behind $\Pi^w \geqslant \Pi^a$ is straightforward. Arbitrage between the agents may hurt the principal since it helps the agents to collude with more degrees of freedom. Hence, the principal can strictly improve her welfare by making sure that arbitrage does not take place since she then faces a less constrained problem.

5 The Optimal Weakly Collusion-Proof Mechanism with Reporting Manipulation Alone

In this section, we assume the monopolist can prohibit resale of the good between the colluding agents. Evidence on this abounds in reality. In the U.S., Electronic Benefit Transfer system makes trafficking of the Food Stamp harder to conduct and easier to detect. Moreover, sellers of information goods can prevent resale indirectly by streaming rentals from places like Netflix and Spotify. With the advent of high speed internet connections and adoption of platforms capable of preventing illegal file sharing, e.g., Kindle, exclusive digital distribution is becoming more feasible. Under this assumption, solving the seller's problem $\mathbf{P}^w$ yields the following three propositions which characterize the weakly collusion-proof mechanisms with respectively negative, positive and zero correlations.

PROPOSITION 4 In the presence of weakly negative correlation, [10] if θ_H is sufficiently large, then $\epsilon^* = 1$ at the optimum, the weakly collusion-proof mechanism $\mathbf{M}^w(\rho) = \{\mathbf{t}^w(\rho), \mathbf{q}^w(\rho)\}$ entails:

[9] Superscripts fb, sb, w and a denote respectively "first-best": "second-best", "without arbitrage" and "arbitrage".

[10] It means that ρ is smaller than and close enough to zero.

- a vector of consumptions $\mathbf{q}^w(\rho) \in \mathbb{R}^4_+$ satisfying $q^w_{LL}(\rho) < q^w_{LH}(\rho) < q^w_{HH}(\rho) < q^w_{HL}(\rho)$, where

$$\left[\frac{\theta_L p_{LH} - p_{HH}\Delta\theta}{\rho + p_{LH}} + \rho\frac{\theta_H - (1-p_{LL})\left(\theta_L - \frac{p_{LH}^2\Delta\theta}{p_{LL}p_{LH}+\rho}\right)}{p_{LL}(\rho + p_{LH})}\right] V'\left[q^w_{LL}(\rho)\right] = c, \tag{25}$$

$$\left[\frac{\theta_L p_{LH} - p_{HH}\Delta\theta}{\rho + p_{LH}} + \rho\frac{(1-p_{LL})\left(\theta_L - \frac{p_{LH}^2\Delta\theta}{p_{LL}p_{LH}+\rho}\right) - p_{HH}\left(\theta_L - \frac{p_{HH}\Delta\theta}{p_{LH}}\right)}{2(\rho + p_{LH})p_{LH}}\right] V'\left[q^w_{LH}(\rho)\right] = c, \tag{26}$$

$$\left[\frac{\theta_H p_{LH}}{\rho + p_{LH}} + \rho\frac{(1-p_{LL})\left(\theta_L - \frac{p_{LH}^2\Delta\theta}{p_{LL}p_{LH}+\rho}\right) - p_{HH}\theta_H}{2(\rho + p_{LH})p_{LH}}\right] V'\left[q^w_{HL}(\rho)\right] = c, \tag{27}$$

$$\left[\frac{\theta_H p_{LH}}{\rho + p_{LH}} + \rho\frac{\theta_H + \left(\theta_L - \frac{p_{HH}\Delta\theta}{p_{LH}}\right)}{2\left(\rho + p_{LH}\right)}\right] V'\left[q^w_{HH}(\rho)\right] = c. \tag{28}$$

- a vector of transfers $\mathbf{t}^w(\rho) \in \mathbb{R}^4$ such that constraints (2), (3), (19) and (22) are binding.

PROOF. See appendix. ■

With weakly negative correlation, the fact that both coalitions (θ_L, θ_L) and (θ_L, θ_H) are prevented from misreporting limits the feasible transfers that could be used by the seller to fully extract the buyers' rents. The principal cannot make t_{LL}, t_{HL} largely positive, t_{LH}, t_{HH} largely negative as they are in the no-collusion outcome without violating the coalitional incentive constraints (19), (20) and (22). A (θ_L, θ_L) coalition would like to avoid bearing this extreme penalty by mimicking a (θ_L, θ_H) or (θ_H, θ_H) one. Similarly, a (θ_L, θ_H) coalition would like to mimic a (θ_H, θ_H) one to get the corresponding large rewards requested in the no-collusion outcome since t_{HH} is then large and negative. Therefore (19), (20) and (22) are likely to be binding. The above analysis shows that the possibility of collusion would amount to protecting the agents with limited liability. Given that θ_H is sufficiently large and monotonicity conditions $q^w_{LL}(\rho) < q^w_{LH}(\rho) < q^w_{HH}(\rho) < q^w_{HL}(\rho)$ hold,[11] local constraints (19) and (22) imply the global constraint (20); therefore only (19) and (22) need to bind at the optimum.

For weakly negative correlation, $\epsilon^* = 1$ at the optimum.[12] Indeed, there is no gain in setting $\epsilon < 1$ since this would only increase the cost of the coalitional incentive constraints (19) and (22). When information is asymmetric, each agent may want to conceal his private information in order to increase his own utility, and this could go against the maximization

[11] This condition can be checked ex post.

[12] Although ϵ belongs to $[0, 1)$, we allow ϵ to take the value equal to one since we are interested in the supremum of the seller's profit.

of their joint utility, so some frictions in side-contract may arise. Moreover, mutual distrust between colluding parties is a further impediment to collusion, since the informal side-contract between agents is usually illegal, implicit, being enforced only by trust, reciprocity or through repeated relationships. With negative correlation, an agent anticipates that his partner's type is more likely to be different from his own. The inherent vigilance against strangers prevents them from communicating and cooperating smoothly with each other.[13] The principal is then able to exploit the agents' mutual distrust to increase the transaction cost of side contracting.

PROPOSITION 5 For a weakly positive correlation, $\epsilon^* = 1$ at the optimum. The optimal weakly-collusion-proof mechanism $\mathbf{M}^w(\rho) = \{\mathbf{t}^w(\rho), \mathbf{q}^w(\rho)\}$ entails

- a vector of consumptions $\mathbf{q}^w(\rho) \in \mathbb{R}^4_+$ satisfying $q^w_{LH}(\rho) < q^w_{LL}(\rho) < q^w_{HL}(\rho) = q^w_{HH}(\rho)$, where

$$\left[\left(\frac{\theta_L p_{LH} - p_{HH}\Delta\theta}{\rho + p_{LH}}\right) + \rho\frac{\theta_H - p_{LH}\left(\theta_L - \frac{p_{HH}\Delta\theta}{p_{LH}} + \theta_H\right) - p_{HH}\theta_H}{(\rho + p_{LH})p_{LL}}\right] V'\left[q^w_{LL}(\rho)\right] = c, \tag{29}$$

$$\left[\left(\frac{\theta_L p_{LH} - p_{HH}\Delta\theta}{\rho + p_{LH}}\right) + \rho\frac{\theta_L - \frac{p_{HH}\Delta\theta}{p_{LH}}}{\rho + p_{LH}}\right] V'\left[q^w_{LH}(\rho)\right] = c, \tag{30}$$

$$\theta_H V'\left[q^w_{HL}(\rho)\right] = \theta_H V'\left[q^w_{HH}(\rho)\right] = c; \tag{31}$$

- a vector of transfers $\mathbf{t}^w(\rho) \in \mathbb{R}^4$ such that (2), (3), (21) and (23) are binding.

The seller cannot make t_{LL}, t_{HL} largely negative and t_{HH}, t_{LH} largely positive without violating the downward coalitional incentive constraints (21), (23) and (24). When (21) holds with equality and $q^w_{LL}(\rho) > q^w_{LH}(\rho)$, the LHS of (23) is larger than that of (24), so only (21) and (23) are binding at the optimum. The fact that $\epsilon^* = 1$ at the optimum implies that the buyers lacking power to reallocate their total purchases cannot collude frictionlessly even if they have positive correlated types.

It is worth noting that our result here is in contrast to LM (2000). In a public good provision environment, they characterize the weakly collusion-proof mechanism with weakly positive correlation. It is shown that $\epsilon^* = 0$ at the optimum (Proposition 5 of LM (2000)). In their model, two agents may consume the same amount of nonexcludable public goods even if they report different valuations to the principal and the quantities are decreasing: $x_{HH} > x_{HL} > x_{LL}$. Then, the lower the transaction cost set by the principal, the higher the virtual valuation of the

[13] As an old saying goes: birds of the same feather flock together.

low-type agent, and the lower incentive a (θ_L, θ_H) coalition may have to misreport (θ_L, θ_L).[14] Therefore, the principal prefers to set $\epsilon^* = 0$, as if there is no friction within side contract. However, things are quite different in our private good environment, in which consumers with different valuations are allocated with different amounts of goods. The consumption allocated to a low-type agent in a heterogeneous coalition is smaller than that in a homogeneous coalition, i.e., $q_{LH} < q_{LL}$. The intuition is as follows. Since (3) binds at the optimum, the expected information rent that a high-type agent can obtain by pretending to be a low-type one is given by $p_{LH}U_{HL} + p_{HH}U_{HH} = \Delta\theta[p_{LH}V(q_{LL}) + p_{HH}V(q_{LH})]$. Because of the well-known trade-off between efficiency and rent extraction, it is optimal to introduce downward distortions of both q_{LL} and q_{LH} compared to their respective first-best levels. For very small ρ, the impact of coalitional incentive constraints on q_{LL} and q_{LH} is negligibly small compared to the impact of the individual incentive constraint. The marginal benefit-marginal cost ratios of these two variables are respectively p_{LL}/p_{LH} and p_{LH}/p_{HH}. With positive correlation, it is clear that the former dominates the latter term, so q_{LL} is less distorted than q_{LH} in equilibrium. Therefore, the higher the value of ϵ, the lower the virtual valuation of a low-type agent, and the weaker incentive a (θ_L, θ_H) coalition may have to misreport (θ_L, θ_L). Then, by setting $\epsilon^* = 1$, the principal is able to exploit the distributional inequality between agents to deter their collusion.

The coalitional incentive constraints are illustrated in the following FIGURE 1 and FIGURE 2. Here and in later diagrams a solid line pointing from type$-ij$ to type$-i'j'$ means that the incentive constraint that ij not be tempted to choose the $i'j'$ contract is binding. A dotted line implies that the corresponding constraint is slack. As stated above, the principal's two-agent unidimensional collusion-preventing problem is equivalent to a single-agent two-dimensional mechanism design problem. The multidimensional mechanism design model differs markedly from and is significantly more complex than its one-dimensional counterpart. It is essentially because different types of agents cannot be unambiguously ordered, therefore the directions in which incentive constraints bind are difficult to determine. The benefit from focusing on a discrete two by two model, however, makes this problem tractable. With weakly negative correlation, the two-dimensional types are ordered decreasingly as: $(\theta^v_{L,1}, \theta^v_{L,1}) \to (\theta^v_{L,2}, \theta_H) \to (\theta_H, \theta_H)$. With weakly positive correlation, they are ordered as: $(\theta_H, \theta_H) \to (\theta^v_{L,1}, \theta^v_{L,1}) \to (\theta^2_{L,2}, \theta_H)$, but in this case both the "highest" and "lowest" types have incentive to misreport the "intermediate"

[14]The CIC$_{HL,LL}$ in LM (2000) is

$$\left(\theta_H + \theta_L - \frac{p_{HH}\epsilon\Delta\theta}{p_{LH}}\right) x_{HL} - t_{HL} - t_{LH} \geqslant \left(\theta_H + \theta_L - \frac{p_{HH}\epsilon\Delta\theta}{p_{LH}}\right) x_{LL} - 2t_{LL}$$

. Given $x_{HL} > x_{LL}$, it is clear that a smaller ϵ will make this constraint easier to be satisfied.

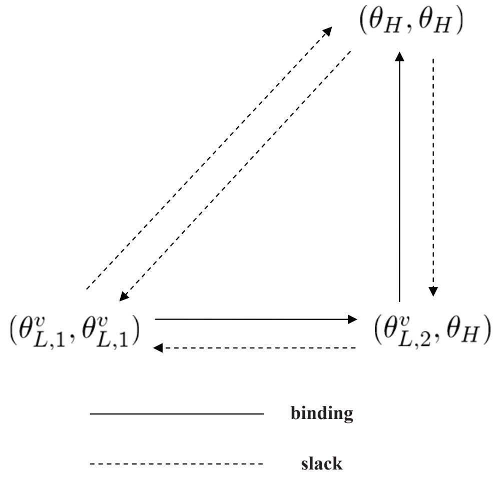

FIGURE 1. $\rho < 0$ without arbitrage

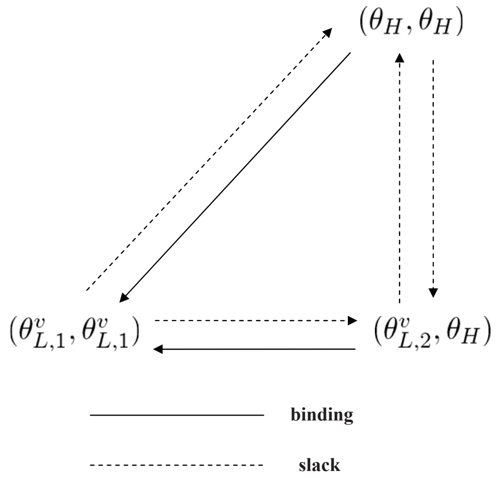

FIGURE 2. $\rho > 0$ without arbitrage

type.

PROPOSITION 6 Assume that types are independently distributed, i.e., $\rho = 0$, then there exists a transfer scheme $\mathbf{t} \in \mathbb{R}^4$ such that (2) and (3) are binding and $\{\mathbf{q}^{sb}(0), \mathbf{t}\}$ is weakly collusion-proof, i.e., $\mathbf{q}^w(0) = \mathbf{q}^{sb}(0)$.

PROOF. See appendix. ■

Summarizing the above discussion, we have the following theorem.

THEOREM 1 *For two-agent nonlinear pricing environments with correlated types, collusive behavior cannot be prevented freely so that reporting manipulation calls for distortions away from the first-best efficiency. Specifically, the optimal mechanism with reporting manipulation alone entails:*

- *downward distortions for the consumptions of low-demand type for correlated types and no distortion for them for uncorrelated types:*

$$q^w_{LL}(\rho)\begin{cases} < q^{sb}_{LL}(\rho) = q^{fb}_{LL}(\rho) & \text{if } \rho > 0 \\ = q^{sb}_{LL}(\rho) < q^{fb}_{LL}(\rho) & \text{if } \rho = 0 \\ < q^{sb}_{LL}(\rho) = q^{fb}_{LL}(\rho) & \text{if } \rho < 0 \end{cases},$$

$$q^w_{LH}(\rho)\begin{cases} < q^{sb}_{LH}(\rho) = q^{fb}_{LH}(\rho) & \text{if } \rho > 0 \\ = q^{sb}_{LH}(\rho) < q^{fb}_{LH}(\rho) & \text{if } \rho = 0 \\ < q^{sb}_{LH}(\rho) = q^{fb}_{LH}(\rho) & \text{if } \rho < 0 \end{cases};$$

- *no distortion for the consumptions of high-demand type for nonnegative correlation and upward distortions for them for negative correlation:*

$$q^w_{HL}(\rho)\begin{cases} = q^{sb}_{HL}(\rho) = q^{fb}_{HL}(\rho) & \text{if } \rho > 0 \\ = q^{sb}_{HL}(\rho) = q^{fb}_{HL}(\rho) & \text{if } \rho = 0 \\ > q^{sb}_{HL}(\rho) = q^{fb}_{HL}(\rho) & \text{if } \rho < 0 \end{cases},$$

$$q^w_{HH}(\rho)\begin{cases} = q^{sb}_{HH}(\rho) = q^{fb}_{HH}(\rho) & \text{if } \rho > 0 \\ = q^{sb}_{HH}(\rho) = q^{fb}_{HH}(\rho) & \text{if } \rho = 0 \\ > q^{sb}_{HH}(\rho) = q^{fb}_{HH}(\rho) & \text{if } \rho < 0 \end{cases};$$

- *the possibility of reporting manipulation reduces strictly the seller's profit for correlated types; it will not hurt the principal if types are uncorrelated:*

$$\Pi^w(\rho)\begin{cases} < \Pi^{sb}(\rho) = \Pi^{fb}(\rho) & \text{if } \rho \neq 0 \\ = \Pi^{sb}(\rho) < \Pi^{fb}(\rho) & \text{if } \rho = 0 \end{cases}; \tag{32}$$

- *the consumptions $\boldsymbol{q}^w(\rho)$ and the efficiency implemented $\Pi^w(\rho)$ are continuous with respect to correlation:*

$$\lim_{\rho\downarrow 0} q^w_{ij}(\rho) = \lim_{\rho\uparrow 0} q^w_{ij}(\rho) = q^w_{ij}(0) = q^{sb}_{ij}(0), \forall i,j \in \{H,L\}$$

$$\lim_{\rho\downarrow 0} \Pi^w(\rho) = \lim_{\rho\uparrow 0} \Pi^w(\rho) = \Pi^w(0) = \Pi^{sb}(0). \tag{33}$$

PROOF. See appendix. ■

In the case of negative correlation, the individual incentive constraint is binding for a downward manipulation while the coalitional incentive constraints are binding for upward manipulations. Hence, collusion creates countervailing incentives and this makes the optimal collusion-proof consumptions exhibit an upward distortion at the top and a downward distortion at the bottom with respect to the optimal scheme without side-contracting.[15] Lacking power to reallocate their total quantities, the agents could collude with a homogeneous transaction cost (i.e., $\epsilon^* = 1$), regardless of the signs of their correlation. This is why the discontinuity at $\rho = 0$ disappears. For nonzero ρ, the collusion-proof constraints prevent the use of the penalty and award system embedded in the FSE mechanism. It offers a rather satisfactory solution of this puzzle which explains to some extent the lack of practical success of explicit yardstick mechanisms. The possibility of collusion enables the two agents to collectively extract rents from the principal, i.e., to succeed in forming a pressure group.

6 The Optimal Weakly Collusion-Proof Mechanism with Both Reporting Manipulation and Arbitrage

If buyers could conduct arbitrage within their cartel, the optimal weakly collusion-proof mechanism could be obtained through solving program $\mathbf{P}^a$. The difficulty, as usual, is to determine the binding constraints. To simplify the constraints system, it is useful to derive the following implementability conditions.

Lemma 1 For a weak correlation, the schedule of collusion-proof implementable consumptions satisfies the following monotonicity condition:

$$[\mathbf{M}] : q_{LL} \leqslant \frac{q_{LH} + q_{HL}}{2} \leqslant q_{HH} \tag{34}$$

for all $\epsilon \in [0, 1)$; if these inequalities hold for all $\epsilon \in [0, 1)$, the local coalitional incentive constraints (11), (14) or (13), (16) are binding, then all the other coalitional incentive constraints are indeed satisfied.

PROOF. See appendix. ■

The relationships among coalitional incentive constraints are depicted in the following FIGURE 3.

Given this result, we could focus in the sequel only on the θ_H agent's Bayesian incentive constraint (2); the θ_L agent's individual rationality constraint (3); the local coalitional incentive constraints (11), (14) or (13), (16); no-arbitrage constraint (18) and the implementability condition (34). Then we can now simplify the principal's problem as the following program $[\mathbf{P}^a_-]$ or

[15] See Lewis and Sappington (1989), Maggi and Rodriguez (1995) and Jullien (1999) for detailed discussion of countervailing incentives.

$$
\begin{array}{ccc}
binding CIC_{LL,LH}(11) & \xrightarrow{2q_{LL}\leqslant q_{LH}+q_{HL}} & CIC_{LH,LL}(13) \\
+ & & + \\
binding CIC_{LH,HH}(14) & \xrightarrow{2q_{HH}\geqslant q_{LH}+q_{HL}} & CIC_{HH,LH}(16) \\
\Big\downarrow{\scriptstyle 2q_{HH}\geqslant q_{LH}+q_{HL}} & & \Big\downarrow{\scriptstyle 2q_{LL}\leqslant q_{LH}+q_{HL}} \\
CIC_{LL,HH}(12) & & CIC_{HH,LL}(15)
\end{array}
$$

$$
\begin{array}{ccc}
binding CIC_{LL,LH}(13) & \xrightarrow{2q_{LL}\leqslant q_{LH}+q_{HL}} & CIC_{LH,LL}(11) \\
+ & & + \\
binding CIC_{LH,HH}(16) & \xrightarrow{2q_{HH}\geqslant q_{LH}+q_{HL}} & CIC_{HH,LH}(14) \\
\Big\downarrow{\scriptstyle 2q_{LL}\leqslant q_{LH}+q_{HL}} & & \Big\downarrow{\scriptstyle 2q_{HH}\geqslant q_{LH}+q_{HL}} \\
CIC_{LL,HH}(15) & & CIC_{HH,LL}(12)
\end{array}
$$

FIGURE 3. Relationships among CICs in the case with arbitrage

$[\mathbf{P}^a_+]$.

$$
[\mathbf{P}^a_-] : \begin{cases} \max\limits_{\{\mathbf{t},\mathbf{q},\epsilon\in[0,1)\}} \Pi(\mathbf{t},\mathbf{q},\epsilon) \\ \text{subject to:} \\ BIR_L, BIC_H, CIC_{LL,LH}, CIC_{LH,HH}, NAC, \mathbf{M} \\ [(2),(3),(11),(14),(18),(34)], \end{cases}
$$

$$
[\mathbf{P}^a_+] : \begin{cases} \max\limits_{\{\mathbf{t},\mathbf{q},\epsilon\in[0,1)\}} \Pi(\mathbf{t},\mathbf{q},\epsilon) \\ \text{subject to:} \\ BIR_L, BIC_H, CIC_{HH,LH}, CIC_{LH,LL}, NAC, \mathbf{M} \\ [(2),(3),(13),(16),(18),(34)]. \end{cases}
$$

The following two propositions show that $[\mathbf{P}^w_+]$ and $[\mathbf{P}^w_-]$ correspond to the cases with weakly positive and negative correlations respectively.

PROPOSITION 7 Assuming that the correlation is weakly negative, then $\epsilon^* = 1$ is the principal's optimal choice. The optimal weakly-collusion-proof mechanism with both reports manipulation and arbitrage entails:

- a schedule of consumptions $\mathbf{q}^a(\rho) \in \mathbb{R}^4_+$ such that

$$
\left(\frac{\theta_H p_{LH}}{\rho + p_{LH}}\right) V'(q_{HH}) + \frac{\rho\theta_H V'\left(\varphi_2(2q_{HH})\right)}{\rho + p_{LH}} = c, \tag{35}
$$

$$
\left(\frac{\theta_H p_{LH}}{\rho + p_{LH}}\right) V'(q_{HL}) + \frac{\rho\left[(1-p_{LL})V'\left(\frac{q_{LH}+q_{HL}}{2}\right)\left(\theta_L - \frac{p^2_{LH}\Delta\theta}{p_{LL}p_{LH}+\rho}\right) - p_{HH}\theta_H V'(q_{HL})\right]}{2(\rho+p_{LH})p_{LH}} = c, \tag{36}
$$

$$
\left(\frac{\theta_L p_{LH} - p_{HH}\Delta\theta}{\rho + p_{LH}}\right) V'(q_{LH}) + \frac{\rho\left[(1-p_{LL})V'\left(\frac{q_{LH}+q_{HL}}{2}\right)\left(\theta_L - \frac{p^2_{LH}\Delta\theta}{p_{LL}p_{LH}+\rho}\right) - p_{HH}\theta_H V'(q_{HL})\right]}{2(\rho+p_{LH})p_{LH}} = c, \tag{37}
$$

$$\left[\frac{\theta_L p_{LH} - p_{HH}\Delta\theta}{\rho + p_{LH}} + \frac{\rho\theta_H}{(\rho + p_{LH})\,p_{LL}} - \frac{\rho(1-p_{LL})\left(\theta_L - \frac{p_{LH}^2\Delta\theta}{p_{LL}p_{LH}+\rho}\right)}{p_{LL}(\rho + p_{LH})}\right] V'(q_{LL}) = c. \tag{38}$$

- a vector of transfers $\mathbf{t}^a \in \mathbb{R}^4$ such that (2), (3), (11), (14) are binding.

- a surplus strictly lower than that attained in the case without arbitrage: $\Pi^a(\rho) < \Pi^w(\rho)$.

PROOF. See appendix. ■

The fact that coalition (θ_L, θ_L) (resp. (θ_L, θ_H)) is prevented from misreporting (θ_L, θ_H) (resp. (θ_H, θ_H)), limits the extreme awards or penalties that could be used by the FSE mechanism. This leads to binding local upward coalitional incentive constraints (11) and (14). Conducting arbitrage will apparently increase the total surplus of a coalition.[16] After optimally allocating the total quantities, a (θ_L, θ_L) (resp. (θ_L, θ_H)) coalition is more likely to be tempted to overreport (θ_L, θ_H) (resp. (θ_H, θ_H)). Hence, it is more reasonable for the principal to disincentive their misreports by setting $\epsilon^* = 1$ and decreasing the virtual valuations of low-type agent.

PROPOSITION 8 For weakly positive correlation, the seller will choose $\epsilon^* = 0$ at the optimum, and the optimal weakly collusion-proof mechanism $\mathbf{M}^a(\rho)$ with both reporting manipulation and arbitrage entails:

- a strictly increasing schedule of consumptions: $q^a_{LL}(\rho) < q^a_{LH}(\rho) < q^a_{HL}(\rho) < q^a_{HH}(\rho)$ given by

$$\left[\frac{\theta_L p_{LH} - p_{HH}\Delta\theta}{\rho + p_{LH}} + \frac{\rho\theta_H}{(\rho + p_{LH})\,p_{LL}}\right] V'(q_{LL}) - \frac{\rho(1-p_{LL})}{p_{LL}(\rho + p_{LH})}\theta_H V'\left(\varphi_2(2q_{LL})\right) = c \tag{39}$$

$$\left(\frac{\theta_L p_{LH} - p_{HH}\Delta\theta}{\rho + p_{LH}}\right) V'\left(q_{LH}\right) + \rho\left[\frac{(1-p_{LL})\theta_L V'(q_{LH}) - p_{HH}\theta_H V'\left(\frac{q_{LH}+q_{HL}}{2}\right)}{2(\rho + p_{LH})p_{LH}}\right] - \lambda\theta_L V''(q_{LH}) = c \tag{40}$$

$$\left(\frac{\theta_H p_{LH}}{\rho + p_{LH}}\right) V'\left(q_{HL}\right) + \rho\left[\frac{(1-p_{LL})\theta_H V'(q_{HL}) - p_{HH}\theta_H V'\left(\frac{q_{HL}+q_{HL}}{2}\right)}{2(\rho + p_{LH})p_{LH}}\right] + \lambda\theta_H V''(q_{HL}) = c, \tag{41}$$

$$\theta_H V'(q_{HH}) = c, \tag{42}$$

  where positive parameter λ is the Lagrangean multiplier of NAC written with $\epsilon = 0$;

- a vector of transfers $\mathbf{t}^a \in \mathbb{R}^4$ such that (2), (3), (13), (16) are binding;

- a profit strictly lower than its counterpart in the case without arbitrage: $\Pi^a(\rho) < \Pi^w(\rho)$.

PROOF. See appendix. ■

The reason why upward coalitional constraints (13) and (16) bind at the optimum is the conflicts between these constraints and the extreme award and penalty used by FSE mechanism. For a weakly positive correlation, $\epsilon^* = 0$ at the optimum. This is a notable difference in proposition 8 compared to propositions 4, 5 and 7, where the principal sets the collusive transaction

[16] The RHS of (11) (resp. (15)) is larger than that of (19) (resp. (23)).

cost at its highest possible level, i.e., $\epsilon^* = 1$. $\theta_H - \theta^v_{L,2} = \Delta\theta\,(1 + p_{HH}\epsilon/p_{LH})$ means the higher the transaction cost of collusion, the larger the divergence of preferences between agents in a (θ_L, θ_H) coalition. Therefore, when reallocation is infeasible, a coalition with a larger ϵ prefers the unequal allocation $\{(q_{LH}, t_{LH}); (q_{HL}, t_{HL})\}$ to the equal one $\{(q_{LL}, t_{LL}), (q_{LL}, t_{LL})\}$ more than a coalition with a smaller ϵ. When the agents could conduct reallocation, however, a coalition may care only about the total quantities obtained rather than that allocated to individual agent by the principal. A smaller ϵ will weaken the incentive of a $LH-$coalition to underreport (θ_L, θ_L). Having a strictly positive ϵ would only make coalitional incentive constraint (13) harder to satisfy. In a word, the presence of asymmetries increases the frictions among the colluding parties; but this effect vanishes when quantities reallocation is feasible. The binding coalitional incentive constraint (13) takes therefore the same form as if consumers could credibly share their information within the coalition. Everything happens as if asymmetric information does not really undermine the agents' ability of colluding, except that their individual incentive and participation constraints are the interim ones.

Ignoring the no-arbitrage constraint at first and then checking it ex post, we find that this constraint cannot be satisfied automatically. This is because the quantities derived from optimization satisfy $(\theta_L - p_{HH}\Delta\theta/p_{LH})\,V(q_{LH}) = \theta_H V(q_{HL})$. However, the NAC written with $\epsilon = 0$ requires $\theta_L V(q_{LH}) = \theta_H V(q_{HL})$. The conflict between (18) and the remaining constraints calls for a further distortion away from the efficiency implemented. The set of binding and slack coalitional incentive constraints with arbitrage is depicted in Figure 4 and 5. For negative correlation, the multidimensional types of the composite agent (coalition) could be ordered decreasingly as $(\theta^v_{L,1}, \theta^v_{L,1}) \to (\theta^v_{L,1}, \theta_H) \to (\theta_H, \theta_H)$, which is identical to the arbitrage-free case. For positive correlation, the multidimensional types are ordered conversely as $(\theta_H, \theta_H) \to (\theta^v_{L,1}, \theta_H) \to (\theta^v_{L,1}, \theta^v_{L,1})$.

In the degenerate case of no correlation, we have the following result.

PROPOSITION 9 When arbitrage are possible and the agents' types are independent ($\rho = 0$), then $\epsilon^* = 1$ and there exists a vector of transfers $\mathbf{t} \in \mathbb{R}^4$ such that $\mathbf{M} = \{\mathbf{t}, \mathbf{q}^{sb}(0)\}$ is the weakly collusion-proof mechanism. That is to say, $\mathbf{M}^a(0) = \mathbf{M}^{sb}(0)$ and $\Pi^a(0) = \Pi^{sb}(0)$.

PROOF. See appendix. ■

This proposition states that an optimal mechanism in the absence of buyer coalition is also weakly collusion-proof with uncorrelated types. It is in line with the main finding of JM (2005).

Summarizing the discussion, we have the following theorem.

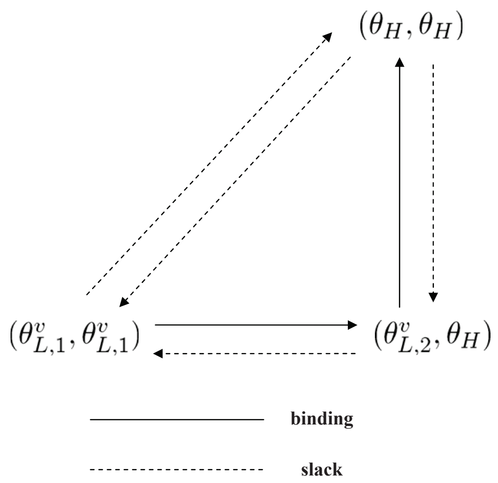

FIGURE 4. $\rho < 0$ with arbitrage

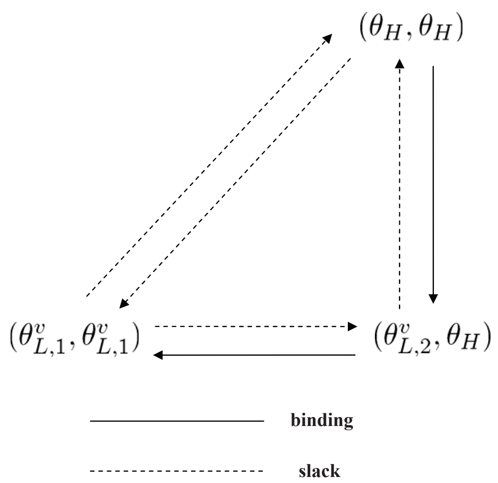

FIGURE 5. $\rho > 0$ with arbitrage

THEOREM 2 *For two-agent nonlinear pricing environments with correlated types, collusive behavior cannot be prevented freely so that arbitrage calls for further distortion away from the first-best efficiency that is lower than the second-best efficiency. Specifically, with both reporting manipulation and arbitrage, the collusion-proof mechanism $\boldsymbol{M}^a(\rho) = (\boldsymbol{t}^a(\rho), \boldsymbol{q}^a(\rho))$ entails the following:*

- *for weakly positive correlation, there is no distortion with respect to the no collusion outcome for $q^a_{HH}(\rho)$, and downward distortions for other quantities; for weakly negative correlation, there are downward distortions for $q^a_{LL}(\rho), q^a_{LH}(\rho)$ and upward distortions for $q^a_{HL}(\rho), q^a_{HH}(\rho)$; for zero correlation, there is no distortion for any $q^a_{ij}(\rho), i, j \in \{H, L\}$,*

$$q^a_{LL}(\rho)\begin{cases} < q^{sb}_{LL}(\rho) = q^{fb}_{LL}(\rho) & \text{if } \rho > 0 \\ = q^{sb}_{LL}(\rho) < q^{fb}_{LL}(\rho) & \text{if } \rho = 0 \\ < q^{sb}_{LL}(\rho) = q^{fb}_{LL}(\rho) & \text{if } \rho < 0 \end{cases},$$

$$q^a_{LH}(\rho)\begin{cases} < q^{sb}_{LH}(\rho) = q^{fb}_{LH}(\rho) & \text{if } \rho > 0 \\ = q^{sb}_{LH}(\rho) < q^{fb}_{LH}(\rho) & \text{if } \rho = 0 \\ < q^{sb}_{LH}(\rho) = q^{fb}_{LH}(\rho) & \text{if } \rho < 0 \end{cases},$$

$$q^a_{HL}(\rho)\begin{cases} < q^{sb}_{HL}(\rho) = q^{fb}_{HL}(\rho) & \text{if } \rho > 0 \\ = q^{sb}_{HL}(\rho) < q^{fb}_{HL}(\rho) & \text{if } \rho = 0 \\ > q^{sb}_{HL}(\rho) = q^{fb}_{HL}(\rho) & \text{if } \rho < 0 \end{cases},$$

$$q^a_{HH}(\rho)\begin{cases} = q^{sb}_{HH}(\rho) = q^{fb}_{HH}(\rho) & \text{if } \rho > 0 \\ = q^{sb}_{HH}(\rho) < q^{fb}_{HH}(\rho) & \text{if } \rho = 0 \\ > q^{sb}_{HH}(\rho) = q^{fb}_{HH}(\rho) & \text{if } \rho < 0 \end{cases};$$

- *consumptions $q^a_{LL}(\rho), q^a_{HH}(\rho)$ are continuous at $\rho = 0$, but $q^a_{LH}(\rho)$ and $q^a_{HL}(\rho)$ are only continuous from the left hand side at $\rho = 0$*

$$\begin{aligned}
&\lim_{\rho\uparrow 0} q^a_{ij}(\rho) = q^a_{ij}(0) = q^w_{ij}(0) = q^{sb}_{ij}(0), \forall i, j \in \{H, L\},\\
&\lim_{\rho\downarrow 0} q^a_{LL}(\rho) = q^a_{LL}(0) = q^w_{LH}(0) = q^{sb}_{LH}(0),\\
&\lim_{\rho\downarrow 0} q^a_{HH}(\rho) = q^a_{HH}(0) = q^w_{HH}(0) = q^{sb}_{HH}(0),\\
&\lim_{\rho\downarrow 0} q^a_{LH}(\rho) > q^a_{LH}(0) = q^w_{LH}(0) = q^{sb}_{LH}(0),\\
&\lim_{\rho\downarrow 0} q^a_{HL}(\rho) < q^a_{HL}(0) = q^w_{HL}(0) = q^{sb}_{HL}(0);
\end{aligned}$$

- *the seller's profit is only left-continuous with respect to ρ at zero correlation,*

$$\lim_{\rho\downarrow 0}\Pi^a(\rho) < \Pi^a(0) = \lim_{\rho\uparrow 0}\Pi^a(\rho). \tag{43}$$

For positive correlation, both individual and coalitional incentive constraints are binding for downward manipulation, a trade-off between efficiency and rent extraction calls for a further downward distortions of q_{LH} and q_{LL} than the case without side contracting, but no distortion for q_{HH}. For negative correlation, however, the individual incentive constraint is binding for a downward manipulation while the coalitional incentive constraints are binding for upward manipulations. Hence, collusion creates countervailing incentives and this calls for two-way distortions for quantities: q_{HL} and q_{HH} are distorted upwardly; q_{LH} and q_{LL} are distorted downwardly compared to the quantities without side-contracting. This is analogous to the argument underlying Corollary 1 in this paper, and the only difference is that now we have $2q^a_{LL}(\rho) \leqslant q^a_{HL}(\rho) + q^a_{LH}(\rho) \leqslant 2q^a_{HH}(\rho)$.

Corollaries 1 and 2 imply that the efficiency distortion originated from arbitrage-preventing gradually vanishes as ρ approaches zero from left side: $\lim_{\rho\uparrow 0}[\Pi^a(\rho) - \Pi^w(\rho)] = 0$; however, a non-vanishing fraction exists when ρ approaches zero from right side: $\lim_{\rho\downarrow 0}[\Pi^a(\rho) - \Pi^w(\rho)] < 0$. Such a discontinuity stems from the heterogeneous transaction costs of collusion. The optimal transaction costs chosen by the principal in different collusion environments could be summarized in the following table.

	without arbitrage	with arbitrage
$\rho < 0$	$\epsilon^* = 1$	$\epsilon^* = 1$
$\rho = 0$	$\epsilon^* \in [0, 1]$	$\epsilon^* = 1$
$\rho > 0$	$\epsilon^* = 1$	$\epsilon^* = 0$

TABLE 1. Transaction costs in different collusion environments

It can be seen that in the case without arbitrage, the transaction costs are homogeneous regardless of the sign of correlation. In the case with arbitrage, however, the transaction costs are heterogeneous. The possibility of arbitrage enables the agents to optimally reallocate their resources and hence to disincentive a (θ_L, θ_H) coalition to underreport (θ_L, θ_L). The principal may thus choose a lowest possible transaction cost to alleviate the coalitional incentive constraint at the cost of breaking the continuity of optimal mechanism with respect to correlation.

7 Conclusion

Applying CM's FSE mechanism, the principal may exploit the correlation between agents' private information to elicit their truthtelling at no cost. For the purpose of protecting their rents, agents may collude at the principal's loss by coordinating their reports and conducting arbitrage on their total resources. As such, the principal needs to fight off collusion by designing her grand mechanism. This raises a natural question concerning how the transaction cost associated with agents' private information can be exploited to overcome collusion. CK (2006) have shown that the the principal can always fight off collusion at no cost in a broad class of economic enviroments with $n \geq 2$ agents for uncorrelated types and $n \geq 3$ agents for correlated types.

We find in this paper that their findings are no longer true if there exist only two agents when types are correlated. In a private good nonlinear pricing environment with correlation and arbitrage, we show that collusion calls for distortion away from the noncollusive efficiency, that is, collusion cannot be preventable at no cost. Moreover, we find that the distortionary ways are quite different for positive and negative correlations. For negative correlation, the distortion vanishes as the correlation goes to zero; for positive correlation, on the contrary, there exists a nonvanishing fraction of distortion. The collusion-proof mechanism is therefore discontinuous with respect to information structure. This discontinuity of mechanism relies on the fact that the principal may choose heterogeneous transaction costs of collusion when arbitrage is possible. As such, it is optimal for her to choose a highest (resp. lowest) possible transaction cost in the presence of negative (resp. positive) correlation.

Appendix

Proof of Proposition 2

Let $\phi_{kl} = \phi(\theta_k, \theta_l), k, l \in \{H, L\}$ for simplicity. Since we are not interested in grand-mechanisms such that the θ_L agents' incentive constraints are binding (these constraints will be satisfied ex post), the third-party's problem can be written as:

$$\max_{\phi(\cdot),x_i(\cdot),y_i(\cdot)} \sum_{(\theta_1,\theta_2)\in\Theta^2} p(\theta_1,\theta_2)\left\{\sum_{i=1}^{2}\Big[\theta_i V\big(x_i(\theta_1,\theta_2,\phi(\theta_1,\theta_2)) + q_i(\phi(\theta_1,\theta_2))\big) - t_i\big(\phi(\theta_1,\theta_2)\big)\Big]\right\}$$

subject to the following constraints.

- Budget balance:

$$(BB:y) \sum_{k=1,2} y_k(\theta_1,\theta_2) = 0, \forall(\theta_1,\theta_2) \in \Theta^2 \tag{44}$$

$$(BB:x) \sum_{k=1,2} x_k(\theta_1,\theta_2,\phi) = 0, \forall(\theta_1,\theta_2) \in \Theta^2, \forall\phi \in \Theta^2. \tag{45}$$

- Bayesian incentive constraints for respectively the θ_H 1 and 2:

$$\begin{aligned}
BIC_1^S(\theta_H) : p_{LH}&\Big[\theta_H V\big(x_1(\theta_H,\theta_L,\phi_{HL}) + q_1(\phi_{HL})\big) + y_1(\theta_H,\theta_L) - t_1(\phi_{HL})\Big]\\
+p_{HH}&\Big[\theta_H V\big(x_1(\theta_H,\theta_H,\phi_{HH}) + q_1(\phi_{HH})\big) + y_1(\theta_H,\theta_H) - t_1(\phi_{HH})\Big]\\
\geqslant p_{LH}&\Big[\theta_H V\big(x_1(\theta_L,\theta_L,\phi_{LL}) + q_1(\phi_{LL})\big) + y_1(\theta_L,\theta_L) - t_1(\phi_{LL})\Big]\\
+p_{HH}&\Big[\theta_H V\big(x_1(\theta_L,\theta_H,\phi_{LH}) + q_1(\phi_{LH})\big) + y_1(\theta_L,\theta_H) - t_1(\phi_{LH})\Big]
\end{aligned} \tag{46}$$

$$\begin{aligned}
BIC_2^S(\theta_H) : p_{LH}&\Big[\theta_H V\big(x_2(\theta_L,\theta_H,\phi_{LH}) + q_2(\phi_{LH})\big) + y_2(\theta_L,\theta_H) - t_2(\phi_{LH})\Big]\\
+p_{HH}&\Big[\theta_H V\big(x_2(\theta_H,\theta_H,\phi_{HH}) + q_2(\phi_{HH})\big) + y_2(\theta_H,\theta_H) - t_2(\phi_{HH})\Big]\\
\geqslant p_{LH}&\Big[\theta_H V\big(x_2(\theta_L,\theta_L,\phi_{LL}) + q_2(\phi_{LL})\big) + y_2(\theta_L,\theta_L) - t_2(\phi_{LL})\Big]\\
+p_{HH}&\Big[\theta_H V\big(x_2(\theta_H,\theta_L,\phi_{HL}) + q_2(\phi_{HL})\big) + y_2(\theta_H,\theta_L) - t_2(\phi_{HL})\Big];
\end{aligned} \tag{47}$$

- Bayesian participation constraints for respectively the θ_H 1 and 2:

$$\begin{aligned}
BIR_1^S(\theta_H) :&p_{LH}\Big[\theta_H V\big(x_1(\theta_H,\theta_L,\phi_{HL}) + q_1(\phi_{HL})\big) + y_1(\theta_H,\theta_L) - t_1(\phi_{HL})\Big]\\
&+p_{HH}\Big[\theta_H V\big(x_1(\theta_H,\theta_H,\phi_{HH}) + q_1(\phi_{HH})\big) + y_1(\theta_H,\theta_H) - t_1(\phi_{HH})\Big]\\
&\geqslant(p_{LH} + p_{HH})U_1^M(\theta_H)
\end{aligned} \tag{48}$$

$$
\begin{aligned}
BIR_2^S(\theta_H):& p_{LH}\Big[\theta_H V\big(x_2(\theta_L,\theta_H,\phi_{LH})+q_2(\phi_{LH})\big)+y_2(\theta_L,\theta_H)-t_2(\phi_{LH})\Big]\\
&+p_{HH}\Big[\theta_H V\big(x_2(\theta_H,\theta_H,\phi_{HH})+q_2(\phi_{HH})\big)+y_2(\theta_H,\theta_H)-t_2(\phi_{HH})\Big]\\
&\geqslant(p_{LH}+p_{HH})U_2^M(\theta_H);
\end{aligned}
\tag{49}
$$

- Participation constraints for respectively the θ_L agents 1 and 2:

$$
\begin{aligned}
BIR_1^S(\theta_L):& p_{LL}\Big[\theta_L V\big(x_1(\theta_L,\theta_L,\phi_{LL})+q_1(\phi_{LL})\big)+y_1(\theta_L,\theta_L)-t_1(\phi_{LL})\Big]\\
&+p_{LH}\Big[\theta_L V\big(x_1(\theta_L,\theta_H,\phi_{LH})+q_1(\phi_{LH})\big)+y_1(\theta_L,\theta_H)-t_1(\phi_{LH})\Big]\\
&\geqslant(p_{LL}+p_{LH})U_1^M(\theta_L)
\end{aligned}
\tag{50}
$$

$$
\begin{aligned}
BIR_2^S(\theta_L):& p_{LL}\Big[\theta_L V\big(x_2(\theta_L,\theta_L,\phi_{LL})+q_2(\phi_{LL})\big)+y_2(\theta_L,\theta_L)-t_2(\phi_{LL})\Big]\\
&+p_{LH}\Big[\theta_L V\big(x_2(\theta_H,\theta_L,\phi_{HL})+q_2(\phi_{HL})\big)+y_2(\theta_H,\theta_L)-t_2(\phi_{HL})\Big]\\
&\geqslant(p_{LL}+p_{LH})U_2^M(\theta_L).
\end{aligned}
\tag{51}
$$

Let us introduce the following multipliers $\rho(\theta_1,\theta_2),\tau(\theta_1,\theta_2),\delta_1,\delta_2,\overline{\nu}_1,\overline{\nu}_2,\underline{\nu}_1,\underline{\nu}_2$, associated with constraints (44) to (51) respectively. We write the Lagrangean function of the above maximization problem as:

$$
\begin{aligned}
L =& E(U_1+U_2)+\sum_{i=1}^{2}\delta_i BIC_i^S(\theta_H)+\sum_{i=1}^{2}\overline{\nu}_i BIR_i^S(\theta_H)+\sum_{i=1}^{2}\underline{\nu}_i BIR_i^S(\theta_L)\\
&+\sum_{(\theta_1,\theta_2)\in\Theta^2}\rho(\theta_1,\theta_2)(BB:y)(\theta_1,\theta_2)+\sum_{(\theta_1,\theta_2)\in\Theta^2}\tau(\theta_1,\theta_2)(BB:x)(\theta_1,\theta_2).
\end{aligned}
$$

- Maximizing with respect to $y_1(\cdot,\cdot),y_2(\cdot,\cdot)$ yields

$$y_1(\theta_L,\theta_L):\quad \rho(\theta_L,\theta_L)-p_{LH}\delta_1+p_{LL}\underline{\nu}_1 \quad =0, \tag{52}$$

$$y_2(\theta_L,\theta_L):\quad \rho(\theta_L,\theta_L)-p_{LH}\delta_2+p_{LL}\underline{\nu}_2 \quad =0, \tag{53}$$

$$y_1(\theta_L,\theta_H):\quad \rho(\theta_L,\theta_H)-p_{HH}\delta_1+p_{LH}\underline{\nu}_1 \quad =0, \tag{54}$$

$$y_2(\theta_L,\theta_H):\quad \rho(\theta_L,\theta_H)+p_{LH}\left(\delta_2+\overline{\nu}_2\right) \quad =0, \tag{55}$$

$$y_1(\theta_H,\theta_L):\quad \rho(\theta_H,\theta_L)+p_{LH}\left(\delta_1+\overline{\nu}_1\right) \quad =0, \tag{56}$$

$$y_2(\theta_H,\theta_L):\quad \rho(\theta_H,\theta_L)+p_{LH}\underline{\nu}_2-p_{HH}\delta_2 \quad =0, \tag{57}$$

$$y_1(\theta_H,\theta_H):\quad \rho(\theta_H,\theta_H)+p_{HH}\left(\delta_1+\overline{\nu}_1\right) \quad =0, \tag{58}$$

$$y_2(\theta_H,\theta_H):\quad \rho(\theta_H,\theta_H)+p_{HH}\left(\delta_2+\overline{\nu}_2\right) \quad =0. \tag{59}$$

Expressions (52) and (53) imply

$$-p_{LH}\delta_1+p_{LL}\underline{\nu}_1=-p_{LH}\delta_2+p_{LL}\underline{\nu}_2. \tag{60}$$

(54) and (55) imply

$$\delta_2 + \overline{\nu}_2 = \underline{\nu}_1 - \frac{p_{HH}}{p_{LH}}\delta_1. \tag{61}$$

(56) and (57) imply

$$\delta_1 + \overline{\nu}_1 = \underline{\nu}_2 - \frac{p_{HH}}{p_{LH}}\delta_2. \tag{62}$$

(58) and (59) imply

$$\delta_1 + \overline{\nu}_1 = \delta_2 + \overline{\nu}_2. \tag{63}$$

In what follows, without loss of generality, we consider the symmetric multipliers $\delta_1 = \delta_2 \equiv \delta, \underline{\nu}_1 = \underline{\nu}_2 \equiv \underline{\nu}, \overline{\nu}_1 = \overline{\nu}_2 \equiv \overline{\nu}$.

- Maximizing with respect to $x_1(\cdot,\cdot,\cdot), x_2(\cdot,\cdot,\cdot)$ yields

$$\tau(\theta_L,\theta_L) + (p_{LL}\theta_L - p_{LH}\delta_1\theta_H + p_{LL}\underline{\nu}_1\theta_L)V'\big(x_1(\theta_L,\theta_L,\phi_{LL}) + q_1(\phi_{LL})\big) = 0, \tag{64}$$

$$\tau(\theta_L,\theta_L) + (p_{LL}\theta_L - p_{LH}\delta_2\theta_H + p_{LL}\underline{\nu}_2\theta_L)V'\big(x_2(\theta_L,\theta_L,\phi_{LL}) + q_2(\phi_{LL})\big) = 0, \tag{65}$$

$$\tau(\theta_L,\theta_H) + (p_{LH}\theta_L - p_{HH}\theta_H\delta_1 + p_{LH}\underline{\nu}_1\theta_L)V'\big(x_1(\theta_L,\theta_H,\phi_{LH}) + q_1(\phi_{LH})\big) = 0, \tag{66}$$

$$\tau(\theta_L,\theta_H) + (p_{LH}\theta_H + p_{LH}\theta_H\delta_2 + p_{LH}\overline{\nu}_2\theta_H)V'\big(x_2(\theta_L,\theta_H,\phi_{LH}) + q_2(\phi_{LH})\big) = 0, \tag{67}$$

$$\tau(\theta_H,\theta_L) + (p_{LH}\theta_H + p_{LH}\theta_H\delta_1 + p_{LH}\overline{\nu}_1\theta_H)V'\big(x_1(\theta_H,\theta_L,\phi_{HL}) + q_1(\phi_{HL})\big) = 0, \tag{68}$$

$$\tau(\theta_H,\theta_L) + (p_{LH}\theta_L - p_{HH}\theta_H\delta_2 + p_{LH}\underline{\nu}_2\theta_L)V'\big(x_2(\theta_H,\theta_L,\phi_{HL}) + q_2(\phi_{HL})\big) = 0, \tag{69}$$

$$\tau(\theta_H,\theta_H) + (p_{HH}\theta_H + p_{HH}\theta_H\delta_1 + p_{HH}\theta_H\overline{\nu}_1)V'\big(x_1(\theta_H,\theta_H,\phi_{HH}) + q_1(\phi_{HH})\big) = 0, \tag{70}$$

$$\tau(\theta_H,\theta_H) + (p_{HH}\theta_H + p_{HH}\theta_H\delta_2 + p_{HH}\theta_H\overline{\nu}_2)V'\big(x_2(\theta_H,\theta_H,\phi_{HH}) + q_2(\phi_{HH})\big) = 0. \tag{71}$$

(64) and (65) imply

$$V'\big(x_1(\theta_L,\theta_L,\phi_{LL}) + q_1(\phi_{LL})\big) = V'\big(x_2(\theta_L,\theta_L,\phi_{LL}) + q_2(\phi_{LL})\big), \forall\phi_{LL} \in \Theta^2.$$

(70) and (71) imply

$$V'\big(x_1(\theta_H,\theta_H,\phi_{HH}) + q_1(\phi_{HH}\big) = V'\big(x_2(\theta_H,\theta_H,\phi_{HH}) + q_2(\phi_{HH})\big), \forall\phi_{HH} \in \Theta^2.$$

Since $x_1(\theta_L,\theta_L,\phi_{LL}) + x_2(\theta_L,\theta_L,\phi_{LL}) = 0, x_1(\theta_H,\theta_H,\phi_{HH}) + x_2(\theta_H,\theta_H,\phi_{HH}) = 0$ from a budget-balance constraint, we have

$$x_1(\theta_L,\theta_L,\phi_{LL}) + q_1(\phi_{LL}) = x_2(\theta_L,\theta_L,\phi_{LL}) + q_2(\phi_{LL}) = \frac{q_1(\phi_{LL}) + q_2(\phi_{LL})}{2}, \forall\phi_{LL} \tag{72}$$

$$x_1(\theta_H,\theta_H,\phi_{HH}) + q_1(\phi_{HH}) = x_2(\theta_H,\theta_H,\phi_{HH}) + q_2(\phi_{HH}) = \frac{q_1(\phi_{HH}) + q_2(\phi_{HH})}{2}, \forall\phi_{HH}. \tag{73}$$

(66) and (67) imply

$$\begin{aligned}&\left(\theta_L - \frac{p_{HH}}{p_{LH}}\theta_H\delta_1 + \underline{\nu}_1\theta_L\right)V'\big(x_1(\theta_L,\theta_H,\phi_{LH}) + q_1(\phi_{LH})\big)\\ &= (1 + \delta_2 + \overline{\nu}_2)\,\theta_H V'\big(x_2(\theta_L,\theta_H,\phi_{LH}) + q_2(\phi_{LH})\big)\end{aligned}$$

Using (61), we obtain

$$\left(\theta_L - \frac{p_{HH}\epsilon}{p_{LH}}\Delta\theta\right) V'\big(x_1(\theta_L,\theta_H,\phi_{LH})+q_1(\phi_{LH})\big) = \theta_H V'\big(x_2(\theta_L,\theta_H,\phi_{LH})+q_2(\phi_{LH})\big), \forall \phi_{LH}$$

where

$$\epsilon = \frac{\delta}{1+\delta+\overline{\nu}}.$$

Similarly, expressions (68), (69) and (62) imply

$$\theta_H V'\big(x_1(\theta_H,\theta_L,\phi_{HL})+q_1(\phi_{HL})\big) = \left(\theta_L - \frac{p_{HH}\epsilon}{p_{LH}}\Delta\theta\right) V'\big(x_2(\theta_H,\theta_L,\phi_{HL})+q_2(\phi_{HL})\big), \forall \phi_{HL}.$$

With budget-balance constraints $x_1(\theta_k,\theta_l,\phi_{kl}) + x_2(\theta_k,\theta_l,\phi_{kl}) = 0, \forall k,l \in \{H,L\}, k \neq l$, the total quantity available to a heterogeneous coalition is split according to the following rule:

$$\begin{aligned}
x_1(\theta_L,\theta_H,\phi_{LH}) + q_1(\phi_{LH}) &= \varphi_1\big(q_1(\phi_{LH}) + q_2(\phi_{LH})\big), && (74)\\
x_2(\theta_L,\theta_H,\phi_{LH}) + q_2(\phi_{LH}) &= \varphi_2\big(q_1(\phi_{LH}) + q_2(\phi_{LH})\big), && (75)\\
x_1(\theta_H,\theta_L,\phi_{HL}) + q_1(\phi_{HL}) &= \varphi_2\big(q_1(\phi_{HL}) + q_2(\phi_{HL})\big), && (76)\\
x_2(\theta_H,\theta_L,\phi_{HL}) + q_2(\phi_{HL}) &= \varphi_1\big(q_1(\phi_{HL}) + q_2(\phi_{HL})\big), && (77)
\end{aligned}$$

for any $\phi_{ij} \in \Theta^2$, where

$$(\varphi_1(x), \varphi_2(x)) = \underset{x_1,x_2:x_1+x_2=x}{\arg\max} \left[\left(\theta_L - \frac{p_{HH}\epsilon\Delta\theta}{p_{LH}}\right) V(x_1) + \theta_H V(x_2)\right].$$

For weakly collusion-proof grand mechanism, $\phi(\theta_1,\theta_2) = (\theta_1,\theta_2), x_i\big(\theta_1,\theta_2,\phi(\theta_1,\theta_2)\big) = 0$, so (72) and (73) are trivially satisfied. Expressions (74) and (75) (or (76) and (77)) imply

$$q_{LH} = \varphi_1\left(q_{LH} + q_{HL}\right), q_{HL} = \varphi_2\left(q_{LH} + q_{HL}\right),$$

or equivalently,

$$\left(\theta_L - \frac{p_{HH}\epsilon}{p_{LH}}\Delta\theta\right) V'(q_{LH}) = \theta_H V'(q_{HL}). \tag{78}$$

This condition states that if the quantities allocated by the principal to agents maximize their total payoff, then the third party has no incentive to conduct reallocation. Therefore, we call it "no-arbitrage constraint (NAC)".

- The optimal manipulation of reports.

– Optimizing with respect to ϕ_{LL} yields

$$\begin{aligned}\phi_{LL}^* \in \arg\max_{\phi_{LL}} \Big\{ & p_{LL}\Big[\theta_L V\big(x_1(\theta_L,\theta_L,\phi_{LL}) + q_1(\phi_{LL})\big) + \theta_L V\big(x_2(\theta_L,\theta_L,\phi_{LL}) + q_2(\phi_{LL})\big) \\ & - t_1(\phi_{LL}) - t_2(\phi_{LL})\Big] \\ & + p_{LL}\underline{\nu}_1\big[\theta_L V\big(x_1(\theta_L,\theta_L,\phi_{LL}) + q_1(\phi_{LL})\big) - t_1(\phi_{LL})\big] \\ & + p_{LL}\underline{\nu}_2\big[\theta_L V\big(x_2(\theta_L,\theta_L,\phi_{LL}) + q_2(\phi_{LL})\big) - t_2(\phi_{LL})\big] \\ & - p_{LH}\delta_1\big[\theta_H V\big(x_2(\theta_L,\theta_L,\phi_{LL}) + q_2(\phi_{LL})\big) - t_1(\phi_{LL})\big] \\ & - p_{LH}\delta_2\big[\theta_H V\big(x_1(\theta_L,\theta_L,\phi_{LL}) + q_2(\phi_{LL})\big) - t_1(\phi_{LL})\big]\Big\}.\end{aligned}$$

Note that at symmetric equilibrium $\delta_1 = \delta_2 = \delta; \underline{\nu}_1 = \underline{\nu}_2 = \underline{\nu}; \overline{\nu}_1 = \overline{\nu}_2 = \overline{\nu}$, then from constraints (60) to (62) and (72), (73) the objective function can be written as

$$\begin{aligned} & (p_{LL} + p_{LL}\underline{\nu}_1 - p_{LH}\delta_1)\big[\theta_L V\big(x_1(\theta_L,\theta_L,\phi_{LL}) + q_1(\phi_{LL})\big) - t_1(\phi_{LL})\big] \\ & +(p_{LL} + p_{LL}\underline{\nu}_2 - p_{LH}\delta_2)\big[\theta_L V\big(x_2(\theta_L,\theta_L,\phi_{LL}) + q_2(\phi_{LL})\big) - t_2(\phi_{LL})\big] \\ & -p_{LH}\Delta\theta\delta_1 V\big(x_1(\theta_L,\theta_L,\phi_{LL}) + q_1(\phi_{LL})\big) - p_{LH}\Delta\theta\delta_2 V\big(x_2(\theta_L,\theta_L,\phi_{LL}) + q_2(\phi_{LL})\big) \\ & =(p_{LL} + p_{LL}\underline{\nu} - p_{LH}\delta) \\ & \times\Bigg\{\left(\theta_L - \frac{p_{LH}^2\epsilon\Delta\theta}{p_{LL}p_{LH} + \rho\epsilon}\right)\Big[V\big(x_1(\theta_L,\theta_L,\phi_{LL}) + q_1(\phi_{LL})\big) + V\big(x_2(\theta_L,\theta_L,\phi_{LL}) + q_2(\phi_{LL})\big)\Big] \\ & -t_1(\phi_{LL}) - t_2(\phi_{LL})\Bigg\} \\ & =(p_{LL} + p_{LL}\underline{\nu} - p_{LH}\delta)\Bigg\{2\left(\theta_L - \frac{p_{LH}^2\epsilon\Delta\theta}{p_{LL}p_{LH} + \rho\epsilon}\right)V\left(\frac{q_1(\phi_{LL}) + q_2(\phi_{LL})}{2}\right) \\ & \quad - t_1(\phi_{LL}) - t_2(\phi_{LL})\Bigg\}.\end{aligned} \tag{79}$$

Hence, we have

$$\phi_{LL}^* \in \arg\max_{\phi_{LL}}\Bigg\{2\left(\theta_L - \frac{p_{LH}^2\epsilon\Delta\theta}{p_{LL}p_{LH} + \rho\epsilon}\right)V\left(\frac{q_1(\phi_{LL}) + q_2(\phi_{LL})}{2}\right) - t_1(\phi_{LL}) - t_2(\phi_{LL})\Bigg\}. \tag{80}$$

– Similarly, optimizing with respect to ϕ_{LH} and ϕ_{HL} yields respectively

$$\begin{aligned}\phi_{LH}^* \in \arg\max_{\phi_{LH}}\Big\{ & \Big(\theta_L - \frac{p_{HH}\epsilon\Delta\theta}{p_{LH}}\Big)V\big(x_1(\theta_L,\theta_H,\phi_{LH}) + q_1(\phi_{LH})\big) \\ & + \theta_H V\big(x_2(\theta_L,\theta_H,\phi_{LH}) + q_2(\phi_{LH})\big) - t_1(\phi_{LH}) - t_2(\phi_{LH})\Big\} \\ = \arg\max_{\phi_{LH}}\Big\{ & \Big(\theta_L - \frac{p_{HH}\epsilon\Delta\theta}{p_{LH}}\Big)V\Big(\varphi_1\big(q_1(\phi_{LH}) + q_1(\phi_{LH})\big)\Big) \\ & + \theta_H V\left(\varphi_2(q_1(\phi_{LH}) + q_2(\phi_{LH}))\right) - t_1(\phi_{LH}) - t_2(\phi_{LH})\Big\}\end{aligned} \tag{81}$$

and

$$\begin{aligned}
\phi_{HL}^* \in \arg\max_{\phi_{HL}} \Bigg\{ & \left(\theta_L - \frac{p_{HH}\epsilon\Delta\theta}{p_{LH}}\right) V\big(x_2(\theta_H, \theta_L, \phi_{HL}) + q_2(\phi_{HL})\big) \\
& + \theta_H V\big(x_1(\theta_H, \theta_L, \phi_{HL}) + q_1(\phi_{HL})\big) - t_1(\phi_{HL}) - t_2(\phi_{HL})\Bigg\} \\
= \arg\max_{\phi_{HL}} \Bigg\{ & \left(\theta_L - \frac{p_{HH}\epsilon\Delta\theta}{p_{LH}}\right) V\Big(\varphi_1\big(q_1(\phi_{HL}) + q_1(\phi_{HL})\big)\Big) \\
& + \theta_H V\left(\varphi_2(q_1(\phi_{HL}) + q_2(\phi_{HL})\right) - t_1(\phi_{HL}) - t_2(\phi_{HL})\Bigg\}.
\end{aligned} \tag{82}$$

– Optimizing with respect to ϕ_{HH} yields

$$\begin{aligned}
\phi_{HH}^* \in \arg\max_{\phi_{HH}} \Bigg\{ & \theta_H V\big(x_1(\theta_H, \theta_H, \phi_{HH}) + q_1(\phi_{HH})\big) + \theta_H V\big(x_2(\theta_H, \theta_H, \phi_{HH}) + q_2(\phi_{HH})\big) \\
& - t_1(\phi_{HH}) - t_2(\phi_{HH})\Bigg\} \\
= \arg\max_{\phi_{HH}} \Bigg\{ & 2\theta_H V\left(\frac{q_1(\phi_{HH}) + q_2(\phi_{HH})}{2}\right) - t_1(\phi_{HH}) - t_2(\phi_{HH})\Bigg\}.
\end{aligned} \tag{83}$$

- In a weakly collusion-proof mechanism $\phi(\theta_1, \theta_2) = (\theta_1, \theta_2)$, inserting into (80), (81), (82) and (83) yields coalitional incentive constraints (11) to (16) in the main text.
- Note that $\epsilon = \frac{\delta}{1+\delta+\bar{\nu}} \in [0, 1)$. Moreover, $\delta > 0$ when the Bayesian incentive constraints (46) and (47) are binding in the third party's optimizing problem.
- Note that participation constraints (48) to (51) are binding for a weakly collusion-proof mechanism. Hence the slackness condition obtained from the Lagrangean optimization does not give any information on ϵ. Therefore, ϵ is a free variable in the principal's programme.

Proof of Corollary 1

In the absence of arbitrage, let $x_i\left(\theta_1, \theta_2; \phi(\theta_1, \theta_2)\right) = 0, i = 1, 2$ in (79), (81), (82)and (83), then we get

$$\begin{aligned}
&\phi_{LL}^* \in \arg\max_{\phi_{LL}} \left\{\left(\theta_L - \tfrac{p_{LH}^2\epsilon\Delta\theta}{p_{LL}p_{LH}+\rho\epsilon}\right)\left[V\left(q_1(\phi_{LL})\right) + V\left(q_2(\phi_{LL})\right)\right] - t_1(\phi_{LL}) - t_2(\phi_{LL})\right\}, \\
&\phi_{LH}^* \in \arg\max_{\phi_{LH}} \left\{\left(\theta_L - \frac{p_{HH}\epsilon\Delta\theta}{p_{LH}}\right) V\big(q_1(\phi_{LH})\big) + \theta_H V\big(q_2(\phi_{LH})\big) - t_1(\phi_{LH}) - t_2(\phi_{LH})\right\}, \\
&\phi_{HL}^* \in \arg\max_{\phi_{HL}} \left\{\left(\theta_L - \frac{p_{HH}\epsilon\Delta\theta}{p_{LH}}\right) V\big(q_2(\phi_{HL})\big) + \theta_H V\big(q_1(\phi_{HL})\big) - t_1(\phi_{HL}) - t_2(\phi_{HL})\right\}, \\
&\phi_{HH}^* \in \arg\max_{\phi_{HH}} \left\{\theta_H V\big(q_1(\phi_{HH})\big) + \theta_H V\big(q_2(\phi_{HH})\big) - t_1(\phi_{HH}) - t_2(\phi_{HH})\right\}.
\end{aligned}$$

Substituting $\phi(\theta_1, \theta_2) = (\theta_1, \theta_2)$ into the above expressions yields constraints (19) to (22).

Proof of Proposition 3

Let

$$\mathcal{D}^a = \left\{(\mathbf{t}, \mathbf{q}) \mid \mathbf{t} \in \mathbb{R}^4, \mathbf{q} \in \mathbb{R}^4_+, s.t : (1) - (4), (11) - (18)\right\}$$

and

$$\mathcal{D}^w = \left\{(\mathbf{t}, \mathbf{q}) \mid \mathbf{t} \in \mathbb{R}^4, \mathbf{q} \in \mathbb{R}^4_+, s.t : (1) - (4), (19) - (24)\right\}$$

represent the feasible regions of the principal's problem with and without arbitrage, respectively. Denote by ℓ_i, r_i, i =11 to 16 and 19 to 24 the left and right hand sides of the above coalitional incentive constraints. Note that under NAC (18), $q_{LH} = \varphi_1(q_{LH} + q_{HL}), q_{HL} = \varphi_2(q_{LH} + q_{HL})$, then $\ell_{11} = \ell_{19}, \ell_{12} = \ell_{20}, \ell_{13} = \ell_{21}, \ell_{14} = \ell_{22}, \ell_{15} = \ell_{23}, \ell_{16} = \ell_{24}$, and it is obvious that $r_{11} > r_{19}, r_{12} = r_{20}, r_{13} > r_{21}, r_{14} > r_{22}, r_{15} = r_{23}, r_{16} > r_{24}$, so $\mathcal{D}^a \subseteq \mathcal{D}^w$, therefore $\Pi^w(\rho) \geqslant \Pi^a(\rho)$. The other two inequalities $\Pi^{fb}(\rho) \geqslant \Pi^{sb}(\rho) \geqslant \Pi^w(\rho)$ are straightforward.

Proof of Proposition 4

If $\rho < 0$, we write the θ_H type's incentive constraint (2), the θ_L type's participation constraint (3), and local upward coalitional incentive constraints (19) and (22) as binding ones by introducing nonnegative parameters $\varepsilon_i, i = 2, 3, 19, 22$.

$$\begin{bmatrix} p_{LL} & p_{LH} & 0 & 0 \\ p_{LH} & p_{HH} & -p_{LH} & -p_{HH} \\ 2 & -1 & -1 & 0 \\ 0 & 1 & 1 & -2 \end{bmatrix} \times \begin{bmatrix} t_{LL} \\ t_{LH} \\ t_{HL} \\ t_{HH} \end{bmatrix} = \begin{bmatrix} \beta_3 - \varepsilon_3 \\ \beta_2 + \varepsilon_2 \\ \beta_{19} - \varepsilon_{19} \\ \beta_{22} - \varepsilon_{22} \end{bmatrix},$$

where

$$\begin{bmatrix} \beta_3 \\ \beta_2 \\ \beta_{19} \\ \beta_{22} \end{bmatrix} \equiv \begin{bmatrix} \theta_L p_{LL} V(q_{LL}) + \theta_L p_{LH} V(q_{LH}) \\ \theta_H p_{LH} \left[V(q_{LL}) - V(q_{HL})\right] + \theta_H p_{HH} \left[V(q_{LH}) - V(q_{HH})\right] \\ \left(\theta_L - \frac{p_{LH}^2 \epsilon \Delta\theta}{p_{LL} p_{LH} + \rho\epsilon}\right) \left[2V(q_{LL}) - V(q_{LH}) - V(q_{HL})\right] \\ \left(\theta_L - \frac{p_{HH} \epsilon \Delta\theta}{p_{LH}}\right) \left[V(q_{LH}) - V(q_{HH})\right] + \theta_H \left[V(q_{HL}) - V(q_{HH})\right] \end{bmatrix}.$$

The expected transfer is therefore

$$\begin{aligned} \mathbb{E}[t(\theta_1, \theta_2)] = \sum_{i \in \{H,L\}} \sum_{j \in \{H,L\}} p_{ij} t_{ij} &= \frac{p_{LH} + p_{HH}}{\rho + p_{LH}} (\beta_3 - \varepsilon_3) - \frac{p_{LH}}{\rho + p_{LH}} (\beta_2 + \varepsilon_2) \\ &- \frac{\rho(1 - p_{LL})}{2(\rho + p_{LH})} (\beta_{19} - \varepsilon_{19}) - \frac{\rho p_{HH}}{2(\rho + p_{LH})} (\beta_{22} - \varepsilon_{22}). \end{aligned} \tag{84}$$

Note that $\rho < 0$ and $2V(q_{LL}) < V(q_{LH}) + V(q_{HL})$(It can be checked ex post.), so the seller will choose $\epsilon^* = 1, \varepsilon_i = 0, i = 2, 3, 19, 22$ at the optimum.

Substituting (84) into the seller's objective function $\Pi(\mathbf{t},\mathbf{q})$ and then optimizing with respect to q_{ij} yields expressions (25)-(28). With weakly negative correlation, the monotonicity of consumptions can be verified as follows. $q^w_{Li}(\rho) < q^w_{Hj}(\rho), \forall (i,j) \in \{H,L\} \times \{H,L\}$ is easily obtained when ρ is sufficiently close to zero. $q^w_{LL}(\rho) < q^w_{LH}(\rho)$ holds when

$$\begin{aligned}
&\frac{\rho\left[\theta_H-(1-p_{LL})\left(\theta_L-\frac{p_{LH}^2\Delta\theta}{p_{LL}p_{LH}+\rho}\right)\right]}{(\rho+p_{LH})p_{LL}} - \frac{\rho\left[(1-p_{LL})\left(\theta_L-\frac{p_{LH}^2\Delta\theta}{p_{LL}p_{LH}+\rho}\right)-p_{HH}\left(\theta_L-\frac{p_{HH}\Delta\theta}{p_{LH}}\right)\right]}{2p_{LH}(\rho+p_{LH})} \\
&= \frac{\Delta\theta\rho}{2p_{LL}p_{LH}(\rho+p_{LH})}\left[2p_{LH}\left(1+\frac{p_{LH}^2}{p_{LL}p_{LH}+\rho}\right) - \frac{\rho p_{HH}p_{LL}(p_{HH}+p_{LH})}{\rho+p_{LL}p_{LH}}\right] < 0.
\end{aligned}$$

This condition is obviously satisfied for weakly negative correlation. $q^w_{HH}(\rho) < q^w_{HL}(\rho)$ is ensured by

$$\begin{aligned}
&\rho\frac{\theta_H+\left(\theta_L-\frac{p_{HH}\Delta\theta}{p_{LH}}\right)}{2(\rho+p_{LH})} - \rho\frac{(1-p_{LL})\left(\theta_L-\frac{p_{LH}^2\Delta\theta}{p_{LL}p_{LH}+\rho}\right)-p_{HH}\theta_H}{2(\rho+p_{LH})p_{LH}} \\
&= \frac{\rho}{2(\rho+p_{LH})p_{LH}}\left[\frac{(p_{LH}+p_{HH})^2\Delta\theta}{p_{LH}} - \rho\frac{(1-p_{LL})\Delta\theta}{p_{LL}p_{LH}+\rho}\right] < 0.
\end{aligned}$$

Hence, the monotonicity of quantities, $q^w_{LL}(\rho) < q^w_{LH}(\rho) < q^w_{HH}(\rho) < q^w_{HL}(\rho)$, is ensured.

The monotonicity of consumptions and the fact that (2), (3), (19), (22) are binding ensure that (1), (4), (21), (23) and (24)are satisfied strictly. The only work left is to check constraint (20). From (19) and (22) written with equality we get

$$\left[2\left(\theta_L - \frac{p_{LH}^2\epsilon\Delta\theta}{p_{LL}p_{LH}+\rho\epsilon}\right)V(q_{LL}) - 2t_{LL}\right] = \left[2\left(\theta_L - \frac{p_{LH}^2\epsilon\Delta\theta}{p_{LL}p_{LH}+\rho\epsilon}\right)V(q_{HH}) - 2t_{HH}\right] + H(\rho)$$

where

$$H(\rho) \equiv \frac{\rho(p_{LH}+p_{HH})}{p_{LH}(\rho+p_{LL}p_{LH})}\left[V(q_{LH})-V(q_{HH})\right] + \Delta\theta\left(1+\frac{p_{LH}^2}{p_{LL}p_{LH}+\rho}\right)\left[V(q_{HH})-V(q_{HL})\right].$$

$$\begin{aligned}
\lim_{\rho\uparrow 0}\frac{H(\rho)}{\rho} &= \lim_{\rho\uparrow 0}\frac{(p_{LH}+p_{HH})\left[V(q_{LH})-V(q_{HH})\right]}{p_{LH}(p_{LL}p_{LH}+\rho)} \\
&\quad + \lim_{\rho\uparrow 0}\Delta\theta\left(1+\frac{p_{LH}^2}{p_{LL}p_{LH}+\rho}\right)\frac{V(q_{HH})-V(q_{HL})}{\rho} \\
&= \frac{p_{LH}+p_{HH}}{p_{LH}^2p_{LL}}\left[V(q^{sb}_{LH}(0))-V(q^{sb}_{HH}(0))\right] \\
&\quad + \Delta\theta\frac{p_{LL}+p_{LH}}{p_{LL}}\lim_{\rho\uparrow 0}\frac{V(q_{HH})-V(q_{HL})}{\rho}
\end{aligned} \tag{85}$$

Letting

$$\begin{aligned}
\alpha(\rho) &\equiv \frac{(1-p_{LL})\left(\theta_L - \frac{p_{LH}^2\Delta\theta}{p_{LL}p_{LH}+\rho}\right) - p_{HH}\theta_H}{2(\rho+p_{LH})p_{LH}} \\
\beta(\rho) &\equiv \frac{\theta_H+\theta_L-\frac{p_{HH}}{p_{LH}}\Delta\theta}{2(\rho+p_{LH})}
\end{aligned}$$

, it can be shown that,

$$
\begin{aligned}
&\lim_{\rho\uparrow 0}\frac{V(q_{HH})-V(q_{HL})}{\rho}=\lim_{\rho\uparrow 0}\left[V'(q_{HH})\frac{dq_{HH}}{d\rho}-V'(q_{HL})\frac{dq_{HL}}{d\rho}\right]\\
=&\lim_{\rho\uparrow 0}\left\{\frac{[V'(q_{HL})]^2\left[\frac{\theta_H p_{LH}}{\rho+p_{LH}}\Big|_\rho'+\alpha(\rho)+\rho\alpha'(\rho)\right]}{\left[\frac{\theta_H p_{LH}}{\rho+p_{LH}}+\rho\alpha(\rho)\right]V''(q_{HL})}-\frac{[V'(q_{HH})]^2\left[\frac{\theta_H p_{LH}}{\rho+p_{LH}}\Big|_\rho'+\beta(\rho)+\rho\beta'(\rho)\right]}{\left[\frac{\theta_H p_{LH}}{\rho+p_{LH}}+\rho\beta(\rho)\right]V''(q_{HH})}\right\}\\
=&\frac{V'[q_{HH}^{sb}(0)]^2[\alpha(0)-\beta(0)]}{\theta_H V''[q_{HH}^{sb}(0)]}=-\frac{V'[q_{HH}^{sb}(0)]c}{\theta_H^2 V''[q_{HH}^{sb}(0)]}\frac{\Delta\theta}{2p_{LH}p_{LL}}.
\end{aligned}
\tag{86}
$$

Inserting expression (86) into (85), we get

$$
\lim_{\rho\uparrow 0}\frac{H(\rho)}{\rho}=\frac{p_{LH}+p_{HH}}{p_{LH}^2 p_{LL}}\left[V(q_{HL}^{sb}(0))-V(q_{HH}^{sb}(0))-\frac{c(\Delta\theta)^2 V'(q_{HH}^{sb}(0))}{2\theta_H^2 V''(q_{HH}^{sb}(0))}\right].
$$

It is easy to see that the last term in the square bracket vanishes as $\theta_H\to+\infty$. Therefore, for sufficiently large θ_H, $\lim_{\rho\uparrow 0}\frac{H(\rho)}{\rho}<0$, which implies that (20) holds strictly, i.e., $H(\rho)>0$ for weakly negative ρ.

Proof of Proposition 5

For weakly positive correlation, the principal could not fully extract the agents' rents without violating coalitional incentive constraints (21), (23) and (24).[17] As usual, IR_L and IC_H are surely binding, and it is obvious that (23) and (24) cannot be simultaneously binding. Therefore, the set of binding constraints consists of either (2), (3), (21), (23) or (2), (3), (21), (24).

- If (2), (3), (21), (24) are binding, then

$$
\begin{aligned}
\mathbb{E}[t(\theta_1,\theta_2)]&=\sum_{i\in\{H,L\}}\sum_{j\in\{H,L\}}p_{ij}t_{ij}\\
&=\frac{p_{LH}+p_{HH}}{\rho+p_{LH}}\beta_3-\frac{p_{LH}}{\rho+p_{LH}}\beta_2+\frac{\rho(1-p_{LL})}{2(\rho+p_{LH})}\beta_{21}-\frac{\rho p_{HH}}{2(\rho+p_{LH})}\beta_{24}
\end{aligned}
\tag{87}
$$

  where

$$
\begin{bmatrix}\beta_3\\ \beta_2\\ \beta_{21}\\ \beta_{24}\end{bmatrix}\equiv\begin{bmatrix}\theta_L p_{LL}V(q_{LL})+\theta_L p_{LH}V(q_{LH})\\ \theta_H p_{LH}\left[V(q_{LL})-V(q_{HL})\right]+\theta_H p_{HH}\left[V(q_{LH})-V(q_{HH})\right]\\ \left(\theta_L-\frac{p_{HH}\epsilon\Delta\theta}{p_{LH}}\right)\left[V(q_{LL})-V(q_{LH})\right]+\theta_H\left[V(q_{LL})-V(q_{HL})\right]\\ \theta_H\left[V(q_{LH})+V(q_{HL})\right]-2\theta_H V(q_{HH})\end{bmatrix}.
$$

[17] Remember that FSE mechanism requires $t_{LL},t_{HL}\to-\infty, t_{LH},t_{HH}\to+\infty$ as ρ is positive and goes to zero.

Substituting it into the principal's objective function and then optimizing with respect to $\epsilon \in [0, 1)$ and $q_{ij}, i, j \in \{H, L\}$ yields $\epsilon^* = 1$ and

$$\left[\frac{p_{LH}+p_{HH}}{\rho+p_{LH}}\theta_L - \frac{p_{LH}^2\theta_H}{(\rho+p_{LH})p_{LL}} + \frac{\rho(1-p_{LL})}{2p_{LL}(p_{LH}+\rho)}\left(\theta_L+\theta_H-\frac{p_{HH}\Delta\theta}{p_{LH}}\right)\right]V'(q_{LL}) = c$$

$$\left[\frac{p_{LH}+p_{HH}}{\rho+p_{LH}}\theta_L - \frac{p_{HH}\theta_H}{(\rho+p_{LH})} - \frac{\rho(1-p_{LL})\left(\theta_L-\frac{p_{HH}\Delta\theta}{p_{LH}}\right)}{2(p_{LH}+\rho)p_{LH}} - \frac{\rho p_{HH}\theta_H}{2(\rho+p_{LH})p_{LH}}\right]V'(q_{LH}) = c$$

$$\theta_H V'(q_{HL}) = \theta_H V'(q_{HH}) = c.$$

It can be verified that $q_{LH} < q_{LL} < q_{HL} = q_{HH}$. Summing (21) and (24) written with equalities yields:

$$[2\theta_H V(q_{HH}) - 2t_{HH}] - [2\theta_H V(q_{LL}) - 2t_{LL}] = \left(\theta_H - \theta_L + \frac{p_{HH}\Delta\theta}{p_{LH}}\right)[V(q_{LH}) - V(q_{LL})] < 0,$$

which contradicts (23).

- If (2), (3), (21), (23) are binding, then

$$\begin{aligned}\mathbb{E}[t(\theta_1,\theta_2)] &= \sum_{i\in\{H,L\}}\sum_{j\in\{H,L\}} p_{ij}t_{ij} \\ &= \frac{p_{LH}+p_{HH}}{\rho+p_{LH}}\beta_3 - \frac{p_{LH}}{\rho+p_{LH}}\beta_2 - \frac{\rho p_{LH}}{\rho+p_{LH}}\beta_{21} - \frac{\rho p_{HH}}{2(\rho+p_{LH})}\beta_{23}\end{aligned}$$

where

$$\begin{bmatrix}\beta_3\\ \beta_2\\ \beta_{21}\\ \beta_{23}\end{bmatrix} \equiv \begin{bmatrix}\theta_L p_{LL}V(q_{LL}) + \theta_L p_{LH}V(q_{LH})\\ \theta_H p_{LH}\left[V(q_{LL}) - V(q_{HL})\right] + \theta_H p_{HH}\left[V(q_{LH}) - V(q_{HH})\right]\\ \left(\theta_L - \frac{p_{HH}\epsilon\Delta\theta}{p_{LH}}\right)\left[V(q_{LL}) - V(q_{LH})\right] + \theta_H\left[V(q_{LL}) - V(q_{HL})\right]\\ 2\theta_H\left[V(q_{LL}) - V(q_{HH})\right]\end{bmatrix}.$$

Substituting it into the principal's objective function $\Pi(\mathbf{t},\mathbf{q})$ and then optimizing with respect to $\epsilon \in [0, 1)$ and $q_{ij}, i, j \in \{H, L\}$ yields $\epsilon^* = 1$ and expressions (29)-(31). The monotonicity of quantities $q^w_{LH}(\rho) < q^w_{LL}(\rho) < q^w_{HL}(\rho) = q^w_{HH}(\rho)$ obtains when

$$\begin{aligned}\theta_H &> \left(\frac{\theta_L p_{LH} - p_{HH}\Delta\theta}{\rho+p_{LH}}\right) + \rho\frac{\theta_H - p_{LH}\left(\theta_L - \frac{p_{HH}\Delta\theta}{p_{LH}} + \theta_H\right) - p_{HH}\theta_H}{(\rho+p_{LH})p_{LL}} \\ &> \left(\frac{\theta_L p_{LH} - p_{HH}\Delta\theta}{\rho+p_{LH}}\right) + \rho\frac{\theta_L - \frac{p_{HH}\Delta\theta}{p_{LH}}}{\rho+p_{LH}}.\end{aligned}$$

The first inequality holds for sufficiently small $\rho > 0$; the second inequality holds since

$$\rho\frac{\theta_H - p_{LH}\left(\theta_L - \frac{p_{HH}\Delta\theta}{p_{LH}} + \theta_H\right) - p_{HH}\theta_H}{(\rho+p_{LH})p_{LL}} - \rho\frac{\theta_L - \frac{p_{HH}\Delta\theta}{p_{LH}}}{\rho+p_{LH}} = \frac{\rho\Delta\theta(p_{LH}+p_LL)(p_{LH}+p_{HH})}{p_{LH}} > 0.$$

Hence, the monotonicity of quantities is ensured and it implies that the other incentive constraints (1), (4), (19), (20), (22) and (24) are all strictly satisfied.

Proof of Proposition 6

With independent types, $p_{LL} = \nu^2, p_{LH} = \nu(1-\nu), p_{HH} = (1-\nu)^2$, then we can find transfers $t_{ij}, i,j \in \{H,L\}$ such that $U_{LL} \equiv \theta_L V(q_{LL}) - t_{LL} = 0, U_{LH} \equiv \theta_L V(q_{LH}) - t_{LH} = 0, U_{HL} \equiv \theta_H V(q_{HL}) - t_{HL} = \Delta\theta V(q_{LL}), U_{HH} \equiv \theta_H V(q_{HH}) - t_{HH} = \Delta\theta V(q_{LH})$. It is easy to verify that IC_H, IR_L are binding and all the coalitional constraints are strictly satisfied for any $\epsilon \in [0,1)$. Substituting $t_{ij}, i,j \in \{H,L\}$ into the principal's objective function and then maximizing with respect to q_{ij} yields $\mathbf{q}^w(0) = \mathbf{q}^{sb}(0)$.

Proof of Corollary 1

- If $\rho > 0$ and is close enough to zero, it can be verified easily from (29), (30) and (31) that $q^w_{LL}(\rho) < q^{sb}_{LL}(\rho), q^w_{LH}(\rho) < q^{sb}_{LH}(\rho)$, $q^w_{HL}(\rho) = q^{sb}_{HL}(\rho), q^w_{HH}(\rho) = q^{sb}_{HH}(\rho)$; if $\rho < 0$ and is close enough to zero, it can be verified from (25)-(28) that $q^w_{LL}(\rho) < q^{sb}_{LL}(\rho), q^w_{LH}(\rho) < q^{sb}_{LH}(\rho)$. The upward distortions for the quantities of high-type consumers, i.e., $q^w_{HL}(\rho) > q^{sb}_{HL}(\rho), q^w_{HH}(\rho) > q^{sb}_{HH}(\rho)$ are ensured by the following two inequalities:

$$\frac{\theta_H p_{LH}}{\rho + p_{LH}} + \rho \frac{(1-p_{LL})\left(\theta_L - \frac{p^2_{LH}\Delta\theta}{p_{LL}p_{LH}+\rho}\right) - p_{HH}\theta_H}{2(\rho+p_{LH})p_{LH}} > \frac{\theta_H p_{LH}}{\rho + p_{LH}} + \frac{\rho\theta_H}{\rho+p_{LH}} = \theta_H$$

$$\frac{\theta_H p_{LH}}{\rho + p_{LH}} + \rho \frac{\theta_H + \left(\theta_L - \frac{p_{HH}\Delta\theta}{p_{LH}}\right)}{2(\rho+p_{LH})} > \frac{\theta_H p_{LH}}{\rho + p_{LH}} + \frac{\rho\theta_H}{\rho+p_{LH}} = \theta_H.$$

- If $\rho \neq 0$, then collusion imposes more constraints on the set of implementable allocations which must now also be coalitionally incentive compatible, and the seller's payoff is thus strictly lower than the first-best level: $\Pi^w(\rho) < \Pi^{sb}(\rho) = \Pi^{fb}(\rho)$. If $\rho = 0$, then $\mathbf{q}^w(\rho) = \mathbf{q}^{sb}(\rho) \neq \mathbf{q}^{fb}(\rho)$, the seller's payoff is thus $\Pi^w(\rho) = \Pi^{sb}(\rho) < \Pi^{fb}(\rho)$, and (32) holds.

- Letting ρ goes to zero in (25)-(28) and (29)-(31), we get the continuity of consumptions: $\lim_{\rho\downarrow 0} q^w_{ij}(\rho) = \lim_{\rho\uparrow 0} q^w_{ij}(\rho) = q^w_{ij}(0), \forall i,j \in \{H,L\}$. Continuity of the seller's profit (33) can be obtained accordingly.

Proof of Lemma 1.

Let

$$f(x) \equiv \left(\theta_L - \frac{p_{HH}\epsilon\Delta\theta}{p_{LH}}\right) V(\varphi_1(x)) + \theta_H V(\varphi_2(x)) - 2\theta_H V\left(\frac{x}{2}\right)$$

$$g(x) \equiv 2\left(\theta_L - \frac{p^2_{LH}\epsilon\Delta\theta}{p_{LL}p_{LH}+\rho\epsilon}\right) V\left(\frac{x}{2}\right) - \left(\theta_L - \frac{p_{HH}\epsilon\Delta\theta}{p_{LH}}\right) V(\varphi_1(x)) - \theta_H V(\varphi_2(x)),$$

then

$$f'(x) = \left(\theta_L - \frac{p_{HH}\epsilon\Delta\theta}{p_{LH}}\right) V'(\varphi_1(x))\varphi_1'(x) + \theta_H V'(\varphi_2(x))\varphi_2'(x) - \theta_H V'\left(\frac{x}{2}\right)$$

$$g'(x) = \left(\theta_L - \frac{p_{LH}^2\epsilon\Delta\theta}{p_{LL}p_{LH}+\rho\epsilon}\right) V'\left(\frac{x}{2}\right) - \left(\theta_L - \frac{p_{HH}\epsilon\Delta\theta}{p_{LH}}\right) V'(\varphi_1(x))\varphi_1'(x) - \theta_H V'(\varphi_2(x))\varphi_2'(x).$$

Since

$$(\varphi_1(x), \varphi_2(x)) = \underset{x_1,x_2:x_1+x_2=x}{argmax} \left[\left(\theta_L - \frac{p_{HH}\epsilon\Delta\theta}{p_{LH}}\right) V(x_1) + \theta_H V(x_2)\right],$$

$\left(\theta_L - \frac{p_{HH}\epsilon\Delta\theta}{p_{LH}}\right) V'(\varphi_1(x)) = \theta_H V'(\varphi_2(x))$ and $\varphi_1'(x) + \varphi_2'(x) = 1$. Therefore,

$$f'(x) = \theta_H V'(\varphi_2(x)) - \theta_H V'\left(\frac{x}{2}\right),$$

$$g'(x) = \left(\theta_L - \frac{p_{LH}^2\epsilon\Delta\theta}{p_{LL}p_{LH}+\rho\epsilon}\right) V'\left(\frac{x}{2}\right) - \left(\theta_L - \frac{p_{HH}\epsilon\Delta\theta}{p_{LH}}\right) V'(\varphi_1(x)).$$

Note that $\varphi_1(x) < \frac{x}{2} < \varphi_2(x)$, hence, $f'(x) < 0, g'(x) < 0$ if ρ is close enough to zero.

- ($\Longrightarrow$) Summing constraints (14) and (16) yields $f(q_{LH} + q_{HL}) \geqslant f(2q_{HH})$; summing constraints (11) and (13) yields: $g(2q_{LL}) \geqslant g(q_{LH} + q_{HL})$. Therefore $q_{LL} \leqslant \frac{q_{LH}+q_{HL}}{2} \leqslant q_{HH}$.

- ($\Longleftarrow$) Assume that $q_{LL} \leqslant \frac{q_{LH}+q_{HL}}{2} \leqslant q_{HH}$ holds. If (14) is binding, then

$$\begin{aligned}\ell_{16} - r_{16} =& [2\theta_H V(q_{HH}) - t_{HH}] - \left[2\theta_H V\left(\frac{q_{LH}+q_{HL}}{2}\right) - t_{HL} - t_{LH}\right] \\ =& f(q_{LH}+q_{HL}) - f(2q_{HH}) \geqslant 0,\end{aligned}$$

  (16) is therefore satisfied. If (11) holds with equality, then

$$\begin{aligned}\ell_{13} - r_{13} =& \left[\left(\theta_L - \frac{p_{HH}\epsilon\Delta\theta}{p_{LH}}\right) V\big(\varphi_1(q_{LH}+q_{HL})\big) + \theta_H V\big(\varphi_2(q_{LH}+q_{HL})\big) - t_{LH} - t_{HL}\right] \\ &- \left[\left(\theta_L - \frac{p_{HH}\epsilon\Delta\theta}{p_{LH}}\right) V\big(\varphi_1(2q_{LL})\big) + \theta_H V\big(\varphi_2(2q_{LL})\big) - 2t_{LL}\right] \\ =& g(2q_{LL}) - g(q_{LH}+q_{HL}) \geqslant 0.\end{aligned}$$

  It follows that (13) is satisfied. Summing (13) and (16) yields:

$$\begin{aligned}\ell_{15} - r_{15} =& [2\theta_H V(q_{HH}) - 2t_{HH}] - [2\theta_H V(q_{LL}) - 2t_{LL}] \\ \geqslant& f(2q_{LL}) - f(q_{LH}+q_{HL}) \geqslant 0,\end{aligned}$$

  (15) is thus satisfied. Summing (11) and (14) written with equalities yields:

$$\begin{aligned}&\ell_{12} - r_{12} \\ =& \left[2\left(\theta_L - \frac{p_{LH}^2\epsilon\Delta\theta}{p_{LL}p_{LH}+\rho\epsilon}\right) V(q_{LL}) - 2t_{LL}\right] - \left[\left(\frac{p_{LH}^2\epsilon\Delta\theta}{p_{LL}p_{LH}+\rho\epsilon}\right) V(q_{HH}) - 2t_{HH}\right] \\ =& g(q_{LH}+q_{HL}) - g(2q_{HH}) \geqslant 0,\end{aligned}$$

  (12) holds. Analogously, given the monotonicity of quantities, binding constraints (13) and (16) imply the other coalitional incentive constraints (11), (12), (14) and (15).

Proof of Proposition 7

If $\rho < 0$, we write the downward individual incentive compatibility constraint (2), the θ_L type's participation constraint (3), the local upward coalitional constraints (11) and (14) as binding ones by introducing nonnegative parameters $\varepsilon_i, i = 2, 3, 11, 14$.

$$\begin{bmatrix} p_{LL} & p_{LH} & 0 & 0 \\ p_{LH} & p_{HH} & -p_{LH} & -p_{HH} \\ 2 & -1 & -1 & 0 \\ 0 & 1 & 1 & -2 \end{bmatrix} \times \begin{bmatrix} t_{LL} \\ t_{LH} \\ t_{HL} \\ t_{HH} \end{bmatrix} = \begin{bmatrix} \beta_3 - \varepsilon_3 \\ \beta_2 + \varepsilon_2 \\ \beta_{11} - \varepsilon_{11} \\ \beta_{14} - \varepsilon_{14} \end{bmatrix},$$

where,

$$\begin{bmatrix} \beta_3 \\ \beta_2 \\ \beta_{11} \\ \beta_{14} \end{bmatrix} \equiv \begin{bmatrix} \theta_L p_{LL} V(q_{LL}) + \theta_L p_{LH} V(q_{LH}) \\ \theta_H p_{LH} \left[V(q_{LL}) - V(q_{HL})\right] + \theta_H p_{HH} \left[V(q_{LH}) - V(q_{HH})\right] \\ 2\left(\theta_L - \frac{p_{LH}^2 \epsilon \Delta\theta}{p_{LL} p_{LH} + \rho\epsilon}\right)\left[V(q_{LL}) - V(\frac{q_{LH}+q_{HL}}{2})\right] \\ \left(\theta_L - \frac{p_{HH}\epsilon\Delta\theta}{p_{LH}}\right)\left[V(\varphi_1(q_{LH}+q_{HL})) - V(\varphi_1(2q_{HH}))\right] + \theta_H \left[V(\varphi_2(q_{LH}+q_{HL})) - V(\varphi_2(2q_{HH}))\right] \end{bmatrix}.$$

The expected transfer is thus obtained from this invertible equations system

$$\begin{aligned} \mathbb{E}[\mathbf{t}(\theta_1, \theta_2)] \equiv & \sum_{i \in \{H,L\}} \sum_{j \in \{H,L\}} p_{ij} t_{ij} \\ = & \frac{p_{LH} + p_{HH}}{\rho + p_{LH}}(\beta_3 - \varepsilon_3) - \frac{p_{LH}}{\rho + p_{LH}}(\beta_2 + \varepsilon_2) \\ & - \frac{\rho(1 - p_{LL})}{2(\rho + p_{LH})}(\beta_{11} - \varepsilon_{11}) - \frac{\rho p_{HH}}{2(\rho + p_{LH})}(\beta_{14} - \varepsilon_{14}). \end{aligned} \tag{88}$$

We use for a moment the monotonicity condition $q_{LL} < \frac{q_{HL}+q_{LH}}{2} < q_{HH}$, which will turn out to be satisfied at the optimum. To maximize the expected transfer, the seller will set $\epsilon^* = 1, \varepsilon_2 = \varepsilon_3 = \varepsilon_{11} = \varepsilon_{14} = 0$. Optimizing with respect to q_{ij} yields (35)-(38). (36) and (37) imply that the NAC (18) is satisfied automatically since $\epsilon^* = 1$.

The only work left is to verify the implementability condition. Since $\lim_{\rho \uparrow 0} q^a_{ij}(\rho) = q^a_{ij}(0) = q^{sb}_{ij}(0), \forall i, j \in \{H, L\}$ and $q^{sb}_{LL}(0) = q^{sb}_{LH}(0) < q^{sb}_{HL}(0) = q^{sb}_{HH}(0)$, $2q^a_{LL}(\rho) < q^a_{LH}(\rho) + q^a_{HL}(\rho) < 2q^a_{HH}(\rho)$ holds when ρ is sufficiently close to zero. Since $\Pi^a(\rho) \leqslant \Pi^w(\rho)$ and $\mathbf{q}^a(\rho) \neq \mathbf{q}^w(\rho)$, we get $\Pi^a(\rho) < \Pi^w(\rho)$.

Proof of Proposition 8

When $\rho > 0$, we write (2), (3), (13) and (16) are binding by introducing nonnegative variables $\varepsilon_2, \varepsilon_3, \varepsilon_{13}$ and ε_{16}.

$$\begin{bmatrix} p_{LL} & p_{LH} & 0 & 0 \\ p_{LH} & p_{HH} & -p_{LH} & -p_{HH} \\ 2 & -1 & -1 & 0 \\ 0 & 1 & 1 & -2 \end{bmatrix} \times \begin{bmatrix} t_{LL} \\ t_{LH} \\ t_{HL} \\ t_{HH} \end{bmatrix} = \begin{bmatrix} \beta_3 - \varepsilon_3 \\ \beta_2 + \varepsilon_2 \\ \beta_{13} + \varepsilon_{13} \\ \beta_{16} + \varepsilon_{16} \end{bmatrix},$$

where,

$$\begin{bmatrix} \beta_3 \\ \beta_2 \\ \beta_{13} \\ \beta_{16} \end{bmatrix} \equiv \begin{bmatrix} \theta_L p_{LL} V(q_{LL}) + \theta_L p_{LH} V(q_{LH}) \\ \theta_H p_{LH} \left[V(q_{LL}) - V(q_{HL})\right] + \theta_H p_{HH} \left[V(q_{LH}) - V(q_{HH})\right] \\ \left(\theta_L - \frac{p_{HH}\epsilon\Delta\theta}{p_{LH}}\right) \left[V(\varphi_1(2q_{LL})) - V(\varphi_1(q_{LH}+q_{HL}))\right] + \theta_H \left[V(\varphi_2(2q_{LL})) - V(\varphi_2(q_{LH}+q_{HL}))\right] \\ 2\theta_H \left[V\left(\frac{q_{LH}+q_{HL}}{2}\right) - V(q_{HH})\right] \end{bmatrix}.$$

Again, the expected transfer is obtained from the binding constraints:

$$\begin{aligned} \mathbb{E}[\mathbf{t}(\theta_1,\theta_2)] \equiv & \sum_{i\in\{H,L\}} \sum_{j\in\{H,L\}} p_{ij} t_{ij} \\ & = \frac{p_{LH}+p_{HH}}{\rho + p_{LH}}(\beta_3 - \varepsilon_3) - \frac{p_{LH}}{\rho + p_{LH}}(\beta_2 + \varepsilon_2) \\ & - \frac{\rho(1-p_{LL})}{2(\rho+p_{LH})}(\beta_{13}+\varepsilon_{13}) - \frac{\rho p_{HH}}{2(\rho+p_{LH})}(\beta_{16}+\varepsilon_{16}). \end{aligned} \tag{89}$$

Considering the monotonicity condition $q_{LL} < \frac{q_{HL}+q_{LH}}{2} < q_{HH}$, it is optimal to set $\varepsilon_3 = \varepsilon_2 = \varepsilon_{13} = \varepsilon_{16} = 0$ and $\epsilon^* = 0$. Optimizing with respect to q_{ij} under the constraint of NAC written with $\epsilon = 0$ yields expressions (39) to (42). $\lambda > 0$ is the Lagrangean multiplier associated with NAC. It can be verified easily that for weakly positive correlation, monotonicity of consumptions $q_{LL} < q_{LH} < q_{HL} < q_{HH}$ is satisfied. Hence the implementability condition $q_{LL} \leqslant \frac{q_{LH}+q_{HL}}{2} \leqslant q_{HH}$ is ensured. Since $\Pi^a(\rho) \leqslant \Pi^w(\rho)$ and $\mathbf{q}^a(\rho) \neq \mathbf{q}^w(\rho)$, the seller's profit is strictly lower than the arbitrage-free case, i.e., $\Pi^a(\rho) < \Pi^w(\rho)$.

Proof of Proposition 9

When $\rho = 0$, let $t_{Lj} = \theta_L V[q^{sb}_{Lj}(0)], t_{Hj} = \theta_H V[q^{sb}_{Lj}(0)] - \Delta\theta V[q^{sb}_{Lj}(0)], \forall j \in \{H, L\}$. It can be verified that for mechanism $(\mathbf{q}^{sb}(0), \mathbf{t})$, (2) and (3) are binding, and all the coalitional incentive constraints hold strictly for any $\epsilon \in [0, 1)$. Since all constraints containing ϵ are slack, any $\epsilon \in [0, 1)$ is indifferent to the principal. Hence, she may choose $\epsilon^* = 1$ to make NAC hold for $\mathbf{q}^{sb}(0)$. The proof is finished.

Proof of Corollary 2

- *The case with weakly positive correlation*

– (39) implies $\lim_{s\downarrow 0} q^a_{LL}(s) = q^a_{LL}(0) = q^{sb}_{LL}(0) < q^{sb}_{LL}(\rho), \forall\rho > 0$. Therefore, $q^a_{LL}(\rho) < q^{sb}_{LL}(\rho)$ when ρ is close enough to zero.

– It follows from (40) that

$$\begin{aligned}
&\theta_H V'[q^{sb}_{HL}(\rho)] = c\\
=&\left(\frac{\theta_H p_{LH}}{\rho + p_{LH}}\right) V'\left[q^a_{HL}(\rho)\right] + \rho\left\{\frac{(1-p_{LL})\theta_H V'\left[q^a_{HL}(\rho)\right] - p_{HH}\theta_H V'\left[\frac{q^a_{LH}(\rho)+q^a_{HL}(\rho)}{2}\right]}{2(\rho+p_{LH})p_{LH}}\right\}\\
&+\lambda V''\left[q^a_{HL}(\rho)\right]\\
<&\left(\frac{\theta_H p_{LH}}{\rho + p_{LH}}\right) V'\left[q^a_{HL}(\rho)\right] + \rho\left\{\frac{(1-p_{LL})\theta_H V'\left[q^a_{HL}(\rho)\right] - p_{HH}\theta_H V'\left[\frac{q^a_{LH}(\rho)+q^a_{HL}(\rho)}{2}\right]}{2(\rho+p_{LH})p_{LH}}\right\}\\
<&\left(\frac{\theta_H p_{LH}}{\rho + p_{LH}}\right) V'\left[q^a_{HL}(\rho)\right] + \rho\left\{\frac{(1-p_{LL})\theta_H V'\left[q^a_{HL}(\rho)\right] - p_{HH}\theta_H V'\left[q^a_{HL}(\rho)\right]}{2(\rho+p_{LH})p_{LH}}\right\}\\
=&\theta_H V'\left[q^a_{HL}(\rho)\right],
\end{aligned}$$

hence, $q^a_{HL}(\rho) < q^{sb}_{HL}(\rho)$.

– Since $q^a_{HL}(\rho) < q^{sb}_{HL}(\rho)$ and $V'[q^a_{LH}(\rho)]/V'[q^a_{HL}(\rho)] = V'[q^{sb}_{LH}(\rho)]/V'[q^{sb}_{HL}(\rho)] = \theta_H/\theta_L$, we have $q^a_{LH}(\rho) < q^{sb}_{LH}(\rho)$.

- *The case with weakly negative correlation*

  – (35) implies

$$\begin{aligned}
\theta_H V'[q^{sb}_{HH}(\rho)] = c &= \left(\frac{\theta_H p_{LH}}{\rho + p_{LH}}\right) V'\left[q^a_{HH}(\rho)\right] + \frac{\rho\theta_H V'\left[\varphi_2(2q^a_{HH}(\rho))\right]}{\rho + p_{LH}}\\
&> \left(\frac{\theta_H p_{LH}}{\rho + p_{LH}}\right) V'\left[q^a_{HH}(\rho)\right] + \frac{\rho\theta_H V'\left[q^a_{HH}(\rho)\right]}{\rho + p_{LH}} = \theta_H V'\left[q^a_{HH}(\rho)\right],
\end{aligned}$$

hence, $q^a_{HH}(\rho) > q^{sb}_{HH}(\rho)$;

  – (36) implies that

$$\begin{aligned}
&\theta_H V'[q^{sb}_{HL}(\rho)] = c\\
=&\left(\frac{\theta_H p_{LH}}{\rho + p_{LH}}\right) V'\left(q^a_{HL}(\rho)\right) + \frac{\rho\left[(1-p_{LL})V'\left(\frac{q^a_{LH}(\rho)+q^a_{HL}(\rho)}{2}\right)\left(\theta_L - \frac{p^2_{LH}\Delta\theta}{p_{LL}p_{LH}+\rho}\right) - p_{HH}\theta_H V'\left(q^a_{HL}(\rho)\right)\right]}{2(\rho+p_{LH})p_{LH}}\\
>&\left(\frac{\theta_H p_{LH}}{\rho + p_{LH}}\right) V'\left(q^a_{HL}(\rho)\right) + \frac{\rho\left[(1-p_{LL})V'\left(q^a_{LH}(\rho)\right)\left(\theta_L - \frac{p_{HH}\Delta\theta}{p_{LH}}\right) - p_{HH}\theta_H V'\left(q^a_{HL}(\rho)\right)\right]}{2(\rho+p_{LH})p_{LH}}\\
=&\left(\frac{\theta_H p_{LH}}{\rho + p_{LH}}\right) V'\left(q^a_{HL}(\rho)\right) + \frac{\rho\left[1-p_{LL}-p_{HH}\right]\theta_H V'\left(q^a_{HL}(\rho)\right)}{2(\rho+p_{LH})p_{LH}} = \theta_H V'\left(q^a_{HL}(\rho)\right)
\end{aligned}.$$

It follows that $q^a_{HL}(\rho) > q^{sb}_{HL}(\rho)$;

– If ρ is close enough to zero, it follows from (37) and (38) that

$$\theta_L V'[q_{LH}^{sb}(\rho)] = c = \left(\frac{\theta_L p_{LH} - p_{HH}\Delta\theta}{\rho + p_{LH}}\right) V'\left(q_{LH}^a(\rho)\right)$$
$$+ \frac{\rho\left[(1-p_{LL})V'\left(\frac{q_{LH}^a(\rho)+q_{LH}^a(\rho)}{2}\right)\left(\theta_L - \frac{p_{LH}^2\Delta\theta}{p_{LL}p_{LH}+\rho}\right) - p_{HH}\theta_H V'\left(q_{LH}^a(\rho)\right)\right]}{2(\rho+p_{LH})p_{LH}}$$
$$< \theta_L V'[q_{LH}^a(\rho)]$$
$$,$$
$$\theta_L V'[q_{LL}^{sb}(\rho)] = c$$
$$= \left[\frac{\theta_L p_{LH} - p_{HH}\Delta\theta}{\rho + p_{LH}} + \frac{\rho\theta_H}{(\rho + p_{LH})\, p_{LL}} - \frac{\rho(1-p_{LL})\left(\theta_L - \frac{p_{LH}^2\Delta\theta}{p_{LL}p_{LH}+\rho}\right)}{p_{LL}(\rho + p_{LH})}\right] V'\left(q_{LL}^a(\rho)\right)$$
$$< \theta_L V'\left(q_{LL}^a(\rho)\right)$$

Therefore, $q_{LL}^a(\rho) < q_{LL}^{sb}(\rho), q_{LH}^a(\rho) < q_{LH}^{sb}(\rho)$.

- *The continuity of quantities* can be verified directly from expressions (39)-(42)and (35)-(38).

- *The continuity of surplus functions.* From (88) and (89), we get

$$\lim_{\rho\downarrow 0}\Pi^a(\rho) = \left[\frac{p_{LH}+p_{HH}}{p_{LH}}\beta_3 - \beta_2 - c\sum_{i\in\{H,L\}}\sum_{j\in\{H,L\}} p_{ij}q_{ij}\right]_{\mathbf{q}=\lim_{\rho\downarrow 0}\mathbf{q}^a(\rho)}$$

$$\lim_{\rho\uparrow 0}\Pi^a(\rho) = \left[\frac{p_{LH}+p_{HH}}{p_{LH}}\beta_3 - \beta_2 - c\sum_{i\in\{H,L\}}\sum_{j\in\{H,L\}} p_{ij}q_{ij}\right]_{\mathbf{q}=\lim_{\rho\uparrow 0}\mathbf{q}^a(\rho)}$$

$$\Pi^a(0) = \left[\frac{p_{LH}+p_{HH}}{p_{LH}}\beta_3 - \beta_2 - c\sum_{i\in\{H,L\}}\sum_{j\in\{H,L\}} p_{ij}q_{ij}\right]_{\mathbf{q}=\mathbf{q}^a(0)}$$

Since $\mathbf{q}^a(0) = \arg\max_{\mathbf{q}\in\mathbb{R}_+^4}\left[\frac{p_{LH}+p_{HH}}{p_{LH}}\beta_3 - \beta_2 - c\sum_{i\in\{H,L\}}\sum_{j\in\{H,L\}} p_{ij}q_{ij}\right]$ and $\lim_{\rho\downarrow 0}\mathbf{q}^a(\rho) \neq \mathbf{q}^a(0)$, $\lim_{\rho\uparrow 0}\mathbf{q}^a(\rho) = \mathbf{q}^a(0)$, it follows that $\lim_{\rho\downarrow 0}\Pi^a(\rho) < \Pi^a(0)$, $\lim_{\rho\uparrow 0}\Pi^a(\rho) = \Pi^a(0)$. (43) is thus verified.

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